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Universal singularities of anomalous diffusion in the Richardson class

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Inhomogeneous environments are rather ubiquitous in nature, often implying anomalies resulting in deviation from Gaussianity of diffusion processes. While sub- and superdiffusion are usually due to contrasting environmental features (hindering or favoring the motion, respectively), they are both observed in systems ranging from the micro- to the cosmological scale. Here we show how a model encompassing sub- and superdiffusion in an inhomogeneous environment exhibits a critical singularity in the normalized generator of the cumulants. The singularity originates directly and exclusively from the asymptotics of the non-Gaussian scaling function of displacement, and the independence from other details confers it a universal character. Our analysis, based on the method first applied in [A. L. Stella *et al.*, arXiv:**2209.02042** (2022) – accepted by Phys. Rev. Lett.], shows that the relation connecting the scaling function asymptotis to the diffusion exponent characteristic of processes in the Richardson class implies a nonstandard extensivity in time of the cumulant generator. Numerical tests fully confirm the results.

Anomalous spatial diffusion occurs when the mean squared displacement $\langle x^2 \rangle \sim t^{2\nu}$ grows non-linearly in time t, yielding by definition subdiffusion for $\nu < 1/2$ and superdiffusion when $\nu > 1/2$ [1]. Deviations from normal diffusion ($\nu = 1/2$) are often found in nature in systems ranging from microscopic to cosmological scales [2]. Subdiffusion ($\nu < 1/2$) is commonly observed in the biological contexts of particles moving inside living cells nuclei, cytoplasm and across membranes [1, 3–13]. Superdiffusion ($\nu > 1/2$) is also rather ubiquitous. It is found in active intracellular transport [14–17], migration processes of cells [18] and more complex organisms and animals [13, 19–22], as well as in the contexts of target search processes [23], particle dispersion in turbulent fluids [24–26], and cosmic rays transport [27, 28].

In many experimental scenarios exhibiting anomalous diffusion [1] the probability density function (PDF) $p(x, t)$ of displacement x satisfies at long times t

$$p(x, t) \sim t^{-\nu} f(x/t^{\nu}) \ , \tag{1}$$

where the scaling function $f(\cdot)$ has a non-Gaussian shape for $\nu \neq 1/2$ [2]. This type of behavior has also been analytically established and numerically conjectured in various models [29–31] and implies an anomalous scaling of displacement in time [32]. This means that, as a consequence of non-Gaussianity, with $f(\cdot)$ integrable on the real axis $\mathbb{R}$ and decaying to zero sufficiently fast for large absolute argument, the n-th order cumulant of displacement diverges as $t^{n\nu}$ for $t \to \infty$. Indeed, this cumulant can be obtained by n-th order differentiation with respect to λ at $\lambda = 0$ of

$$\begin{aligned} \log G(\lambda, t) = & \ \log \left(\int_{\mathbb{R}} dx e^{\lambda x} P(x, t) \right) \\ \sim & \ \log \left(\int_{\mathbb{R}} dz e^{\lambda t^{\nu} z} f(z) \right) \ , \end{aligned} \tag{2}$$

where G is a moment generating function, and we put $z = x/t^{\nu}$. In Ref. [33] it was shown that for a variety of models with non-Gaussian scaling f can be proven [34] to have the asymptotic (large $|z|$) shape:

$$f(z) \sim |z|^{\psi} e^{-c|z|^{\delta+1}} \tag{3}$$

for some positive constant c and exponents δ and ψ, which for the paradigmatic continuous time random walk (CTRW) model [35] was verified exactly.

Two known classes of anomalous diffusion processes, determined through specific relations between the exponents δ and ν, are expected to exhibit the stretched exponential decay in Eq. 3 [32, 33]. The Fisher class is characterized by the relation $\delta = \nu/(1-\nu)$, first established in the context of polymers with excluded volume in equilibrium [36], while the Richardson class relation, $\delta = (1-\nu)/\nu$, stems from a seminal paper dealing with particles dispersion in turbulent fluids [37]. The latter is expected to apply when diffusion steps have certain dependencies on space position [38].

Anomalous scaling is also directly responsible for universal features of diffusion processes [33]. By the Laplace

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estimate, the generating function $G(\lambda, t)$ can be shown to grow asymptotically as $\sim \exp(t^{\zeta}\varepsilon(\lambda))$ for some $\zeta > 0$, defining a scaling cumulant generating function (SCGF)

$$\varepsilon(\lambda) = \lim_{t\to\infty} \frac{1}{t^{\zeta}} \log G(\lambda, t) \ , \tag{4}$$

which exhibits a power-law singularity $\propto |\lambda|^{\frac{1+\delta}{\delta}}$ around $\lambda = 0$ [33]. Universality of the singular behavior is expected since the derivation shows that the singularity is determined by the asymptotic large $|z|$ behavior of the scaling function, which can be common to different processes. Of such universality, the model we are going to consider below (Eq. 5), provides an explicit example. The exponent ζ in Eq. 4 determines the extensivity in time of the logarithm of the generating function. The Fisher class is consistent with a standard definition of the SCGF, in which $\log(G)$ is simply divided by t in Eq. 4 (hence $\zeta = 1$). This extensivity in time reminds the extensivity in size one encounters when dealing with equilibrium critical phenomena, so that the $t \to \infty$ limit yields the analogue of a difference of equilibrium free energy densities, with time playing the role of size [39, 40]. Indeed, the whole discussion of the consequences of anomalous scaling presented in the case of diffusion in the Fisher class [33] can be applied also to critical systems in equilibrium. Consider for example a finite Ising model on a regular lattice box (in two or more dimensions) with N spins at the critical temperature and in zero magnetic field. The role of displacement is played by the total magnetization, which, normalized by an appropriate power of N (acting as time) becomes a continuous variable analogous to our z in the $N \to \infty$ limit. The probability distribution of the total magnetization obeys a scaling with N of the form in Eq. 1, and the scaling function is not known exactly, but a behavior like in Eq. 3 has been conjectured [41, 42]. As we show below, for the Richardson class the method foresees a non-standard extensivity in time and the necessity to divide the generator by a power t^{ζ}, with $\zeta \neq 1$ depending on the diffusion exponent [33]. In spite of the different extensivity involved, also our derivation for Richardson processes should be regarded as a way of establishing a parallel between equilibrium criticality and dynamics [33], according to a general strategy on which much of our understanding of non-equilibrium is based [43–45].

The approach of Ref. [33] was explicitly applied and shown to predict exact results for the continuous time random walk (CTRW) model and fractional drift diffusion equations [4, 35, 46]. Both free and biased models exhibited sub-diffusion, while only in the biased case super-diffusion could be encompassed. Moreover, all such applications implied adoption of standard extensivity of the cumulant generator ($\zeta = 1$ in Eq. 15), as appropriate for processes in the Fisher class. It remains an open issue to test the validity of this analysis for processes belonging to the Richardson class and possibly displaying both sub- and super-diffusion regimes. The present work is devoted to the exploration of a specific diffusion model with both such features.

The process we consider in this work was introduced in Ref. [38] to model a scenario of inhomonogenous diffusion, in which the diffusion constant has an explicit dependence on the position [47–50]. We show how this model can exhibit anomalous scaling at all times, implying that Eq. 1 holds as an equality. However, unlike in the case of the CTRW model a direct analytical evaluation of the SCGF is not feasible for this process. We show how the method of Ref. [33] allows to circumvent this problem and to correctly estimate the leading singular term of the SCGF, proven to abide by a non-trivial Richardson-like extensivity. We highlight the existence of a universal singularity for the SCGF, as in the case of CTRW and fractional diffusion equations. Through large deviation theory [43, 44] we show how the PDF in the long-time limit is modulated by a non standard singular rate function, related to the extensivity t^{ζ} of the SCGF (Eq. 4). Ultimately, numerical evaluations of the integrals in the asymptotic regime corroborate the correctness of the predictions of the method first implemented in Ref. [33].

Following Ref. [38] we start from the Langevin equation for a particle moving on a one-dimensional axis:

$$\frac{dx}{dt} = \sqrt{2D(x)}\xi(t) \tag{5}$$

where ξ is a δ-correlated white Gaussian noise $\langle \xi(t)\xi(t')\rangle = \delta(t-t')$, while the diffusion coefficient has a power-law spatial dependence $D(x) = D_0|x|^q$ for some $D_0 > 0$ and any $q < 2$. Adopting Stratonovich prescription, the corresponding Fokker-Plank equation is:

$$\partial_t p(x,t) = \partial_x \left[\sqrt{D(x)}\partial_x \left[\sqrt{D(x)} p(x,t)\right]\right] \tag{6}$$

Given an initial condition $p(x, t = 0) = \delta(x)$, the probability density function regulating the process can be shown to be [38]

$$p(x,t) = \frac{|x|^{-q/2}}{\sqrt{4\pi D_0 t}} e^{-\frac{|x|^{2-q}}{(2-q)^2 D_0 t}} \tag{7}$$

yielding a mean squared displacement

$$\langle x^2(t)\rangle = \frac{\Gamma\left(\frac{6-q}{2(2-q)}\right)}{\pi^{1/2}} (2-q)^{\frac{4}{2-q}} (D_0 t)^{\frac{2}{2-q}} \tag{8}$$

where $\Gamma(\cdot)$ is the complete Gamma function. It is therefore clear how this model provides subdiffusion in the case $q < 0$ and superdiffusion for $0 < q < 2$, with the following relation connecting the spatial dependence of the diffusion constant with the diffusion exponent ν:

$$\nu = \frac{1}{2-q} \ . \tag{9}$$

The PDF of the process can be easily seen to abide by the scaling form of Eq. 1 with

$$f(z) = \frac{|z|^{\frac{1-2\nu}{2\nu}}}{\sqrt{4\pi D_0}} e^{-\frac{\nu^2 |z|^{1/\nu}}{D_0}} \tag{10}$$

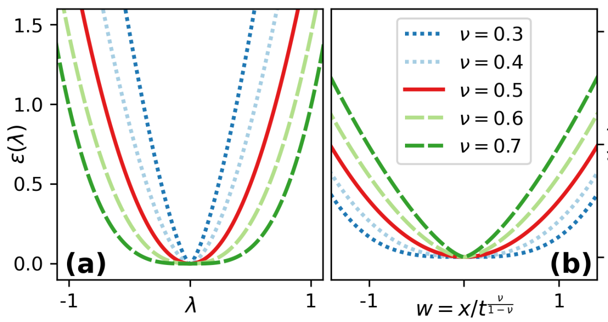

FIG. 1. Examples of SCGFs $\varepsilon(\lambda)$ (a) and rate functions $I(w)$ (b) for different regimes of anomalous diffusion: subdiffusion (dotted blue shades), superdiffusion (dashed green shades) and normal diffusion (solid red). Both exhibit the expected power-law singularity predicted in Eqs. 15 and 17 for $\lambda = 0$ and $w = 0$, respectively.

as scaling function, where we remind that $z = x/t^\nu$. We stress again that the scaling in Eq. 7 holds exactly at all times, not only asymptotically as requested by Eq. 1. Another remarkable fact is that the behavior of the scaling function in Eq. 3 holds on the whole z axis. It can be shown that both these circumstances are determined by the particular initial condition chosen for the process [51]. Setting $p(x, 0) = \delta(x - x_0)$ with some nonzero x_0 would lead to the validity of the scaling form in Eqs. 7 and 10 only for large t and large $|z|$ [51, 52].

For every $0 < \nu < 1$ the generating function of the moments can be found through the two-sided Laplace transform $G(\lambda, t) = \int_{-\infty}^{+\infty} dx\; e^{\lambda x} p(x, t)$ [53], which in terms of the rescaled displacement z reads:

$$G(\lambda, t) = \frac{1}{\sqrt{4\pi D_0}} \int_{-\infty}^{+\infty} dz\; |z|^{\frac{1-2\nu}{2\nu}} e^{\lambda z t^\nu - \frac{\nu^2 |z|^{1/\nu}}{D_0}} \tag{11}$$

An exact evaluation of this integral for long t is not feasible, so that application of the Laplace's maximization method of Ref. [33] for its estimate, besides being suggested by the form of the tails, appears mandatory.

As time increases, the integrand in Eq. 11 concentrates around some specific value $\bar{z}$ that maximizes the argument of the exponential. Separating the analysis for positive and negative values of z we find

$$\bar{z} = \mathrm{sgn}(\lambda) \left(\frac{1}{\nu} D_0 |\lambda| t^\nu \right)^{\frac{\nu}{1-\nu}} \tag{12}$$

where $\mathrm{sgn}(\cdot)$ represents the sign function, implying that $\bar{z}$ and λ have the same sign. Moreover, for long times $\bar{z}$ diverges to $+\infty$ and $-\infty$ as a power of t for $\lambda > 0$ and $\lambda < 0$, respectively. Substituting such value in the exponential form and performing the Gaussian integration centered in $\bar{z}$ allows to obtain asymptotically [33]

$$\begin{aligned} \log G(\lambda, t) = \quad & \lambda t^\nu \bar{z} - \tfrac{\nu^2}{D_0} \bar{z}^{1/\nu} + \\ & + \tfrac{1}{2} \log\!\left(\tfrac{1}{2(1-\nu)}\right) + \mathcal{O}(\bar{z}^{-1/\nu}) \end{aligned} \tag{13}$$

where a term proportional to $\log \bar{z}$ turns out to have prefactor equal to zero. The cancellation of this term $\propto \log \bar{z}$ is due to the fact that, with reference to the notations adopted in Eq. 3, the exponents characterizing the tails of $f(z)$ satisfy $\psi = (\delta - 1)/2$, which is also valid for the cases of anomalous diffusion studied in Ref. [33].

Taking into account Eq. 12, we can eventually write

$$\begin{aligned} \log G(\lambda, t) &= (1-\nu) \left(\tfrac{D_0}{\nu} t |\lambda|^{1/\nu} \right)^{\frac{\nu}{1-\nu}} + \\ &\quad + \tfrac{1}{2} \log\!\left(\tfrac{1}{2(1-\nu)}\right) + \mathcal{O}(t^{-\frac{\nu}{1-\nu}}) \end{aligned} \tag{14}$$

implying an extensivity appropriate for the Richardson class [37] with $\zeta = \nu/(1-\nu)$. Consequently, a SCGF can be defined as

$$\varepsilon(\lambda) = \lim_{t \to \infty} \frac{\log G(\lambda, t)}{t^{\frac{\nu}{1-\nu}}} = (1-\nu) \left(\frac{D_0}{\nu} \right)^{\frac{\nu}{1-\nu}} |\lambda|^{\frac{1}{1-\nu}} \tag{15}$$

which exhibits a power-law singularity of order $1/(1-\nu)$ around $\lambda = 0$ as shown above (Fig. 1a), implying a divergence of the n-th derivative as soon as n exceeds $1/(1-\nu)$. In the case $\nu = 1/2$ the SCGF of the free Brownian diffusion is recovered, finding also consistency with the SCGF of a free Markovian (memory-less) CTRW [54, 55].

In Eq. 14 appears a constant term $C(\nu) = -\frac{1}{2} \log(2(1-\nu))$ independent of time, which is negative for sub-, positive for super- and zero for normal diffusion. In the context of equilibrium critical phenomena this type of term was obtained in [41] by applying a Laplace maximization method to an integral analogous to the one we used for Eq. 11. This integral was expected to express, for an N spin Ising model at the critical temperature and zero magnetic field, the so called Privman-Fisher anomaly [56, 57], i.e. the N-independent term of the total free energy of interest in the context of finite size scaling theory [39]. The analogy of the calculation follows from the fact that, as mentioned above, the large argument behavior of the scaling function of the total magnetization with size was postulated to have the same form derived in Ref. [33] for the displacement and given in Eq. 3. The role of λ in Eq. 2 was played there by an auxiliary nonzero magnetic field. In our context time takes the place of size, but it appears remarkable that the constant term $C(\nu)$ is nonzero only in case anomalous scaling holds ($\nu \neq 1/2$) and its sign marks a distinction between super- and subdiffusion. The parallel of the approach of Ref. [33] with studies of anomalous scaling in equilibrium critical phenomena certainly acquires motivation for deeper investigation in light of the presence of this analogue of Privman-Fisher term.

Integration of our results within the framework of large deviation theory [43, 44] shows how the singularity of the SCGF translates into a singularity of the rate function $I(w)$ modulating the probability of observing fluctuations of the rescaled position $w = x/t^{\frac{\nu}{1-\nu}}$ [33]. In the case of normal diffusion ($\nu = 1/2$), w has the meaning of a velocity, while for $\nu < 1/2$ and $\nu > 1/2$ can be interpreted as a sub- and super-velocity, respectively. For simplicity, we will refer to w as an "anomalous velocity" in this

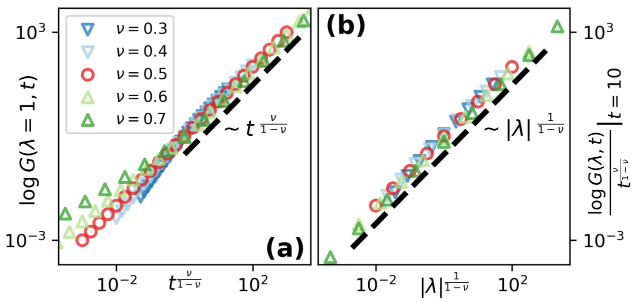

FIG. 2. (a) Numerical evaluation of the cumulant generating function $\log G(\lambda = 1, t)$ for different values of ν (including sub-, normal and superdiffusion). Plotting against the rescaled time $t^{\frac{\nu}{1-\nu}}$ shows an excellent collapse already at times $t > 1$. (b) Numerical evaluation of the SCGF through the normalized cumulant generating function $t^{-\frac{\nu}{1-\nu}} \log G(\lambda, t)$ at $t = 10$, hinting the presence of a Richardson kind of scaling for the cumulants. An excellent collapse for 6 decades hints that the SCGF $\varepsilon(\lambda) \sim |\lambda|^{1/(1-\nu)}$, implying a power-law singularity of such order around $\lambda = 0$.

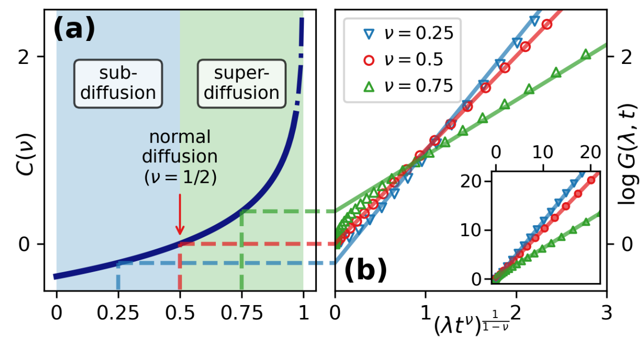

FIG. 3. (a) Constant term $C(\nu) = -\frac{1}{2}\log(2(1-\nu))$ appearing in $\log G$ as a result of the Laplace approximation (Eq. 14). The constant is negative for sub-diffusion ($\nu < 1/2$) and positive for super-diffusion ($\nu > 1/2$) while it is zero only for normal diffusion ($\nu = 1/2$), in agreement with the fact that in the last case the scaling function f is Gaussian-shaped and the Laplace approximation becomes exact. (b) Numerical integration of Eq. 11 shows a linear dependence of $\log G(\lambda, t)$ for large values of $(\lambda t^{\nu})^{\frac{1}{1-\nu}}$ (inset). Examples for sub- (blue down-pointing triangles), normal (red circles), and super- (green up-pointing triangles) diffusion are provided. A linear-fitting of the asymptotic part returns an intercept that matches very accurately the constant $C(\nu)$, showing that our Laplace approximation is able to capture exactly an analogue of the Privman-Fisher term.

manuscript. The probability of observing a certain deviation from the typical value $w = 0$ – expected given the absence of any form of drift in the model – in the long-time limit follows a large deviation principle

$$p(x/t^{\frac{\nu}{1-\nu}} = w, t) \sim e^{-t^{\frac{\nu}{1-\nu}} I(w)} . \qquad (16)$$

The convexity and differentiability of the SCGF (Eq. 15) ensures the validity of the Gärtner-Ellis theorem [58, 59] which allows to express the rate function as Legendre-Fenchel transform of ε [60, 61]:

$$I(w) = \sup_{\lambda \in \mathbb{R}} [w\lambda - \varepsilon(\lambda)] = \frac{\nu^2 |w|^{1/\nu}}{D_0} \qquad (17)$$

Thus, the anomalous scaling induces a singular behavior in the rate function (Fig. 1b), as already observed for processes in the Fisher class [33]. It is worth to stress here that the above result showing the consequences of anomalous scaling of the displacement distribution on the rate function is not related to what in the recent literature is referred to as "anomalous scaling of dynamical large deviations" [62–64]. Indeed, by this last expression the authors refer to situations in which the exponential decay of the PDF in Eq. 16 occurs with a power of time different from the one needed to obtain the normalized observable w.

Finally, let us validate all the above results with numerical calculations. Contrary to the CTRW and fractional drift diffusion examples presented in Ref. [33], this inhomogeneous diffusion model does not allow for an exact evaluation of the cumulant generating function $\log G$. The integral defining the generating function in Eq. 11 cannot be expressed in terms of explicit functions for any arbitrary value of the diffusion exponent $0 < \nu < 1$. Therefore, we need to proceed with a numerical estimation of such integral and extrapolate from the results its asymptotic dependence on time to verify that the extensivity of the cumulant generating function is the one predicted for the Richardson class. In Fig. 2a we report the numerical evaluation of $\log G(\lambda = 1, t)$ as a function of time, for different diffusion exponents ranging from $\nu = 0.3$ (sub-diffusion) to $\nu = 0.7$ (super-diffusion) including the case of normal diffusion $\nu = 1/2$. Plotting $G(\lambda, t)$ against $t^{\nu/(1-\nu)}$ in log-log scale, shows an excellent collapse on the bisector line already for $t \sim 1$, quickly consolidating as time increases. This corroborates the validity of the approach in estimating an extensivity of the Richardson class through the Laplace method (Eq. 14).

This result hints that for large enough times one should be able to normalize the cumulant generating function over $t^{\nu/(1-\nu)}$ and obtain a finite SCGF for all values of λ (Eq. 15). We do so by evaluating numerically $\log G(\lambda, t)$ at $t = 10$ as a function of the Laplace variable λ, again for different values of ν encompassing sub-, normal and super-diffusion. Normalizing such integral over $t^{\nu/(1-\nu)}$ as suggested by the previous analysis, we obtain an estimation of the SCGF, which is formally reached only in the $t \to \infty$ limit. Plotting in log-log scale against $\lambda^{1/(1-\nu)}$ (Fig. 2b) we find a perfect collapse on the bisector line for all values of λ, simultaneously corroborating the full shape of the SCGF predicted in Eq. 15 and the existence of power-law singularities in the origin as those reported in Fig. 1.

It is of particular interest also to check the consistency of our Laplace estimate of the analogue of the Privman-Fisher term $C(\nu)$ in Eq. 14 with the numerical evaluation of G. In Fig. 3b we plot the result of numerical integrations of the generating function of the cumulants (logarithm of Eq. 11) against $(\lambda t^\nu)^{\frac{1}{1-\nu}}$ for values of ν providing different diffusive regimes. A linear slope is expected for large values of the ordinate (shown in the inset), as predicted by the leading order term produced by the Laplace approximation (Eq. 14). Remarkably, the intercept obtained by fitting such slope matches very accurately the constant term $C(\nu)$ obtained in the approximation, suggesting that our method not only allows to capture correctly the leading order singularities of the SCGF, but also yields an exact estimate of an analogue of the Privman-Fisher anomaly [56, 57]. We also note how, consistently with these results, in Fig. 2(a) we are able to appreciate how $\log G$ for short times approaches the bisector line from below (negative constant) for subdiffusive motions and from above (positive constant) for super-diffusive motions, while in the case of normal diffusion (zero-costant) the collapse holds at any time.

Summarizing, we showed that the method of Ref. [33] applies to a diffusion process in the Richardson class, predicting correctly the nonstandard extensivity in time of the cumulants generator $\log G(\lambda, t)$ and the singularity of the scaling cumulant generating function $\varepsilon(\lambda)$ in the Laplace variable λ. The model considered is remarkable in several respects. In first place it satisfies scaling for all t and presents the form in Eq. 1 of the scaling function on the whole z axis. The fact that these properties become only asymptotic for initial conditions different from $p(x, 0) = \delta(x)$ provides a concrete example of the way universality mechanisms operate in the approach. Indeed, the results of Ref. [51] allow to easily verify that adoption of $p(x, 0) = \delta(x - x_0)$ leaves scaling valid for $t \to \infty$ with the same form of scaling function at large $|z|$. Thus, the leading singular behavior does not change for these modified initial conditions [33]. Another remarkable feature of the model is the simple ν-dependent form of the analogue of the Privman-Fisher term, which distinguishes with its sign between sub- and super-diffusion. Once verified that the approach of Ref. [33] works successfully for processes in both the Fisher and the Richardson class, it is legitimate to ask if, in view of its flexibility, the range of applications could encompass also diffusions outside these classes. The formalism leading to equations like Eq. 13 in fact leaves room for different relations linking ν and δ, only at the cost of adjusting the extensivity in time of $\log G$. The exploration of such possibilities, or a deeper understanding of the reason why Fisher and Richardson relations play a special role is left for future investigations.

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