Interconnections between networks acting like an external field in a first-order percolation transition

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Interconnections between networks act like an external field in a first-order percolation transition

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Many interdependent, real-world infrastructures involve interconnections between different communities or cities. Here we show how the effects of such interconnections can be described as an external field for interdependent networks experiencing a first-order percolation transition. We find that the critical exponents γ and δ, related to the external field can also be defined for first-order transitions but that they have different values than those found for second-order transitions. Surprisingly, we find that both sets of different exponents (for first and second order) can even be found within a single model of interdependent networks, depending on the dependency coupling strength. Nevertheless, in both cases both sets satisfy the Widom's identity, $\delta - 1 = \gamma/\beta$ which further supports the validity of their definitions. Furthermore, we find that both Erdős-Rényi and scale-free networks have the same values of the exponents in the first-order regime implying that these models are in the same universality class. In addition, we find that in k-core percolation the values of the critical exponents related to the field are the same as for interdependent networks suggesting that these systems also belong to the same universality class.

I. INTRODUCTION

In the past two decades, network theory has successfully described and predicted collective phenomena of many complex systems such as the brain, climate and infrastructures [1–4]. The resilience of such networks is often studied under a percolation process where a fraction $1-p$ of nodes are removed randomly from the network and the size of the largest connected component, S (the order parameter of the system) is measured [5–10]. Several generalizations of percolation-like processes have also been developed and these processes also affect the values of the critical exponents, and the nature of the transition e.g., whether it is continuous or abrupt. Specifically, *interdependent* networks [11–15] where one network depends on another, have drawn much interest. In these systems there exist several networks with the ordinary connectivity links within the networks, yet dependency links between the networks imply that if a node at one end of a dependency link fails than the node at the other end will also fail, even if it is still connected in its own network. This process leads percolation on interdependent networks to result in cascading failures resulting in abrupt, first-order percolation transitions.

The Ising model is one of the fundamental examples used for studying a system under the effect of an external field and it is characterized by an up-down symmetry [16]. In the absence of an external field as the system cools down and goes from the disordered to the ordered phase, a spontaneous symmetry breaking to the up or down states occurs at a critical temperature. However, this symmetry can be broken by an external field. As the system heats up and the system goes from the ordered to the disordered phase an external field can keep the system ordered even above the critical temperature. In contrast, percolation does not have such symmetry and as the system crosses the critical point an infinite cluster emerges. Nonetheless, an analogy to the external field can be found in term of keeping the system in the ordered phase even at the critical point.

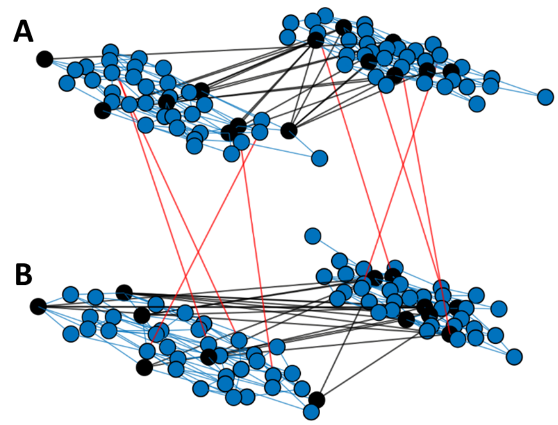

FIG. 1: **Model illustration.** The model is composed of two networks A and B, each contains two modules (blue nodes) with a small fraction r of nodes (called here the core) that have interlinks connecting the two modules (black nodes). M_{inter} links are assigned randomly between the cores of the modules (black links). The two networks A and B depend on each other via a fraction q of dependency links shown in red. The dependency links are assumed to be within the same community in each network.

Several researchers have studied 'interconnected networks' where two networks each with many connections inside their own network, also have a smaller number of links between them [17–23]. Such networks have also been described in the literature as networks with community structure, since each network can be regarded as a separate community [24–27].

Recently, a new realistic model of community structure has been proposed where only some small fraction, r, of nodes are assumed *a priori* to be capable of having interlinks to other communities [28]. This model is realistic for cases where additional resources are needed at a node in order to accommodate inter links. However, once such infrastructure exists, adding additional interlinks is of low cost. For example, international airports may need longer runways for transoceanic flights. Nonetheless, once these runways exist, adding more transoceanic flights is easy. Similarly, power stations that transfer large load to long distances may require additional infrastructure in order to handle such load. It was found that in this model, the fraction r affects the *continues second order* percolation transition analogously to an external field in spin systems or a ghost field in percolation.

The analogy to an external field is best characterized through the key exponents β, δ, and γ describing the behavior of the system near (and at) criticality [5, 29–31], which fulfill Widom's identity $\delta - 1 = \gamma/\beta$ implying that there are only 2 degrees of freedom in determining these exponents. We propose here to generalize these critical exponents for both *first and second order* phase transitions when the control parameter p is near (and at) the percolation threshold p_c, with respect to $S(0, p_c)$ that can be zero (in 2nd order) or non-zero (in 1st order):
i) The critical exponent β describes the behavior of the order parameter (S) near the critical point with zero-field ($r = 0$) and is given by

$$S(0, p) - S(0, p_c) \sim (p - p_c)^{\beta} \quad . \tag{1}$$

ii) *At the critical point,* ($p = p_c$), the increase of the order parameter with the magnitude of the field, r, is given by the critical exponent δ as

$$S(r, p_c) - S(0, p_c) \sim r^{1/\delta} \quad . \tag{2}$$

iii) The susceptibility of the system, χ, is given by the partial derivative of the order parameter with respect to the field, r, and scales near the critical point with the exponent γ as

$$\chi \equiv \left(\frac{\partial S(r, p)}{\partial r}\right)_{r \to 0} \sim |p - p_c|^{-\gamma} \quad . \tag{3}$$

Here we study analytically and via simulations the percolation of community structure in interdependent networks as shown in Fig. 1 (described later in detail) with q fraction of interdependent nodes i.e., $1 - q$ fraction of nodes in each network are autonomous. We observe two distinct regimes in this same model, characterizing different values of the critical exponents and universality classes. For small values of q, the network undergoes a continuous second-order phase transition as for isolated ER networks and has the corresponding critical exponent values. However, for large values of q this system undergoes a first-order phase transition with a different set of exponent values [15]. Moreover we find here that the fraction of interconnected nodes, r, also for the case of an abrupt transition, can be analogized to an external field.

II. MODEL

Our network model (demonstrated in Fig. 1) assumes two communities where only a small fraction r of nodes in each community are capable of having interlinks [28]. A total of M_{inter} links are then assigned among this small subset of nodes. Next, the two networks constructed in this model are set to be partially interdependent with q fraction of nodes in each network depending on nodes in the other network, as seen in Fig. 1.

III. RESULTS

A. Analytic solution

We begin by developing an analytic solution for the effect of interlinks on percolation of interdependent networks of the type described above. We start by defining the generating functions for the degree distribution of intra- and inter- connected nodes. For intra nodes we obtain $G_0^{intra}(x) = \sum_k p_k^{intra} x^k$ and $G_1^{intra}(x) = \sum_k q_k^{intra} x^k$ where p_k^{intra} is the probability for a node to have k intra links and $q_k^{intra} = \frac{(k+1)p_{k+1}^{intra}}{z}$ is the intra excess degree distribution with z being the average intra degree [10, 32]. We assume that interlinks are always assigned randomly and thus their generating functions are given by $G_0^{inter}(x) = G_1^{inter}(x) = e^{-\kappa(1-x)}$ where $\kappa = \frac{M_{inter}}{rN}$ is the average inter-degree of the r fraction of nodes in the core and N is the total number of nodes.

We next define u and v, the probability that after removal of $1-p$ fraction of nodes from *each* network, an intra and inter edge respectively do not lead to a node connected to the giant component. They satisfy the equations:

$$u = 1 - p\left[1 - G_1^{intra}(u)(1 - r + rG_0^{inter}(v))\right] \times$$
$$\times \left[1 - q + qp(1 - G_0^{intra}(u)(1 - r + rG_0^{inter}(v))\right],$$
$$v = 1 - p\left[1 - G_0^{intra}(u)G_1^{inter}(v)\right] \times$$
$$\times \left[1 - q + qp(1 - G_0^{intra}(u)G_0^{inter}(v))\right].$$

For ER networks (i.e., $p_k^{intra} = \frac{z^k e^{-z}}{k!}$), $G_0^{intra}(u) = G_1^{intra}(u) = e^{-z(1-u)}$ and $S = 1 - u$, leading to a single transcendental equation relating S, q, and r

$$\frac{1-q+2qp-\sqrt{(1-q)^2+4qS}}{2qp}e^{zS}+(r-1)=r\exp\left(\frac{\kappa p}{r}\left[(r-1)(e^{-zS}-1)\frac{1-q+\sqrt{(1-q)^2+4qS}}{2}-\frac{S}{p}\right]\right). \quad (4)$$

As seen in Fig. 2, for large values of q the system undergoes an abrupt first order transition while for small values of q (Fig. 2 inset and [28, 33]) it experiences a continuous second-order transition. We note that for high q (the first-order regime), even for $r > 0$, there is still an abrupt phase transition (Fig. 2). However, as r increases, we observe that at the value of $p_c(r = 0)$, there is a scaling behavior between $S(r, p)$ and r (see Fig. 2b,c,d and below) suggesting that r can be analogized to an external field. Two sets of different critical exponents arise from Eq. (4). For strong dependency (i.e. large values of q) we obtain $\delta = 2$ and $\beta = \gamma = 1/2$ while for weak coupling we find $\delta = 2$ and $\beta = \gamma = 1$ as in no coupling, see [28]. Both sets of critical exponents satisfy Widom's identity $\delta - 1 = \gamma/\beta$.

In Fig. 3, the two sets of different critical exponents for strong and weak dependency are obtained from numerical analysis of Eq. (4). Later we also present an analytic derivation of the two sets of exponents obtained from Eq. (4). It can be seen that for large values of q we obtain $\gamma = 1/2$ and the system undergoes an abrupt transition (Fig. 2). In contrast, for small values of q we obtain $\gamma = 1$ and the system undergoing a continuous second-order transition (Fig. 2 inset and [28, 33]). This shows that even in a single model (represented by a single equation (Eq. (4))) one can obtain both sets of exponents. For the case of removing $1-p$ fraction of nodes from *network A only*, an implicit equation similar to Eq. (4) can be obtained for the limits $q = 0, 1$ (See Appendix A and [28]). In order to be consistent with [12, 34], Figs. 2 and 4 are presented for this specific case.

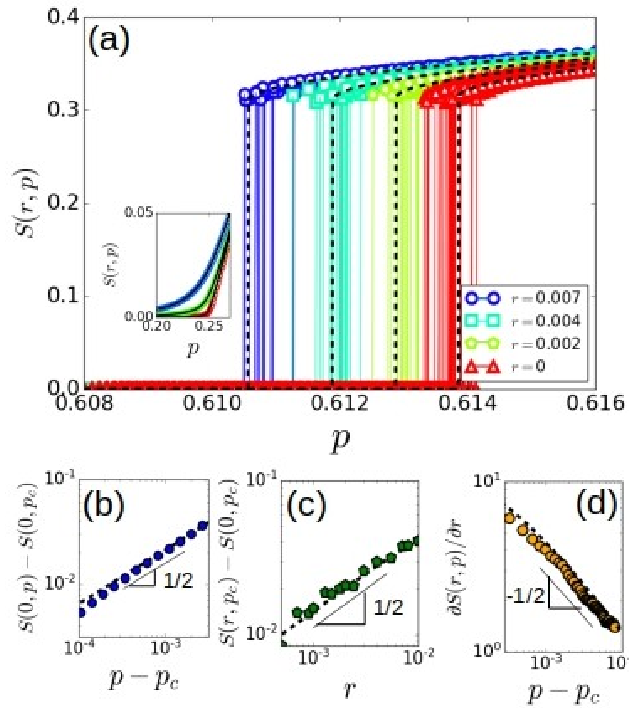

FIG. 2: **(a) The size of the mutual giant connected component $S(r, p)$ for interdependent networks with large q ($q = 1$) whose communities each have an ER structure.** Simulations of single realizations (colored connected symbols) and theory (dashed lines) are shown. At the critical point of the system with no field, $p_c(r = 0)$, we observe that as r increases, $S(r, p_c)$ increases as well. The power-law scaling with r of this increase suggests that r can be analogized to an external field. The inset shows the effect of the external field, r, for a single network composed of two communities and a continuous transition (not interdependent, i.e., $q = 0$). Simulations are shown for $N = 10^7$ nodes, mean degree $z = 4$ and $M_{inter} = N/20$. We find that the critical exponents have the values **(b)** $\beta = 1/2$, **(c)** $\delta = 2$ and **(d)** $\gamma = 1/2$ and the Widom's identity $\delta - 1 = \gamma/\beta$ is satisfied. The simulations are plotted with symbols and the theory as dashed lines. The simulations for γ are shown for $N = 10^8$ and $r = 0.0007$.

B. Analytic Derivation of Critical Exponents

Having solved the model and demonstrated that scaling relationships analogous to those of an external field can be found for first order transitions, here we seek to extract these scaling exponents analytically. To derive the exponents for ER networks, we let $f(S, p, r, q)$ be

$$f(S,p,r,q)=\left[(1-r)(1-e^{-zS})\cdot \cdot\frac{1-q+\sqrt{(1-q)^2+4qS}}{2}-\frac{S}{p}\right], \quad (5)$$

and thus Eq. (4) takes the form

$$r+\frac{2f(S,p,r,q)}{1-q+\sqrt{(1-q)^2+4qS}}= =r\exp\left[\frac{\kappa p f(S,p,r,q)}{r}-zS\right]. \quad (6)$$

In the limit of $r \to 0$, we recover $f(S, p, 0, q) = 0$ defining the giant component of two interdependent ER networks. Likewise, at criticality, $f_S(S_c, p_c, 0, q) = 0$, where

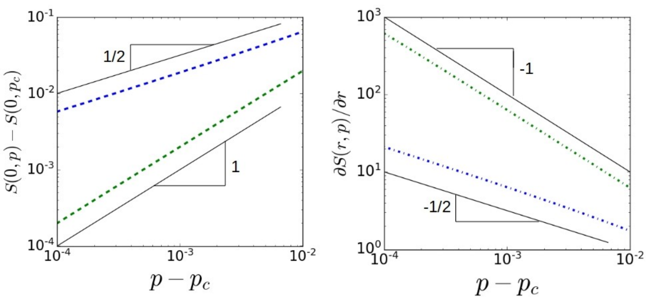

FIG. 3: **The critical exponents β (left) and γ (right) for $q = 0.2$ (green) and 0.8 (blue).** It can be seen that for small values of q we get $\beta = \gamma = 1$, while for large values of q we get $\beta = \gamma = 1/2$. Here $r = 10^{-8}$.

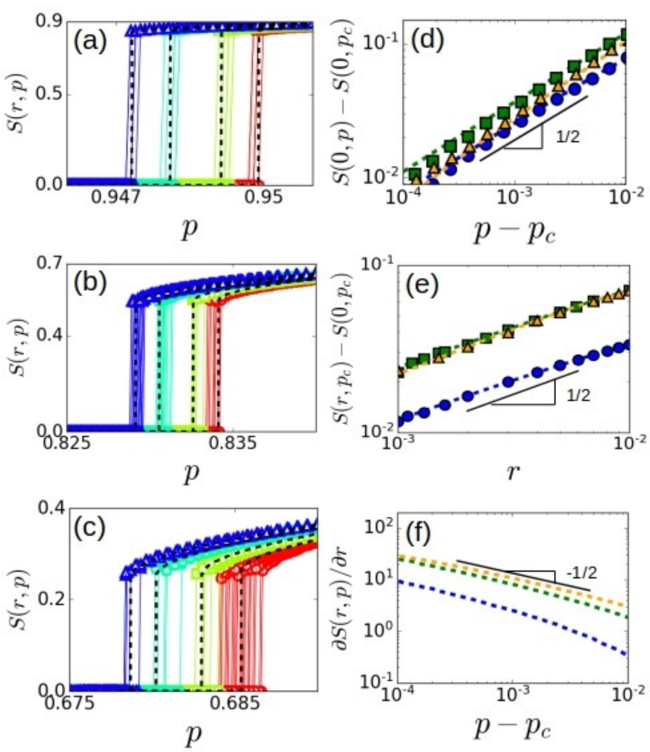

FIG. 4: **The size of the mutual giant connected component $S(r, p)$ for interdependent SF networks with large q ($q = 1$). (a)** $\lambda = 4.5$, **(b)** $\lambda = 3.35$ and **(c)** $\lambda = 2.8$. Here $r = 0.01$ (blue connected triangles), $r = 0.007$ (cyan connected pentagons), $r = 0.003$ (yellow connected squares) and $r = 0$ (red connected circles). The critical exponents **(d)** $\beta = 1/2$, **(e)** $\delta = 2$ and **(f)** $\gamma = 1/2$. Colors and symbols are $\lambda = 4.5$ - blue circles, $\lambda = 3.35$ - green squares and $\lambda = 2.8$ - orange triangles. In all cases β, δ and γ are the same as for ER (see Fig. 2) suggesting that for interdependent networks ER and SF are in the same universality class. Simulations are shown for $N = 10^7$ and $M_{inter} = N/20$. The theory in all the plots is shown as dashed lines. For γ the theory is shown for $r = 0.0001$.

f_S refers to the derivative of f with respect to S. Expanding $f(S, p, 0, q)$ around $S = S_c$ and $p = p_c$ gives

$$f(S,p,0,q) = f(S_c,p_c,0,q) + f_S(S_c,p_c,0,q)(S-S_c)+ \\ + f_p(S_c,p_c,0,q)(p-p_c) + ... = 0.$$

Further developing analytically this expansion leads to $\beta = 1/2$ for large q and $\beta = 1$ for small q. The exponents δ and γ require different approximations in the expansion of $f(S, p, r, q)$ in order to arrive at their values analytically (we find $\delta = 2$ and $\gamma = 1/2$ for large q and $\delta = 2$ and $\gamma = 1$ for small q), see Appendix B for the detailed derivation.

Simulations and theory for the critical exponents for ER networks with large q are in excellent agreement and are shown in Fig. 2b-d. The simulations support our analytical derivation that $\delta = 2$ and $\beta = \gamma = 1/2$. These values also satisfy the Widom's identity. Simulations for the critical exponents for $q = 0$ have been shown elsewhere [28] and also show excellent agreement with our finding that for small q, $\delta = 2$ and $\beta = \gamma = 1$. These values also satisfy Widom's identity.

Lee. et al Ref [33] found $\gamma = 1$ for large values of q using finite size scaling analysis of the susceptibility χ measured from the fluctuations. This finding seems to be in contrast to $\gamma = 1/2$ measured as a response to the field (Eq. (3)) found here. This might indicate that the fluctuation dissipation theorem (FDT) which has been studied widely [35–37] is violated. To resolve this question, we also measured for large q, γ directly based on the fluctuations as $\chi = N(\langle S^2\rangle - \langle S\rangle^2)$ with respect to the *meta-stable* state of *each* realization with respect its distance to p_c (see Zhou et al [38]) and find $\gamma = 1/2$, thus validating the FDT (see Appendix C). This demonstrates an interesting point how measuring γ in different ways could lead to two different exponent values which can be related to each other [38].

C. Scale free networks

Fig. 4 shows $S(r, p)$ for interdependent SF networks (i.e., $p_k^{intra} \sim k^{-\lambda}$) for different values of λ with large q, showing excellent agreement between the theory and the simulations. The critical exponents can also be measured and clear scaling relations are observed. For $\lambda > 4$ we have low heterogeneity and indeed as expected we find similar results as for ER networks (i.e, $\delta = 2$ and $\beta = \gamma = 1/2$). For high heterogeneity (i.e. $3 < \lambda < 4$ and $2 < \lambda < 3$) the system has different exponents than ER for the case of a single layer (corresponding to $q = 0$) [28], yet for interdependent networks for large q we find that the critical exponents are the same as for interdependent ER networks, $\delta = 2$ and $\beta = \gamma = 1/2$. Thus, our results suggest that interdependent ER networks and interdependent SF networks with large q are in the same universality class in contrast to small q at which the exponents are different [28]. The reasoning is most probably due to the fact that the random spread of damage due to interdependence does not distinguish between high and low degree nodes.

D. k-core percolation

To further assess the effects of an external field on first-order transitions, we also consider here the case of k-core percolation. The k-core percolation is an iterative process in which one removes randomly $1-p$ fraction of nodes as in regular percolation where nodes with less than k neighbors are also considered failed. Thus, in the final giant component all remaining nodes have at least k links to other surviving nodes [39–42] as illustrated in Fig. 5. We study k-core percolation on two communities where only a fraction r of nodes in each community are capable of having interlinks (See Fig. 5). M_{inter} links are assigned randomly between pair of nodes one from each community that are capable of having interlinks. We denote by z the average degree in each community and by κ the average degree between the communities. To solve k-core percolation on our model, we generalize the approach of *Dorogovtsev et al.* [39] for two networks or communities with r interconnected nodes. We denote the probabilities R and T that following an intra and inter edge respectively, we will not reach a node which is connected to the giant k-core. The probabilities R and T satisfy the coupled equations

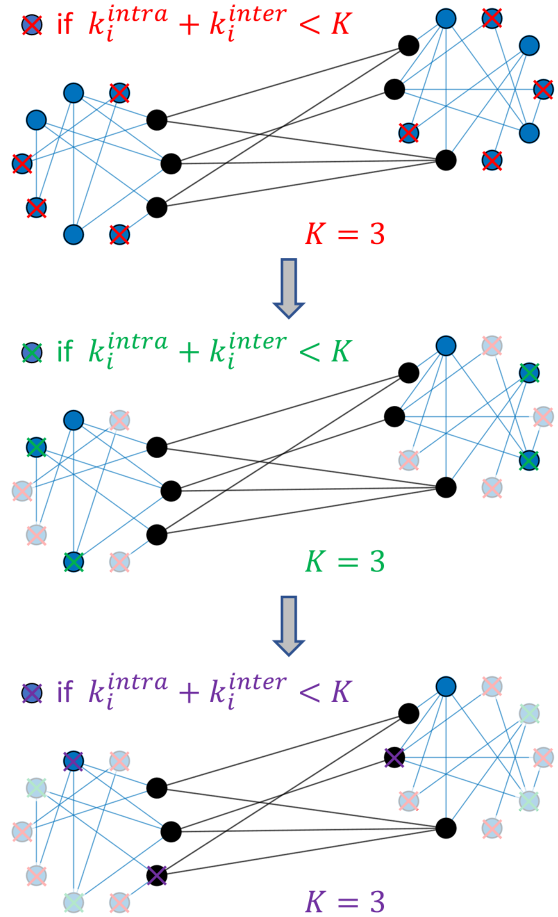

FIG. 5: **Model illustration - k-core.** The model is composed of two interconnected networks. The black nodes have both interlinks and intralinks while the blue nodes have only intralinks. In k-core percolation a node with degree smaller then k fail in iterative process until a final giant component remains.

$$R = 1 - p + p\sum_{n=0}^{k-2}\{(1-r)[\frac{(1-R)^n}{n!}\frac{d^n}{dR^n}G_1^{intra}(R)] + r\sum_{j=0}^{n}[\frac{(1-R)^j}{j!}\frac{d^j}{dR^j}G_1^{intra}(R)]\cdot \cdot[\frac{(1-T)^{(n-j)}}{(n-j)!}\frac{d^{(n-j)}}{dT^{(n-j)}}G_0^{inter}(T)]\} \quad , \quad (7)$$

and

$$T = 1 - p + p\sum_{n=0}^{k-2}\sum_{j=0}^{n}[\frac{(1-R)^j}{j!}\frac{d^j}{dR^j}G_0^{intra}(R)]\cdot \cdot[\frac{(1-T)^{(n-j)}}{(n-j)!}\frac{d^{(n-j)}}{dT^{(n-j)}}G_1^{inter}(T)]. \quad (8)$$

Here the generating functions for the intra and inter nodes are the ones defined in Sec IIIA. The size of the giant k-core will be:

$$S_k(r,p) = p\sum_{n=k}^{\infty}\left\{(1-r)\left[\frac{(1-R)^n}{n!}\frac{d^n}{dR^n}G_0^{intra}(R)\right] + r\sum_{j=0}^{n}[\frac{(1-R)^j}{j!}\frac{d^j}{dR^j}G_0^{intra}(R)]\cdot \cdot[\frac{(1-T)^{(n-j)}}{(n-j)!}\frac{d^{(n-j)}}{dT^{(n-j)}}G_0^{inter}(T)]\right\} \quad . \quad (9)$$

For simplicity we denote $\phi_a^n(R) = \frac{(1-R)^n}{n!}\frac{d^n}{dR^n}G_a^{intra}(R)$, $\psi_a^n(T) = \frac{(1-T)^n}{n!}\frac{d^n}{dT^n}G_a^{inter}(T)$ and $\phi_{ab}^n(R,T) = \sum_{j=0}^{n}\phi_a^j(R)\psi_b^{n-j}(T)$. Then Eqs. (7)-(9) become,

$$R = 1 - p + p\sum_{n=0}^{k-2}[(1-r)\phi_1^n(R) + r\phi_{10}^n(R,T)] \quad , \quad (10)$$

$$T = 1 - p + p\sum_{n=0}^{k-2} \phi_{01}^n(R,T) \quad (11)$$

and

$$S_k(r,p) = p\sum_{n=k}^{\infty}[(1-r)\phi_0^n(R) + r\phi_{00}^n(R,T)], \quad (12)$$

where ϕ_{00}^n and ϕ_0^n are the probabilities that a randomly selected node has n neighbors connected to the giant k-core if it is or is not interconnected, respectively. Here $a, b \in \{0, 1\}$. For the case of ER networks the generating functions are the same as the ones in the main text and thus,

$$\phi_a^n(R) = \psi_a^n(R) = \frac{(z(1-R))^n}{n!}e^{-z(1-R)}$$

and

$$\phi_{ab}^n(R,T) = \sum_{j=0}^{n} \frac{(z(1-R))^j}{j!}\frac{(\kappa(1-T))^{(n-j)}}{(n-j)!}e^{-z(1-R)-\kappa(1-T)}$$

In this case $\phi_0^n(R) = \phi_1^n(R)$ and $\phi_{00}^n(R,T) = \phi_{01}^n(R,T) = \phi_{10}^n(R,T)$.
Simulations and theoretical results for 3-core percolation on ER networks can be seen in Fig. 6a. Figs. 6b-d show that the critical exponents for k-core percolation are the same as those found for interdependent percolation (i.e. $\delta = 2, \beta = 1/2$ and $\gamma = 1/2$) which further suggests that interdependent percolation and k-core percolation are in the same universality class.

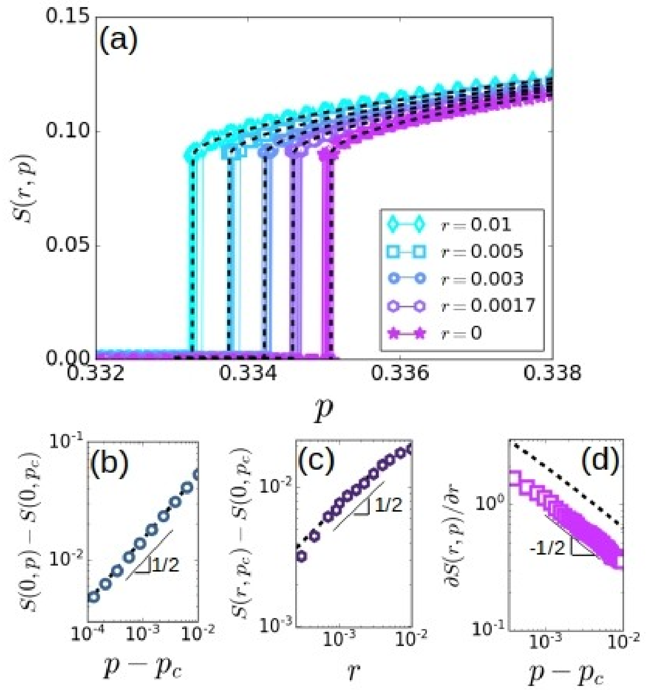

FIG. 6: **(a) The fractional size of the giant k-core $S(r,p)$ for interconnected ER networks with $k = 3$ and $z = 10$.** The theory (dashed lines) is compared with simulations of different realizations (colored connected symbols) for several values of r. Note that the limit of $r = 0$ is the result of ordinary k-core percolation ($k = 3$) on an ER network. The critical exponents: we find that **(b)** $\beta = 1/2$, **(c)** $\delta = 2$ and **(d)** $\gamma = 1/2$ which are the same as those we found for interdependent percolation. Note, also the Widom's identity $\delta - 1 = \gamma/\beta$ is satisfied.

IV. SUMMARY

In summary, we have shown the effects of an external field represented by interconnected nodes on first-order percolation phase transitions. This is done by analyzing analytically and numerically, interdependent networks with interconnections. We find that a *single* model of interdependent networks possesses two different sets of exponent values depending on the level of interdependence coupling, q. For high-values of q the critical exponents are the same for both ER and SF networks ($\delta = 2$ and $\beta = \gamma = 1/2$) suggesting a common universality class. Moreover, we find that k-core percolation has similar exponents suggesting that interdependent percolation and k-core percolation belong to a common universality class as well. These exponents satisfy Widom's identity $\delta - 1 = \gamma/\beta$ and their common value suggests the existence of a single universality class describing these cascading phenomenon. We hope our study will encourage more researchers to apply an external field in systems experiencing first order transitions and study their critical behaviour.

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Appendix A ANALYTIC DERIVATION FOR THE CASE OF REMOVAL FROM NETWORK A ONLY FOR $q = 1$

For the specific case of fully interdependent networks ($q = 1$), we consider the case of removal of $1 - p$ fraction of nodes from *network A only* in order to be consistent with [12, 34]. In this case u and v satisfy the equations [28]:

$$u = 1 - p + pG_1^{intra}(u)[1 - r + rG_0^{inter}(v)],$$
$$v = 1 - p + pG_0^{intra}(u)G_1^{inter}(v).$$

For the case of fully interdependent ER networks we use the framework from [12, 34] to arrive at the following equation for the mutual giant component,

$$e^{-zS}(r-1)+1-\sqrt{\frac{S}{p}}=$$
$$=r\exp\left[\frac{\kappa\sqrt{Sp}(e^{-zS}(r-1)+1-\sqrt{\frac{S}{p}}-r)}{r}-zS\right]. \quad \text{(A.1)}$$

Note that for $r=0$, Eq. (A.1) recovers the well-known result for two interdependent networks [12] $S=p(1-e^{-zS})^2$ and Eq. (4) in the main text is recovered with $p\to\sqrt{p}$ since there we removed p fraction of nodes from both networks. Fig. 2 in the main text shows excellent agreement between the theory of Eq. (A.1) and the simulations.

Appendix B CRITICAL EXPONENTS - ANALYTIC DERIVATION FOR PARTIAL INTERDEPENDENT NETWORKS

In the main manuscript, we defined $f(S,p,r,q)$ which allows us to analytically find the critical exponents

$$f(S,p,r,q)=\Bigg[(1-r)(1-e^{-zS})\cdot$$
$$\cdot\frac{1-q+\sqrt{(1-q)^2+4qS}}{2}-\frac{S}{p}\Bigg]. \quad \text{(B.1)}$$

In the limit of $r\to 0$, we recover $f(S,p,0,q)=0$ defining the giant component of two interdependent ER networks. Likewise, at criticality, $f_S(S_c,p_c,0,q)=0$, where f_S refers to the partial derivative of f with respect to S. In order to find β we expand $f(S,p,0,q)$ around $S=S_c$ and $p=p_c$

$$f(S,p,0,q)=f(S_c,p_c,0,q)+f_S(S_c,p_c,0,q)(S-S_c)+$$
$$+f_p(S_c,p_c,0,q)(p-p_c)+...=0.$$

rearranging and keeping the dominant terms gives:

$$(S-S_c)^2\sim -2\frac{f_p(S_c,p_c,0,q)}{f_{SS}(S_c,p_c,0,q)}(p-p_c)-$$
$$-2\frac{f_{Sp}(S_c,p_c,0,q)}{f_{SS}(S_c,p_c,0,q)}(p-p_c)(S-S_c)+...$$

the scaling depends on the value of $f_p(S_c,p_c,0,q)=S_c/p_c^2$. Small values of q exhibit a second order phase transition with $S_c=0$ leading to $\beta=1$ while large values of q lead to an abrupt phase transition with $S_c>0$ giving $\beta=1/2$. It can be demonstrated that $f_{Sp}(S_c,p_c,0,q)=1/p_c^2\neq 0$ and $f_{SS}(S_c,p_c,0,q)\neq 0$.

In order to find δ we take the limit of small r. We define $x=f(S,p,r,q)/r$ which satisfies the equation:

$$1+\frac{2x}{1-q+\sqrt{(1-q)^2+4qS}}=\exp(\kappa px-zS). \quad \text{(B.2)}$$

The dominant balance of the equation for x and the conditions $0<S,p<1$, imply that in the limit $r\to 0$, x approaches a constant whose value depends on p and q, denoted here as $C(p,q)$. We let the superscript ^{c} denote that f is evaluated at $p=p_c, S=S_c, r=0$. Expanding f around $S=S_c, r=0$ (with fixed $p=p_c$), keeping the dominant terms and making use of the prior equation, we obtain

$$x=f_r^c+\frac{1}{2}f_{SS}^c\frac{(S-S_c)^2}{r}+...$$

whose RHS must equal to $C(p_c,q)$. In addition, $f_r^c=-S_c/p_c$ and $f_{Sr}^c=-1/p_c$. For large values of q the transition is abrupt, therefore $S_c>0$ and thus $x=0$ is not a solution. By plugging $x=-S_c/p_c$ into Eq. (B.2) at criticality one can see that $C(p_c,q)\neq -S_c/p_c$ and thus $(S-S_c)^2\sim\frac{2[C(p_c,q)-f_r^c]}{f_{SS}^c}r$ giving $\delta=2$. For small values of q the transition is continues, $S_c=0$ and thus

$$x=\frac{f}{r}=f_{SS}^c\frac{S^2}{2r}+f_{Sr}^cS+...=C(p_c,q)+o(1).$$

If $C(p_c,q)<0$ then simply $S\sim r^{1/2}$ meaning $\delta=2$. If $C(p_c,q)=0$ then $x(r\to 0)=o(1)$. Assuming small x and S in Eq. (B.2) gives

$$1+\frac{1}{1-q}x+o(x)=1+\kappa p_cS-zS+o(x)+o(S),$$

where $o(x)$ is the little-O notation meaning terms much smaller than x. Therefore,

$$f_{SS}^c\frac{S^2}{2r}-\frac{1}{p_c}S+...=\frac{z}{\kappa p_c-1/(1-q)}S+... \quad .$$

Since $\kappa\gg z$ the condition $\kappa p_c-1/(1-q)>0$ is always satisfied, the equation has no balance and therefore we ignore the solution $x(p_c,r\to 0)=0$ and thus $\delta=2$ concluding that $\delta=2$ for all value of q.

In order to find γ we recall that the solution of Eq. (B.2) can be written as $x=C(p_c,q)+o(1)<0$ where $r\to 0$. Thus,

$$f(S,p,r,q)=rC(p,q)+o(r).$$

Taking $\partial_r|_p$ on both sides, $r\to 0$ and rearranging for $\frac{\partial S}{\partial r}$ gives,

$$\frac{\partial S}{\partial r}=\frac{C(p,q)-f_r(S,p,0,q)}{f_S(S,p,0,q)}.$$

Expanding f_r and f_S, and substituting the result for the scaling of $(S - S_c)$ with $p - p_c$ gives

$$\frac{\partial S}{\partial r} = \frac{C(p_c, q) - f_r^c + ...}{f_{SS}^c A(p - p_c)^\beta + 1/p_c^2(p - p_c) + ...}.$$

For large values of q we find that $\beta = 1/2$ leading to $\gamma = 1/2$ while for small values of q, $\beta = 1$ giving $\gamma = 1$.

Appendix C CRITICAL EXPONENTS - ANALYTIC DERIVATION FOR FULLY INTERDEPENDENT NETWORKS ($q = 1$)

In order to derive analytically the critical exponents from Eq. (A.1) we define

$$f(S, p, r) = e^{-zS}(r - 1) + 1 - \sqrt{\frac{S}{p}} \qquad \text{(C.1)}$$

and thus Eq. (A.1) takes the form

$$f(S, p, r) = r \exp\left[\kappa\sqrt{Sp}(f(S, p, r)/r - 1) - zS\right]. \qquad \text{(C.2)}$$

The analytic derivation of the critical exponents is based on several assumptions regarding $f(S, p, r)$ and its derivative at criticality. Here we validate these assumptions:

1) $\boldsymbol{f(S_c, p_c, 0) = 0}$. We get: $-e^{-zS_c} + 1 - \sqrt{\frac{S_c}{p_c}} = 0$ which yields the well known result [12] $S_c = p_c(e^{-zS_c} - 1)^2$.

2) $\boldsymbol{f_S(S_c, p_c, 0) = 0}$. We obtain: $ze^{-zS_c} - \frac{1}{2\sqrt{S_c p_c}} = 0$ as shown in Fig. C.1.

3) $\boldsymbol{f_{SS}(S_c, p_c, 0)} = -z^2 e^{-zS_c} + \frac{1}{4S_c\sqrt{S_c p_c}} = -z^2 e^{-zS_c} + ze^{-zS_c}\frac{1}{2S_c} = ze^{-zS_c}\left(-z + \frac{1}{2S_c}\right) \neq 0$.

Inequality *3)* is valid due to the fact that if we assume $zS_c = 1/2$ and we substitute that in the equations obtained from *1)* and *2)* we get wrong equality $-e^{-zS_c} + 1 = 2zS_c e^{-zS_c}$.

4) $\boldsymbol{f_p(S_c, p_c, 0)} = \frac{1}{2p_c}\sqrt{\frac{S_c}{p_c}} \neq 0$.

5) $\boldsymbol{f_r(S_c, p_c, 0)} = e^{-zS_c} \neq 0$.

6) $\boldsymbol{C(p_c)} \neq f_r^c = e^{-zS_c}$.

Inequality *6)* is valid because if we assume $x = C(p_c) = e^{-zS_c}$ when $r \to 0$ and $p = p_c$ and substitute into $x = \exp\left(\kappa\sqrt{Sp}(x - 1) - zS\right)$ we get: $e^{-zS_c} = \exp\left(\kappa\sqrt{S_c p_c}(e^{-zS_c} - 1) - zS_c\right)$ which yields $S_c = 0$.

To obtain β we assume $S \to S_c$ and $p \to p_c$ in $f_S(S_c, p_c, 0) = 0$. Expanding $f(S, p, 0)$ around $S = S_c$ and $p = p_c$, noting that $f_S(S_c, p_c, 0) = f(S_c, p_c, 0) = 0$, and keeping the dominant terms gives

$$(S - S_c)^2 = -\frac{2f_p(S_c, p_c, 0)}{f_{SS}(S_c, p_c, 0)}(p - p_c) + ...$$

from which we obtain

$$(S - S_c) \sim (p - p_c)^{\frac{1}{2}}.$$

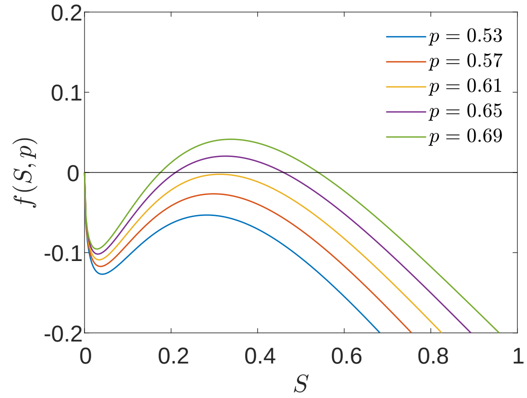

FIG. C.1: **$f(S, p)$ as a function of S for different values of p.** Here one can see that at $p = p_c \approx 2.4554/4 \approx 0.61135$ the condition $f_S(S_c, p_c, 0) = 0$ is satisfied.

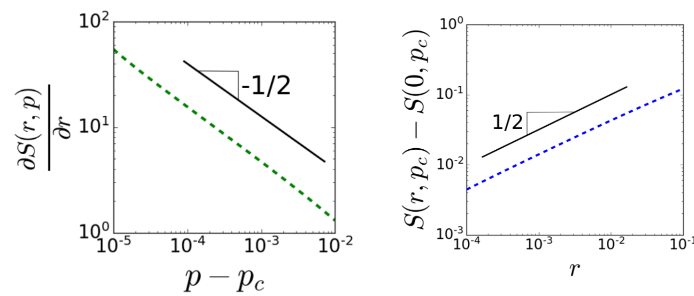

FIG. C.2: **The critical exponents γ (left) and δ (right) for Eq. 4 in the main text with κ independent of r.** The derivation of the critical exponents is validated by solving Eq. 4 in the main text which gives $\delta = 2$ and $\gamma = 1/2$. Here $\kappa = 10$.

Thus, $\beta = \frac{1}{2}$.

To find δ we set $p = p_c$ in Eq. (C.2) and analyze the limit of $r \to 0$. We consider $x = \frac{f(S,p,r)}{r}$, and rewrite Eq. (C.2) as $x = \exp\left[\kappa\sqrt{Sp}(x - 1) - zS\right]$. The dominant balance of the equation for x and the conditions $0 <$

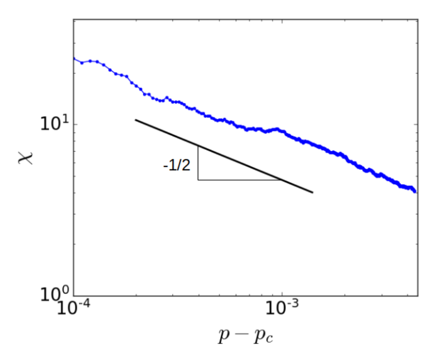

FIG. C.3: **The critical exponents γ measured directly from the fluctuations**. We find that $\gamma = 1/2$ and thus the FDT is not violated.

$S,p<1$, imply that in the limit $r\to 0$, x approaches a constant whose value depends on p, denoted here $C(p)$. We let the superscript ^{c} denote that f is evaluated at $p=p_c, S=S_c, r=0$. Expanding f around $S=S_c, r=0$ (with fixed $p=p_c$) and making use of the prior equation, we obtain

$$x = f_r^c + \frac{1}{2} f_{SS}^c \frac{(S-S_c)^2}{r} + ...$$

whose RHS must be equal to $C(p_c)$. Thus, as long as $C(p_c) \neq f_r^c$ it follows that

$$(S-S_c) \sim r^{\frac{1}{2}},$$

meaning $\delta = 2$ as confirmed numerically in Fig. C.2.

To find γ we note that for $r\to 0^+$, $x = C(p) + o(1)$. Thus

$$f(S,p,r) = rC(p) + o(r).$$

Taking $\partial_r|_p$ on both sides, $r\to 0$ and rearranging for $\frac{\partial S}{\partial r}$ gives,

$$\frac{\partial S}{\partial r} = \frac{C(p) - f_r(S,p,0)}{f_S(S,p,0)}.$$

Expanding f_r and f_S, and substituting the result for scaling of $(S-S_c)$ with $p-p_c$ gives

$$\frac{\partial S}{\partial r} = \frac{C(p_c) - f_r^c + ...}{f_{SS}^c \sqrt{-2 f_p^c / f_{SS}^c} (p-p_c)^{\frac{1}{2}} + ...}.$$

Therefore, to the leading term

$$\frac{\partial S}{\partial r} \sim (p-p_c)^{-\frac{1}{2}},$$

giving $\gamma = \frac{1}{2}$ as shown numerically in Fig. C.2.

In order to validate the FDT we measured γ directly from the fluctuations as $\chi = N(\langle S^2\rangle - \langle S\rangle^2) \sim (p-p_c)^{-\gamma}$. We find that $\gamma = 1/2$ and the FDT is not violated as can be clearly seen in Fig. C.3.

Appendix D CRITICAL EXPONENTS - ANALYTIC DERIVATION FOR A SINGLE NETWORK ($q=0$)

For the case of a single network the equation for the giant component S is [28]

$$e^{-zS}(r-1) + 1 - \frac{S}{p} = \\ = r\exp\left(\frac{\kappa p(e^{-zS}(r-1) + 1 - \frac{S}{p} - r)}{r} - zS\right). \quad \text{(D.1)}$$

In order to derive analytically the critical exponents β, δ and γ we let $f(S,p,r)$ denote the LHS of Eq. (D.1). Thus

$$f(S,p,r) = r\exp\left(\kappa p(f(S,p,r)/r - 1) - zS\right). \quad \text{(D.2)}$$

In the limit of $r\to 0$ we find $f(S,p,0)=0$ which recovers the equation for a single ER network. Likewise, at criticality $f_S(S_c,p_c,0)=0$. These yield $S_c=0$ and $p_c=\frac{1}{z}$. In addition, one can demonstrate the following:

1) $\boldsymbol{f_{SS}(S_c,p_c,0)} = -z^2 \neq 0$.
2) $\boldsymbol{\partial_p^n f(S_c,p_c,0)} = 0$.
3) $\boldsymbol{\partial_S \partial_p^n f(S_c,p_c,0)} = -\partial_p^n(1/p) \neq 0$.
4) $\boldsymbol{\partial_S \partial_p f(S_c,p_c,0)} = z^2$.
5) $\boldsymbol{f_r(S_c,p_c,0)} = 1 \neq 0$.

To obtain β we assume $S\to S_c$ and $p\to p_c$ in $f(S,p,0)=0$. Expanding $f(S,p,0)$ around $S=S_c$ and $p=p_c$ gives

$$f(S,p,0) = f(S_c,p_c,0) + f_S(S_c,p_c,0)S + f_p(S_c,p_c,0)(p-p_c) + \\ + \frac{1}{2} f_{SS}(S_c,p_c,0)S^2 + f_{Sp}(S_c,p_c,0)S(p-p_c) + ... = 0$$

Noting *1)*, *2)* and *4)* and keeping the dominant terms gives

$$S = -\frac{2f_{Sp}^c}{f_{SS}^c}(p-p_c) + ... = 2(p-p_c) + ...$$

from which we find

$$S \sim (p-p_c)^1.$$

meaning $\beta = 1$.

To find δ we substitute in Eq. (D.1) $p=p_c$ and analyze the limit of $r\to 0$. We now consider $x = \frac{f(S,p,r)}{r}$ and rewrite Eq. (D.1) as $x = \exp(\kappa p(x-1) - zS)$. Assuming $p=p_c$ and $r\to 0$ yields $S\to 0$, thus, we can find x for the zero-order term from $x = \exp(\kappa/z(x-1))$ which has two solutions. The first solution is $0<x_1<1$ and the second is $x_2=1$. Regarding the solution $x_2=1$ we can also find the first-order term by substituting $x = 1+\epsilon$, which yields,

$$1+\epsilon = \exp(\kappa\epsilon/z - zS) = 1 + \kappa\epsilon/z - zS + ...$$

and gives $\epsilon \sim zS/(\kappa/z - 1)$. Expanding $f(S,p,r)$ at $p=p_c$ around $S=S_c$ and $r=0$ gives

$$f(S,p_c,r) = f(S_c,p_c,0) + f_S(S_c,p_c,0)S + \\ + f_r(S_c,p_c,0)r + \frac{1}{2} f_{SS}(S_c,p_c,0)S^2 + ...$$

Using $x = \frac{f(S,p,r)}{r}$ with $x_2 = 1+\epsilon$, noting *1)* and *5)* and keeping the dominant terms gives

$$1 + \frac{zS}{(\kappa/z - 1)} \sim 1 - \frac{(zS)^2}{2r}.$$

which yields $S \sim \frac{2z}{z-\kappa} r$. However, this can not be true since $\kappa \gg z$ and $S > 0$ implying that x_2 is a spurious solution. Therefore we will use x_1 which gives

$$x = 1 - \frac{(zS)^2}{2r} + ... = x_1 + ...$$

and keeping the dominant terms we find

$$S \sim r^{\frac{1}{2}},$$

i.e. $\delta = 2$.

To find γ we note that for $r \to 0^+$, $x = C(p) + o(1)$. Thus

$$f(s, p, r) = rC(p) + o(r).$$

Taking $\partial_r|_p$ on both sides and $r \to 0$ gives

$$\frac{\partial S}{\partial r} f_S(S, p, 0) + f_r(S, p, 0) = C(p).$$

Rearranging for $\frac{\partial S}{\partial r}$ gives,

$$\frac{\partial S}{\partial r} = \frac{C(p) - f_r(S, p, 0)}{f_S(S, p, 0)}.$$

Expanding f_r and f_S, substituting the result for scaling of $(S - S_c)$ with $p - p_c$ and using *1*), *4*) and *5*) gives

$$\frac{\partial S}{\partial r} = \frac{C(p_c) - f_r^c + ...}{f_S^c + f_{SS}^c S + f_{Sp}^c (p - p_c) + ...} = \\ = \frac{1 - x_1}{z^2}(p - p_c)^{-1} + ...$$

Therefore the leading term is

$$\frac{\partial S}{\partial r} \sim (p - p_c)^{-1}.$$

giving $\gamma = 1$.

Appendix E ANALYTIC DERIVATION OF q_c

In the main manuscript we showed that for low values of q the transition is continues while for high values of q the transition is abrupt. Here we will derive analytically the transition point q_c between these two behaviours. For $r = 0$ Eq. 4 in the main text take the form

$$-S + (1 - q)p[1 - \exp(-zS)] + qp^2[1 - \exp(-zS)]^2 = 0$$

and we denote the LHS as $f(S, p, q)$. At the transition point q_c the following conditions are satisfied

$$\begin{cases} f_S(0, p_c, q_c) = 0 \\ f_{SS}(0, p_c, q_c) = 0 \end{cases}$$

and explicitly

$$\begin{cases} -1 + (1 - q_c)p_c z = 0. \\ -z^2(1 - q_c)p_c + 2z^2 q_c p_c^2 = 0. \end{cases}$$

leading to

$$q_c = \frac{(z + 1) - \sqrt{2z + 1}}{z}. \tag{E.1}$$

Appendix F SUMMARY OF THE CRITICAL EXPONENTS

In this table we summarize the values of the critical exponents β, δ and γ for various percolation processes on random network structures.

process	β	δ	γ
regular percolation (ER)	1	2	1
regular percolation (SF) $\lambda > 4$	1	2	1
regular percolation (SF) $4 > \lambda > 3$	$1/(\lambda - 3)$	depends on λ	depends on λ
regular percolation (SF) $3 > \lambda > 2$	$1/(3 - \lambda)$	depends on λ	depends on λ
interdependent percolation(ER,SF)	1/2	2	1/2 (1 according to the definition of [33])
k-core percolation ($k \geq 3$)	1/2	2	1/2

[1] David Meunier, Renaud Lambiotte, and Edward T Bullmore. Modular and hierarchically modular organization of brain networks. *Frontiers in neuroscience*, 4:200, 2010.

[2] Flaviano Morone, Kevin Roth, Byungjoon Min, H Eugene Stanley, and Hernán A Makse. Model of brain activation predicts the neural collective influence map of the brain. *Proceedings of the National Academy of Sciences*, 114(15):3849–3854, 2017.

[3] Cornelis Jan Stam, Arjan Hillebrand, Huijuan Wang, and Piet Van Mieghem. Emergence of modular structure in a large-scale brain network with interactions between dynamics and connectivity. *Frontiers in computational neuroscience*, 4:133, 2010.

[4] Kazuko Yamasaki, Avi Gozolchiani, and Shlomo Havlin. Climate networks around the globe are significantly affected by el nino. *Physical Review Letters*, 100(22):228501, 2008.

[5] Armin Bunde and Shlomo Havlin. *Fractals and disordered systems*. Springer Science & Business Media, 2012.

[6] Réka Albert, Hawoong Jeong, and Albert-László Barabási. Error and attack tolerance of complex networks. *Nature*, 406(6794):378–382, 2000.

[7] Reuven Cohen et al. Resilience of the Internet to Random Breakdowns. *Phys. Rev. Lett.*, 85:4626–4628, Nov 2000.

[8] Reuven Cohen and Shlomo Havlin. *Complex networks: structure, robustness and function*. Cambridge university press, 2010.

[9] Mark Newman, Albert-Laszlo Barabasi, and Duncan J Watts. *The structure and dynamics of networks*, volume 19. Princeton University Press, 2011.

[10] Duncan S Callaway, Mark EJ Newman, Steven H Strogatz, and Duncan J Watts. Network robustness and fragility: Percolation on random graphs. *Physical Review Letters*, 85(25):5468, 2000.

[11] Sergey V. Buldyrev et al. Catastrophic cascade of failures in interdependent networks. *Nature*, 464(7291):1025–1028, Apr 2010.

[12] Jianxi Gao, Sergey V Buldyrev, H Eugene Stanley, and Shlomo Havlin. Networks formed from interdependent networks. *Nature Physics*, 8(1):40, 2012.

[13] S.M. Rinaldi, J.P. Peerenboom, and T.K. Kelly. Identifying, understanding, and analyzing critical infrastructure interdependencies. *Control Systems, IEEE*, 21(6):11–25, 2001.

[14] Jianxi Gao et al. Robustness of a network of networks. *Physical Review Letters*, 107(19):195701, 2011.

[15] Roni Parshani, Sergey V. Buldyrev, and Shlomo Havlin. Interdependent Networks: Reducing the Coupling Strength Leads to a Change from a First to Second Order Percolation Transition. *Phys. Rev. Lett.*, 105:048701, Jul 2010.

[16] Ernst Ising. Beitrag zur theorie des ferromagnetismus. *Zeitschrift für Physik*, 31(1):253–258, 1925.

[17] Anna Saumell-Mendiola, M Ángeles Serrano, and Marián Boguná. Epidemic spreading on interconnected networks. *Physical Review E*, 86(2):026106, 2012.

[18] Filippo Radicchi and Alex Arenas. Abrupt transition in the structural formation of interconnected networks. *Nature Physics*, 9(11):717–720, Sep 2013.

[19] Manlio De Domenico, Albert Solé-Ribalta, Sergio Gómez, and Alex Arenas. Navigability of interconnected networks under random failures. *Proceedings of the National Academy of Sciences*, 111(23):8351–8356, 2014.

[20] Charles D Brummitt et al. Suppressing cascades of load in interdependent networks. *Proceedings of the National Academy of Sciences*, 109(12):E680–E689, 2012.

[21] L D Valdez et al. The role of bridge nodes between layers on epidemic spreading. *New Journal of Physics*, 20(12):125003, dec 2018.

[22] Louis M Shekhtman et al. Critical field-exponents for secure message-passing in modular networks. *New Journal of Physics*, 20(5):053001, 2018.

[23] Jingfang Fan et al. Structural resilience of spatial networks with inter-links behaving as an external field. *New Journal of Physics*, 20(9):093003, 2018.

[24] Michelle Girvan and Mark EJ Newman. Community structure in social and biological networks. *Proceedings of the National Academy of Sciences*, 99(12):7821–7826, 2002.

[25] Gergely Palla, Imre Derényi, Illés Farkas, and Tamás Vicsek. Uncovering the overlapping community structure of complex networks in nature and society. *Nature*, 435(7043):814–818, 2005.

[26] Andrea Lancichinetti, Santo Fortunato, and János Kertész. Detecting the overlapping and hierarchical community structure in complex networks. *New Journal of Physics*, 11(3):033015, 2009.

[27] Peter J Mucha et al. Community structure in time-dependent, multiscale, and multiplex networks. *Science*, 328(5980):876–878, 2010.

[28] Gaogao Dong et al. Resilience of networks with community structure behaves as if under an external field. *Proceedings of the National Academy of Sciences*, page 201801588, 2018.

[29] Kerson Huang. *Introduction to statistical physics*. Chapman and Hall/CRC, 2009.

[30] H Eugene Stanley. *Phase transitions and critical phenomena*. Clarendon Press, Oxford, 1971.

[31] D. Stauffer and A. Aharony. *Introduction to percolation theory: revised second edition*. CRC press, 2014.

[32] Mark EJ Newman, Steven H Strogatz, and Duncan J Watts. Random graphs with arbitrary degree distributions and their applications. *Physical Review E*, 64(2):026118, 2001.

[33] Deokjae Lee, S Choi, M Stippinger, J Kertész, and B Kahng. Hybrid phase transition into an absorbing state: Percolation and avalanches. *Physical Review E*, 93(4):042109, 2016.

[34] Sergey V Buldyrev et al. Catastrophic cascade of failures in interdependent networks. *Nature*, 464(7291):1025, 2010.

[35] A Crisanti and F Ritort. Violation of the fluctuation–dissipation theorem in glassy systems: basic notions and the numerical evidence. *Journal of Physics A: Mathematical and General*, 36(21):R181, 2003.

[36] Tomás S Grigera and NE Israeloff. Observation of fluctuation-dissipation-theorem violations in a structural glass. *Physical Review Letters*, 83(24):5038, 1999.

[37] Enzo Marinari, Giorgio Parisi, Federico Ricci-Tersenghi, and Juan J Ruiz-Lorenzo. Violation of the fluctuation-dissipation theorem in finite-dimensional spin glasses. *Journal of Physics A: Mathematical and General*, 31(11):2611, 1998.

[38] Dong Zhou et al. Simultaneous first-and second-order percolation transitions in interdependent networks. *Physical Review E*, 90(1):012803, 2014.

[39] Sergey N Dorogovtsev, Alexander V Goltsev, and Jose Ferreira F Mendes. K-core organization of complex networks. *Physical Review Letters*, 96(4):040601, 2006.

[40] Alexander V Goltsev, Sergey N Dorogovtsev, and Jose Ferreira F Mendes. k-core (bootstrap) percolation on complex networks: Critical phenomena and nonlocal effects. *Physical Review E*, 73(5):056101, 2006.

[41] Shai Carmi et al. A model of internet topology using k-shell decomposition. *Proceedings of the National Academy of Sciences*, 104(27):11150–11154, 2007.

[42] JM Schwarz, Andrea J Liu, and LQ Chayes. The onset of jamming as the sudden emergence of an infinite k-core cluster. *EPL (Europhysics Letters)*, 73(4):560, 2006.