Complete initial value spacetimes containing black holes in general relativity: Application to black hole-disk systems

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Complete initial value spacetimes containing black holes in general relativity: Application to black hole-disk systems

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We present a new initial data formulation to solve the *full set* of Einstein equations for spacetimes that contain a black hole under general conditions. The method can be used to construct complete initial data for spacetimes (the full metric) that contain a black hole. Contrary to most current studies the formulation requires minimal assumptions. For example, rather than imposing the form of the spatial conformal metric we impose 3 gauge conditions adapted to the coordinates describing the system under consideration. For stationary, axisymmetric spacetimes our method yields Kerr-Schild black holes in vacuum and rotating equilibrium neutron stars. We demonstrate the power of our new method by solving for the first time the whole system of Einstein equations for a nonaxisymmetric, self-gravitating torus in the presence of a black hole. The black hole has dimensionless spin $J_{\rm bh}/M_{\rm bh}^2 = 0.9918$, a rotation axis tilted at a 30° angle with respect to the angular momentum of the disk, and a mass of $\sim 1/5$ of the disk.

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I. INTRODUCTION

Although the term "black hole" was coined fairly recently by John Wheeler 52 years ago, and its physical significance was questioned earlier by Einstein himself, it turns out that 21^{st} century physics will be dominated by these extraordinary objects. The spectacular first detection by the Laser Interferometer Gravitational-Wave Observatory (LIGO) of a merging binary black hole system [1], as well as the nine follow-up detections, unequivocally confirmed their existence and many of their properties.

Accurate modelling of black hole spacetimes requires initial data that precisely describe the systems under consideration. Over the years three main different ideas have been heavily employed to address this problem. These are the conformal transverse traceless (CTT) decomposition [2–4], the puncture method [5], and the conformal thin-sandwich (CTS) approach [6] (see [7] for summary and discussions). Many variants of these formulations exist but of particular importance is the so-called Isenberg-Wilson-Mathews [8] formulation, whose strength stems from its simplicity and versatility, as it is used for black hole as well as neutron star spacetimes.

All methods described above solve a subset of the Einstein equations. One common characteristic is that they assume the form of the spatial conformal metric which is associated with the true dynamical degrees of freedom of the gravitational field [3]. In [9] the authors used a constrained scheme presented in [10] for the Einstein equations to solve for the conformal metric as well. In doing so they discovered solutions that better satisfy the Einstein equations (at least in comparison to conformally flat solutions) and reach dimensionless spins up to $J_{\rm bh}/M_{\rm bh}^2 \sim 0.85$. Their solutions displayed Kerr-like properties but they were not expressed in any of the well-known coordinates, like the Kerr-Schild ones [11], which are known to yield high-spin initial data and exhibit good behaviour in evolution simulations.

In this paper we present a new formulation and a new code within the COCAL (Compact Object CALculator) project [12] that solves *all* the Einstein equations in a self-consistent manner and achieves the following: (1) In the absence of matter our code can reproduce the exact Kerr-Schild solution, even for high spins. No assumptions on axisymmetry are imposed and therefore this is the first *generic* 3-d method that obtains an exact Kerr solution and can be applied with minimal changes to a broad range of *nonaxisymmetric* problems, such as tilted disks or binary systems. (2) The domain of the solution extends inside the apparent horizon, which is well-suited for evolution simulations. (3) In the presence of massless disks around the black hole our code reproduces well-known solutions (e.g. [13, 14]). (4) The first self-consistent, tilted black hole-torus solutions are presented that solve for the *total* spacetime metric. In addition, these are the highest mass ratio and black hole spin solutions constructed for black hole-torus systems to date. We present a solution with a spinning black hole whose dimensionless spin is $J_{\rm bh}/M_{\rm bh}^2 = 0.9918$, has an angle with respect to the angular momentum of the torus of $\theta = 30^\circ$, while the torus has rest mass approximately five times the black hole mass.

Tilted disk-black hole systems can be produced in the merger of black hole-neutron star systems where the spin of the black hole is tilted with respect to the total angular momentum of the system [15, 16]. Tilted black holes may also arise in massive disks in active galactic nuclei and quasars [17].

In the following, greek letters denote spacetime indicies while latin letters indicate spatial ones. We adopt units with $G = c = M_\odot = 1$, unless otherwise stated.

II. FORMULATION FOR GRAVITY

We use the standard $3+1$ formalism to express spacetime $\mathcal{M} = \mathbb{R} \times \Sigma_t$ as a foliation of three dimensional spacelike hypersurfaces Σ_t (t labels the hypersurface) with spatial coordinates x^i and unit normal vector n^μ. Points with the same values of x^i in neighboring hypersurfaces are connected with a timelike vector t^μ that can be decomposed as $t^\mu := \alpha n^\mu + \beta^\mu$, where α is the lapse and β^μ is the (spatial) shift vector. The first fundamental form of the hypersurfaces is $\gamma_{\alpha\beta} := g_{\alpha\beta} + n_\alpha n_\beta$, and the full spacetime line element is $ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$. A conformal geometry is introduced by setting $\gamma_{ij} := \psi^4 \tilde{\gamma}_{ij}$. We further define $\tilde{\gamma}_{ij} := f_{ij} + h_{ij}$, where f_{ij} is the flat metric in arbitrary coordinates and h_{ij} the non-flat contributions *which we wish to evaluate* together with the rest of the potentials, ψ, α, β^i, that are computed traditionally. For the conformal geometry we assume $\det(\tilde{\gamma}_{ij}) = \tilde{\gamma} = \det(f_{ij})$.

The initial data are the 3-metric γ_{ij} and the extrinsic curvature $K_{ij} = -\frac{1}{2}\mathcal{L}_{\mathbf{n}}\gamma_{ij}$ ($\mathcal{L}_{\mathbf{n}}$ denotes the Lie derivative with respect to the unit normal n^α) which is further decomposed as $K_{ij} = A_{ij} + \frac{1}{3}\gamma_{ij}K$, where K is its trace and A_{ij} its tracefree part. The conformal tracefree part of the extrinsic curvature is defined as $\tilde{A}_{ij} = \psi^{-4}A_{ij}$ and we introduce the decomposition

$$\tilde{A}_{ij} = \tilde{A}^{\text{KS}}_{ij} + \tilde{\sigma}(\tilde{\mathbb{L}}\tilde{W})_{ij}, \tag{1}$$

where $\tilde{A}^{\text{KS}}_{ij}$ is the Kerr-Schild part, $\tilde{W}_i$ an unknown spatial vector, and $\tilde{\sigma}$ a scalar. $\tilde{\mathbb{L}}$ is the conformal Killing operator: $(\tilde{\mathbb{L}}\tilde{W})_{ij} = \tilde{D}_i\tilde{W}_j + \tilde{D}_j\tilde{W}_i - \frac{2}{3}\tilde{\gamma}_{ij}\tilde{D}_k\tilde{W}^k$.

A Kerr black hole spacetime in Kerr-Schild coordinates [11] can be written as $ds^2 = (\eta_{\alpha\beta} + 2Hl_\alpha l_\beta)dx^\alpha dx^\beta$, where $H = mr^3/(r^4 + (a_i x^i)^2)$, $r^2 = (x^i x_i - a^i a_i)/2 + \sqrt{(x^i x_i - a^i a_i)/4 + (a^i x_i)^2}$, and $l_\alpha = (1, l_i)$ with $l_i = x^j(r^2\delta_{ij} + r\epsilon_{ijk}a^k + a_i a_j)/(r(r^2+a^2))$. Note $r^2 \neq x_i x^i = \hat{r}^2$. Here $x_i = x^i$ and a^i is the spin of the black hole, $a^2 = a_i a^i$, and l_α is a null vector, both with respect to the spacetime metric $g_{\alpha\beta}$ as well as the Minkowski metric $\eta_{\alpha\beta}$. The $3+1$ quantities of an arbitrarily spinning black hole in Kerr-Schild coordinates are $\psi_{\text{KS}} = (1+2H)^{1/12}$, $\alpha_{\text{KS}} = 1/\sqrt{1+2H}$, $\beta^i_{\text{KS}} = 2H\alpha^2 l^i$, $\gamma^{\text{KS}}_{ij} = \delta_{ij} + 2Hl_i l_j$ and therefore $h^{\text{KS}}_{ij} = \psi^{-4}_{\text{KS}}(\delta_{ij} + 2Hl_i l_j) - \delta_{ij}$. Using the 3+1 quantities above one can compute $\tilde{A}^{\text{KS}}_{ij}$ which appears in Eq. (1). The trace of the extrinsic curvature in Kerr-Schild coordinates is

$$K_{\text{KS}} = \frac{2H\alpha^3_{\text{KS}}}{r}\left(1 + H + \frac{2H^2 r}{m}\right). \tag{2}$$

and in our calculations we assume $K = K_{\text{KS}}$.

Taking combinations of the projections of the Einstein equations ($(G_{\mu\nu} - 8\pi T_{\mu\nu})n^\mu n^\nu = 0$, $(G_{\mu\nu} - 8\pi T_{\mu\nu})n^\mu \gamma^\nu{}_i = 0$, $(G_{\mu\nu} - 8\pi T_{\mu\nu})\gamma^\mu{}_i\gamma^\nu{}_j = 0$) onto the spatial hypersurface [18, 19] one can arrive at a set of elliptic equations

$$\mathring{\triangle}\psi = -h^{ij}\mathring{D}_i\mathring{D}_j\psi + \tilde{\gamma}^{ij}C^k{}_{ij}\mathring{D}_k\psi + \frac{1}{8}\psi\tilde{\mathcal{R}} - \frac{\psi^5}{8}\left(\tilde{A}_{ij}\tilde{A}^{ij} - \frac{2}{3}K^2\right) - 2\pi\rho_{\text{H}}\psi^5, \tag{3}$$

$$\begin{aligned}\mathring{\triangle}\tilde{W}_i = &-h^{ab}\mathring{D}_a\mathring{D}_b\tilde{W}_i + \tilde{\gamma}^{ab}[\mathring{D}_a(C^m{}_{bi}\tilde{W}_m) + C^m{}_{ab}\tilde{D}_m\tilde{W}_i + C^m{}_{ai}\tilde{D}_b\tilde{W}_m] - \frac{1}{3}\mathring{D}_i\mathring{D}_j\tilde{W}^j - \tilde{\mathcal{R}}_{ij}\tilde{W}^j \\ &- \frac{1}{\tilde{\sigma}}\tilde{D}^j\tilde{A}^{\text{KS}}_{ij} - \frac{1}{\tilde{\sigma}}\tilde{D}^j\tilde{\sigma}(\tilde{\mathbb{L}}\tilde{W})_{ij} - \frac{6}{\psi\tilde{\sigma}}\tilde{D}^j\psi\tilde{A}_{ij} + \frac{2}{3\tilde{\sigma}}\mathring{D}_iK + \frac{8\pi}{\tilde{\sigma}}j_i,\end{aligned} \tag{4}$$

$$\begin{aligned}\mathring{\triangle}\tilde{\beta}_i = &-h^{ab}\mathring{D}_a\mathring{D}_b\tilde{\beta}_i + \tilde{\gamma}^{ab}[\mathring{D}_a(C^m{}_{bi}\tilde{\beta}_m) + C^m{}_{ab}\tilde{D}_m\tilde{\beta}_i + C^m{}_{ai}\tilde{D}_b\tilde{\beta}_m] - \frac{1}{3}\mathring{D}_i\mathring{D}_j\tilde{\beta}^j - \tilde{\mathcal{R}}_{ij}\tilde{\beta}^j - \frac{2\alpha^2}{\psi^6}\tilde{A}_i{}^j\tilde{D}_j\left(\frac{\psi^6}{\alpha}\right) \\ &+ \frac{4\alpha}{3}\mathring{D}_iK + \tilde{\gamma}^{jm}\tilde{D}_j\tilde{u}_{im} + 16\pi\alpha j_i,\end{aligned} \tag{5}$$

$$\mathring{\triangle}(\alpha\psi) = -h^{ij}\mathring{D}_i\mathring{D}_j(\alpha\psi) + \tilde{\gamma}^{ij}C^k{}_{ij}\mathring{D}_k(\alpha\psi) + \frac{\alpha\psi}{8}\tilde{\mathcal{R}} + \alpha\psi^5\left(\frac{7}{8}\tilde{A}_{ij}\tilde{A}^{ij} + \frac{5}{12}K^2\right) - \psi^5\mathcal{L}_{\alpha\mathbf{n}}K + 2\pi\alpha\psi^5(\rho_{\text{H}} + 2S), \tag{6}$$

$$\begin{aligned}\mathring{\triangle}h_{ij} = &-\frac{1}{3}\tilde{\gamma}_{ij}\mathring{D}_kh_{ab}\mathring{D}^kh^{ab} + \frac{2}{3}\tilde{\gamma}_{ij}\mathring{D}^kC^a{}_{ak} + 2\Bigg[\tilde{\mathcal{R}}^{\text{KS}}_{ij} + \tilde{\mathcal{R}}^{\text{NL}}_{ij} - 8\pi S_{ij} + \psi^4\left(\frac{1}{3}K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k{}_j\right) \\ &+ \frac{1}{\alpha\psi^2}\left(-\mathring{D}_i\mathring{D}_j(\alpha\psi^2) + C^k{}_{ij}\mathring{D}_k(\alpha\psi^2) + 4\mathring{D}_i(\alpha\psi)\mathring{D}_j\psi + 4\mathring{D}_i\psi\mathring{D}_j(\alpha\psi)\right) - \frac{1}{\alpha}\mathcal{L}_{\alpha\mathbf{n}}(\psi^4\tilde{A}_{ij})\Bigg]^{\text{TF}},\end{aligned} \tag{7}$$

for the eleven metric potentials $\psi, \alpha, \tilde{\beta}_i, h_{ij}$ and the three auxiliary components of $\tilde{W}_i$. We define h^{ij} through $\tilde{\gamma}^{ij} = f^{ij} + h^{ij}$ where $\tilde{\gamma}^{ij}, f^{ij}$ the inverses of $\tilde{\gamma}_{ij}, f_{ij}$. The covariant derivatives associated with $\gamma_{ij}, \tilde{\gamma}_{ij}, f_{ij}$ are respectively $D, \tilde{D}$, and $\mathring{D}$. The symbol $\mathring{\triangle}$ means $\mathring{\triangle} = \mathring{D}_k\mathring{D}^k$. It is $D_i\beta^k = \tilde{D}_i\beta^k + \tilde{C}^k{}_{ij}\beta^j$ and $\tilde{D}_i\beta^k = \mathring{D}_i\beta^k + C^k{}_{ij}\beta^j$ where $\tilde{C}^k{}_{ij} = \frac{2}{\psi}(\tilde{\gamma}^k{}_i\tilde{D}_j\psi + \tilde{\gamma}^k{}_j\tilde{D}_i\psi - \tilde{\gamma}_{ij}\tilde{\gamma}^{km}\tilde{D}_m\psi)$ and $C^k{}_{ij} = \frac{1}{2}\tilde{\gamma}^{km}(\mathring{D}_ih_{mj} + \mathring{D}_jh_{mi} - \mathring{D}_mh_{ij})$. Contraction on the first two indices results to $C^k{}_{kj} = \frac{1}{2\tilde{\gamma}}\mathring{D}_j\tilde{\gamma}$ and

$\tilde{C}^k{}_{kj} = \frac{1}{2\gamma}\tilde{D}_j\gamma$. For $\tilde{\gamma} = 1$, as in our computations, $C^k{}_{kj} = 0$. In the case where Cartesian coordinates are used for the flat metric, $f_{ij} = \delta_{ij}$ then $\mathring{D}$ is the usual partial derivative ∂, and $\mathring{\Delta}$ the Laplacian in Cartesian coordinates. The superscript $^{\rm TF}$ means the trace-free part. The conformal shift is defined as $\tilde{\beta}^i = \beta^i$ and thus $\tilde{\beta}_i = \tilde{\gamma}_{ij}\tilde{\beta}^j = \psi^{-4}\beta_i$. The matter sources that appear on the right-hand side of Eqs. (3)-(7) are $\rho_{\rm H} = T_{\mu\nu}n^\mu n^\nu$, $j_i = -T_{\mu\nu}\gamma^\mu{}_i n^\nu$, $S = T_{\mu\nu}\gamma^{\mu\nu}$, and $S_{ij} = T_{\mu\nu}\gamma^\mu{}_i\gamma^\nu{}_j$.

Eq. (3) is the Hamiltonian constraint, Eq. (4) as well as Eq. (5) are the momentum constraints, Eq. (6) is the spatial trace of the Einstein's equation for $\partial_t K_{ij}$ combined with the Hamiltonian constraint, and Eq. (7) is the spatial tracefree part of Einstein's equation. Eqs. (4), (5) imply that we solve for the momentum constraint twice. This idea has been used succesfully in [20] and there are two reasons for adopting this method. First by introducing Eq. (1) one has now two expressions for the conformal traceless extrinsic curvature, the second being

$$\tilde{A}_{ij} = \frac{1}{2\alpha}[(\tilde{\mathbb{L}}\tilde{\beta})_{ij} - \tilde{u}_{ij}], \qquad \tilde{u}_{ij} = [\partial_t\tilde{\gamma}_{ij}]^{\rm TF}, \tag{8}$$

which involves the shift vector. Solving for β^i is necessary since it will be used in the computation of $\mathcal{L}_{\alpha\mathbf{n}}K$ in Eq. (6) and $\mathcal{L}_{\alpha\mathbf{n}}(\psi^4\tilde{A}_{ij})$ in Eq. (7). The second reason is that the introduction of Eq. (1) (i.e. resolving the momentum constrain for $\tilde{W}_i$) enables us to obtain *apparent horizon penetrating solutions*. In particular since in our method we use excision, the use of this extra decomposition makes possible the use of grids that excise a region *inside the apparent horizon*, which facilitates the evolution of our systems. Without decomposition (1) the system of Eqs. (3), (5), (6), (7) with Kerr-Schild inner boundary conditions and extrinsic curvature given by Eq. (8) converges only when the excised region is *outside* the apparent horizon. The faster the black hole spins the further out one has to perform the excision.

In this work we choose $\partial_t\tilde{\gamma}_{ij} = 0$ and therefore $\tilde{u}_{ij} = 0$. Similarly we assume $\partial_t\tilde{A}_{ij} = \partial_t K = 0$. This is consistent with stationary systems like rotating stars or a Kerr black hole. In binary systems where one typically assumes a helical symmetry, $k^\alpha = t^\alpha + \Omega\phi^\alpha$, a better choice would be $\mathcal{L}_{\mathbf{k}}\tilde{\gamma}_{ij} = 0 = \mathcal{L}_{\mathbf{k}}\tilde{A}_{ij}$ which results to $\tilde{u}_{ij} = -\Omega(\tilde{\mathbb{L}}\phi)_{ij}$.

Another important term in our system is the one that involves $\tilde{\mathcal{R}}_{ij}$, the 3-d Ricci tensor associated with the conformal geometry $\tilde{\gamma}_{ij}$. One can show [18] that

$$\tilde{\mathcal{R}}_{ij} = -\frac{1}{2}\mathring{\Delta}h_{ij} + \tilde{\mathcal{R}}^{\rm KS}_{ij} + \tilde{\mathcal{R}}^{\rm NL}_{ij}, \tag{9}$$

where

$$\tilde{\mathcal{R}}^{\rm KS}_{ij} = -\frac{1}{2}(f_{ik}\mathring{D}_jF^k + f_{jk}\mathring{D}_iF^k), \tag{10}$$

$$\begin{aligned}\tilde{\mathcal{R}}^{\rm NL}_{ij} = &-\frac{1}{2}(h^{ab}\mathring{D}_a\mathring{D}_bh_{ij} + \mathring{D}_ih^{ab}\mathring{D}_bh_{aj} + \mathring{D}_jh^{ab}\mathring{D}_ah_{ib})\\ &-\frac{1}{2}[\mathring{D}_i(h_{kj}F^k) + \mathring{D}_j(h_{ik}F^k)]\\ &-\mathring{D}_iC^k{}_{kj} + C^k{}_{km}C^m{}_{ij} + F^kC_{kij} - C^k{}_{im}C^m{}_{kj},\end{aligned} \tag{11}$$

and $F^i = \mathring{D}_a\tilde{\gamma}^{ia}$. Notice that the terms $\tilde{\mathcal{R}}^{\rm KS}_{ij}$, $\tilde{\mathcal{R}}^{\rm NL}_{ij}$ also enter into Eq. (7), which we discuss below. The nonlinear term $\tilde{\mathcal{R}}^{\rm NL}_{ij}$ is second order in h_{ij} and therfore smaller than the first order terms $\tilde{\mathcal{R}}^{\rm KS}_{ij}$ and $\mathring{\Delta}h_{ij}$ in Eq. (9). The term $\tilde{\mathcal{R}}^{\rm KS}_{ij}$ involves the gauge functions F^i which are identical to the $\tilde{\Gamma}^i$ in the Baumgarte-Shapiro-Shibata-Nakamura (BSSN) formulation [7]. For initial data, in order for the whole system (3-7) to converge, *these functions must be fixed* [18]. For rotating stars the Dirac gauge condition $F^i = 0$ was used [21, 22], which in the case of stationary and axisymmetric problems is also known to yield solutions numerically identical to the exact ones [21]. For binary neutron star systems the same Dirac gauge condition $F^i = 0$ was used in [19, 23] to produce the most accurate initial data, especially for the late inspiral binaries. Similarly [9] applied that gauge for single black hole spacetimes. Here, since we want to be able to retrieve the Kerr-Schild black hole, we set

$$F^j = \mathring{D}_ih^{ij}_{\rm KS}. \tag{12}$$

where $h^{ij}_{\rm KS}$ are the exact Kerr-Schild potentials. Eq. (12) are gauge conditions that are related to our freedom in choosing spatial coordinates. Setting $F^i = 0$ for black holes may yield solutions qualitatively close to Kerr-Schild, but spins only up to 0.85 [9].

Imposing conditions (12) to the solutions of the system Eq. (3)-(7), and thereby having a self-consistent iteration scheme, an adjustment is necessary for the h_{ij}. Following [19], (or [22] Eq. (29-32)) gauge vector potentials ξ^a introduced in the transformation

$$\delta\gamma^{ab} \to \delta\gamma^{ab} - \mathring{D}^a\xi^b - \mathring{D}^b\xi^a, \tag{13}$$

are used to adjust h^{ab} as

$$h^{ab\prime} = h^{ab} - \mathring{D}^a\xi^b - \mathring{D}^b\xi^a + \frac{2}{3}f^{ab}\mathring{D}_c\xi^c, \tag{14}$$

where now $h^{ab\prime}$ are chosen to satisfy the condition $\mathring{D}_bh^{ab\prime} = F^a$ given by (12). The gauge vector potentials ξ^a are solved from the elliptic equations,

$$\mathring{\Delta}\xi^a = \mathring{D}_bh^{ab} - \frac{1}{3}\mathring{D}^a\mathring{D}_b\xi^b - F^a, \tag{15}$$

and then h_{ab} are replaced by Eq. (14).

In our method we use excision [24] with inner boundary conditions being the exact Kerr-Schild values on some excised sphere, chosen inside the outer black hole horizon. For outer boundary conditions we use ones that lead to an asymptotically flat spacetime. The augmented system of the 17 elliptic equations (3)-(7), (15) with $\tilde{\sigma} = 1$ and zero boundary conditions for the gauge potentials and the vector $\tilde{W}_i$ converges smoothly in vaccum or in the presence of matter (like a massive disk), even for near maximally-spinning black holes. In a typical iteration we first solve Eq. (4) to obtain $\tilde{W}_i$, then $\tilde{A}_{ij}$ is constructed through Eq. (1) which is then used in the right hand side of Eqs. (3)-(7) to compute the rest of the potentials. We want to emphasize that any solution of our method not only satifies the constraint equations but it *also solves for the conformal geometry, thus providing a way to control the gravitational wave content of the initial data in a self-consistent way.*

III. FORMULATION FOR THE FLUID

As a first application of our new formulation we compute massive disks in the presence of tilted black holes. Such systems will inevitably have rich behaviour as they *cannot* be in equilibrium [25–27].

We assume that the stress energy tensor is decribed by a perfect fluid with 4-velocity u^α: $T^{\alpha\beta} = \rho h u^\alpha u^\beta + p g^{\alpha\beta}$, where h is the specific enthalpy, ρ the rest-mass density and p the fluid pressure. It is $\rho h = \rho + p$. Bianch identity together with the 1^{st} law of thermodynamics $\rho dh = \rho T ds + dp$, implies $\nabla_\alpha T^{\alpha\beta} = \rho u^\beta \omega_{\alpha\beta} + h u_\alpha \nabla_\beta(\rho u^\beta) - \rho T \nabla_\alpha s = 0$. Here $\omega_{\alpha\beta} = \nabla_\alpha(h u_\beta) - \nabla_\beta(h u_\alpha)$ is the relativistic vorticity. By assuming conservation of rest-mass $\nabla_\alpha(\rho u^\alpha) = 0$ and an isentropic flow one arrives at the relativistic Euler equation $u^\beta \omega_{\alpha\beta} = 0$ [22].

The approximate symmetries that will be invoked will determine the fluid motion. One approximation that can be adopted is to extend the quasi-stationarity condition of the gravitational fields to the initial *fluid variables* as well. Thus assume $\mathcal{L}_{\mathbf{t}}(h u_\alpha) = \mathcal{L}_{\mathbf{t}}\rho = 0$. We can also assume that the fluid *motion* is axisymmetric about the fluid axis, which we take to be the z-axis. Thus $\mathcal{L}_{\boldsymbol{\phi}}(h u^\alpha) = \mathcal{L}_{\boldsymbol{\phi}}\rho = 0$, where ϕ^i the generator of rotations around the z-axis. Under these assumptions, and the fact that the matter 4-velocity can be written as $u^\alpha = u^t k^\alpha$ with $k^\alpha = t^\alpha + \Omega\phi^\alpha$, the Euler equation becomes $\nabla_\alpha \ln \frac{h}{u^t} + u^t u_\phi \nabla\Omega = 0$.

Assuming that the combination $u^t u_\phi$ is a function of the angular velocity Ω one arrives at

$$\frac{h}{u^t} e^{\int j(\Omega) d\Omega} \;=\; \mathcal{E}, \qquad j(\Omega) = u^t u_\phi, \tag{16}$$

where $\mathcal{E}$ a constant. In terms of the gravitational variables it is $j(\Omega) = \gamma_{ij}\omega^i\phi^j/(\alpha^2 - \gamma_{ij}\omega^i\omega^j)$, with $\omega^i = \beta^i + \Omega\phi^i$.

For this work we assumed a barotropic fluid EoS with $p = K\rho^\Gamma$, where K, Γ are constants, but our code can compute more exotic EoSs as in [28] or even in a tabulated form. Given an EoS, as well as a differential rotation form for $j(\Omega)$ [29], the algebraic equation (16) must be solved together with (3)-(7), (15), and (4) in order to compute the fluid variables u^t, ρ (alternatively h or p) and the constant $\mathcal{E}$ together with the gravitational variables.

IV. NUMERICAL IMPLEMENTATION

For the numerical solution of the Poisson-type of equations, Eqs. (3)-(7), (15), and (4), we use the Komatsu-Eriguchi-Hachisu (KEH) method for black holes, which was first developed in [24] and implemented within the COCAL code in [12]. The Green's functions used in the representation formula match the boundary conditions that we impose on our variables, $\{\psi, \tilde\beta_i, \alpha, h_{ij}, \xi^i, \tilde W_i\}$, and in the present calculation are the Dirichlet-Dirichlet functions (for all variables), Eq. (B8) in [24]. A single spherical $(\hat r, \hat\theta, \hat\phi)$ grid is used, identical to the black-hole grids of [24], with uniform intervals in $\hat\theta, \hat\phi$ and non-uniform intervals in $\hat r$. In the solutions presented here we used $N_r \times N_\theta \times N_\phi = 660 \times 48 \times 48$ intervals that cover the whole space $\hat r \in [\hat r_a, \hat r_b], \hat\theta \in [0,\pi]$, and $\hat\phi \in [0, 2\pi]$. Here $\hat r_a$ denotes the excised sphere inside the horizon and $\hat r_b = 10^5 m$. Convergence studies in the new formulation will be presented elsewhere [30].

Apart from the isolated Kerr solution we have computed as a check massless axisymmetric disks in the pressence of a Kerr black hole using the differential law $\Omega = k\ell^\alpha$ [13] where k, α constants and $\ell = -u_\phi/u_t$ the specific angular momentum. Note that $j(\Omega) = u_t u^\phi = \ell/(1 - \Omega\ell)$. For a black hole spin $a/m = 0.9$, differential law parameter $\alpha = -17/3$, polytropic index $\Gamma = 1.\bar{4}$, and inner point disk characteristics $\ell_{\rm in}/m = 3.313$, $r_{\rm in}/m = 6$ our solution shows excellent agreement (maximum density agrees to four significant digits) with a calculation of [31], used to generate an equilibrium solution prescribed in [14] and constructed via the ILLINOIS GRMHD code [32].

V. TILTED BLACK-HOLE-TORUS SYSTEM

The first self-gravitating black hole-toroidal systems have been computed by Nishida & Eriguchi, [33], (see also Stergioulas [34]), while more recently, using different methods, by Ansorg & Petroff [35], as well as Shibata [36]. All authors computed equilibria by solving the 2-d problem of stationary and axisymmetric Einstein equations.

With our new method we computed sequences of full 3-d nonaxisymmetric solutions of self-gravitating tori around tilted black holes. In order to do that we fix the inner point of the torus along the x-axis (here we used $\hat r_{\rm in} = 8m$) and the maximum rest-mass density inside the torus (but not its position). No assumptions are made regarding the shape or the outer boundary of the torus. Solving the equation of hydrostatic equilibrium (16) together with (3)-(7), (15), and (4) we obtain one black hole-toroid model. Then we slightly increase the rest-mass density and recompute the same equations. In this way a sequence of black hole-toroids with increasing mass of the torus is obtained. For low mass ratios and low black hole spins one model needs ~ 100 itera-

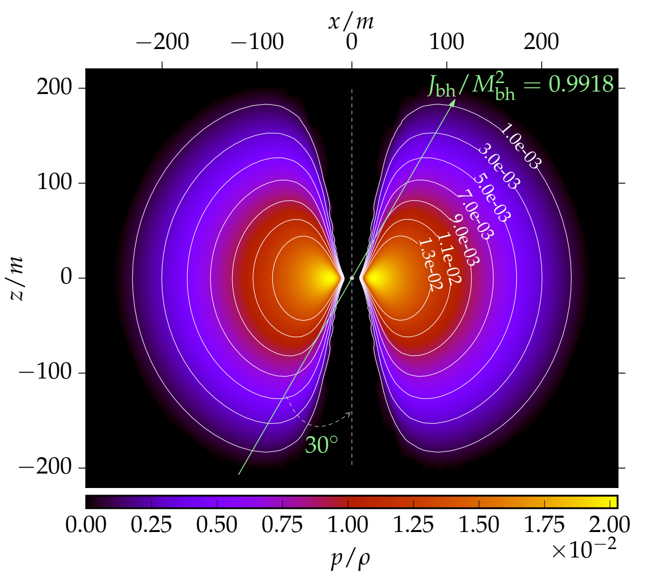

FIG. 1. Density plot of the function p/ρ (pressure over rest-mass density) on the $x-z$ plane for the solution presented in the (*Tilted black-hole-torus system*) section. All units are in $G = c = M_\odot = 1$. The inner part of the torus corresponds to $\hat{r}_{\rm in} = 8m$.

tions, while for high mass ratios and high spins ~ 1500 iterations are required. A model is computed when all gravitational and fluid variables have a difference $\sim 10^{-7}$ between two successive iterations.

For the particular example shown in Fig. 1 we have black hole parameters, $a/m = 0.95$ tilted at an angle $\hat{\theta} = 30°, \hat{\phi} = 0$ (these parameters determine the 3 + 1 quantities of the initial background solution $\alpha_{\rm KS}$, $\beta^i_{\rm KS}$, $\psi_{\rm KS}$, $h^{\rm KS}_{ij}$), a barotropic EoS with $K = 123.6$, $\Gamma = 2$, and a rotation law of the form $j(\Omega) = A^2\Omega\left[\left(\frac{\Omega_c}{\Omega}\right)^q - 1\right]$ with $A = 0.1$, $q = 1$ [22]. The two constants that appear in our equations, $(\mathcal{E}, \Omega_c)$, are computed by evaluating Eq. (16) at two points (the inner fixed point of the torus and the point of maximum density) and solving the resulting nonlinear system at each iteration.

The disk has maximum height of $200m$ and maximum width of $242m$ with inner point at $\hat{r} = 8m$ as seen in Fig. 1. Its rest mass is $M_0 = 5.181m$. The apparent horizon of the vacuum Kerr black hole is at $r_+/m = 1 + \sqrt{1-(a/m)^2} = 1.312$. It intersects the x-axis at $\hat{r}_+ = 1.5233m$ while its intersection with the z-axis happens at $\hat{r}_+ = 1.3726m$. Our grid covers the region $\hat{r} \in [1.2498, 10^5]m$ and all angles, and excises the region $\hat{r} < 1.2498$. The system has Arnowitt-Deser-Misner (ADM) mass $M = 6.144m$ and angular momentum $J^i = (0.4908, -0.001519, 31.19)m^2$.

The angular momentum of the black hole is calculated through the isolated horizon formalism [38], $J^i_{\rm bh} = (0.5169, -0.0006792, 0.8925)m^2$. Using the apparent horizon finder described in [24] we calculated the mass of the black hole [37] to be $M_{\rm bh} = 1.0198m$. Thus the dimensionless spin of the black hole in the black hole-toroid system is $J_{\rm bh}/M^2_{\rm bh} = 0.9918$. The angular momentum of the torus is then $J^i_{\rm t} = J^i - J^i_{\rm bh} = (-0.02614, -0.0008393, 30.29)m^2$. If one uses a Komar integral to calculate the angular momentum of the torus the result is $J^i_{\rm t,Komar} = (0, 0, 30.17)m^2$ which shows good agreement in the z-component. This model was the last member of a sequence of black hole-toroids with increasing rest-masses starting from an infinitesimal disk of rest mass $\sim 10^{-4}m$ around a Kerr black hole of dimensioneless spin $a/m = 0.95$. As the torus gains mass and angular momentum it spins-up the black hole. The last model computed here with $M_0 = 5.181m$ has spinned up the black hole to almost maximal spin. A further increase in the angular momentum and mass of the torus in a quasi-equilibrium state is impossible since it will drive the spin of the black hole beyond the maximum value.

VI. DISCUSSION

In this work we present a new formulation for the initial value problem in general relativity for spacetimes that contain a black hole and the first nonaxisymmetric black hole-disk solution. Here the disk is ~ 5 times more massive than the black hole and the hole has near-extremal spin.

Our formulation provides a good starting point for numerical evolution calculations. Unlike other methods it does not assume a conformal metric (6 components) but instead 3 gauge conditions (3 components) chosen to match known, closely related, physical models (e.g. Kerr-Schild black holes or axisymmetric stars). For stationary axisymmetric spacetimes our formulation yields the unique equilibrium solutions. For nonaxisymmetric spacetimes our solutions are not equilibria, but in contrast to other commonly adopted formulations, they provide a way of controling the gravitational wave content in a self-consistent way.

Although in the present article we used excision, it would not be difficult using the same decompositions to solve also for puncture initial data (by decomposing the conformal factor and solving the Hamiltonian for the regular part), which are widely also used. We think that our method will be useful in the gravitational wave detection-multimessenger astronomy era since it can compute more accurate initial values needed for simulations similarly to what the original waveless formulation did for binary neutron stars [19, 23]. Problems such as junk radiation, better imposition of helical symmetry, or more accurate resolution of tidal effects are examples where our new method can be more appropriate than current studies.

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