Exciting the scalar ghost mode through time evolution

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Do the Spirits Rise?

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ABSTRACT

A nonlocal gravity model based on $\frac{1}{\Box}R$ achieves the phenomenological goals of generating cosmic acceleration without dark energy and of suppressing the growth of perturbations compared to the ΛCDM model. Although the localized version of this model possesses a scalar ghost, the nonlocal version does not suffer from any obvious problem with ghosts. Here we study the possibility that the scalar ghost mode might be uncontrollably excited through time evolution, even though it is initially absent. We present strong evidence that this does not happen, so the analogy is to the conformal mode of general relativity which can be excited but only in a controlled way.

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1 Introduction

The evidence is strong that our universe is currently undergoing a phase of accelerated expansion [1–8], however, there is no similarly strong indication concerning the cause. The simplest solution is the ΛCDM model, in which acceleration is driven by a very small cosmological constant. This model fits the observed expansion history when the fractional energy densities of the cosmological constant, of nonrelativistic matter and of radiation take the respective values $\Omega_\Lambda \approx 0.7$, $\Omega_m \approx 0.3$, $\Omega_r \approx 8.5 \times 10^{-5}$. However, the ΛCDM model raises some theoretical concerns:

1. Why is the energy density of the cosmological constant $\rho_\Lambda^{\rm obs} \sim (10^{-3}{\rm eV})^4$ so small compared to the natural energy densities of fundamental theory?
2. Why does $\rho_\Lambda^{\rm obs}$ have a value which causes it to become dominant so recently in cosmic history?

These are, respectively, the old and new problems of the cosmological constant [9–12].

There have also been extensive efforts to explain cosmic acceleration by modified gravity [13–15]. The only local, metric-based, generally coordinate invariant and potentially stable class of models are based on generalizing R in the Einstein-Hilbert Lagrangian to $f(R)$ [16]. However, these models can only reproduce the ΛCDM expansion history for the ΛCDM choice of $f(R) = R - 2\Lambda$ [17]. Among the three remaining options of using fields other than the metric to carry part of the gravitational force, breaking general covariance or abandoning locality [18, 19], we consider a metric-based, invariant, nonlocal modification based on distorting the Einstein-Hilbert Lagrangian by an algebraic function of the nonlocal scalar $\Box^{-1}R$ defined with retarded boundary conditions [20],

$$S_{\rm DW} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[R + R f\Big(\frac{1}{\Box} R\Big) \right] . \tag{1}$$

Because the nonlocal scalar $\Box^{-1}R$ is dimensionless this class of models avoids the introduction of the new mass scale which is so problematic for the ΛCDM model. It also incorporates two features which naturally delay the onset of cosmic acceleration to very late times:

- Nothing happens during radiation domination because $R = 0$; and
- Even after matter domination the growth of $\Box^{-1}R$ is only logarithmic in the co-moving time, so that its current value is about -14.

The algebraic function $f(X)$ can be chosen for negative X to reproduce the ΛCDM expansion history [21–23]. By taking $f(X)$ to vanish for positive X one completely avoids the problems inside gravitationally bound systems that are so challenging for $f(R)$ models.

Closely related nonlocal models have also exploited the delayed response of $\Box^{-1}R$ [24–26]:

$$S_{\rm MM} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[R - m^2 R\frac{1}{\Box^2}R\right] , \tag{2}$$

$$S_{\rm VAAS} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[R + m^2 \frac{1}{\Box}R\right] , \tag{3}$$

$$S_{\rm ABN} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[R - m^4 \frac{1}{\Box^2}R\right] . \tag{4}$$

These models approximately reproduce the ΛCDM expansion history, however, they all require a new mass parameter m^2 which of the same order as the ΛCDM cosmological constant. In each of the models (1-4) perturbations about the cosmological background show deviations from the ΛCDM model. For (1) the growth rate on the largest scales is reduced, relative to the ΛCDM model, which improves the fit to existing data [27,28]. The trend is opposite for (2), although not enough to falsify the model [29–34].

Each of the nonlocal models (1-4) can be re-cast in a localized form by the introduction of auxiliary scalar fields. For the original model (1) the localized version employs scalar fields X and U [35–42],

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\Big[R + Rf(X) + g^{\mu\nu}\partial_\mu X\partial_\nu U + UR\Big] . \tag{5}$$

Varying with respect to U and X and substituting the solutions in (5) seems to recover the original, nonlocal form (1),

$$\frac{16\pi G}{\sqrt{-g}}\frac{\delta S}{\delta U} = -\Box X + R = 0 \quad\Rightarrow\quad X = \frac{1}{\Box}R + X_{\rm homo} , \tag{6}$$

$$\frac{16\pi G}{\sqrt{-g}}\frac{\delta S}{\delta X} = Rf'(X) - \Box U = 0 \quad\Rightarrow\quad U = \frac{1}{\Box}\Big(Rf'(X)\Big) + U_{\rm homo} , \tag{7}$$

where $\Box X_{\rm homo} = 0 = \Box U_{\rm homo}$. Whereas the localized model (5) could be regarded as a fundamental theory, which might be subjected to quantization, the presence of the inverse d'Alembertian in the original, nonlocal model (1) means that it can only be treated as an effective field theory. In fact it was proposed to represent the most cosmologically significant part of the quantum gravitational effective action induced by graviton loops during primordial inflation [20].

Another important difference between the localized model and its nonlocal ancestor concerns degrees of freedom. The localized version (5) contains two scalar degrees of freedom, corresponding to the arbitrary initial value data which determine the homogeneous solutions $X_{\rm homo}$ and $U_{\rm homo}$ in relations (6) and (7). In the original, nonlocal version (1) the fields X and U obey retarded boundary conditions, that is, both they and their first time derivatives vanish on the initial value surface. Hence the nonlocal model (1) lacks the two extra scalar degrees of freedom which are present in its localized counterpart (5). This difference is crucial because the field redefinition $A_\pm = \frac{1}{2}(X \pm U)$ reveals that A_+ is a ghost field,

$$g^{\mu\nu}\partial_\mu X \partial_\nu U = g^{\mu\nu}\Big[\partial_\mu A_+ \partial_\nu A_+ - \partial_\mu A_- \partial_\nu A_-\Big] \; . \tag{8}$$

Relation (8) has two important consequences [19, 43]:

- The original, nonlocal model (1) is a constrained version of the localized model (5) in which the scalars X and U and their first derivatives vanish on the initial value surface; and
- The localized model (5) suffers from a kinetic energy instability whereas the original, nonlocal model (1) may be stable.

In a stable theory one can only excite one degree of freedom by lowering the excitation of some other degree of freedom. Because there is only a finite amount of energy available in any given initial system, there is an upper limit to the wave number of a mode which can be excited. In contrast, interacting field theories with a kinetic energy instability, such as (5), are driven to a peculiar time evolution in which negative energy modes of arbitrarily high wave number are excited, along with corresponding positive energy degrees of freedom. The conformal mode of general relativity would engender precisely such a kinetic instability were it not constrained to be nondynamical. The original, nonlocal model (1) has a chance of avoiding the kinetic instability because the ghost mode of its local counterpart (5) is similarly constrained to be nondynamical. The purpose of this paper is to check that it stays that way. That is, we seek to confirm that the evolution of permitted perturbations does not lead to explosive excitation of the ghost mode.

Note that we are *not* claiming the ghost mode remains zero, any more than stability proofs of general relativity require the conformal factor to remain unity. In fact the ghost mode is nonzero even in the background solution [21–23], just as the conformal factor of general relativity expands in the cosmological background.[1] We will show that perturbations of the ghost field also become nonzero but that they do so in a controlled way.

If we had an energy functional the task would be simple: we would merely establish that the Hamiltonian is bounded below. Unfortunately, there is no energy functional for gravitating systems in cosmology. What we will do instead is to follow the evolution of scalar plane wave perturbations about the cosmological background, both with X and U obeying retarded boundary conditions and with them obeying a variety of more general initial conditions. Of course retarded boundary conditions correspond to the original, nonlocal model (1), whereas more general initial conditions access the ghost mode. The radical contrast between these two cases provides strong evidence that no ghost appears in the original, nonlocal model (1).

[1] Another parallel between the ghost mode and the conformal factor of general relativity is that they can both be fixed by a gauge choice. Of course employing such a gauge in no way avoids the instability that would result without initial value constraints.

This paper has four sections, of which this Introduction is the first. In section 2 we give the linearized field equations for scalar plane wave perturbations in cosmology. Section 3 presents the results of numerical evolution from various initial conditions. Our conclusions comprise section 4.

2 Cosmological scalar perturbations

The field equations of any metric-based modification to gravity can be expressed as,

$$G_{\mu\nu} + \Delta G_{\mu\nu} = 8\pi G T_{\mu\nu} \ , \tag{9}$$

where $G_{\mu\nu}$ and $T_{\mu\nu}$ are the usual is Einstein tensor and stress-energy tensor, respectively. The modification appropriate to (1) is,

$$\Delta G_{\mu\nu} = \Bigl[G_{\mu\nu} + g_{\mu\nu}\Box - D_\mu D_\nu\Bigr]\Bigl\{f\Bigl(\frac{1}{\Box}R\Bigr) + \frac{1}{\Box}\Bigl[R f'\Bigl(\frac{1}{\Box}R\Bigr)\Bigr]\Bigr\} \\ + \Bigl[\delta^{(\rho}_{\mu}\delta^{\sigma)}_{\nu} - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\Bigr]\partial_\rho\Bigl(\frac{1}{\Box}R\Bigr)\partial_\sigma\Bigl(\frac{1}{\Box}\Bigl[R f'\Bigl(\frac{1}{\Box}R\Bigr)\Bigr]\Bigr) \ , \tag{10}$$

where $\Box^{-1}$ is always defined with retarded boundary conditions. The analogous localized form is,

$$\Delta G_{\mu\nu} = \Bigl[G_{\mu\nu} + g_{\mu\nu}\Box - D_\mu D_\nu\Bigr]\Bigl[f(X) + U\Bigr] + \Bigl[\delta^{(\rho}_{\mu}\delta^{\sigma)}_{\nu} - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\Bigr]\partial_\rho X \partial_\sigma U \ . \tag{11}$$

Recall again that (11) only agrees with (10) when the scalars X and U and their first derivatives vanish on the initial value surface. We will use this version of the theory, first with retarded boundary conditions and then with more general initial conditions.

We consider scalar metric perturbations (in Newtonian gauge) about a homogeneous, isotropic and spatially flat background,

$$ds^2 = -\Bigl[1 + 2\widetilde{\Psi}(t,\vec{x})\Bigr]dt^2 + a^2(t)\Bigl[1 + 2\widetilde{\Phi}(t,\vec{x})\Bigr]d\vec{x}\cdot d\vec{x} \ . \tag{12}$$

The corresponding auxiliary scalars take the form,

$$X(t,\vec{x}) = \overline{X}(t) + \widetilde{X}(t,\vec{x}) \qquad , \qquad U(t,\vec{x}) = \overline{U}(t) + \widetilde{U}(t,\vec{X}) \ . \tag{13}$$

Each of the tilde-carrying perturbation fields can be decomposed into spatial plane waves,

$$\widetilde{\Psi}(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}}\Psi(t,\vec{k}) \qquad , \qquad \widetilde{\Phi}(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}}\Phi(t,\vec{k}) \ , \tag{14}$$

$$\widetilde{X}(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}}\delta X(t,\vec{k}) \qquad , \qquad \widetilde{U}(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}}\delta U(t,\vec{k}) \ . \tag{15}$$

We use a slightly different notation for the energy density,

$$T_{00}(t,\vec{x}) = \rho(t) + \widetilde{\rho}(t,\vec{x}) = \rho(t)\left[1 + \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}}\delta(t,\vec{k})\right] . \tag{16}$$

Because each spatial plane wave evolves independently at linearized order we will simply give equations for the Fourier components $\Psi(t,\vec{k})$, $\Phi(t,\vec{k})$, $\delta X(t,\vec{k})$, $\delta U(t,\vec{k})$ and $\delta(t,\vec{k})$.

It remains to give the equations for the background quantities, and for the linearized perturbation fields. We first define the background nonlocal distortion function and its first derivative,

$$\overline{f} \equiv f\Big(\overline{X}(t)\Big) \qquad , \qquad \overline{f}' \equiv f'\Big(\overline{X}(t)\Big) . \tag{17}$$

The modified background metric field equations (9) are,

$$3H^2 + [3H^2 + 3H\partial_t](\overline{f} + \overline{U}) + \frac{1}{2}\partial_t\overline{X}\partial_t\overline{U} = 8\pi G\rho , \tag{18}$$

$$-(2\dot{H} + 3H^2) - [2\dot{H} + 3H^2 + 2H\partial_t + \partial_t^2](\overline{f} + \overline{U}) + \frac{1}{2}\partial_t\overline{X}\partial_t\overline{U} = 8\pi G p , \tag{19}$$

and the background auxiliary scalar field equations (6-7) are,

$$-(\partial_t^2 + 3H\partial_t)\overline{X} = 6(\dot{H} + 2H^2) , \tag{20}$$

$$-(\partial_t^2 + 3H\partial_t)\overline{U} = 6(\dot{H} + 2H^2)\overline{f}' . \tag{21}$$

The background fields $\overline{X}(t)$ and $\overline{U}(t)$, and their first derivatives, vanish on the initial value surface. In the sub-horizon regime of $k \gg Ha$ the equations for linearized perturbations are [28],

$$k^2\Phi + k^2\left[\Phi(\overline{f} + \overline{U}) + \frac{1}{2}(\overline{f}'\delta X + \delta U)\right] = 4\pi G a^2\rho\delta , \tag{22}$$

$$(\Phi + \Psi) + (\overline{f}'\delta X + \delta U) + (\Phi + \Psi)(\overline{f} + \overline{U}) = 0 , \tag{23}$$

$$\ddot{\delta} + 2H\dot{\delta} = -\frac{k^2}{a^2}\Psi , \tag{24}$$

$$\Big(-\partial_t^2 - 3H\partial_t - \frac{k^2}{a^2}\Big)\delta X = 2\frac{k^2}{a^2}(\Psi + 2\Phi) , \tag{25}$$

$$\Big(-\partial_t^2 - 3H\partial_t - \frac{k^2}{a^2}\Big)\delta U = 2\frac{k^2}{a^2}(\Psi + 2\Phi)\overline{f}' . \tag{26}$$

3 Perturbation growth with and without the ghost

The purpose of this section is to compare the evolution of scalar perturbations in the original, nonlocal model (1) — which may be stable — with perturbations in the localized theory (5) — which is certainly not stable. In both cases the evolution equations are (22-26); the difference between the two models is the initial conditions obeyed by $\delta X(t,\vec{k})$ and $\delta U(t,\vec{k})$. The initial conditions corresponding to the original, nonlocal model (1) are that these fields and their first derivatives vanish on the initial value surface. We first evolve from retarded boundary conditions, then explore a variety of more general conditions, and finally contrast the results.

The actual evolution is performed with respect to the cosmological redshift,

$$1+z \equiv \frac{a_{\rm now}}{a(t)} \Longrightarrow \frac{d}{dt} = -(1+z)H(z)\frac{d}{dz} \quad \text{where} \quad H(z) = H_0\sqrt{\Omega_r(1+z)^4 + \Omega_m(1+z)^3 + \Omega_\Lambda}\ . \quad (27)$$

The function $f(X)$ was chosen to make the expansion history exactly that of the ΛCDM model [21–23]. We keep the initial conditions for the perturbations $\Phi(t,\vec{k})$, $\Psi(t,\vec{k})$ and $\delta(t,\vec{k})$ the same for all the cases:

$$\Phi(z_i) \,=\, \Phi_{\rm GR}(z_i)\ ,\ \Psi(z_i) = \Psi_{\rm GR}(z_i) = -\Phi_{\rm GR}(z_i)\ , \quad (28)$$

$$\delta(z_i) \,=\, \delta_{\rm GR}(z_i) = \frac{2k^2 a(z_i)}{3H_0^2\Omega_m}\Phi_{\rm GR}(z_i)\ ,\ \delta'(z_i) = \delta'_{\rm GR}(z_i)\ . \quad (29)$$

We set $z_i = 9$ (corresponding to $t_i = 0.55$ Gyrs) and $k = 100H_0 = 0.03h\text{Mpc}^{-1}$ as in [28, 44–46]. By choosing $z_i = 9$ we can safely set the initial conditions for Φ, Ψ and δ the same as in general relativity (GR) because the nonlocal modifications are negligible for $z > 5$ [21]. The scale of $k = 100H_0$ is small enough to take the subhorizon limit (or the quasi-static limit) and large enough to keep the perturbations linear [44, 45].

3.1 Perturbations without the ghost

The perturbation equations of the original nonlocal version are equivalent to these localized equations as long as the initial conditions for δX and δU are

$$\text{IC0}:\ \delta X(z_i) = 0\ ,\ \delta U(z_i) = 0\ ,\ \delta X'(z_i) = 0\ ,\ \delta U'(z_i) = 0\ . \quad (30)$$

Here the $z_i = 9$ is the redshift corresponding to the initial time $t_i = 0.55$ Gyrs. We denote this set of initial conditions by "IC0". Figure 1 presents the various results. Note the absence of large fluctuations in $\Phi(t,\vec{k})$, $\Psi(t,\vec{k})$ and $\delta(t,\vec{k})$.

3.2 Perturbations with the ghost

Of course there are infinitely many variations of the retarded boundary conditions IC0 (30). In order not to prejudice the theory towards strong growth it makes sense to parameterize initial conditions

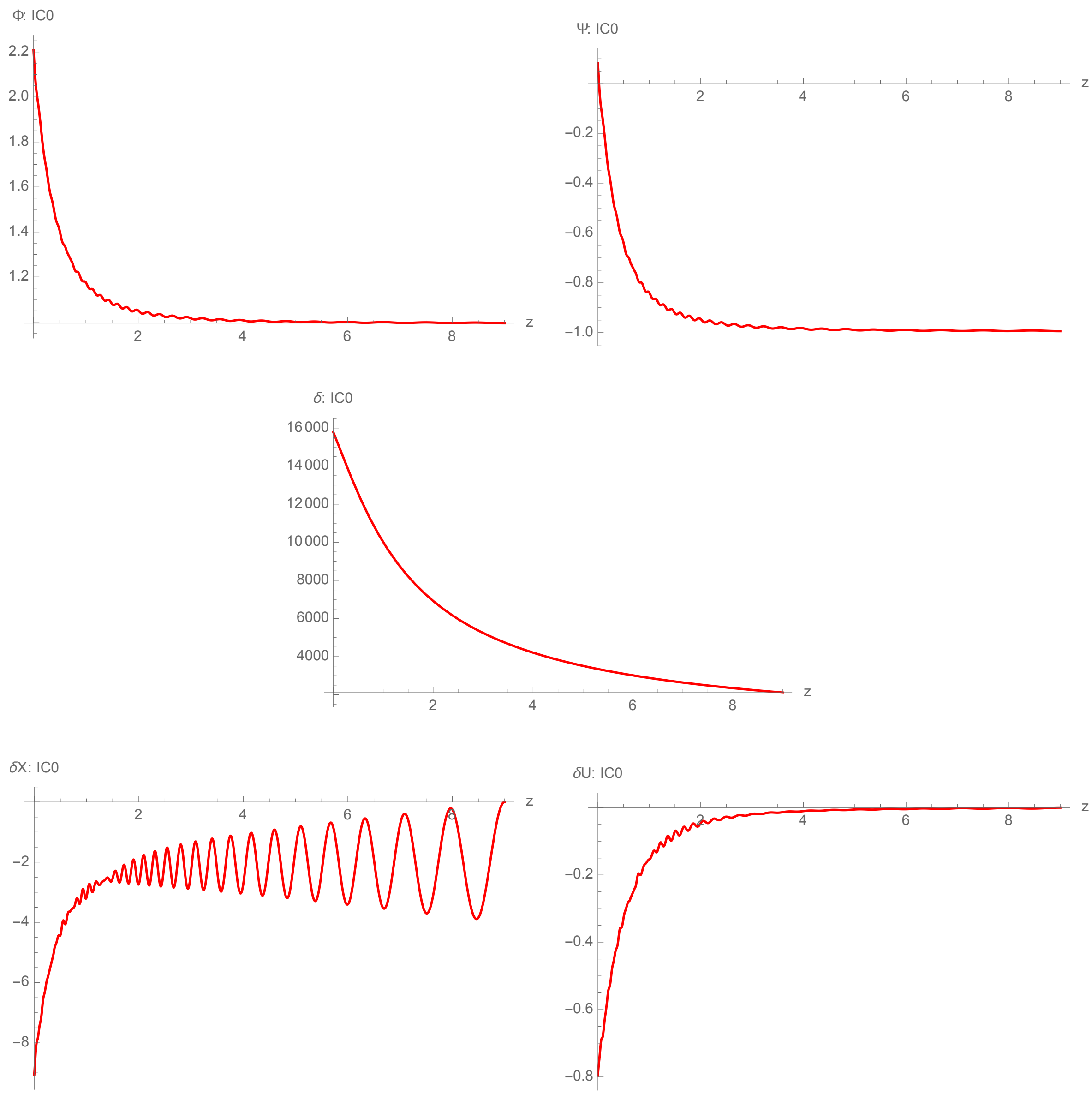

Figure 1: The evolution (as a function of redshift) of $\Phi(t,\vec{k})$, $\Psi(t,\vec{k})$, $\delta(t,\vec{k})$, $\delta X(t,\vec{k})$ and $\delta U(t,\vec{k})$ starting from IC0 initial conditions (30).

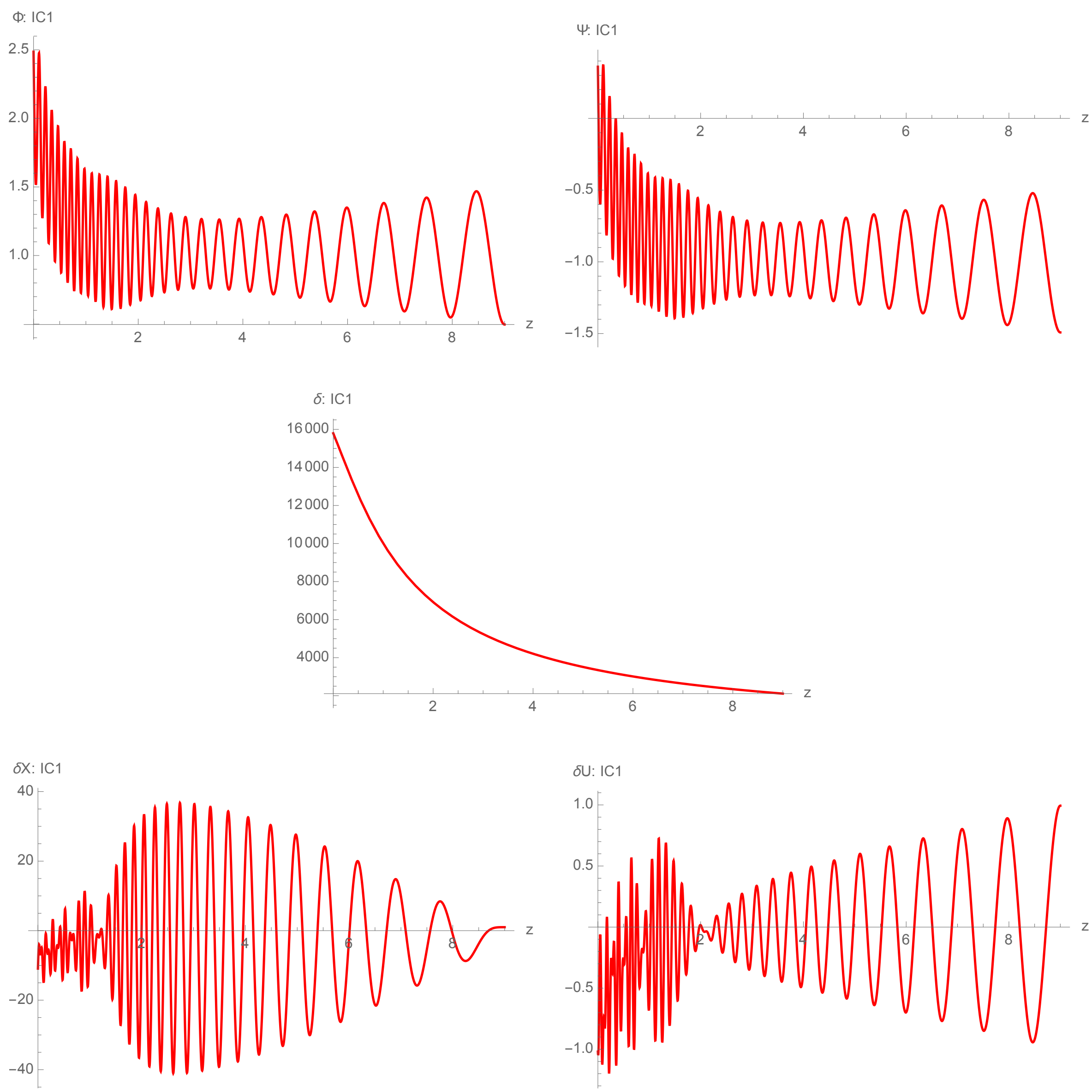

Figure 2: The evolution (as a function of redshift) of $\Phi(t,\vec{k})$, $\Psi(t,\vec{k})$, $\delta(t,\vec{k})$, $\delta X(t,\vec{k})$ and $\delta U(t,\vec{k})$ starting from IC1 initial conditions (31).

for $\delta X(t,\vec{k})$ and $\delta U(t,\vec{k})$ in terms of the metric potentials and the density perturbation. Because the latter fields initially agree with general relativity, for which $\Psi(t,\vec{k}) = -\Phi(t,\vec{k})$, we are reduced to just $\Phi(t,\vec{k})$ and $\delta(t,\vec{k})$. A simple condition based on $\Phi(t,\vec{k})$ is,

$$\text{IC1}: \ \delta X(z_i) = \Phi(z_i) \ , \ \delta U(z_i) = \Phi(z_i) \ , \ \delta X'(z_i) = \Phi'(z_i) \ , \ \delta U'(z_i) = \Phi'(z_i) \ , \tag{31}$$

We call this "IC1" and the results for it are given in Fig. 2. Note that the fluctuations in $\delta X(t,\vec{k})$ have about 20 times the amplitude of those with the IC0 initial condition of Fig. 1. Note also that these fluctuations are communicated to the metric potentials $\Phi(t,\vec{k})$ and $\Psi(t,\vec{k})$.

A reasonable initial condition involving the density perturbation $\delta(t,\vec{k})$ is,

$$\text{IC2}: \ \delta X(z_i) = \delta(z_i) \ , \ \delta U(z_i) = \delta(z_i) \ , \ \delta X'(z_i) = \delta'(z_i) \ , \ \delta U'(z_i) = \delta'(z_i) \ . \tag{32}$$

We call this "IC2" and the results for it are given in Fig. 3. Because the density perturbation is so much larger than the metric potentials the resulting fluctuations in $\delta X(t,\vec{k})$ have about $40,000$ times the amplitude of those with the IC0 initial condition of Fig. 1! The fluctuations of the metric potentials $\Phi(t,\vec{k})$ and $\Psi(t,\vec{k})$ are similarly enhanced with respect to those of IC0.

We explored many other initial conditions, for example,

$$\text{IC1-a}: \ \delta X(z_i) = \Phi(z_i) \ , \ \delta U(z_i) = \Phi(z_i) \ , \ \delta X'(z_i) = -\Phi'(z_i) \ , \ \delta U'(z_i) = -\Phi'(z_i) \ , \tag{33}$$

$$\text{IC1-b}: \ \delta X(z_i) = \Phi(z_i) \ , \ \delta U(z_i) = \Phi(z_i) \ , \ \delta X'(z_i) = \Phi(z_i) \ , \ \delta U'(z_i) = \Phi(z_i) \ . \tag{34}$$

We have not reported them because the results are very similar to those of IC1 (31). The same is true for the variants of IC2 (32) which involve the density perturbation $\delta(t,\vec{k})$.

3.3 Comparing IC0 with IC1 and IC2

Figure 1 shows the evolution of normal perturbations, whereas Figures 2 and 3 depict the evolution of perturbations in which the ghost field $\delta A_+ = \frac{1}{2}[\delta X(t,\vec{k}) + \delta U(t,\vec{k})]$ is excited. The contrast between IC0 (without the ghost) and the other conditions IC1-2 (with the ghost) is striking. It becomes even more so in Figure 4, which displays just $\delta A_\pm$ for all three cases, both as functions of redshift z and as functions of the co-moving time t. Recall that a kinetic instability manifests through the ghost field (δA_+) experiencing a wild time evolution, and conserving energy by dragging along the normal fields, in this case δA_-. The IC1 and IC2 initial conditions show this quite clearly, whereas that behavior is not at all apparent with the IC0 initial conditions.

Of course the amplitudes $\delta A_\pm$ have no immediate physical meaning. Because the ghost instability is associated with kinetic energy we would like a measure of how much kinetic energy resides in $A_\pm$. Of course there is no true energy functional for gravity in cosmology so one cannot expect complete precision but a rough measure of the kinetic energy in $A_\pm$ derives from their stress tensors

$$\mp T^{\pm}_{\mu\nu} = \partial_\mu A_\pm \partial_\nu A_\pm - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} \partial_\rho A_\pm \partial_\sigma A_\pm \ . \tag{35}$$

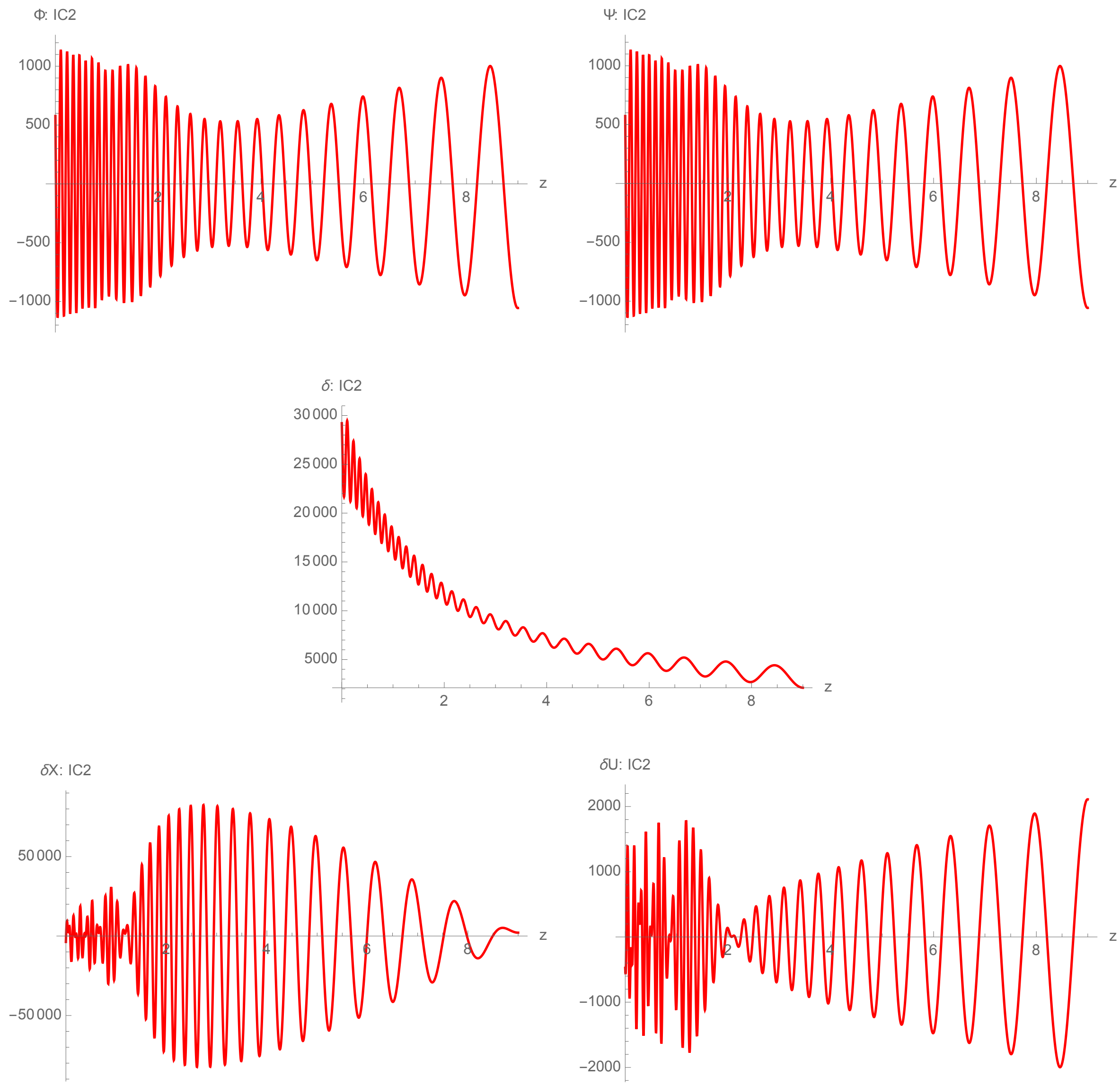

Figure 3: The evolution (as a function of redshift) of $\Phi(t,\vec{k})$, $\Psi(t,\vec{k})$, $\delta(t,\vec{k})$, $\delta X(t,\vec{k})$ and $\delta U(t,\vec{k})$ starting from IC2 initial conditions (32).

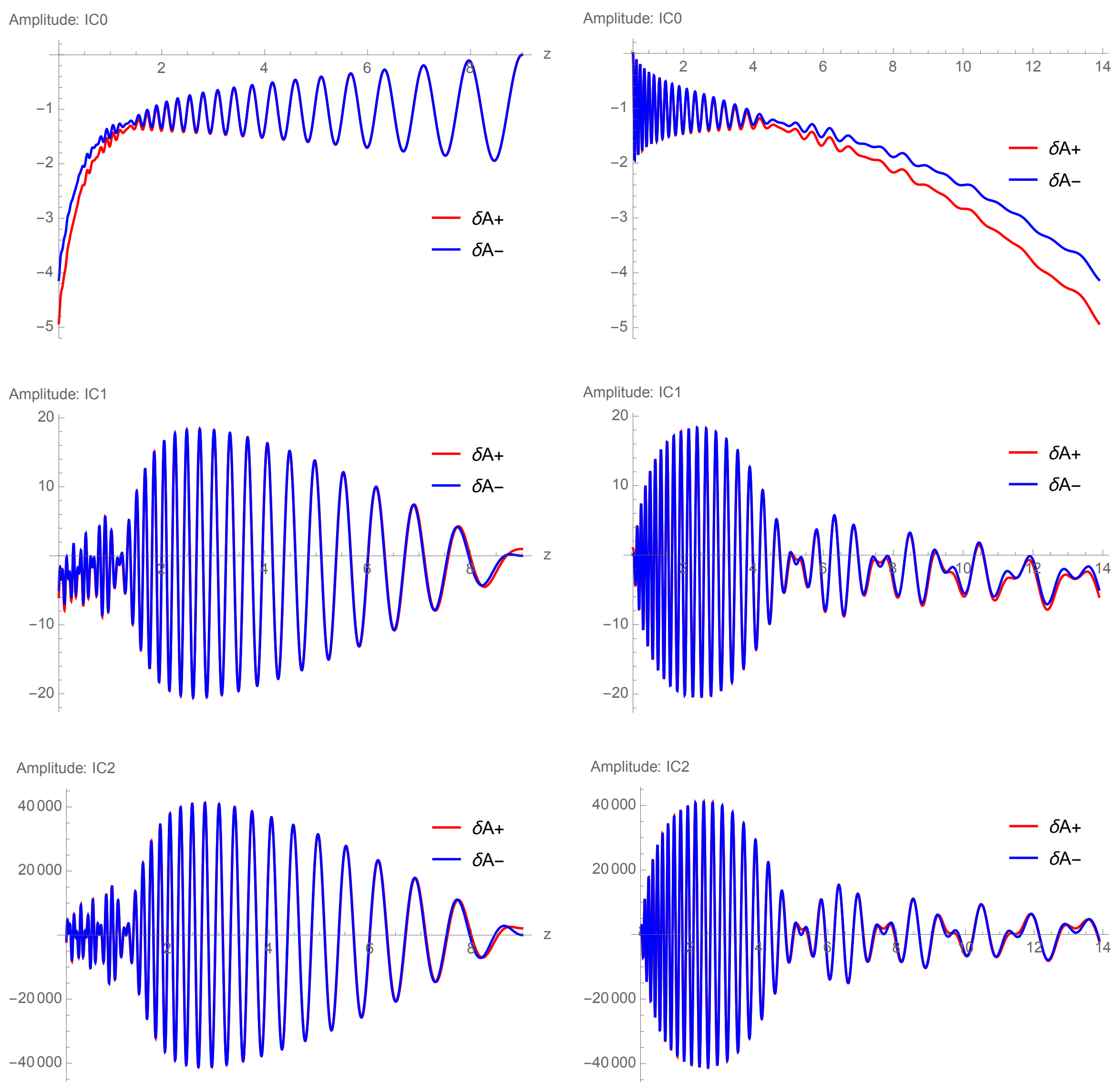

Figure 4: Amplitudes of the perturbations $\delta A_{\pm} = \frac{1}{2}(\delta X \pm \delta U)$ versus the redshift z (on the left) and versus the co-moving time t (in the right) for the initial conditions IC0 (30), IC1 (31), and IC2 (32).

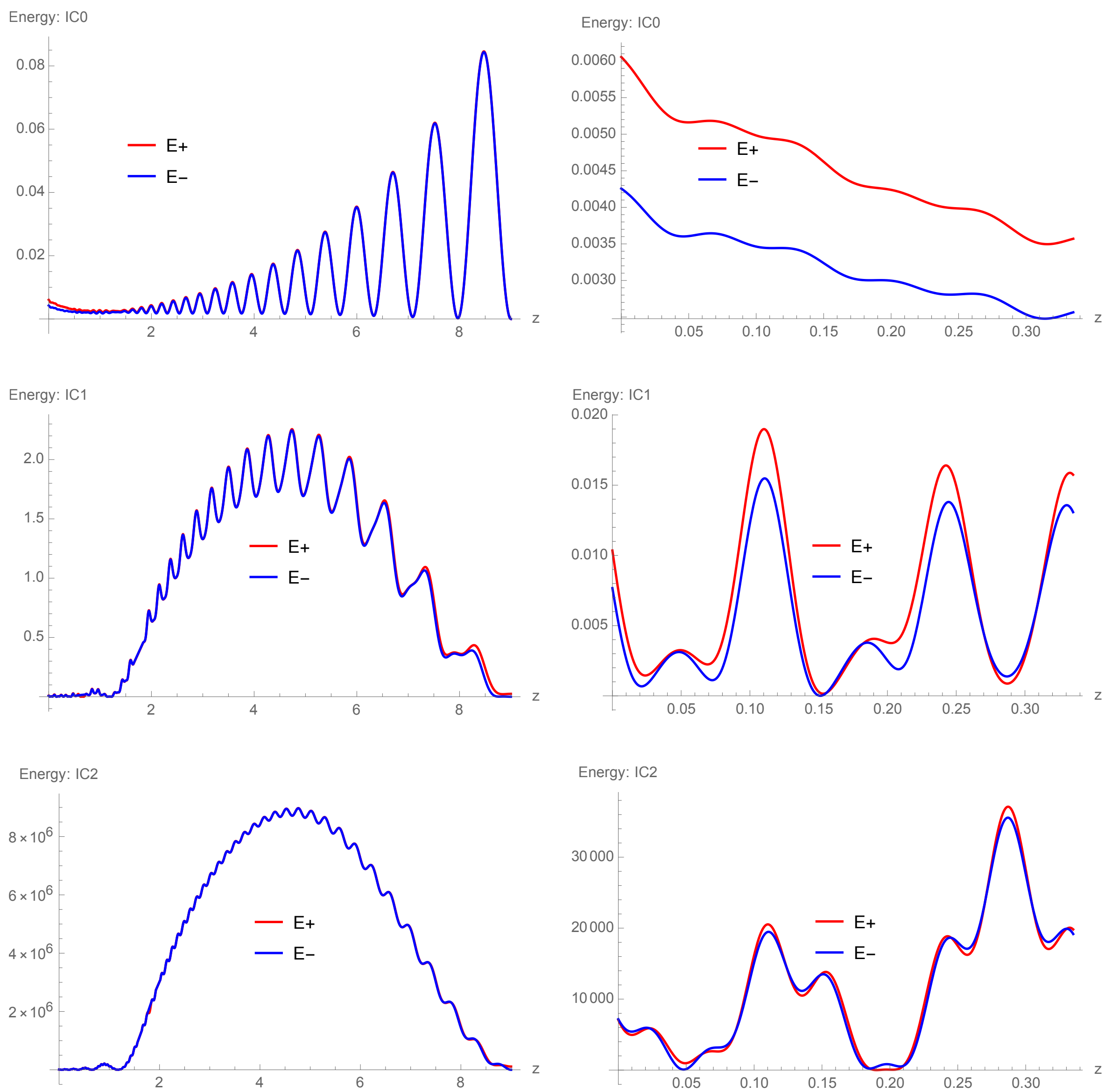

Figure 5: The kinetic energies $E_{\pm}$ (36) versus redshift z for the initial conditions IC0 (30), IC1 (31), and IC2 (32). In each case the left hand graphs show the full range $0 < z < 9$, whereas the right hand graphs provide an expanded view of the late time regime $0 < z < 0.34$.

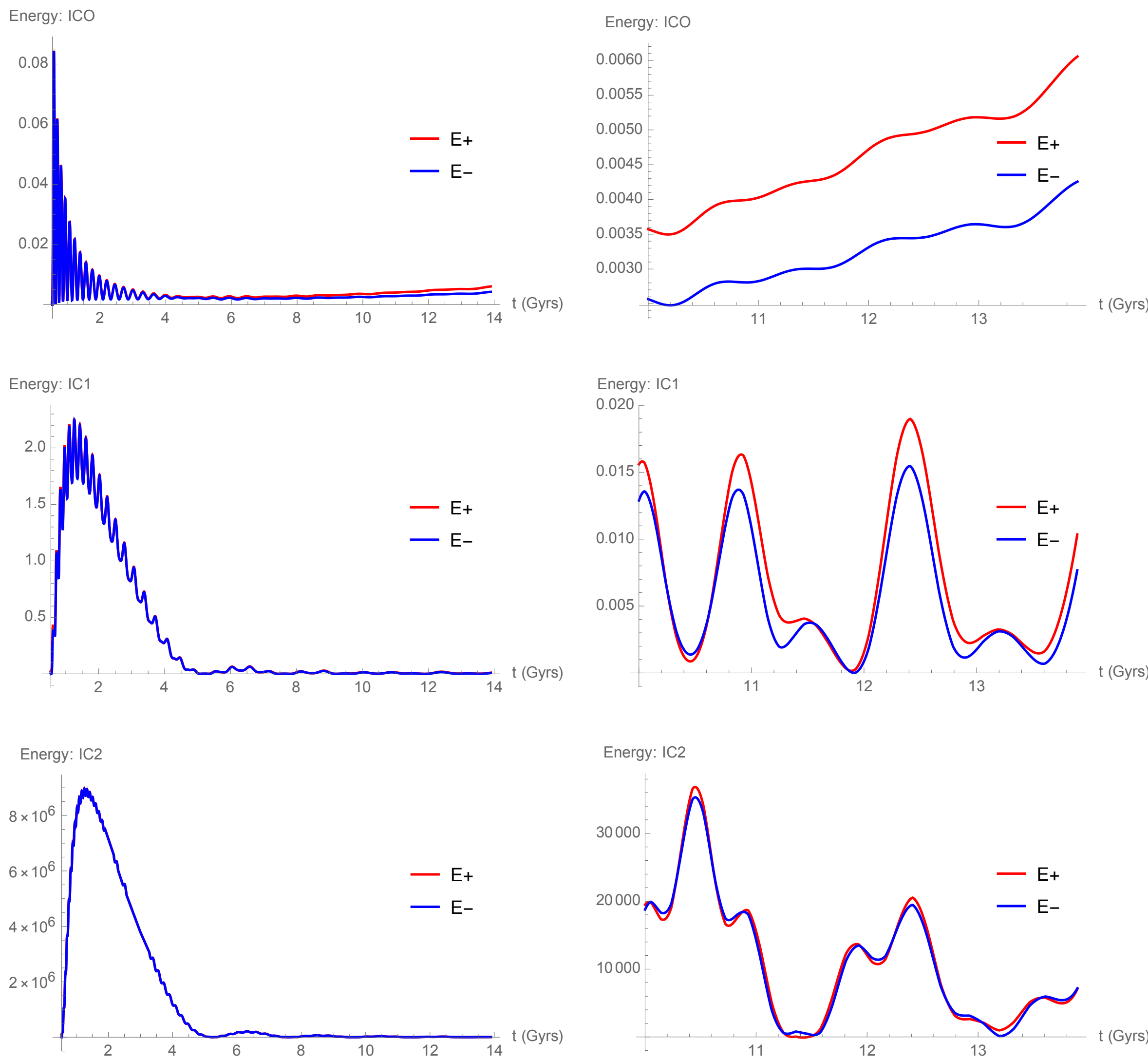

Figure 6: The kinetic energies $E_{\pm}$ (36) versus the co-moving time t for the initial conditions IC0 (30), IC1 (31), and IC2 (32). In each case the left hand graph shows the full range 0.55 Gyr $< t <$ 13.89 Gyr whereas the right hand graphs provide an expanded view of the late time regime 10 Gyr $< t <$ 13.89 Gyr.

One can recognize $T^+_{\mu\nu} + T^-_{\mu\nu}$ as the final term in relation (11) for the localized version of the modified Einstein tensor. Perturbing $T^{\pm}_{\mu\nu}$ around the cosmological background induces linearized spatial plane wave contributions which drop out of equations (22-23) in the sub-horizon regime of $k \gg Ha$. At quadratic order there are diagonal terms and mixings between the metric perturbations and the auxiliary scalars. A rough measure of how much kinetic energy resides in $A_\pm$ comes from the diagonal contributions,

$$E_\pm \equiv \frac{1}{2}\delta\dot{A}^2_\pm + \frac{1}{2}\frac{k^2}{a^2}\delta A^2_\pm \; . \tag{36}$$

Because the actual kinetic energy of the ghost mode is $-E_+$ we see again the terrible instability associated with ghosts. It costs *zero* total energy to start with arbitrarily large values of $\delta A_+(0,\vec{k}) = \delta A_-(0,\vec{k})$, at arbitrarily large wave numbers. That is all precluded by the retarded initial conditions of the original, nonlocal theory (1) but it is a fatal problem for the localized model (5).

Figures 5 and 6 show $E_\pm$ for each of the three initial conditions, first as functions of the redshift z and then in terms of the co-moving time t. For the non-ghost initial condition IC0 the energies E_+ and E_- have distinct evolutions, and are quite small. In contrast, $E_\pm$ are almost identical for the ghost conditions IC1 and IC2, and they are much larger than for IC0. For the non-ghost condition IC0 the energies steadily fall until very late times. For each of the two ghost conditions the energy increases to the point (about $z = 4.7$) at which the cosmological redshift begins to dissipate it. It must be remembered that these results follow from the *linearized* field equations

The peak in $E_\pm$ comes earlier, and is much higher, for larger wave numbers k. Figure 7 shows the result for $k = 500H_0$, at which the peak occurs at about $z = 5$. We confirmed the general trend by runs at $k = 300H_0$, $k = 700H_0$ and $k = 1000H_0$, but there is no point in presenting these graphs.

4 Discussion

Nonlocal cosmology (1) is not an attempt to replace general relativity but rather to provide a phenomenological representation of quantum infrared effects which grew nonperturbatively strong during the epoch of primordial inflation. The idea is that general relativity is the fundamental theory of gravity, but what we observe is the nonlocal effective field equations, just as quantum electrodynamics is the fundamental theory of charged matter, but the observed running of charge manifests in solutions to the nonlocal effective field equations. The degrees of freedom of nonlocal cosmology are the same as those of general relativity [19,43]. This is apparent from the fact that the inverse scalar d'Alembertian is defined with retarded boundary conditions on an initial value surface corresponding roughly to the epoch of primordial inflation.

In sharp contrast, the localized model (5) represents an alternate gravity theory in which two fundamental scalars figure. One of these scalars is a ghost, which means the localized theory suffers from a virulent kinetic instability that causes the ghost to be more and more highly excited, with a

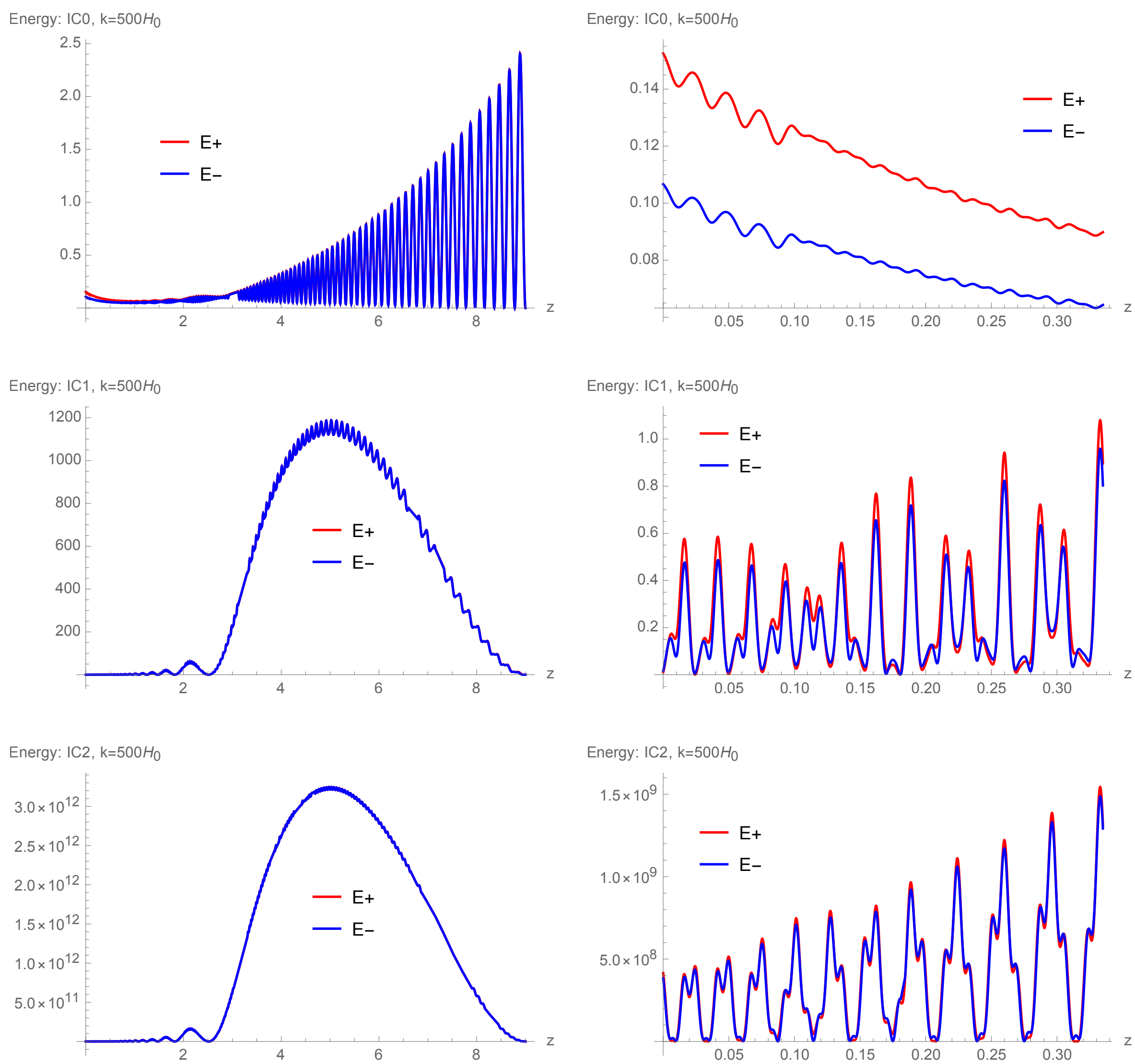

Figure 7: The $k = 500H_0$ kinetic energies $E_\pm$ (36) versus redshift z for the initial conditions IC0 (30), IC1 (31), and IC2 (32). In each case the left hand graphs show the full range $0 < z < 9$, whereas the right hand graphs provide an expanded view of the late time regime $0 < z < 0.34$.

consequent excitation of the positive energy degrees of freedom. That is apparent from the relative signs of the scalar stress tensors (35). Hence the localized model cannot possibly be acceptable. The worrisome thing for nonlocal cosmology is that one can view its Lagrangian (1) as a constrained version of (5) in which the two scalars and their first time derivatives vanish on the initial value surface. (Note that this constraint already precludes the worst instability of having arbitrarily large values of $\delta A_+(0,\vec{k}) = \delta A_-(0,\vec{k})$ at arbitrarily high wave numbers.) That does not necessarily condemn nonlocal cosmology to suffer the kinetic instability; the familiar conformal factor of unmodified general relativity would also be a ghost were it not for the Hamiltonian constraint. But it is prudent to check that the constraint of nonlocal cosmology is effective in controlling the ghost.

Note that the constraint does not compel the ghost field to remain zero, any more than the constraint of general relativity requires the conformal factor to remain unity. Both the ghost mode of nonlocal cosmology and the conformal factor of general relativity evolve even in the cosmological background. What we seek to show is rather that the constraint protects against explosive growth.

If we had an energy functional for nonlocal cosmology the check would be simple. In the absence of such an energy functional we have instead studied the evolution of linearized spatial plane wave perturbations about the cosmological background, both starting from the retarded boundary conditions (30) of nonlocal cosmology and with more general initial conditions (31) and (32). In Fig. 1 we see that the perturbations of nonlocal cosmology show no sign of the kinetic instability. Although the scalar $\delta X(t,\vec{k})$ does experience some decaying oscillations at early times, they are not communicated to the other fields. Evolutions from more general conditions are shown in Figures 2 and 3. In both cases the oscillations of $\delta X(t,\vec{k})$ are much larger, they grow, and they are communicated to the other perturbation fields. This is how a kinetic instability manifests.

Figure 4 gives the ghost and normal scalars, δA_+ and δA_-, respectively, for the three initial conditions. With retarded boundary conditions (IC0) the two experience some decaying oscillations at first and go on to distinct evolutions at late times. For the other boundary conditions (IC1 and IC2) the oscillations are much larger, they grow, and they are coupled. Recall that the ghost dragging along the other fields is what characterizes a kinetic instability. Figures 5 and 6 show the same thing using the magnitudes of the kinetic energies.

One thing we cannot do with the linearized field equations is exhibit the explosive instability associated with mixing from different wave vectors, when each one starts with general initial value data. However, within the limitations of what is easy to study numerically, our analysis has provided strong evidence against nonlocal cosmology suffering from the kinetic instability of its localized cousin. It also demonstrates why the localized version is so problematic.

Devising a full stability proof would require an energy functional, which does not exist for gravitating systems in cosmology. We suspect that this may not be as big an obstacle as it might seem because nonlocal cosmology approaches de Sitter at late times. So we propose adapting the famous result of Abbott and Deser [47] for general relativity with a positive cosmological constant. Instead of the Hilbert action we would use the localized Lagrangian (5) with the scalars constrained to obey retarded initial conditions. And instead of de Sitter providing the asymptotic conditions, it would be the background solution for nonlocal cosmology. Then we would try to prove positivity of the energy for sub-horizon fluctuations, just as Abbott and Deser did. That seems a worthy project for the future.

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