Theoretical and observed quality factor of gravitational quadrupoles

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Theoretical and Observed Quality Factor of Gravitational Quadrupoles

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Useful insights into the complicated physics of gravitational waves can often be drawn from approximations to analogous physics of electromagnetic waves. Here, we present a gravitational form of the electromagnetic Chu limit that sets bounds on the achievable Q (quality factor) relating the ratio of stored energy to radiated energy in the underlying fields. In particular, we answer two fundamental physics questions: 1) what is the theoretical Q for gravitational quadrupole radiation sources, and 2) can gravitational Q be observed from recent measured astronomical data? Gravitational Q is shown to follow an inverse seventh-order power law, and gravitational-wave data is used to find observed values of Q for GW170817. Inasmuch as electromagnetic Q serves a central role in design and analysis of electrically-small antennas, the proposed gravitational$Q$ offers the potential for a similar utility in the design and analysis of gravitationally-small detectors and quadrupoles.

I. INTRODUCTION

Einstein's 1916 general relativity field equations predicted the gravitational waves first observed in 2015, much as Maxwell's electromagnetic theory in the 1860's predicted the radio waves first observed by Hertz in 1880's and later applied by Marconi in the 1890's [1–4]. Even before Einstein's general relativity theory, similarities between electrostatics and Newtonian gravity lead Heaviside to propose a gravitational analog to electromagnetics [5]. Such parallels between gravitational and electromagnetic phenomena continue to be useful in understanding and developing concepts within general relativity [6]. Therefore, we similarly explore the development of the gravitational counterpart to a fundamental and useful concept in the theory of electromagnetic fields and antennas, the Chu limit [7–10].

For over 50 years, the Chu limit has been used to provide bounds on the Q, or quality factor, of antennas as a function of antenna size, where Q is related to the ratio of stored energy to radiated energy in electromagnetic fields. For antennas, Q is of further practical significance, since it corresponds to the ratio of antenna center frequency to bandwidth. Thus, Q is of fundamental importance in the design and analysis of antennas. Furthermore, the Chu limit has also recently been shown to roughly predict the quantum 1s-2p spontaneous emission relaxation time of hydrogen atoms [11].

This wide-ranging utility of the Chu limit, from quantum emission to antenna physics, raises the question of whether a similar Q can be derived and observed for gravitational quadrupole sources of gravitational waves. A second question is whether observation of such a gravitational Q may be out of reach, given the difficulty of gravitational wave measurements.

Here, we show that it is not only possible to analytically derive Q for gravitational quadrupole radiation sources, but we also show that LIGO (Laser Interferometer Gravitational-Wave Observatory) GW170817 data can yield observed values for Q over a range of gravitational quadrupole sizes [1, 12–15]. The proposed gravitational Q depends on gravitational-wave frequency and quadrupole size, and is shown to exhibit a different power law than the electromagnetic antenna Chu limit [16].

In the following, the theoretical Q of gravitational quadrupoles is derived, and observed values of gravitational Q are computed from gravitational-wave data. The combination of stored energy and radiation considered here is distinct from earlier results by Marengo and Ziolkowski for purely non-radiating sources and purely radiating sources in electromagnetics and general relativity [17]. As noted in earlier work by the author [18], the notion of gravitational Q developed here is distinct from the notion of mechanical Q of cryogenic spherical gravitational antennas in Merkowitz *et al.* [19]. Lastly, it should also be noted that recent non-Foster and metamaterial experiments show results much better than the electromagnetic Chu limit would permit, suggesting the potential that the proposed gravitational Q concepts could lead to similar gravitational detector enhancements [20–22].

II. THEORY

Before proceeding with the derivation of the Q of gravitational quadrupoles, it is helpful to first briefly review the Chu limit for electromagnetic antennas. The Chu limit (also known as Wheeler-Chu limit or Chu-Harrington limit) sets a lower bound on lossless linearly polarized electromagnetic antenna Q (quality factor) of [7, 10]

$$Q = \frac{1}{B} = \frac{f_0}{\Delta} = \frac{1}{k\,a} + \frac{1}{(k\,a)^3} \approx \frac{1}{(k\,a)^3} \text{ for } k\,a \ll 1\ , \quad (1)$$

where $B = \Delta/f_0$ is fractional bandwidth, Δ is antenna bandwidth in Hz, f_0 is antenna center frequency in Hz, a is radius of a sphere that would enclose the antenna, wavenumber $k = 2\pi f_0/c = 2\pi f_0/3 \times 10^8 = 2\pi/\lambda_0$

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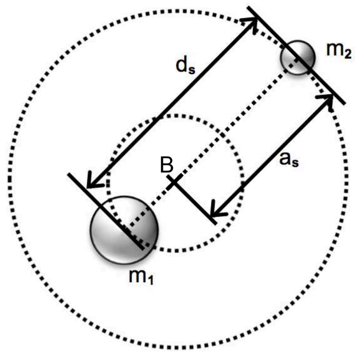

FIG. 1. Binary star system with circular orbit, solar masses $m_1 > m_2$, with barycenter denoted "B," orbital separation d_s, and the larger orbital radius being a_s.

rad/m, vacuum wavelength is λ_0 m, and size-parameter $k\,a \ll 1$ radian would constitute an electrically-small antenna. Extending the result to certain circular polarization cases results in a slightly altered cubic term of $0.5/(k\,a)^3$, as noted by Pozar [23]. Chu derived the limit from the energy of components of the electromagnetic field, and a summary in Sievenpiper [10] shows many decades of experimental results where the Chu limit is never violated.

Next, we derive the theoretical Q of gravitational quadrupoles. In this, it is first useful to express some gravitational-wave properties as a function of a wavenumber-size product, similar to the $k\,a$ parameter in the electromagnetic Chu limit. To begin, consider the binary circular orbit of Fig. 1. The luminosity in watts of a gravitational wave produced by the inspiral and merger of such a binary neutron star pair, or pair of black holes, is [13, 24, 25]

$$\mathcal{L} = \frac{32}{5}\frac{(m_1+m_2)^5\nu^2G^4}{d_s^5c^5} \tag{2}$$

where the binary star masses are m_1 and m_2 in kg, $\nu = m_1m_2/(m_1+m_2)^2$, and $G = 6.7\times10^{-11}$ N·(m/kg)^2 is the gravitational constant. The orbital separation d_s of the two stars in meters is

$$d_s = \left(\frac{(m_1+m_2)G}{\omega_{orb}^2}\right)^{1/3}, \tag{3}$$

where orbital frequency $\omega_{orb} = 2\pi f_{orb}$ in rad/s is

$$\omega_{orb} = \frac{\omega_{gw}}{2} = \frac{2\pi f_{gw}}{2} = \left(\frac{(m_1+m_2)G}{d_s^3}\right)^{1/2}, \tag{4}$$

for observed gravitational wave frequency ω_{gw} rad/s.

Recalling that Chu parameter "a" is the radius of a sphere enclosing an antenna, this would then correspond to the larger orbital radius a_s in the binary star circular orbit of Fig. 1, with

$$a_s = d_s\frac{m_1}{m_1+m_2} = \left(\frac{m_1^3G}{\omega_{orb}^2(m_1+m_2)^2}\right)^{1/3}, \tag{5}$$

where $m_1 > m_2$ and the smallest sphere enclosing the binary star orbits is of radius a_s m. Then, rearranging (5), the orbital frequency becomes

$$\omega_{orb} = \left(\frac{m_1^3G}{a_s^3(m_1+m_2)^2}\right)^{1/2}. \tag{6}$$

The proposed gravitational-antenna size parameter "$k\,a_s$" is then

$$k\,a_s = \frac{\omega_{gw}}{c}a_s = \frac{2\omega_{orb}}{c}a_s = \left(\frac{4m_1^3G}{a_sc^2(m_1+m_2)^2}\right)^{1/2} \tag{7}$$

It is also useful to rearrange (7) and solve for a_s as a function of $k\,a_s$, giving

$$a_s = \frac{4m_1^3G}{(k\,a_s)^2c^2(m_1+m_2)^2}, \tag{8}$$

and substituting for a_s in (6) yields orbital frequency ω_{orb} as a function of the $k\,a_s$ size parameter:

$$\omega_{orb} = \frac{(k\,a_s)^3c^3}{8G}\frac{(m_1+m_2)^2}{m_1^3}. \tag{9}$$

Alternatively, rearranging (9) gives size parameter $k\,a_s$ as a function of orbital frequency:

$$\begin{aligned} k\,a_s &= \left(\frac{8\,\omega_{orb}\,G\,m_1^3}{c^3(m_1+m_2)^2}\right)^{1/3} \\ &= \frac{2m_1}{c}\left(\frac{\omega_{orb}\,G}{(m_1+m_2)^2}\right)^{1/3}. \end{aligned} \tag{10}$$

Similarly, first substituting $d_s = a_s\,(m_1+m_2)/m_1$ in (2), and then taking a_s from (8), the luminosity as a function of size parameter $k\,a_s$ is

$$\begin{aligned} \mathcal{L} &= \frac{32}{5}\frac{(m_1+m_2)^5\nu^2G^4}{(a_s(m_1+m_2)/m_1)^5c^5} \\ &= \frac{\nu^2c^5}{160\,G}\left(\frac{m_1+m_2}{m_1}\;k\,a_s\right)^{10}, \end{aligned} \tag{11}$$

where $\mathcal{L}$ determines the gravitational radiation component later used in determining gravitational Q.

Next, the total available potential and kinetic energy of the gravitational system is determined. To begin, the Newtonian orbital energy is [24]

$$E_N = -\frac{m_1m_2G}{2d_s} = -\frac{m_1^2m_2G}{2a_s(m_1+m_2)}. \tag{12}$$

However, we propose that the total remaining available energy is determined by the difference between the orbital energy at any given orbital separation d_s and the final orbital energy at final orbital separation d_{min} at coalescence. Accordingly, the available orbital energy is defined as

$$\begin{aligned} E_A &= -\left(\frac{m_1m_2G}{2d_s} - \frac{m_1m_2G}{2d_{min}}\right) \\ &= -\frac{m_1}{m_1+m_2}\left(\frac{m_1m_2G}{2a_s} - \frac{m_1m_2G}{2a_{min}}\right) \\ &= \frac{m_1^2m_2G}{m_1+m_2}\left(\frac{1}{2a_{min}} - \frac{(k\,a_s)^2c^2(m_1+m_2)^2}{8m_1^3G}\right), \end{aligned} \tag{13}$$

where the minimum orbital separation can be determined from the observed gravitational wave data as $d_{min} = \left[(m_1+m_2)G/\omega_{max}^2\right]^{1/3}$ m, where ω_{max} rad/s is the maximum observed orbital frequency. For GW170817, $\omega_{max} \approx 600\pi$ rad/s, and $d_{min} \approx 47$ km.

Using the definition $Q = \omega E/(dE/dt)$ noted by Sievenpiper *et al.* [10], and substituting the foregoing results gives

$$Q = \frac{\omega_{orb} E_A}{dE_A/dt} = \frac{\omega_{orb} E_A}{\mathcal{L}} \tag{14}$$
$$= \frac{20 m_1^7 G}{m_2 c^2 (m_1+m_2)^5} \left(\frac{(k\,a_s)^{-7}}{2a_{min}} - \frac{(k\,a_s)^{-5} c^2 (m_1+m_2)^2}{8 m_1^3 G} \right),$$

which is the desired result giving theoretical gravitational quadrupole Q as a function of gravitational quadrupole wavenumber-size parameter $k\,a_s$. In particular, note that for $k\,a_s \ll 1$, that Q tends to become inversely proportional to the seventh power of $k\,a_s$. This inverse seventh-order proportionality differs significantly from the inverse cubic behavior of the electromagnetic Chu limit. Preliminary results in in [18] did not provide the equation for Q given above in (2). Since the luminosity in (2) and orbital energy in (13) account for all radiation losses, Q would include both gravitational quadrupole radiation polarizations, plus "+" and cross "×."

The result for gravitational Q in (14) is an equality rather than a theoretical bound, since it was derived for the particular binary orbit scenario of Fig. 1, rather than an arbitrary gravitational-wave source. However, for a given total mass $m_1 + m_2$ with $m_1 \geq m_2$ and for some given $k\,a_s$ during inspiral, it can be shown that Q from (14) is minimized when $m_1{=}m_2$ for $k\,a_s \ll 1$. In addition, analogies with electromagnetics would suggest that higher-order gravitational multipoles would tend to increase Q, although general relativity leads to considerably more complicated range of multipole expansions as noted by Thorne in [26]. Notwithstanding, the result in (14) is also limited by underlying slow-motion, weak-field, post-Newtonian assumptions in the derivation of (2), and is subject to non-relativistic limitations inherent in (12) for orbital energy [24]. Therefore, scenarios may exist where even lower values of Q are possible, and so it remains to be seen whether a more general result or lower bound for Q can be found, given the complexities of general relativity. Despite the foregoing limitations, the result for Q in (14) would seem appropriate for the most common scenarios of gravitational wave events, in light of Thorne's [26] comment that all gravitational-wave sources in the universe today are probably isolated sources.

III. OBSERVED GRAVITATIONAL Q

The foregoing equations show that gravitational quadrupole size parameter $k\,a_s$ can be determined from the gravitational wave frequency $\omega_{gw} = 2\omega_{orb}$, and that

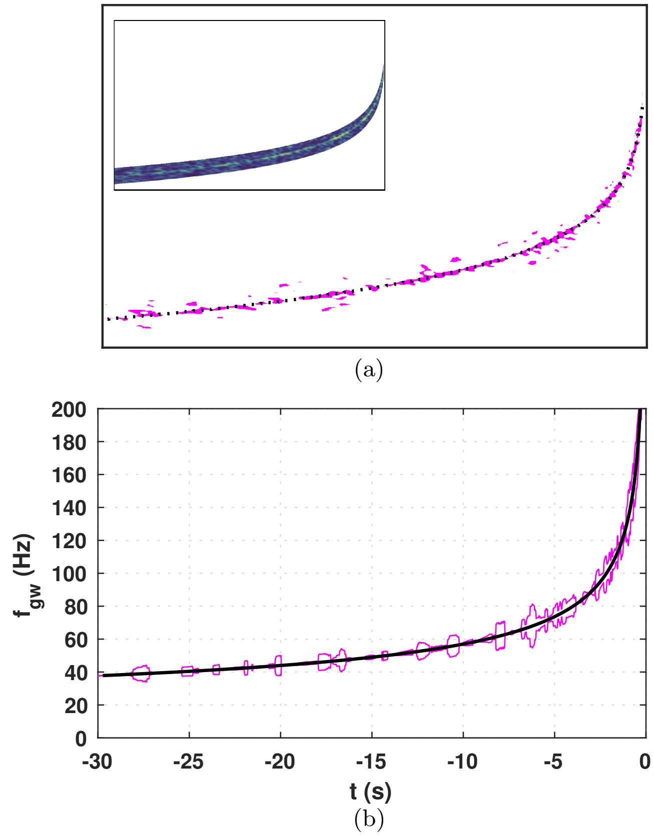

FIG. 2. Observed gravitational wave GW170817 frequency f_{gw} as a function of time t. (a) An extracted and thresholded portion of the time-frequency representation from Abbott *et al.* [16] where gray (magenta in color prints) regions are signals above threshold, the dotted curve is fitted f_{gw}, and the inset shows portion from Abbott *et al.* used for threshold. (b) Plot of final fitted f_{gw} shown as solid curve, with thin gray (magenta in color prints) curves indicating a two-standard-deviation error. Coalescence is at $t = 0$.

gravitational quadrupole Q can then be determined from $k\,a_s$. Therefore, estimation of ω_{gw} from observed gravitational data will yield both $k\,a_s$ and Q. We estimate $\omega_{gw} = 2\pi f_{gw}$ from the known time response of the frequency chirp as [25]

$$\frac{1}{f_{gw}} = \frac{8\pi}{125^{1/8}} \left(\frac{G^{5/3} m_1 m_2}{c^5 \left(m_1+m_2\right)^{1/3}}\, t' \right)^{3/8}, \tag{15}$$

where the time before coalescence is $t' = -t$. Taking the logarithm yields

$$\ln(f_{gw}) = 4.88 - 3\ln(t')/8\,, \tag{16}$$

for published GW170817 masses $m_1 \approx 3.6 \times 10^{30}$ kg and $m_2 \approx 2.2 \times 10^{30}$ kg [13].

Fig. 2(a) shows a thresholded version of the LIGO-Livingston time-frequency representation of gravitational wave GW170817 from Abbott *et al.* [16], extracting the

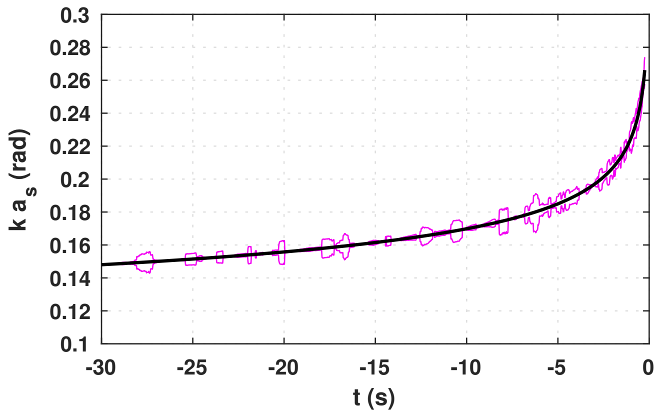

FIG. 3. Observed gravitational wave GW170817 gravitational quadrupole size parameter $k\,a_s$ as a function of time. Solid curve is for fit to data, with thin gray (magenta in color prints) curves indicating error in $k\,a_s$ that would correspond to the two-standard-deviation error in f_{gw} of Fig. 2(b). Coalescence is at $t = 0$.

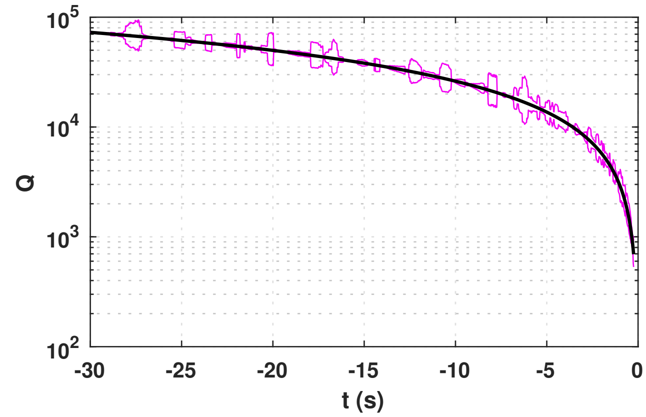

FIG. 4. Observed gravitational wave GW170817 quality factor Q as a function of time. Solid curve is for fit to data, with thin gray (magenta in color prints) curves indicating error in Q that would correspond to the thin gray (magenta in color prints) curves of error for $k\,a_s$ in Fig. 3. Coalescence is at $t = 0$.

strongest portions of the signal shown as the gray (magenta in color prints) regions near the dotted black curve. The upper left inset of Fig. 2(a) shows the portion of the LIGO-Livingston time-frequency data that was thresholded and used to generate Fig. 2(a). The vertical height of the sample shown in the inset is approximately 70 pixels at every frequency, even though the curvature of the chirp gives the illusion of a thinner vertical slice toward the right side of the inset.

The dotted black curve of Fig. 2(a) results from a cubic polynomial fit to the logarithm of gravitational frequencies f_{gw} above threshold in gray (magenta in color prints) regions. A cubic fit was chosen to provide extra degrees of freedom, in case observed data deviated significantly from the theoretical formula. The cubic fit shown is found to be $\ln(f_{gw}) = 4.884 - 0.3607\ln(t') - 0.001378\,[\ln(t')]^2 + 0.0002857\,[\ln(t')]^3$, in good agreement with the theoretical relation of (16). In fact, the last two terms of the polynomial only contribute 2.8% to $f_{gw} \approx 38$ Hz at $t = -30$ s, and only contribute 0.2% to $f_{gw} \approx 218$ Hz at $t = -0.25$ s.

The solid black curve in Fig. 2(b) shows the fitted curve to f_{gw} on linear scales for the 30 seconds before coalescence, along with thin gray (magenta in color prints) curves that indicate a two standard deviation error. The thin gray (magenta in color prints) curves in Fig. 2(b) are two standard deviations from the fitted curve at each time point, using the standard deviation of frequencies above threshold at each time point. At time instants where there are less than 6 frequency samples above threshold in Fig. 2(a), the standard deviation is set to zero in Fig. 2(b). However, the reader should note that that the two-standard-deviation error indicators do not include other error sources, such as uncertainty in masses m_1 and m_2. Coalescence occurs at $t = 0$ in the plot of Fig. 2(b).

To find the variation in $k\,a_s$ as a function of time, substituting $\omega_{orb} = \omega_{gw}/2$ in (10) gives

$$k\,a_s = \left(\frac{4\omega_{gw} G m_1^3}{c^3(m_1+m_2)^2}\right)^{1/3} . \qquad (17)$$

Applying this to Fig. 2(b), the solid black curve in Fig. 3 shows $k\,a_s$ of GW170817 as a function of time, corresponding to the solid black curve for the cubic fit to f_{gw} in Fig. 2(b). The thin gray (magenta in color prints) curves provide an indicator of error, and are directly computed from the thin gray (magenta in color prints) curves of error for f_{gw} in Fig. 2(b).

Lastly, the Q of the GW170817 gravitational quadrupole can be computed from $k\,a_s$ of Fig. 3, using (14). The solid black curve in Fig. 4 shows Q as a function of time, corresponding to the solid black curve for $k\,a_s$ in Fig. 3. The thin gray (magenta in color prints) curves provide an indicator of error, and are directly computed from the thin gray (magenta in color prints) curves of error for $k\,a_s$ of Fig. 3. At times where there are no frequency samples, the error is set to zero. Thus, the Q of the gravitational quadrupole is seen to decrease from approximately $Q \approx 7.1 \times 10^4$ where $k\,a_s \approx 0.15$ at 30 seconds before coalescence, to $Q \approx 1.2 \times 10^3$ where $k\,a_s \approx 0.25$ just before coalescence. As noted earlier, the thin gray (magenta in color prints) error indicators do not include other possibly significant error sources, such as uncertainty in masses m_1 and m_2.

IV. CONCLUSION

The theoretical Q of a gravitational quadrupole has been derived, and Q has been shown to be proportional to $(k\,a_s)^{-7}$ for gravitationally-small sources having $k\,a_s \ll 1$ rad. As for electromagnetic antennas, $k\,a_s$ is the size of the gravitational source in radians at the gravitational-wave frequency. The gravitational wave data was used to find time-dependent observed values of $k\,a_s$ for gravitational-wave GW170817. Importantly, $k\,a_s$ was shown to vary over a range of values from $k\,a_s \approx 0.15$ rad to $k\,a_s \approx 0.25$ rad using measured GW170817 data. This, in turn, allowed the variation of Q for GW170817 to be observed as $k\,a_s$ changed during the inspiral of the binary neutron star. A somewhat counterintuitive observation is that $k\,a_s$ of GW170817 increased over time, even as a_s decreased during the inspiral. Because $k\,a_s = \omega_{gw} a_s/c$, this counterintuitive result shows that ω_{gw} increases faster than a_s decreases during the inspiral of GW170817. We also note that alternative attempts to determine Q by independently estimating the luminosity numerator and the changing energy in the denominator of (14) tended to lead toward more noisy Q estimates, and such alternatives ultimately seemed to depend on the observable of gravitational-wave frequency and the fundamental result in (14).

The degree to which electrically-small antenna engineering insights may be applied to the understanding of binary inspiral gravitational fields and the design of gravitational-wave detectors remains an open question. In the results presented for GW170817, the binary neutron star quadrupole remained gravitationally small with $k\,a_s \ll 1$ rad during the observed 30-second inspiral. The analogous notion of electrically-small antennas with $k\,a \ll 1$ is an important concept for analyzing and understanding electromagnetic fields, and is a useful engineering tool for designing better antennas, for enhancing antenna signal strength, and for improving antenna bandwidth. Since Q provides a measure of the ratio of stored energy to radiated energy in electromagnetic fields of antennas, it is hoped that gravitational Q may offer similar utility in gravitational field analysis. In addition, Q is useful in the design of methods to enhance signal coupling from electromagnetic antennas, such as non-Foster, metamaterial, and passive-tuning methods [20–22]. Although the focus of the present work has been on gravitational radiation sources, similar results should apply to gravitational detectors under reciprocity considerations and the gravitationally-small dimensions of terrestrial detectors. Finally, the proposed gravitational Q concepts may lead to future gravitational detector design approaches, since recent non-Foster and metamaterial experiments have shown results much better than the electromagnetic Chu limit would permit [20–22].

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