



CHORUS

This is the accepted manuscript made available via CHORUS. The article has been published as:

Correlation patterns from massive phonons in 1+1 dimensional acoustic black holes: A toy model

Richard A. Dudley, Paul R. Anderson, Roberto Balbinot, and Alessandro Fabbri

Phys. Rev. D **98**, 124011 — Published 12 December 2018

DOI: [10.1103/PhysRevD.98.124011](https://doi.org/10.1103/PhysRevD.98.124011)

Correlation patterns from massive phonons in 1+1 dimensional acoustic black holes: a toy model

Richard A. Dudley* and Paul R. Anderson†

Department of Physics, Wake Forest University, Winston-Salem, North Carolina 27109, USA

Roberto Balbinot‡

*Dipartimento di Fisica dell'Università di Bologna and INFN sezione di Bologna, Via Irnerio 46, 40126 Bologna, Italy
Centro Fermi - Museo Storico della Fisica e Centro Studi e Ricerche Enrico Fermi, Piazza del Viminale 1, 00184 Roma, Italy*

Alessandro Fabbri§

*Centro Fermi - Museo Storico della Fisica e Centro Studi e Ricerche Enrico Fermi, Piazza del Viminale 1, 00184 Roma, Italy
Dipartimento di Fisica dell'Università di Bologna and INFN sezione di Bologna, Via Irnerio 46, 40126 Bologna, Italy
Laboratoire de Physique Théorique, CNRS UMR 8627,
Bât. 210, Université Paris-Sud 11, 91405 Orsay Cedex, France
Departamento de Física Teórica and IFIC, Universidad de Valencia-CSIC, C. Dr. Moliner 50, 46100 Burjassot, Spain*

Transverse excitations in analogue black holes induce a mass like term in the longitudinal mode equation. With a simple toy model we show that correlation functions display a rather rich structure characterized by groups of parallel peaks. For the most part the structure is completely different from that found in the massless case.

PACS numbers:

I. INTRODUCTION

Recent years have witnessed a growing interest in the so called analogue (gravity) models. These are condensed matter systems (Bose-Einstein condensates are the most studied) where one can mimic classical and quantum features present in a gravitational black hole (BH) or in cosmology [1].

The major effort has been devoted to the study of the analogue of Hawking's quantum BH radiation [2] that should appear in fluids when the flow undergoes a subsonic to supersonic transition [3]. This theoretical prediction has been confirmed for the first time by experiments performed by Steinhauer [4, 5], who was able to catch in a Bose-Einstein condensate (BEC) the characteristic imprint in the correlation function of the Hawking pair creation [6], namely the presence of a peak.

Both a full quantum mechanical calculation [7] (see also [8, 9]) and one using a quantum field theory in curved space approach which includes backscattering [10], predicted the existence of two further peaks beyond this primary one.

Most of the work for BEC analogue BHs assumed the flow to be one dimensional in which only longitudinal modes are excited and the corresponding phonons are massless. If one or more transverse modes are excited then a mass-like term appears in the mode equation which is still a 1+1 dimensional equation. Thus the phonons acquire a mass. A preliminary investigation [11, 12] showed hints of a much richer and more complex structure for the density density correlation function than occurs in the massless case. In particular it was predicted that undulations should occur.

Undulations were first theoretically discovered for massless phonons in analogue white hole BECs in the supersonic region of the fluid [13]. Undulations are zero frequency standing waves generated at the sonic horizon, the locus where the flow turns from subsonic to supersonic. The Bogoliubov dispersion relation for a Bose-Einstein condensate (BEC) undergoing a one dimensional stationary flow along the negative x direction is obtained by solving the Bogoliubov-de

*Electronic address: dudlra13@wfu.edu

†Electronic address: anderson@wfu.edu

‡Electronic address: balbinot@bo.infn.it

§Electronic address: afabbri@ific.uv.es

Gennes equation in the asymptotic regions where the flow speed and the sound speed are constant.¹ It is given by

$$\omega - vk = \pm \sqrt{c^2 k^2 (1 + \xi^2 k^2)} \quad (1.1)$$

where ω is the conserved frequency, k the longitudinal wave vector, c the speed of sound and

$$\xi = \frac{\hbar}{m_a c} , \quad (1.2)$$

the healing length with m_a the mass of an atom in the BEC. The \pm signs refer to the positive or negative branch.

One sees that nontrivial zero frequency roots exist only if $v > c$. They are

$$k_0 = \pm \frac{1}{c_L \xi} \sqrt{v_L^2 - c_L^2} , \quad (1.3)$$

where v_L and c_L are the asymptotic ($x \rightarrow -\infty$) constant values of the velocity and the speed of sound in the supersonic region that we name L . The existence of these roots relies on the dispersive character of Eq. (1.1). Considering the group velocity $v_g = \frac{1}{\partial_\omega k}$ one can show that they are directed against the flow. This explains why the k_0 mode can be generated at the sonic horizon for white hole (WH) flows only.

However, when one allows for transverse excitations of the BEC, the dispersion relation changes to

$$\omega - vk = \pm \sqrt{c^2 (k^2 + k_\perp^2) [1 + \xi^2 (k^2 + k_\perp^2)]} , \quad (1.4)$$

where k_\perp is the transverse wave vector. One sees that for $k_\perp^2 \xi^2 \ll 1$ the quantity $c^2 k_\perp^2$ acts as a mass squared term and in this regime we have two new zero frequency solutions of Eq. (1.4); for v_L^2/c_L^2 not too close to one they are

$$k_0^m = \pm \frac{c_L k_\perp}{\sqrt{v_L^2 - c_L^2}} . \quad (1.5)$$

The novelty is that these lie in the phonon (linear) branch of the dispersion relation and furthermore their group velocity is now directed along the flow. This means that the presence of the mass-like term induces undulations that can be generated at the sonic horizon in BH flows (not in WH) and that these are well described by the hydrodynamical approximation underlying the condensed matter-gravity analogy.

As shown in Ref. [13], in WH flows the undulation characterized by the wave vector k_0 , Eq. (1.3), is responsible for the checkerboard pattern appearing when considering the BEC density-density correlation function when both points are located in the supersonic region of the WH flow. One expects a similar behaviour to emerge for BH flows inside the horizon, triggered by the presence of the transverse modes [11].

The motivation for this work was to investigate the details of the undulations using a simple model that should capture all the essential features of the underlying physics. What we have found is that, while undulations exist, they do not dominate the behavior of the two point correlation function. Instead an equally complex structure of correlation peaks is found. It is shown that the most prominent peaks can be approximated using the stationary phase approximation for the mode integral.

In Sec. II the details of our model are given. In Sec. III explicit expressions for the mode functions and the two point correlation function are given for the Unruh state. Our results for the two point correlation function are presented in Sec. IV along with a brief review of what happens in the massless case. Sec. V contains a brief summary and our conclusions. One appendix gives details of the relationship between our model and that of [6, 10] and the other gives details of the normalization of some of the mode functions.

II. THE MODEL

We shall consider a minimally coupled massive scalar field which satisfies the equation

$$(\square - m^2)\hat{\phi} = 0 , \quad (2.1)$$

¹ See for example Eqs. (2.5) and (2.6) of Ref. [10]. Note that the definition of the healing length ξ there differs by a factor of two from that used here.

where $\square = \nabla_\mu \nabla^\mu$ is the covariant d'Alembertian. The field propagates the 1+1 dimensional space-time described by the metric

$$ds^2 = -c^2 dT^2 + (dx + v_0 dT)^2. \quad (2.2)$$

We use this Painlevé-Gullstrand form for the metric because we wish to make a connection with a BEC analogue black hole. In that case $\hat{\phi}$ would describe phase fluctuations in a BEC and the mass term would be related to the momentum k_\perp of transverse excitations. There are subtleties related to the connection which are discussed in Appendix A.

In the BEC analogue the flow is assumed to be uniform in the negative x direction with $\vec{v} = -v_0 \hat{x}$, $v_0 > 0$. The speed of sound is assumed to vary uniformly along the same direction as the flow from an asymptotic value of c_R for $x \rightarrow +\infty$ to an asymptotic value of c_L for $x \rightarrow -\infty$, with $c_R > v_0 > c_L$. The transition from subsonic to supersonic occurs at $x = 0$ where $c = v_0$. The L region ($x < 0$) describes a sonic BH and the R region ($x > 0$) is outside the sonic horizon. Typical functions used for the sound speed in the literature are \tanh or \tan^{-1} . For the numerical computations in this paper we use the sound speed profile

$$\begin{aligned} c(x) &= \sqrt{c_L^2 + \frac{1}{2}(c_R^2 - c_L^2) \left[1 + \frac{2}{\pi} \tan^{-1} \left(\frac{x+b}{\sigma_v} \right) \right]}, \\ b &= \sigma_v \tan \left[\frac{\pi}{c_R^2 - c_L^2} \left(v_0^2 - \frac{1}{2}(c_R^2 + c_L^2) \right) \right], \end{aligned} \quad (2.3)$$

which was also used in [7, 10]. Here σ_v is related to the width of the profile.

Throughout we use units such that $\hbar = c_R = 1$, although we will frequently include c_R in expressions for clarity. Lengths are given in units of the healing length evaluated in the limit $x \rightarrow \infty$ which is $\xi_1 = \frac{\hbar}{m_a c_R}$. Masses are given in units of m_a which is equivalent in these units to ξ^{-1} . For the plots of the two point correlation function for massive phonons shown in Sec. IV we use the following values for the parameters:

$$v_0 = \frac{3}{4}, \quad c_L = \frac{1}{2}, \quad c_R = 1 \quad \text{and} \quad \sigma_v = 8\xi_1. \quad (2.4)$$

These are the same parameters that were used for the plots of the density density correlation function in [10] in the massless case.

With standard manipulations one can rewrite the metric (2.2) in Schwarzschild-like coordinates

$$ds^2 = -(c^2 - v_0^2) dt^2 + \frac{c^2}{c^2 - v_0^2} dx^2 = -(c^2 - v_0^2)(dt^2 - dx^{*2}) \quad (2.5)$$

where

$$t = T - \int_{x_1}^x dy \frac{v_0}{c(y)^2 - v_0^2}, \quad (2.6)$$

and

$$x^* = \int_{x_2}^x dy \frac{c(y)}{c(y)^2 - v_0^2}. \quad (2.7)$$

Here x_1 and x_2 are arbitrary constants. For the numerical calculations we use

$$x_1 = x_2 = \pm 10.5\xi_1. \quad (2.8)$$

Note that in R the interval $(0, +\infty)$ in x is mapped to $(-\infty, +\infty)$ in x^* , while in L the interval $(-\infty, 0)$ in x is mapped to $(+\infty, -\infty)$ in x^* .

Recalling that the horizon is at $x = 0$, the surface gravity is given by

$$\kappa = \left(\frac{dc}{dx} \right)_{x=0}. \quad (2.9)$$

For the parameters we are using

$$\kappa\xi_1 \approx 0.0185617. \quad (2.10)$$

In these coordinates the Klein Gordon equation (2.1) becomes

$$[-\partial_t^2 + \partial_{x^*}^2 - m^2(c^2 - v_0^2)]\hat{\phi} = 0. \quad (2.11)$$

One should note that near the horizon the modes behave as effectively massless and that modes propagating in R towards $x = +\infty$ are totally reflected back to the horizon when $\omega < m\sqrt{c_R^2 - v_0^2}$.

Given these considerations, we shall use a rather crude approximation for the mass-like term in Eq. (2.11) which makes a simple analytical treatment possible while maintaining the essential physical behavior of the system. We approximate the mass term by a step function, both in R and in L so that

$$\begin{aligned} c^2 - v_0^2 &\rightarrow (c_R^2 - v_0^2)\theta(x^* - {}_R x_0^*), & x > 0 \\ &\rightarrow (c_L^2 - v_0^2)\theta(x^* - {}_L x_0^*), & x < 0. \end{aligned} \quad (2.12)$$

We choose ${}_R x_0^*$ and ${}_L x_0^*$ to be zero for simplicity. For the numerical calculations the arbitrary constants in the definitions of x^* in the R and L regions (2.8) were chosen so that (2.12) provides a reasonable approximation of $c^2 - v_0^2$ for the sound speed parameters used for our plots (2.4); see Fig. 1. Then Eq. (2.11) simplifies to

$$[-\partial_t^2 + \partial_{x^*}^2 - m_R^2\theta(x^* - {}_R x_0^*)]\hat{\phi} = 0, \quad (2.13)$$

for $x > 0$ (region R) with $m_R^2 \equiv m^2(c_R^2 - v_0^2)$, and

$$[-\partial_t^2 + \partial_{x^*}^2 + m_L^2\theta(x^* - {}_L x_0^*)]\hat{\phi} = 0, \quad (2.14)$$

for $x < 0$ (supersonic region L) with $m_L^2 \equiv m^2(v_0^2 - c_L^2)$. The classical solutions of Eqs. (2.13) and (2.14) can therefore

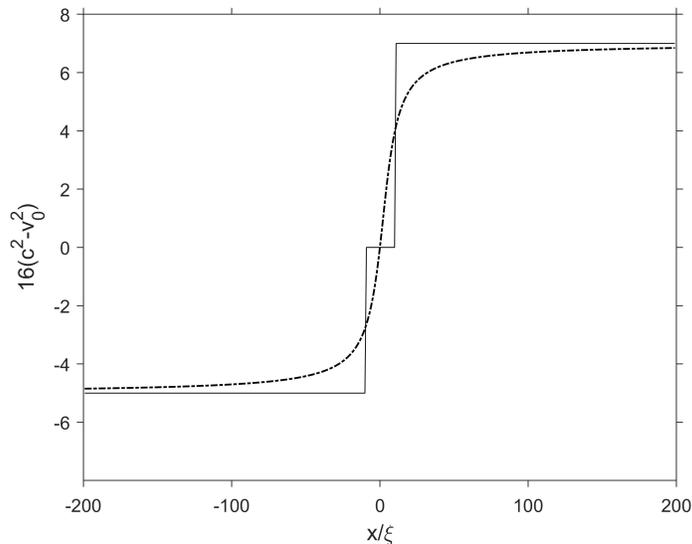


FIG. 1: Shown is a plot of the coefficient of m^2 in (2.11) for the sound speed profile (2.3) using the parameters (2.4)(dashed) and our approximation to that coefficient (2.12)(solid).

be expressed simply in terms of massless (where the step functions vanish) or massive plane waves.

III. THE UNRUH STATE

Hawking's BH evaporation can be understood as a pair production phenomenon triggered by the formation of a horizon in the gravitational collapse of a star leading to a BH. Unruh [14] showed that the late time behavior of this

dynamical process can be mimicked in a stationary BH metric by imposing appropriate boundary conditions on the quantum state of the field on the past horizon of the Kruskal analytical extension of the BH metric. In our case the corresponding Penrose diagram is given in Fig. 2.

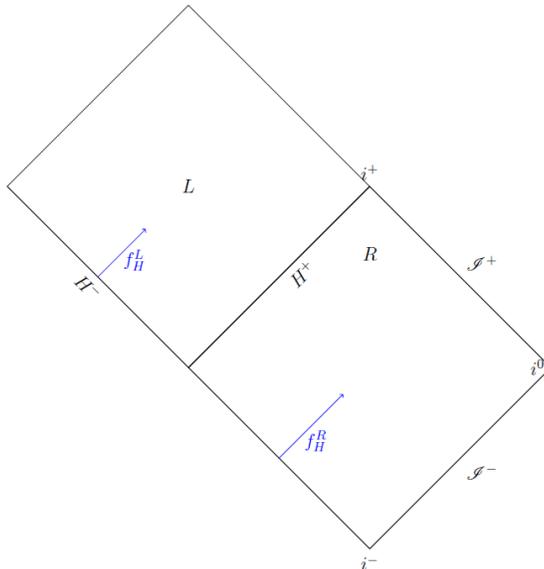


FIG. 2: Penrose diagram with modes which are nonzero on H_- in the interior, L and exterior, R .

For the Unruh state, one requires the modes, which we denote by $f_H^K(\omega_K)$, originating on the past horizon H_- to be positive frequency with respect to the Kruskal U coordinate defined as

$$U_K = \pm \frac{e^{-\kappa u}}{\kappa} . \quad (3.1)$$

Here $-$ and $+$ refer to R and L respectively, and

$$u = t - x^* \quad (3.2)$$

is Eddington-Finkelstein (E.F.) retarded null coordinate. Also needed are the modes which propagate downstream from $x = +\infty$. We denote them by $f_I(\omega)$ and they are positive frequency with respect to the time t . Note that, given the supersonic character of region L , there are no modes coming from $x = -\infty$ and propagating upstream.

The field operator $\hat{\phi}$ is then expanded as

$$\hat{\phi}(x) = \int_0^\infty d\omega_K [\hat{a}_{\omega_K} f_H^K(\omega_K) + h.c.] + \int_{m_R}^\infty d\omega [\hat{b}_\omega f_I(\omega) + h.c.] . \quad (3.3)$$

Here \hat{a}_{ω_K} and \hat{b}_ω are annihilation operators which together with their creation counterparts $\hat{a}_{\omega_K}^\dagger$ and \hat{b}_ω^\dagger satisfy the usual boson commutation rules. Note the lower bound in the last integral: there are no incoming modes with $\omega < m_R$.

The modes are normalized as usual by the conserved scalar product

$$(f_1(\omega_1, x), f_2(\omega_2, x)) = -i \int f_1(\omega_1, x) \overleftrightarrow{\partial}_\mu f_2^*(\omega_2, x) [-g_\Sigma(x)]^{\frac{1}{2}} d\Sigma^\mu \quad (3.4)$$

with $d\Sigma^\mu = n^\mu d\Sigma$, where Σ is a Cauchy surface, g_Σ the determinant of the induced metric and n^μ a future directed unit vector perpendicular to Σ .

Near the horizon ($x^* <_R x_0^*$ in R and $x^* <_L x_0^*$ in L) the modes are massless. Therefore the correctly normalized modes emerging from the past horizon, according to the Unruh vacuum prescription, are

$$f_H^K(\omega_K) = \frac{1}{\sqrt{4\pi\omega_K}} e^{-i\omega_K U_K} . \quad (3.5)$$

These modes are normalized on the past horizon, which is a Cauchy surface for these modes.

In R the massless modes (3.5) propagate freely until they reach ${}_R x_0^*$ where they encounter the step function barrier. They are partially (or totally) reflected back and partially transmitted becoming massive for $x^* > {}_R x_0^*$. To find the form of the reflected and (eventually) transmitted part one has to impose junction conditions for the modes at the discontinuity point $x^* = {}_R x_0^*$. As usual, one requires continuity of the modes and their first derivative. Since the boundary is expressed in the x^* coordinate, it is analytically complicated to rewrite the $f_H^K(\omega_K)$ modes of Eq. (3.5) in (t, x^*) coordinates. It is much simpler to use a Bogoliubov transformation to expand these Unruh modes in terms of the usual Eddington-Finkelstein retarded modes defined in R and L (see Fig. 2). On the past horizon they are

$$f_H^R = \frac{e^{-i\omega u}}{\sqrt{4\pi\omega}}, \quad f_H^L = \frac{e^{i\omega u}}{\sqrt{4\pi\omega}}. \quad (3.6)$$

Both of them are correctly normalized on the past horizon and positive frequency, f_H^R with respect to the time t and f_H^L with respect to the interior time x^* . Therefore

$$f_H^K(\omega_K) = \int_0^\infty d\omega [\alpha_{\omega_K\omega}^L f_H^L(\omega) + \beta_{\omega_K\omega}^L f_H^{L*}(\omega)] + \int_0^\infty d\omega [\alpha_{\omega_K\omega}^R f_H^R(\omega) + \beta_{\omega_K\omega}^R f_H^{R*}(\omega)], \quad (3.7)$$

with α and β are given by²

$$\alpha_{\omega_K,w}^L = (f_H^K(\omega_K, x), f_H^L(\omega)) \quad (3.8)$$

$$\beta_{\omega_K,w}^L = (f_H^K(\omega_K, x), f_H^{L*}(\omega)) \quad (3.9)$$

$$\alpha_{\omega_K,w}^R = (f_H^K(\omega_K, x), f_H^R(\omega)) \quad (3.10)$$

$$\beta_{\omega_K,w}^R = (f_H^K(\omega_K, x), f_H^{R*}(\omega)). \quad (3.11)$$

All scalar products are evaluated on the past horizon.

The exact expressions for the E.F. modes f_H^R can be obtained in the R region by noting that the mode equation there is exactly the same as the Schrodinger equation for scattering in one dimension off a step function potential. To see this note that we can write

$$f_H^R = \frac{e^{-i\omega t}}{\sqrt{4\pi\omega}} \chi_H^R(x^*). \quad (3.12)$$

Then the mode equation (2.13) becomes

$$\frac{d^2 \chi_H^R}{dx^{*2}} + [\omega^2 - m_R^2 \theta(x^* - x_0^*)] \chi_H^R = 0. \quad (3.13)$$

Enforcing the continuity conditions at the barrier $x^* = {}_R x_0^*$ is very simple. The f_H^R modes are incident from the past horizon where they are given by (3.6). A mode propagates upstream in R until the barrier at $x^* = {}_R x_0^* = 0$ is reached. If $\omega > m_R$ the mode is partially transmitted towards $x^* = +\infty$ in the form $\chi_H^R = A_\omega e^{i\sqrt{\omega^2 - m_R^2} x^*}$ as a massive u mode³ and partially reflected towards the horizon in the massless form $D_\omega e^{-i\omega x^*}$. Continuity of the radial mode functions and their first derivatives at the barrier gives

$$A_\omega = \frac{2\omega}{\omega + k_R}, \quad (3.14a)$$

$$D_\omega = \frac{\omega - k_R}{\omega + k_R}, \quad (3.14b)$$

$$k_R \equiv \sqrt{\omega^2 - m_R^2}. \quad (3.14c)$$

² Their explicit values are given in eqs. (4.14b)-(4.14e) of Ref. [10].

³ Although the modes are massive and so don't propagate along null geodesics, we use the "u v" notation to denote right moving and left moving waves respectively.

Thus for $x > 0$

$$f_H^R = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega t} \left[e^{i\omega x^*} + \frac{\omega - k_R}{\omega + k_R} e^{-i\omega x^*} \right], \quad x^* < {}_R x_0^* = 0, \quad (3.15a)$$

$$= \frac{1}{\sqrt{4\pi k_R}} \left(\frac{2\sqrt{k_R\omega}}{\omega + k_R} \right) e^{-i\omega t} e^{ik_R x^*}, \quad x^* > {}_R x_0^* = 0. \quad (3.15b)$$

If $0 \leq \omega < m_R$ there is total reflection (see also [18]): the reflection coefficient for the mode is one and the mode is exponentially damped to the right of the barrier.

The reflected portions $\frac{D_\omega}{\sqrt{4\pi\omega}} e^{-i\omega(t+x^*)}$ of the $f_H^R(\omega)$ modes enter the horizon and travel unaffected in the interior region L ($x < 0$) until they encounter the step function at ${}_L x_0^*$ where particle production occurs.⁴ By applying the same type of matching procedure as before and setting ${}_L x_0^* = 0$ we have in L

$$f_H^R(\omega) = \frac{1}{\sqrt{4\pi\omega}} \left(\frac{\omega - k_R}{\omega + k_R} \right) e^{-i\omega t} e^{-i\omega x^*}, \quad x^* < {}_L x_0^* = 0, \quad (3.16a)$$

$$= \frac{1}{\sqrt{4\pi k_L}} \left(\sqrt{\frac{k_L}{\omega}} \frac{\omega - k_R}{\omega + k_R} \right) e^{-i\omega t} \left[\frac{k_L + \omega}{2k_L} e^{-ik_L x^*} + \frac{k_L - \omega}{2k_L} e^{ik_L x^*} \right], \quad x^* > {}_L x_0^* = 0, \quad (3.16b)$$

$$k_L \equiv \sqrt{\omega^2 + m_L^2}. \quad (3.16c)$$

Note that k_L is real for all values of ω .

For the $f_H^L(\omega)$ modes one similarly obtains

$$f_H^L(\omega) = \frac{1}{\sqrt{4\pi\omega}} e^{i\omega t} e^{-i\omega x^*}, \quad x^* < {}_L x_0^* = 0 \quad (3.17a)$$

$$= \frac{1}{\sqrt{4\pi k_L}} e^{i\omega t} \left[\sqrt{\frac{k_L}{\omega}} \frac{k_L + \omega}{2k_L} e^{-ik_L x^*} + \sqrt{\frac{k_L}{\omega}} \frac{k_L - \omega}{2k_L} e^{ik_L x^*} \right], \quad x^* > {}_L x_0^* = 0. \quad (3.17b)$$

The basis is completed by including the $f_I(\omega)$ modes coming from $x = +\infty$ and traveling downstream. These modes exist only for $\omega > m_R$ and originate as massive incoming v modes. Their normalization is discussed in Appendix B. In region R ($x > 0$) they are

$$f_I(\omega) = \frac{1}{\sqrt{4\pi k_R}} e^{-i\omega t} \left[\frac{k_R - \omega}{k_R + \omega} e^{ik_R x^*} + e^{-ik_R x^*} \right], \quad x^* > {}_R x_0^* = 0, \quad (3.18a)$$

$$= \frac{1}{\sqrt{4\pi\omega}} \frac{2\sqrt{k_R\omega}}{k_R + \omega} e^{-i\omega t} e^{-i\omega x^*}, \quad x^* < {}_R x_0^* = 0. \quad (3.18b)$$

The $f_I(\omega)$ modes maintain the form of Eq. (3.18b) in the L ($x < 0$) region until they encounter the step discontinuity at $x^* = {}_L x_0^* = 0$ where they undergo particle production. Since the L region is supersonic, both the v mode and the u mode travel downstream towards $x = -\infty$. Noting that x^* is a time coordinate in this region, the modes are given by the expressions

$$f_I(\omega) = \frac{1}{\sqrt{4\pi k_R}} \frac{2k_R}{k_R + \omega} e^{-i\omega t} e^{-i\omega x^*}, \quad x^* < {}_L x_0^*, \quad (3.19a)$$

$$= \frac{1}{\sqrt{4\pi k_L}} \frac{2\sqrt{k_R k_L}}{k_R + \omega} e^{-i\omega t} \left[\frac{k_L + \omega}{2k_L} e^{-ik_L x^*} + \frac{k_L - \omega}{2k_L} e^{ik_L x^*} \right], \quad x^* > {}_L x_0^*. \quad (3.19b)$$

Having found the complete set of modes that corresponds to the Unruh state, we turn next to the construction of the symmetric two-point function for the field, also called the Hadamard function. A general expression for it is

$$G^{(1)}(x, x') = \langle 0 | \{ \hat{\phi}(T, x), \hat{\phi}(T', x') \} | 0 \rangle. \quad (3.20)$$

⁴ One could also think of this as scattering. However in L , x^* is the time coordinate so the scattering is anomalous because it involves negative norm modes (see for instance section 9.6 of [15]).

An explicit expression for the Unruh state in was derived in [10]. One first substitutes (3.3) into (3.20) which gives

$$G(x, x') = I + J , \quad (3.21)$$

with

$$I = \int_0^\infty d\omega_K [f_K^H(\omega_K, t, x)f_K^{H*}(\omega_K, t', x') + f_K^{H*}(\omega_K, t, x)f_K^H(\omega_K, t', x')] , \quad (3.22a)$$

$$J = \int_0^\infty d\omega [f_I(\omega, t, x)f_I^*(\omega, t', x') + f_I^*(\omega, t, x)f_I(\omega, t', x')] . \quad (3.22b)$$

Expressing the f_H^K modes in terms of the $f_H^{L,R}(\omega)$ ones by means of the Bogoliubov transformation (3.7) one finds that I can be rewritten as

$$\begin{aligned} I = \int_0^\infty d\omega \frac{1}{\sinh\left(\frac{\pi\omega}{\kappa}\right)} [& f_L^H(\omega, t, x)f_R^H(\omega, t', x') + f_L^{H*}(\omega, t, x)f_R^{H*}(\omega, t', x') \\ & + f_R^H(\omega, t, x)f_L^H(\omega, t', x') + f_R^{H*}(\omega, t, x)f_L^{H*}(\omega, t', x') \\ & + \cosh\left(\frac{\pi\omega}{\kappa}\right) [f_L^H(\omega, t, x)f_L^{H*}(\omega, t', x') + f_L^{H*}(\omega, t, x)f_L^H(\omega, t', x') \\ & + f_R^H(\omega, t, x)f_R^{H*}(\omega, t', x') + f_R^{H*}(\omega, t, x)f_R^H(\omega, t', x')]] . \end{aligned} \quad (3.23)$$

It is important to note that while our toy model is simple enough that analytic expressions have been obtained for all of the mode functions, the resulting integrands in the above expressions are complicated enough that we have computed them numerically rather than analytically. Since the points are split no renormalization counterterms are necessary and because the field has a mass, there are no infrared divergences as there are in the massless case.

IV. RESULTS

Our main result is that the existence of a mass term in the mode equation, however small it may be, fundamentally changes the nature of the solutions, leading to a much more complex structure than occurs in the massless case. To provide some context for this structure we first investigate the behavior of the two point function in the Minkowski vacuum state in two dimensions for both massless and massive scalar fields. Then we review the massless case for both our two dimensional analogue BH model and the two dimensional model in [10] which is obtained via dimensional reduction. The primary difference between the two is the existence of an effective potential in the latter case resulting from the dimensional reduction. This effective potential causes scattering of the mode functions and particle production to occur. After that we discuss the results for our model and finish with a discussion of what happens as the mass becomes very small.

A. Correlation function in flat space

To gain insight into the features seen in the black hole analogue it is useful to investigate the behavior of the two point correlation function in flat 2D spacetime in both the massless and massive cases. The solutions to the mode equations which correspond to the Minkowski vacuum are

$$\begin{aligned} f_k &= \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega t + ikx} , \\ \omega &= \sqrt{k^2 + m^2} . \end{aligned} \quad (4.1)$$

The two point function is

$$\langle\{\hat{\phi}(t, x), \hat{\phi}(t', x')\}\rangle = \frac{1}{4\pi} \int_{-\infty}^\infty \frac{dk}{\omega} \left(e^{-i\omega(t-t') + ik(x-x')} + e^{i\omega(t-t') - ik(x-x')} \right) \quad (4.2a)$$

$$= \frac{1}{\pi} \int_0^\infty \frac{dk}{\omega} \cos[\omega(t-t')] \cos[k(x-x')] , \quad (4.2b)$$

In the massless case $\omega = k$ and there is an infrared divergence which can be removed by imposing a lower limit cutoff λ . Using the identity

$$\cos[k(t-t')] \cos[k(x-x')] = \frac{1}{2} (\cos[k(|t-t'| + |x-x'|)] + \cos[k(|t-t'| - |x-x'|)]) , \quad (4.3)$$

it is easy to show that

$$\langle \{\hat{\phi}(t, x), \hat{\phi}(t', x')\} \rangle = -\frac{1}{2\pi} (\text{ci}[\lambda(|t-t'| + |x-x'|)] + \text{ci}[\lambda(|t-t'| - |x-x'|)]) , \quad (4.4)$$

where $\text{ci}(x)$ is a cosine integral. For large enough separations of the points there will be structure in the two point correlation function but it depends on the cutoff λ . Further if one or more derivatives with respect to the spacetime coordinates are taken, as they would be for the density density correlation function, then the infrared divergence disappears so one can take the limit $\lambda \rightarrow 0$. It is easy to see that in this case all of the structure disappears except for the peak that occurs when the points come together and the peaks that occur when $|x-x'| = |t-t'|$.

The massive case is completely different. Computing the integral in (4.2b) gives

$$\langle \{\hat{\phi}(t, x), \hat{\phi}(t', x')\} \rangle = -\frac{1}{2} N_0(m\sqrt{(t-t')^2 - (x-x')^2}) , \quad |t-t'| > |x-x'| , \quad (4.5a)$$

$$= \frac{1}{\pi} K_0(m\sqrt{(x-x')^2 - (t-t')^2}) , \quad |x-x'| > |t-t'| , \quad (4.5b)$$

where N_0 and K_0 are Bessel functions. If $|t-t'| > |x-x'|$ then there is a series of parallel correlation peaks which occur because N_0 is an oscillating function with the amplitude of the oscillations damped as its argument gets larger in magnitude. But if $|x-x'| > |t-t'|$ then the only correlation peak is the one where the points come together. The boundary between the two regions, $|t-t'| = |x-x'|$, is also a correlation peak in both the massless and massive cases. For the analogue black hole, evidence that the correlation peaks in the massless case correspond to boundaries in the massive case when both points are inside the horizon is given below.

One can find the locations of the peaks along with the boundary between where multiple peaks do and do not occur using the method of stationary phase on the integral in (4.2a). It is easily seen that there are stationary phase points for real values of k only if $|t-t'| > |x-x'|$. Further, if stationary phase points do exist, one can use the method to reproduce to leading order in $(t-t')^2 - (x-x')^2$ the result in (4.5a).

When $|t-t'| < |x-x'|$ there is no series of parallel peaks as can be seen from Eq. (4.5b). However, if one breaks up the contribution to the two point function in this case into right and left moving modes and considers them separately then one finds a series of peaks that are parallel to the main peak when the points come together. The integrals in question are

$$\begin{aligned} J_{(left-moving)} &= \frac{1}{4\pi} \int_0^\infty \frac{dk}{\omega} e^{-i\omega(t-t') - ik(x-x')} , \\ J_{(right-moving)} &= \frac{1}{4\pi} \int_{-\infty}^0 \frac{dk}{\omega} e^{-i\omega(t-t') - ik(x-x')} , \end{aligned} \quad (4.6)$$

In Fig. 3 the real parts of the solutions are plotted for the case that $t-t' = \frac{1}{2}(x-x')$. It is clear that what happens is that cancellations cause the structure to disappear when these integrals are added together. Similar effects occur in the BH analogue model. In this case the integrands are slightly different so the contributions from the initially left moving $f_H^{R,L}$ modes to these peaks is not completely cancelled by the contribution from the initially right moving f_I modes. However, the cancelation is good enough that the remaining structure is smaller by at least three orders of magnitude than the other more prominent structures discussed below.

Since multiple correlation peaks occur for any $m > 0$, in some sense there is an abrupt transition between the two point function for a massless and a massive field. However, it is instructive to see what happens as the mass decreases in the region where there are multiple peaks. Because of the factor of m^2 in the Hankel function in (4.5a) the separations between the peaks gets larger as m gets smaller. Thus if one restricts attention to a finite region near the origin, then one finds that as the mass decreases the structure effectively moves out of this region. As discussed below, we find something similar in the BH analogue case.

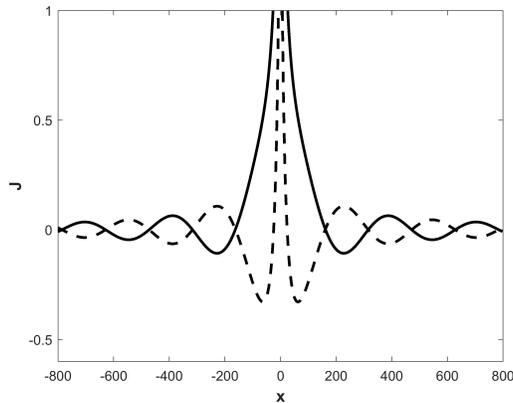


FIG. 3: The real part of solutions for $J_{(left-moving)}$ (dashed) and $J_{(left-moving)}$ (solid). Note that when added together there is clear cancellation of the smaller peaks and only the single large peak that occurs when the points come together survives.

B. Review of the massless BH analogue case

Before discussing the structure of the two point function in the massive case it is useful to review what happens in the massless case for a BH analogue, i.e. when one sets $m = 0$ in the mode equation (2.11). This case was investigated in the pioneering work [6]. There it was found that, along with the usual peak which occurs when the points come together, the density density correlation function in a BEC (which is simply related to $G(x, x')$, see [6]) has a peak when one point is inside and one point is outside the horizon, revealing the correlations between the Hawking particles and the negative energy partners inside. This is the smoking gun of the Hawking effect which has been found in the experiments of Steinhauer [4, 5].

Numerical studies using a full quantum treatment of the BEC [7] and studies using QFT in curved space techniques [10] confirmed the existence of the in-out peak and showed the existence of two others.⁵ One was a weaker, in-out peak and the other a correlation peak that occurs when both points are inside the horizon. Fig. 4 contains a plot from Ref. [10] which shows these peaks. We have superposed the numbers 1, 2, and 3 on this plot to label the main in-out peak, the secondary in-out peak, and the in-in peak respectively. Penrose diagrams which sketch the

⁵ The reason the extra peaks show up in the treatment of [10] is that the effective potential which results from a dimensional reduction from four to two dimensions is included and this results in scattering and particle production effects which do not occur when the effective potential is not present.

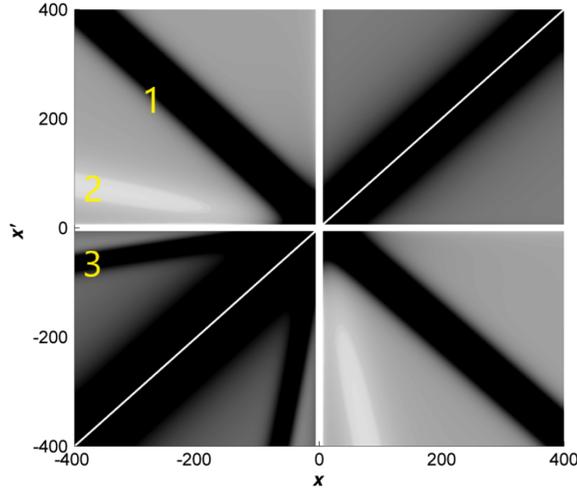


FIG. 4: Density-density correlation function for 1+1D BEC BH analog with no mass-like term which shows 3 peaks. The figure is reproduced from [10] with the numbers superposed on that figure.

corresponding correlations between the modes for the three peaks are given in Fig 5. One sees that peak 1) is of the

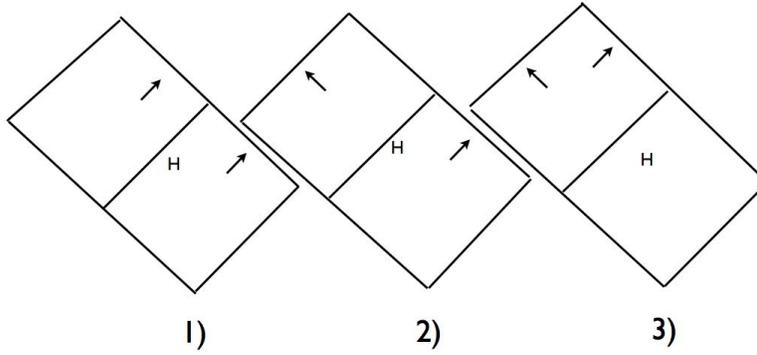


FIG. 5: Correlations between the modes found in the massless case.

$u - u$ type whereas peaks 2) and 3) are of the type $u - v$ and require for their existence a scattering of the massless modes.

C. Primary results for the massive case

The most prominent structures we have found for the two point correlation function are shown in Fig. 6. In each quadrant there is also a group of correlation peaks which are so small that they do not show up on the scale of the plot. Because they are insignificant compared to the more prominent peaks we do not discuss them further in this subsection but they are briefly discussed in Sec. IV E below.

In contrast with the massless case, the structures in the massive case generally consist of groups of roughly parallel peaks which are sometimes separated by boundaries. The locations of the boundaries when both points are inside the

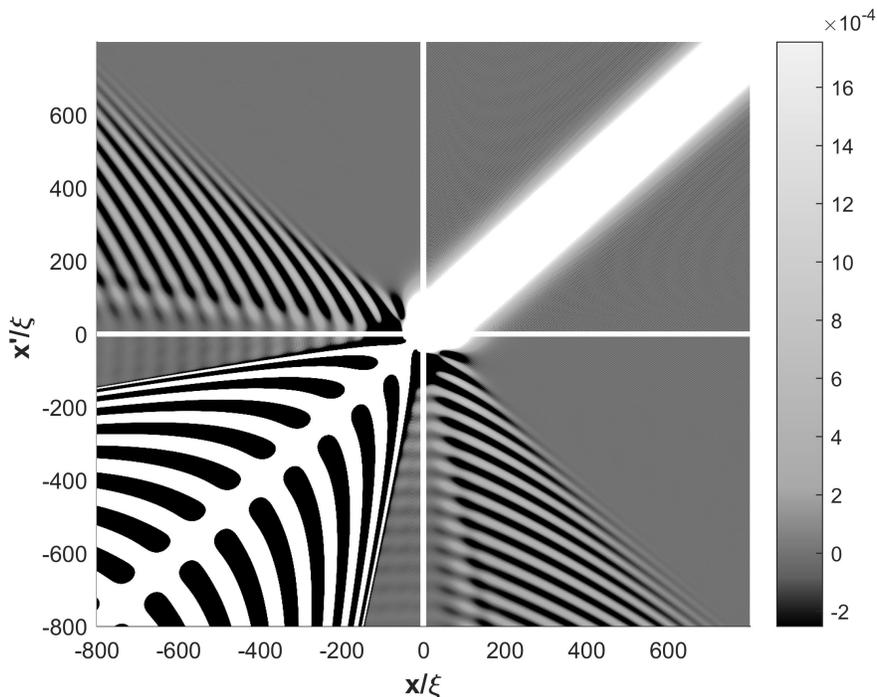


FIG. 6: Plot for the two point function for $m = 4 \times 10^{-2}$. The horizontal and vertical white stripes correspond to regions near the horizon which are excluded from the plot. The contrast has been arranged to show the primary structures that have been observed in this case. As a result regions which appear to be pure white aside from the exclusions are regions where the two point function is larger than 1.7×10^{-3} . For the pure black regions the two point function is smaller than -2×10^{-4} .

horizon correspond to the locations of the peaks that would occur in our simple model in the massless case⁶. When one point is inside and one point is outside the horizon the boundaries appear to be close to the location of the peaks in the massless case. By examining (3.22b) and (3.23) it is not hard to show that for our simple model peak 1) occurs when the contributions to the phase of a right moving mode in R and a right moving mode in L cancel. For point x in the R region and x' in the L region this results in the condition

$$(t - t') - (x^* - x^{*\prime}) = 0. \quad (4.7)$$

Similarly peak 2) occurs when the contributions to the phase of a right moving mode in R and a left moving mode in L cancel. For point x in the R region and x' in the L region this corresponds to

$$(t - t') - (x^* + x^{*\prime}) = 0. \quad (4.8)$$

Peak 3) occurs when the contributions to the phase of a left moving mode in L and a right moving mode in L cancel. In some cases this occurs when

$$(t - t') + (x^* + x^{*\prime}) = 0, \quad (4.9a)$$

while in others it occurs when

$$(t - t') - (x^* + x^{*\prime}) = 0. \quad (4.9b)$$

⁶ We show the peaks here in the case that there is no potential so that solutions to the radial mode equation in the massless case are linear combinations of $e^{i\omega x^*}$ and $e^{-i\omega x^*}$. The solutions are more complicated if there is a potential and the locations of the peaks may be altered somewhat. Of course strictly speaking in the massless case there is no scattering or particle production if there is no potential so peaks 2) and 3) don't occur. However, they would occur if there was scattering, although their locations might be somewhat different.

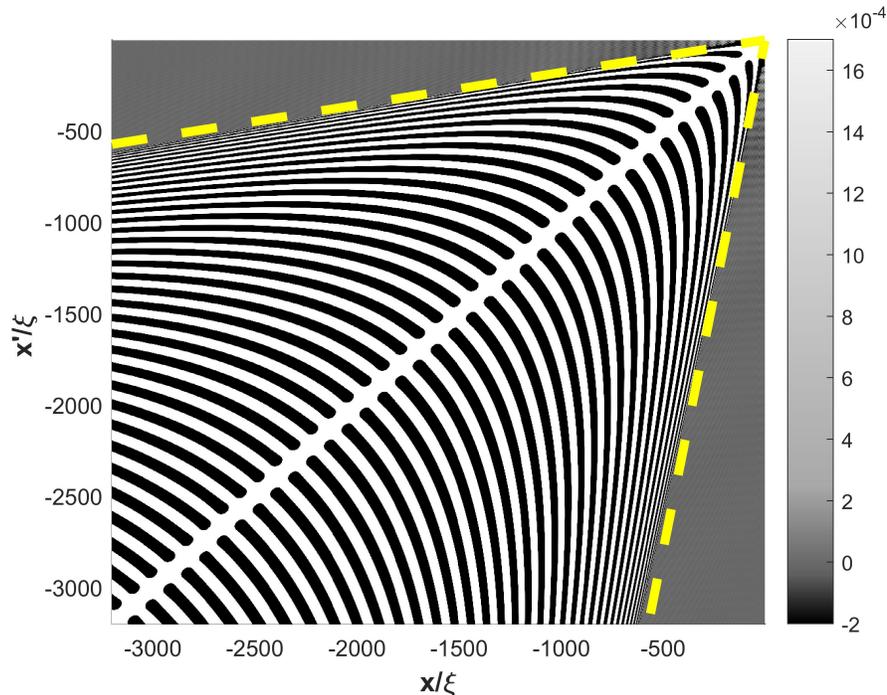


FIG. 7: Plot of the two point function for $m = 4 \times 10^{-4}$. The region shown is the lower left quadrant of Fig. 6, where both points are in the interior of the analogue BH. Here the range of the plot has been extended to better illustrate the features in this region. The dashed lines indicate the correlation peaks in the massless case with locations specified by (4.9a). Note that the less pronounced set of peaks that is clearly visible in Fig. 6 is only visible here near the upper right corner of the plot. The reason is that the amplitudes of these peaks decrease as the separation between the points x and x' increases.

Using (2.6) with $T' = T$ and (2.7) one can find the relationships between x and x' for these peaks. These are plotted as the dashed lines in Figs. 6, 7, and 8.

Arguably the most interesting part of Fig. 6 is the lower left quadrant where both points are inside the horizon. This is where the most prominent peaks occur and it is also the location of the undulations which are discussed in the next subsection. When both points are inside the horizon, the most prominent peaks have a well defined boundary. This boundary is at the location of the peaks that occur in the massless case when both points are inside the horizon. The details of this region along with the locations of the two peaks in the massless case are shown in Fig. 8.

The two regions in Fig. 6 for which one point is inside and one point is outside the horizon are the top left and bottom right quadrants. They also have a rich structure. There are two groups of peaks, partially superposed, which are both less prominent than the group of large peaks which occur when both points are inside the horizon. One group consists of a series of parallel peaks which are roughly diagonal. In both quadrants the upper boundary of these peaks is approximately, but not exactly, where peak 1) occurs in the massless case, see Eq. (4.7) and Fig. 4. The other group consists of a series of peaks that are roughly parallel to the horizon. The upper boundary of these peaks is in the approximate location of peak 2) in the massless case, see Eq. (4.8) and Fig. 4. There is an overlapping region where both groups of peaks are superposed. The details of this region along with the locations of the two peaks in the massless case are shown in Fig. 8.

The top right quadrant in Fig. 6 is the region where both points are outside of the event horizon. The only peak visible in the plots is the correlation peak that occurs when the points come together. As mentioned above, in each quadrant including this one, there is at least one group of very small peaks that do not show up on the scale of the plot.

D. Undulations

As discussed in the Introduction, undulations are expected to occur in the region where both points are far enough inside the horizon that the particle creation event has occurred. The undulations were predicted in [11] for 2D spacetimes with an event horizon when a mass term is present in the mode equation. It was shown there that the

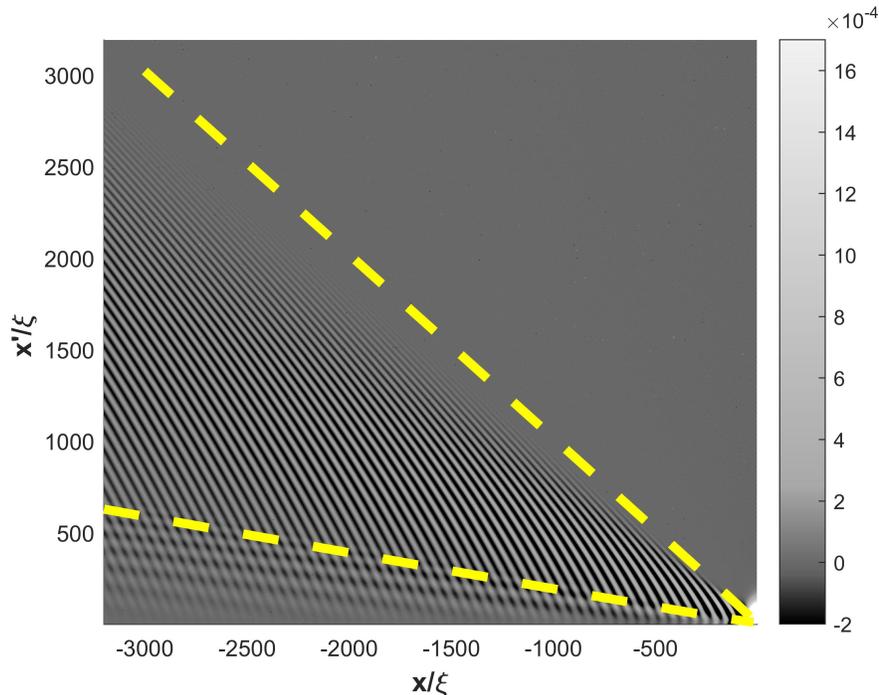


FIG. 8: Plot of the two point function for $m = 4 \times 10^{-4}$. The region shown is the upper left quadrant of Fig. 6, where one point is in the interior and the other point is in the exterior of the analogue BH. Here the range of the plot has been extended to better illustrate the features in this region. The dashed lines indicate the locations of the correlation peaks in the massless case with locations specified by (4.7) and (4.8).

contributions of these modes to the two-point function will go like $\cos(p_m x)\cos(p_m x')$, with $p_m \propto m$. This leads to a cross hatched pattern with a periodicity that is proportional to the mass.

It is clear by inspection of the lower left quadrant of Fig. 6 that no such pattern occurs. To investigate the question of whether undulations are present we show in Fig. 9 the patterns obtained when certain the integrals in (3.22b) and (3.23) are computed for certain frequency ranges. The top plots are only for the contributions from the low frequency modes, $0 \leq \omega \leq \frac{m}{100}$, and they reproduce the expected cross hatched pattern. Thus the undulations are there. The middle plots include all modes with $0 \leq \omega \leq \frac{m_R}{10}$ and their structure is essentially the same as that for the bottom plots which include all values of ω . Thus, when compared to the prediction in [11], we find that the low frequency contribution does not dominate the total two point function and instead a much more complicated relationship is found for the total two point function.⁷

E. Stationary phase approximation

The stationary phase approximation can be used to gain some insight into the primary group of peaks which occur in the two point function when both points are inside the horizon. Not all terms in the integrand in this case have stationary phase points for positive real values of ω , but those with exponentials whose arguments are of the forms

$$\mp i\omega(t-t') \pm ik_L(x^* + x^{*'}) \quad (4.10a)$$

$$\mp i\omega(t-t') \mp ik_L(x^* + x^{*'}), \quad (4.10b)$$

⁷ We must note, however, that in our idealized model scattering coefficients depend only on ω , m_R , m_L and not on the surface gravity κ . In more realistic models those scattering coefficients that do not go to 1 as $\omega \rightarrow +\infty$ will typically go to 0 more rapidly than those in this model. This could affect the results for the total two-point function presented here.

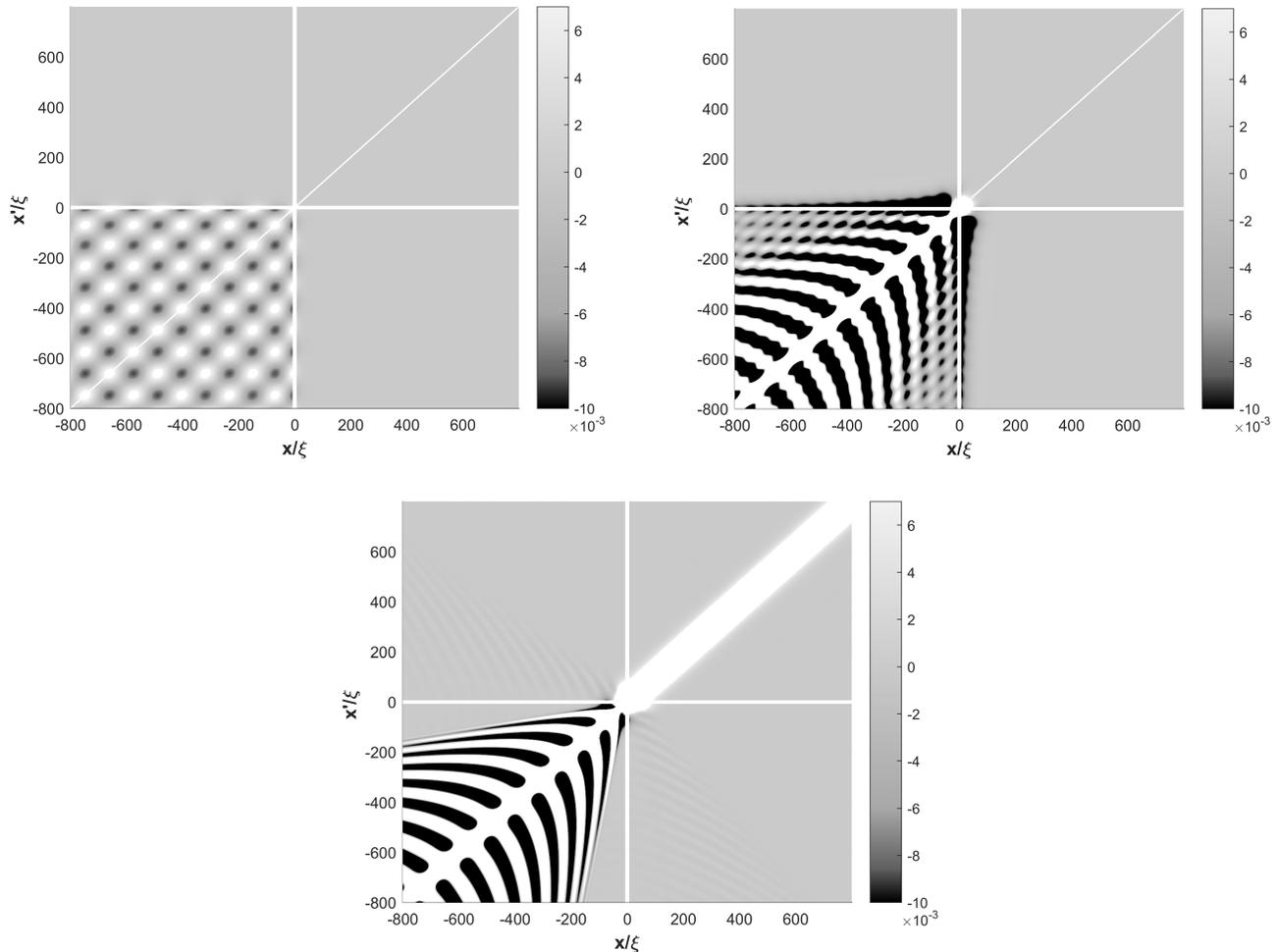


FIG. 9: Plots of the two point function for the case $m = 4 \times 10^{-2}$ for various frequency bands. Top: Low frequency contribution to the two-point function for $0 \leq \omega \leq \frac{mR}{100}$. Middle: Contributions from frequencies in the range $0 \leq \omega \leq \frac{mR}{10}$. Bottom: Contributions from all frequencies. Note that if the scale for the color bar is decreased significantly then additional structure becomes visible in each of the plots.

have stationary phase points at

$$\omega = \frac{m_L(t-t')}{\sqrt{(x^* + x^{*'})^2 - (t-t')^2}}, \quad (4.11)$$

if $|x^* + x^{*'}| > |t - t'|$ and $t - t' > 0$. Thus the boundary between where a stationary phase point exists for positive values of ω and where one does not is given by the condition $|x^* + x^{*'}| = |t - t'|$. This is the condition (4.9b) which corresponds to the location of correlation peak (3) in Fig. 4.

The unusual structure of the most prominent group of peaks when both points are inside the horizon, along with their boundaries, can be reproduced by the stationary phase approximation for some of the integrands which contribute to the two point function.⁸ The stationary phase approximation for this case is plotted in Fig. 10. Comparison with

⁸ The range of values of x and x' for which there is a stationary phase approximation is more limited for the f_I modes than is given by the conditions (4.11) because the minimum value of ω for those modes is m_R . When the integrals are computed exactly there are partial cancellations which occur between the contributions from the f_H and f_I modes. Because of the extra limitation in the range of the stationary phase approximation for the f_I modes, there are ranges of values of x and x' for which such cancellations cannot occur for the stationary phase approximation. As a result the structure that appears due to the full stationary phase approximation looks

Fig. 7 shows that the large peaks are reproduced by the approximation along with their boundaries. The other group of peaks in Fig. 7 are not reproduced by the stationary phase approximation.

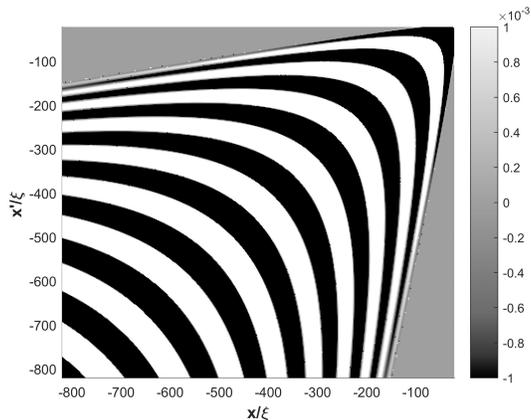


FIG. 10: Stationary phase result for the contributions to the two point correlation function when both points are inside the horizon which come from the two terms in (3.23). One term has an interior upstream mode at point x and an interior downstream mode at point x' . The other has the spatial points for these modes reversed.

There is no stationary phase approximation for the integrals when both points are outside the horizon and far from it for the values of $t - t'$ that we are using. Recall these are obtained from (2.6) with the condition $T = T'$. This is consistent with the lack of prominent correlation peaks in this case as shown in Fig. 6 where the only peak visible is that which occurs when both points come together.

When one point is inside and one point is outside the horizon the stationary phase approximation reproduces the group of peaks which are approximately parallel to the horizon. The other group of peaks which are approximately diagonal are not reproduced by this approximation.

As mentioned previously there is some structure in each case, in-in, in-out, and out-out that is too small to show up on the plots. The contributions to this structure come from terms in the mode integrals for which there is no stationary phase approximation in the integration range. In most cases we find that there is a partial cancellation between the contributions from the $f_H^{R,L}$ and the f_I modes.

F. Behavior as $m \rightarrow 0$

The two point correlation function for several values of the mass m of the phonons is shown in Fig. 11 for a fixed range of values of x and x' .

A careful examination of the sequence shows that two things occur. One is that, for large values of $|x|$ there appears to be a scaling of the coordinates of the form $x \rightarrow x/m$. The other is that as m gets small, in the region that is not too far from the horizon, the primary peak appears that occurs in the massless case when one point is inside and one point is outside the horizon.

The apparent scaling can be seen from the fact that in Fig. 11 the spacing between the peaks increases as m decreases. As a result the structure moves out of the region shown in the plots. In Fig. 12 the two point correlation function for $m = 10^{-4}$ is shown for a much larger range of values of x and one can see that the structure is still there. Comparison with the plots in Fig. 11 shows that the locations of the peaks appear to scale like x/m .

Examination of the mode equation in regions R and L , (2.13) and (2.14) respectively, shows that the mass can be scaled out of these equations if we let $t \rightarrow t/m$, $x^* \rightarrow x^*/m$, and $\omega \rightarrow m\omega$. This is reflected in the solutions to the mode equation that are shown in Sec. III. If in those solutions we set $t = \bar{t}/m$, $x^* = \bar{x}^*/m$, and $\omega = m\bar{\omega}$ then they have the behavior

$$f(\omega, m, t, x) = \frac{f(\bar{\omega}, m = 1, \bar{t}, \bar{x})}{\sqrt{m}}, \quad (4.12)$$

significantly different from that which is due to the full numerical calculations in these regions.

where the function f on the left and the function f on the right are the same functions. Substituting into (3.22b) one finds that the contribution to the two point function from the integral J is

$$J(m, t - t', x^*, x^{*'}) = \frac{J(m = 1, \bar{t} - \bar{t}', \bar{x}^*, \bar{x}^{*'})}{m}, \quad (4.13)$$

where the function on the right and on the left is the same function but with different arguments. Thus there is an exact scaling of the contribution from J with the mass.

As discussed at the end of Sec. III, having used the coordinate system t and x^* related to the metric (2.5) to compute the two point function we then change back to the original lab coordinates T and x because for a BEC analogue black hole the density density correlation function (which is computed from the two point function) is measured experimentally at equal lab times T and T' . It is clear from the relations (2.6) and (2.7) that so long as the sound speed varies as a function of position there is no simple linear relationship between t and x or x^* and x . Thus the scaling does not work in the lab coordinates T and x . However, if both points in the correlation function are far from the horizon then the sound speed is approximately constant. For example if x is in the R region far from the horizon then

$$c(x) = c_R + \frac{\sigma_v (c_L^2 - c_R^2)}{2\pi c_R x} + O\left(\frac{1}{x}\right)^2. \quad (4.14)$$

Multiplying (2.6) and (2.7) by a factor of m and using (4.14) one finds

$$\begin{aligned} \bar{t} &= m \left[T - \frac{v_0}{c_R^2 - v_0^2} x - v_0 \frac{\sigma_v (c_R^2 - c_L^2)}{\pi (c_R^2 - v_0^2)} \log x + C_1 + O\left(\frac{1}{x}\right) \right] \\ \bar{x}^* &= m \left[\frac{c_R}{c_R^2 - v_0^2} x + \frac{\sigma_v (c_R^2 - c_L^2) (c_R^2 + v_0^2)}{2\pi c_R (c_R^2 - v_0^2)^2} \log x + C_2 + O\left(\frac{1}{x}\right) \right], \end{aligned} \quad (4.15)$$

with C_1 and C_2 constants. One finds similar expressions if x is in the L region. Clearly it is not possible to eliminate the m dependence of the right hand sides through a simple scaling of x . However, if one changes to $x = \bar{x}/m$ and $T = \bar{T}/m$, then the leading order terms on the right hand sides of these equations do not depend on m . Further all of the other terms decrease as m does for fixed \bar{x} . If these expansions are substituted into the expression for $J(m = 1, \bar{t}, \bar{x}^*)$ then it is found that the dependence on the mass becomes weaker for fixed \bar{x} as the mass decreases. So we have an approximate scaling of this contribution to the two point function for the point separations that we are interested in.

If (4.12) is substituted into (3.23) one sees that there is no simple scaling for the contribution to the two point function I . The reason is that we are not changing the sound speed profile (2.3) as we change the mass and therefore the surface gravity κ does not change. However the locations of the correlation peaks should depend on the arguments of the exponentials of the mode functions and not very much on the functions $\text{csch}(\pi\omega/\kappa)$ and $\text{coth}(\pi\omega/\kappa)$ which do not oscillate. Therefore we would expect that the locations of the peaks would scale with x in the same approximate way for I as they do for J .

In the limit $m \rightarrow 0$, the primary correlation peak with one point inside and one point outside the horizon is the only one that should occur in our model since the effective potential is zero. In fact in Fig. 11 it appears that in the finite sized region shown, this peak appears once the mass is small enough. However, as seen in Fig. 13 this peak does not remain a peak when both points are far enough away from the horizon. Thus in some sense the two point function for very small masses is qualitatively similar to the two point function in the massless case only if one's attention is restricted to a finite region centered around the horizon.

To understand why this peak appears in a finite region containing the horizon for small m it is sufficient to consider the primary contribution to the peak which comes from the f_R^H and the f_L^H modes. Recall that in our calculations the scattering occurs at $x^* = 0$ in R and the particle creation occurs at $x^* = 0$ in L . For $x^* \geq 0$ f_R^H is given by (3.15b) and f_L^H is given by (3.16b). The part of f_L^H that contributes to the primary in-out peak if $m = 0$ is the part proportional to $e^{-ik_L x^{*'}}$. The contribution of both of these modes to the integral I in (3.23) for $x^* > 0$ and $x^{*' > 0}$ is

$$I_1 = \frac{1}{2\pi} \int_{m_R}^{\infty} \frac{d\omega}{\sinh\left(\frac{\pi\omega}{\kappa}\right)} \frac{(k_L + \omega)}{k_L(k_R + \omega)} \cos[\omega(t - t') + k_R x^* - k_L x^{*'}]. \quad (4.16)$$

In the massless case $k_L = k_R = \omega$ and the correlation peak occurs at points where the argument of the cosine

vanishes.⁹ These are given by the condition $t - t' = -(x^* - x^{*\prime})$. In the low mass limit

$$\begin{aligned} k_R &= \omega - \frac{m_R^2}{2\omega} + \dots \\ k_L &= \omega + \frac{m_L^2}{2\omega} + \dots \end{aligned} \quad (4.17)$$

Thus for modes with $m^2 \ll \omega$ the argument of the cosine in (4.16) does not vary rapidly with ω near the position of the peak in the massless case for values of x^* and $x^{*\prime}$ such that $\frac{m_R^2}{2\omega} x^* \ll 1$ and $\frac{m_L^2}{2\omega} x^{*\prime} \ll 1$. Thus we would expect the peak to occur for values of x^* and $x^{*\prime}$ which satisfy these conditions. However for significantly larger values of x^* and $x^{*\prime}$ these terms in the argument will vary significantly with ω and this change in the phase of the cosine in (4.16) with respect to ω will result in cancellations that eliminate this peak when the integral is computed. This is exactly what is observed for large distances from the horizon in Fig. 13.

V. CONCLUSIONS

For an acoustic black hole we have presented a pedagogical toy model that shows the catalytic effect which the presence of excitations transverse to the flow has in producing a rather rich and complex structure in the correlation functions, in particular in the interior (BH) region. Transverse excitations induce a mass like term in the equation for the longitudinal modes which not only causes backscattering of the modes but also allows the existence of zero frequency standing waves (undulations) inside the horizon.

We have found a few different groups of correlation peaks that lie along approximately parallel curves. In contrast, there are only three prominent correlation peaks in the density density correlation function in the massless case. In the massive case their locations are not peaks for the most part. However when both points are inside the horizon the locations of the correlation peaks in the massless case serve as boundaries for a group of prominent peaks in the massive one. When one point is inside and one point is outside the horizon, for the calculations we have done, there are boundaries at the approximate, but not exact, locations of the correlation peaks in the massless case.

The most prominent group of peaks in the massive case occur when both points are inside the horizon. They have approximately the same distances between them as the undulations do, but they have a different pattern. This means that the contribution to the two point function from the undulations is effectively erased by the contributions of modes with higher frequencies. The boundary of this prominent group of peaks on both sides corresponds to the locations of the peak that occurs in the massless case when both points are inside the horizon.

If one point is inside and one point is outside the horizon one group of peaks is close to the horizon and roughly parallel to it. Their upper boundary is, for the calculations we did, in close proximity with the location of the secondary correlation peak that occurs in the massless case. A second group of peaks which are roughly diagonal has an upper boundary which is in close proximity with the location of the primary correlation peak in the massless case.

There are also very small correlation peaks which do not show up in our plots. These come from terms in the two point function for which there is no stationary phase approximation for the given integration range. They occur in each quadrant of Fig. 6. Because they are so small it would be extremely difficult to observe them experimentally and it is likely that other effects which are not accounted for in our simple model would be more important.

We have provided strong evidence that there is a discontinuity between the types of structure that occur in the massless and massive cases if one considers the behavior of the two point correlation function over the entire analogue spacetime. However there appears to be a smooth transition to the massless case if attention is restricted to a finite sized region. There are two reasons for this. First we have provided evidence that all of the structures observed scale like $1/m$ so that the peaks get farther apart as the mass decreases. Second we have also shown that a single correlation peak appears near the horizon when the mass gets small enough and one point is inside one is outside the horizon. At large distances from the horizon this peak disappears. However, if attention is restricted to a finite region that contains the origin, then for small enough masses this peak is qualitatively similar to the primary correlation peak found in the massless case.

On physical grounds we expect the correlation peaks that appear in our plots to be quite general and not to be an artifact of the simple model used. For BEC analogue black holes, experimental evidence has been reported only

⁹ In the massless case one must impose an infrared cutoff on the integral in (4.16). The cutoff will impose structure in the correlation function for large enough separations of the points, but if the density density correlation function is computed then the cutoff can be taken to zero and the extraneous structure disappears.

for the main peak which occurs when one point is inside and one point is outside the horizon. However in future experiments, where the sensitivity in the correlation function measurements will be increased, one expects the various patterns we have discussed to show up in systems with excitations of one or more transverse modes, putting the QFT in curved space approach to this peculiar type of condensed matter system on even more solid ground.

Acknowledgments

We would like to thank Jeff Steinhauer and Germain Rousseaux for interesting and useful comments. This work is partially supported by the Spanish Mineco grant FIS2014-57387-C3-1-P, the Generalitat Valenciana project SEJI/2017/042, the Severo Ochoa Excellence Center Project SEV-2014-0398, the National Science Foundation under Grants No. PHY-0856050, No. PHY-1308325, and No. PHY-1505875 to Wake Forest University, and a Wake Forest University Bridge Grant. Some of the numerical work was done using the WFU DEAC cluster; we thank the WFU Provost's Office and Information Systems Department for their generous support.

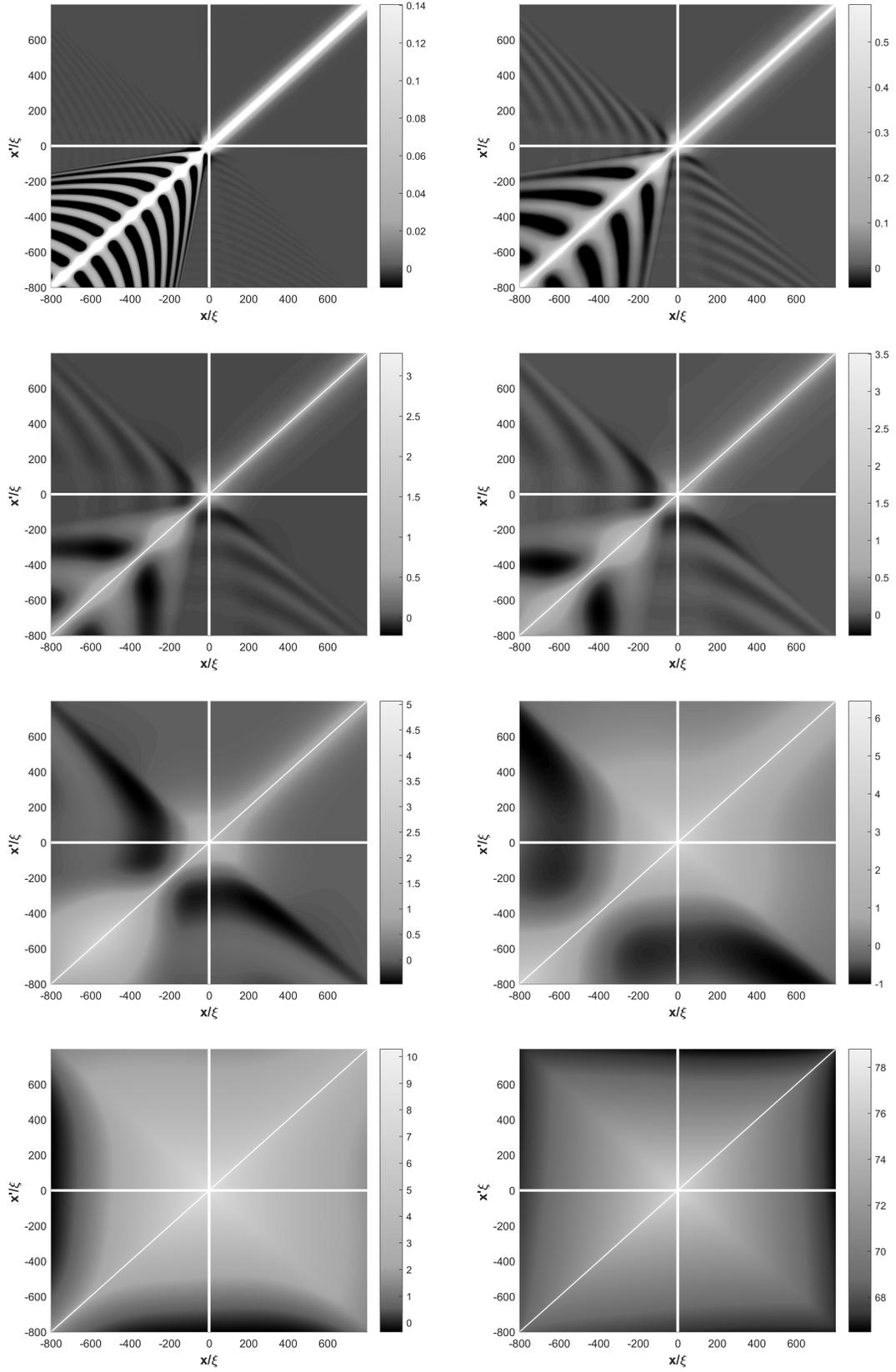


FIG. 11: Shown are plots for a range of values of the mass m of the phonon field. From left to right and top to bottom and starting in the upper left the masses are: $m = 4 \times 10^{-2}$, $m = 2 \times 10^{-2}$, $m = 10^{-2}$, $m = 8 \times 10^{-3}$, $m = 6 \times 10^{-3}$, $m = 4 \times 10^{-3}$, $m = 2 \times 10^{-3}$, $m = 10^{-3}$ and $m = \times 10^{-4}$.

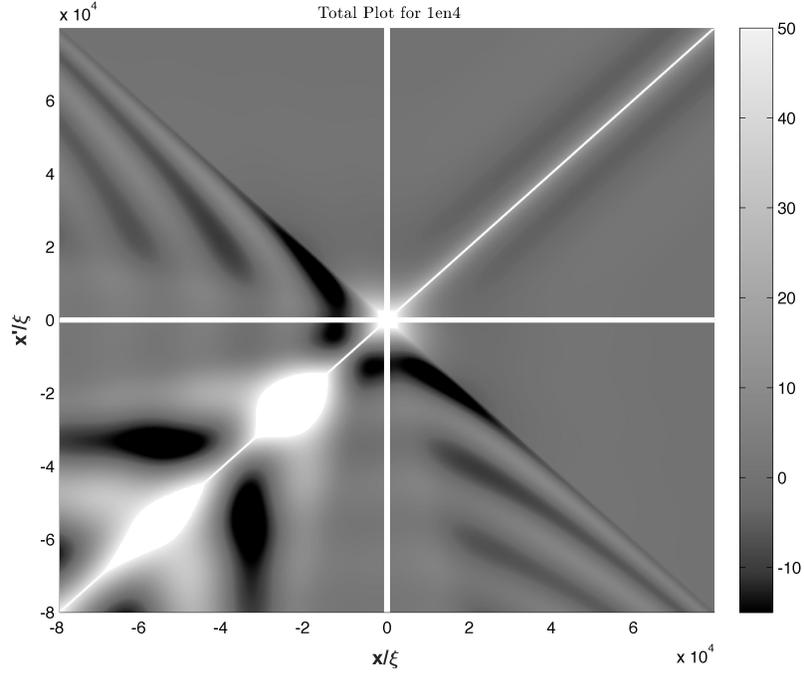


FIG. 12: Plot of the two point function for $m = 10^{-4}$, which is the lowest mass case shown in Fig. 11(lower right) . The range of this plot has been extended significantly when compared to previous plot to show the structure due to the mass.

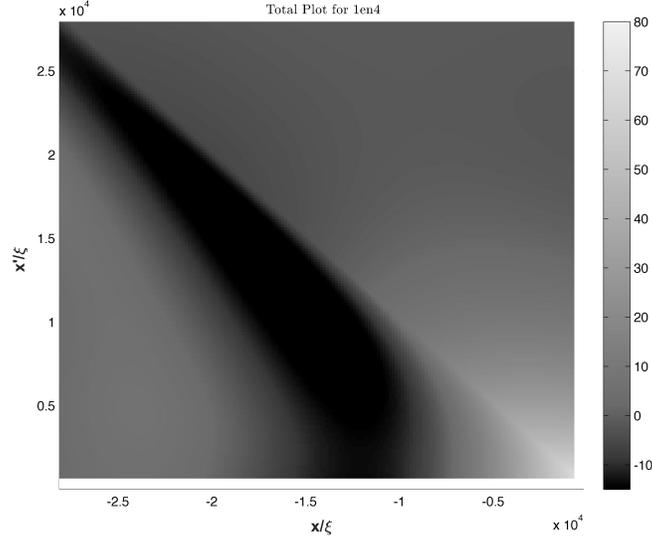


FIG. 13: Plot of the two point function for $m = 10^{-4}$, which is the lowest mass case shown in Fig. 11(lower right). It is also the same mass as was used for the plot in Fig. 12. The region shown is where one point is in the interior and one point is in the exterior of the analogue BH. Near the horizon a peak can be seen at the same approximate location as peak 1) in the massless case, see Eq. (4.7) and Fig. 4. This peak disappears for points which are further from the horizon.

Appendix A: Connection with a BEC analogue black hole

In [10] a detailed derivation is given for the equation satisfied by the phase fluctuation operator $\hat{\theta}_1$ which along with the density fluctuation operator \hat{n} describes the noncondensed part of the BEC. In the hydrodynamical approximation (see [10]) the equation is

$$\square \hat{\theta}_1 = 0, \quad (\text{A1})$$

with the d'Alembertian evaluated using the four dimensional metric

$$ds^2 = \frac{n}{m_a c} [-c^2 dT^2 + (dx + v_0 dT)^2 + dy^2 + dz^2]. \quad (\text{A2})$$

Here n is the density of the condensate (assumed to be constant), m_a is the mass of one of the atoms in the condensate, and we have written the form of the metric for the case we are considering which is $\vec{v} = -v_0 \hat{x}$. With the change of variable (2.6) the metric and the wave equation are

$$ds^2 = \frac{n}{m_a c} \left[-(c^2 - v_0^2) dt^2 + \frac{c^2}{c^2 - v_0^2} dx^2 + dy^2 + dz^2 \right], \quad (\text{A3a})$$

$$\square \hat{\theta}_1 = \left[-\frac{c}{c^2 - v_0^2} \partial_t^2 + c \partial_x \left(\frac{c^2 - v_0^2}{c^2} \partial_x \right) + c(\partial_y^2 + \partial_z^2) \right] \hat{\theta}_1 = 0. \quad (\text{A3b})$$

Next we perform a dimensional reduction by defining a new field $\hat{\theta}_1^{(2)}(x, t)$ such that

$$\hat{\theta}_1 = \sqrt{\frac{m_a c}{n \ell_\perp^2}} \hat{\theta}_1^{(2)}(x, t) e^{i(p_y^\perp y + p_z^\perp z)}, \quad (\text{A4})$$

with ℓ_\perp the transverse dimension of the BEC. Substituting into (A3b) gives

$$\left[-\frac{c^{3/2}}{c^2 - v_0^2} \partial_t^2 + c \partial_x \left(\frac{(c^2 - v_0^2)}{c^2} \partial_x \sqrt{c} \right) - c^{3/2} (p^\perp)^2 \right] \hat{\theta}_1^{(2)} = 0, \quad (\text{A5})$$

which can be written in the form

$$[\square - c m^2 + V(x)] \hat{\theta}_1^{(2)} = 0, \quad (\text{A6})$$

with

$$\square = -\frac{c}{c^2 - v_0^2} \partial_t^2 + \frac{c^2 - v_0^2}{c} \partial_x^2 + \frac{dc}{dx} \left(1 + \frac{v_0^2}{c^2} \right) \partial_x, \quad (\text{A7})$$

the two dimensional d'Alembertian for the metric

$$ds^2 = -\frac{c^2 - v_0^2}{c} dt^2 + \frac{c}{c^2 - v_0^2} dx^2, \quad (\text{A8})$$

and

$$m^2 = (p^\perp)^2$$

$$V(x) = \frac{1}{2} \frac{d^2 c}{dx^2} \left(1 - \frac{v_0^2}{c^2} \right) - \frac{1}{4c} \left(\frac{dc}{dx} \right)^2 + \frac{5v_0^2}{4c^3} \left(\frac{dc}{dx} \right)^2. \quad (\text{A9})$$

The final form of the metric and the wave equation is obtained by using (2.7) to obtain

$$\left[-\partial_t^2 + \partial_{x^*}^2 - m^2 (c^2 - v_0^2) + \frac{(c^2 - v_0^2)}{c} V(x) \right] \hat{\theta}_1^{(2)} = 0. \quad (\text{A10a})$$

$$ds^2 = \frac{c^2 - v_0^2}{c} [-\partial_t^2 + \partial_{x^*}^2], \quad (\text{A10b})$$

The two dimensional metrics (A10b) and (2.5) differ by an overall factor of c^{-1} . However they both have horizons at the location where $c = v_0$ and they both have the same surface gravity, (2.9), if the horizon is at $x = 0$. Further if we set $V = 0$ then the equation satisfied by $\hat{\theta}_1^{(2)}$ (A10a) is the same as that satisfied by $\hat{\theta}$ (2.11). Since the relationship between x and x^* is also the same, this means that the equations for the two point function in the Unruh state (3.22a), (3.22b), and (3.23) are the same for both fields if the same sound speed profile is used. This is what allows us to use our simple model for a massive scalar field propagating in a two dimensional spacetime to obtain information that is relevant for a BEC analogue BH when there are transverse excitations of the phonon modes.

Appendix B: f_I mode normalization

The normalization for the modes f_I is nontrivial because the mass-like term in the mode equation varies with position. For our toy model, in both regions R and L this term is zero for $x^* < 0$ and a nonzero constant for $x^* > 0$.

The f_I modes can be written in the form

$$f_I = N e^{-i\omega t} \chi_I(x), \quad (\text{B1})$$

with N a normalization constant. χ_I satisfies the same equation (3.13) as χ_R^H . In R the modes can be thought of as v modes which originate at $x = +\infty$ and initially move inward. In terms of χ_I , a massive incoming v mode, $e^{-ik_R x^*}$, is partially reflected back to $x = +\infty$ by the boundary at $x^* = 0$ in the form $B_\omega e^{ik_R x^*}$ (u mode) and is partially transmitted as a massless mode towards the horizon in the form $C_\omega e^{-i\omega x^*}$ with k_R defined in (3.14c). Continuity of χ_I and its first derivative at $x^* = 0$ then gives

$$B_\omega = \frac{k_R - \omega}{k_R + \omega}, \quad (\text{B2a})$$

$$C_\omega = \frac{2\sqrt{\omega k_R}}{k_R + \omega}. \quad (\text{B2b})$$

We normalize by choosing the $t = 0$ surface in region R along with the part of the past horizon H^- which is in region L as the Cauchy surface along which we evaluate the scalar product (3.4). Note that the f_I modes vanish on H^- . Then

$$(f_I(\omega, x), f_I(\omega', x)) = -i \int_{-\infty}^{\infty} f_I(\omega, x^*) \overleftrightarrow{\partial}_t f_I^*(\omega', x^*) dx^* \quad (\text{B3a})$$

$$\begin{aligned} &= (\omega + \omega') \left[\int_{-\infty}^0 C_\omega N_\omega C_{\omega'}^* N_{\omega'}^* e^{-ix^*(\omega - \omega')} dx^* \right. \\ &\quad + \int_0^{\infty} \left(B_\omega N_\omega B_{\omega'}^* N_{\omega'}^* e^{ix^*(k_R - k'_R)} + N_\omega N_{\omega'}^* e^{-ix^*(k_R - k'_R)} \right. \\ &\quad \left. \left. + B_\omega N_\omega N_{\omega'}^* e^{ix^*(k_R + k'_R)} + N_\omega B_{\omega'}^* N_{\omega'}^* e^{-ix^*(k_R + k'_R)} \right) dx^* \right]. \end{aligned} \quad (\text{B3b})$$

The first integral in (B3b) can be computed using an integrating factor of the form $e^{\epsilon x^*}$, with $0 < \epsilon \ll 1$, along with the relation [17]

$$\frac{1}{x - x_0 \mp i\epsilon} = P \frac{1}{x - x_0} \pm i\pi\delta(x - x_0), \quad (\text{B4})$$

with P denoting the principle value. Dropping the ‘‘P’’ we find

$$\begin{aligned} \int_{-\infty}^{x^*=0} C_\omega N_\omega C_{\omega'}^* N_{\omega'}^* e^{-ix^*(\omega - \omega')} dx^* &= \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{x^*=0} C_\omega N_\omega C_{\omega'}^* N_{\omega'}^* e^{-ix^*(\omega - \omega' + i\epsilon)} dx^* \\ &= \lim_{\epsilon \rightarrow 0} \left[C_\omega N_\omega C_{\omega'}^* N_{\omega'}^* \left(\frac{1}{-i(\omega - \omega' + i\epsilon)} \right) \right] \\ &= \left[C_\omega N_\omega C_{\omega'}^* N_{\omega'}^* \left(\frac{i}{(\omega - \omega')} + \pi\delta(\omega - \omega') \right) \right]. \end{aligned} \quad (\text{B5})$$

The second integral can be computed in a similar manner.

Using the identity $\delta(k_R - k'_R) = \frac{k_R}{\omega} [\delta(\omega - \omega') - \delta(\omega + \omega')]$ along with the fact that $\delta(\omega + \omega') = 0 = \delta(k_R + k'_R)$ it is easy to show that

$$\begin{aligned}
(f_I(\omega, x), f_I(\omega', x)) &= (\omega + \omega') \left[C_\omega N_\omega C_{\omega'}^* N_{\omega'}^* \left(\frac{i}{(\omega - \omega')} + \pi \delta(\omega - \omega') \right) \right. \\
&\quad + B_\omega N_\omega B_{\omega'}^* N_{\omega'}^* \left(\frac{i}{(\omega - \omega')} + \pi \frac{k_R}{\omega} \delta(\omega - \omega') \right) \\
&\quad + N_\omega N_{\omega'}^* \left(\frac{-i}{(\omega - \omega')} + \pi \frac{k_R}{\omega} \delta(\omega - \omega') \right) + B_\omega N_\omega N_{\omega'}^* \left(\frac{i}{(\omega - \omega')} \right) \\
&\quad \left. + N_\omega B_{\omega'}^* N_{\omega'}^* \left(\frac{-i}{(\omega - \omega')} \right) \right]. \tag{B6}
\end{aligned}$$

Using Eqs. (B2) for B_ω and C_ω gives

$$\begin{aligned}
(f_I(\omega, x), f_I(\omega', x)) &= (\omega + \omega') N_\omega N_{\omega'}^* \left\{ \left(\frac{2}{1 + \frac{\omega}{k_R}} \right) \left(\frac{2}{1 + \frac{\omega'}{k'_R}} \right) \left[\frac{i}{(\omega - \omega')} + \pi \delta(\omega - \omega') \right] \right. \\
&\quad + \left(\frac{1 - \frac{\omega}{k_R}}{1 + \frac{\omega}{k_R}} \right) \left(\frac{1 - \frac{\omega'}{k'_R}}{1 + \frac{\omega'}{k'_R}} \right) \left[\frac{i}{(k_R - k'_R)} + \pi \frac{k_R}{\omega} \delta(\omega - \omega') \right] + \left[\frac{-i}{(k_R - k'_R)} + \pi \frac{k_R}{\omega} \delta(\omega - \omega') \right] \\
&\quad \left. + \left(\frac{1 - \frac{\omega}{k_R}}{1 + \frac{\omega}{k_R}} \right) \left[\frac{i}{(k_R - k'_R)} \right] + \left(\frac{1 - \frac{\omega'}{k'_R}}{1 + \frac{\omega'}{k'_R}} \right) \left[\frac{-i}{(k_R - k'_R)} \right] \right\}. \tag{B7}
\end{aligned}$$

The right hand side of this equation can be broken into a part which is a factor of $\delta(\omega - \omega')$ and a part which is not. The latter is

$$\begin{aligned}
(\omega + \omega') N_\omega N_{\omega'}^* &\left\{ \left(\frac{2}{1 + \frac{\omega}{k_R}} \right) \left(\frac{2}{1 + \frac{\omega'}{k'_R}} \right) \left[\frac{i}{(\omega - \omega')} \right] + \left(\frac{1 - \frac{\omega}{k_R}}{1 + \frac{\omega}{k_R}} \right) \left(\frac{1 - \frac{\omega'}{k'_R}}{1 + \frac{\omega'}{k'_R}} \right) \left[\frac{i}{(k_R - k'_R)} \right] \right. \\
&\quad \left. + \left[\frac{-i}{(k_R - k'_R)} \right] + \left(\frac{1 - \frac{\omega}{k_R}}{1 + \frac{\omega}{k_R}} \right) \left[\frac{i}{(k_R - k'_R)} \right] + \left(\frac{1 - \frac{\omega'}{k'_R}}{1 + \frac{\omega'}{k'_R}} \right) \left[\frac{-i}{(k_R - k'_R)} \right] \right\}. \tag{B8}
\end{aligned}$$

After some algebra one finds that this expression is equal to zero. The part of the Eq. B7 which is proportional to $\delta(\omega - \omega')$ is,

$$\pi(\omega + \omega') N_\omega N_{\omega'}^* \left\{ \left(\frac{2}{1 + \frac{\omega}{k_R}} \right) \left(\frac{2}{1 + \frac{\omega'}{k'_R}} \right) + \left(\frac{1 - \frac{\omega}{k_R}}{1 + \frac{\omega}{k_R}} \right) \left(\frac{1 - \frac{\omega'}{k'_R}}{1 + \frac{\omega'}{k'_R}} \right) \frac{k_R}{\omega} \frac{k_R}{\omega} \right\}. \tag{B9}$$

Since $(f_I(\omega, x), f_I(\omega', x)) = \delta(\omega - \omega')$ the above expression must be equal to 1 when $\omega = \omega'$. With this constraint we find that

$$N_\omega = \sqrt{\frac{\frac{2\omega^2}{k_R} + \frac{\omega^3}{k_R^2} + \omega}{(4\pi\omega) \left(\frac{\omega^2}{k_R} + k_R + 2\omega \right)}} = \frac{1}{\sqrt{4\pi k_R}}. \tag{B10}$$

-
- [1] C. Barcelo, S. Liberati and M. Visser, Living Rev. Rel. **8**, 12 (2005).
 - [2] S.W. Hawking, Nature **248**, 30 (1974); Comm. Math. Phys. **43** 199 (1975).
 - [3] W. Unruh, Phys. Rev. Lett. **46**, 1351 (1981).
 - [4] J. Steinhauer, Nature Phys. **12** (2016), 959.
 - [5] J.R.M. de Nova, K. Golubkov, V.I. Kolobov, J. Steinhauer, arXiv:1809.00913.

- [6] R. Balbinot, A. Fabbri, S. Fagnocchi, A. Recati, and I. Carusotto, *Phys. Rev. A* **78**, 021603 (2008).
- [7] I. Carusotto, S. Fagnocchi, A. Recati, R. Balbinot, and A. Fabbri, *New J. Phys.* **10**, 103001 (2008).
- [8] A. Recati, N. Pavloff and I. Carusotto, *Phys. Rev. A* **80**, 043603 (2009).
- [9] J. Macher and R. Parentani, *Phys. Rev. A* **80**, 043601 (2009).
- [10] P.R. Anderson, R. Balbinot, A. Fabbri, R. Parentani, *Phys. Rev.* **D87**, 124018 (2013).
- [11] A. Coutant, A. Fabbri, R. Parentani, R. Balbinot, P.R. Anderson, *Phys. Rev.* **D86**, 064022 (2012).
- [12] A. Coutant and R. Parentani, *Phys. Fluids* **26**, 044106 (2014).
- [13] C. Mayoral, A. Recati, A. Fabbri, R. Parentani, R. Balbinot and I. Carusotto, *New J. Phys.* **13**, 025007 (2011).
- [14] W. Unruh, *Phys. Rev. D* **14**, 870 (1976).
- [15] R. Balbinot, I. Carusotto, A. Fabbri, C. Mayoral, A. Recati, *Lect. Notes Phys.* **870**, 181-219 (2013).
- [16] N. Birrell and P. Davies, *Quantum fields in curved space*, Cambridge University Press (1982).
- [17] see e.g. J. Mathews and R. L. Walker, *Mathematical Methods of Physics*, Second Edition , W.A. Benjamin, Inc. (1970), p. 481.
- [18] G. Jannes, P. Maïssa, T.G. Philbin and G. Rousseaux, *Phys. Rev. D.* **83**, 104028 (2011).