# Gravitational backreaction near cosmic string kinks and cusps

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# Gravitational back-reaction near cosmic string kinks and cusps

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## Abstract

We find the leading-order effect of gravitational back-reaction on cosmic strings for points near kinks and cusps. Near a kink, the effect diverges as the inverse cube root of the distance to the kink, and acts in a direction transverse to the worldsheet. Over time the kink is rounded off, but only regions fairly close to the kink are significantly affected. Near cusps, the effect diverges inverse linearly with the distance to the cusp, and acts against the direction of the cusp motion. This results in a fractional loss of string energy that diverges logarithmically with the distance of closest approach to the cusp.

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## I. INTRODUCTION

Cosmic strings are one-dimensional topological defects which may form dynamically at a symmetry breaking phase transition in the early universe [1, 2]. Models of string theory also suggest the possibility that fundamental strings (and D1-branes) can be stretched by the cosmic expansion in the early universe and form a cosmic superstring network [3, 4]. As massive objects generically in motion, the strings radiate gravitational waves, and a network of cosmic string loops would produce a stochastic background (E.g., see [5] and references therein). They are therefore of great interest to gravitational wave observatories, many of which are actively searching for cosmic strings [6–8].

The emission of gravitational waves is accompanied by back-reaction: cosmic strings self-interact gravitationally, which generically changes their shape and has the potential to affect the stochastic gravitational wave background. However, owing to the complexity of a typical cosmic string loop's shape [9], it is generally infeasible to solve analytically for the evolution of a cosmic string undergoing gravitational back-reaction. Analytic solutions are known only for a few simple loop shapes [10, 11].

Instead we focus here on the self-interaction process very near features of the cosmic string loop of particular interest to its overall evolution: kinks and cusps. Kinks are persistent points on a loop where there is a discontinuity in the the tangent vector to the loop [12]; cusps are transient points that recur once per oscillation period where the string moves (formally) at the speed of light [13].

The pioneering work in cosmic string back-reaction was done by Quashnock and Spergel [14]. They found that there were no divergences in the gravitational back-reaction due to nearby points on a smooth string. However, in the case of kinks and cusps, the string is not smooth, so their argument does not apply, and there is the possibility of effects that become unboundedly large at points arbitrarily close to these features.

Indeed, we find that points on cosmic strings very near to kinks and cusps experience a divergent self-force. This corrects the claim made by two of us (J.M.W and K.D.O) in Ref. [15] that the back-reaction near kinks was not divergent and thus that kinks would not be rounded off. The error in the analysis of Ref. [15] is discussed in Ref. [16].

In Sec. II, we frame the problem and establish our methodologies. In Sec. III, we find the self-interaction for a generic point far from kinks or cusps, reproducing a result of Ref. [14]. In Sec. IV, we solve for the self-interaction very near to a kink, and in Sec. V for very near to a cusp. We conclude in Sec. VI.

We work in linearized gravity, which is accurate because the string's coupling to gravity is very small. Our metric signature is $(- + ++)$, and we work in units where the speed of light is one.

## II. SETUP

### A. The string worldsheet

We first consider a string following the Nambu-Goto equations of motion in flat space. As usual, we will describe the string in the conformal gauge and choose the timelike parameter on the string equal to the spacetime coordinate $t$. Then the string motion is given by [2],

$$X^\gamma = \frac{A^\gamma(v) + B^\gamma(u)}{2}\,, \tag{1}$$

where $u$ and $v$ are null coordinates and $A' = dA/dv$ and $B' = dB/dv$ are null vectors tangent to the string worldsheet and with unit time component. In terms of the usual spacelike string coordinate $\sigma$ that parameterizes energy, $u = t + \sigma$ and $v = t - \sigma$.

The gravitational effect of the string will give rise to a small perturbation to the metric, which will in turn give a small correction to the string motion. We will compute that correction and apply it after a complete oscillation by changing the functions $A$ and $B$. We will see below that this approximation is very accurate in realistic situations.

The tangent vectors $A'$ and $B'$ have unit spatial length, and so we commonly represent their spatial parts, $\mathbf{A}'(v)$ and $\mathbf{B}'(u)$, as curves on the unit sphere [17]. In this representation, we may easily identify kinks and cusps: kinks are discontinuous jumps of a tangent vector from one point on the unit sphere to another, while cusps are points on the unit sphere where the tangent vector curves cross. Kinks are a phenomenon due to one of the two tangent vectors, and are present at any time slice of the loop, while cusps involve both tangent vectors and only appear at a specific moment in each oscillation. This representation of kinks and cusps demonstrates that kinks inhibit cusps: a discontinuous jump in a tangent vector's curve allows it to avoid an intersection with the other tangent vector. For a closed loop in the rest frame, the "center of mass" of the tangent vector curves must lie at the center of the unit sphere, and so string loops will generically have cusps unless they contain kinks.

We are interested in the back-reaction on some point on a cosmic string, which we will refer to as the *observation point* or simply the *observer*. We will indicate observer quantities by an overbar, i.e. the observer is located at $\bar{X}$.

In most cases we place the origin of coordinates at the observer, but for observers near a cusp, we will use the cusp itself as the origin. Quantities at the origin will be denoted by subscript 0, and we will expand around that point,

$$A(v) = vA'_0 + \frac{v^2}{2}A''_0 + \frac{v^3}{6}A'''_0\,, \tag{2a}$$

$$B(u) = uB'_0 + \frac{u^2}{2}B''_0 + \frac{u^3}{6}B'''_0\,. \tag{2b}$$

In order for the vectors to be null ($A' \cdot A' = B' \cdot B' = 0$), we must introduce the constraints

$$A'_0 \cdot A'_0 = 0\,, \tag{3a}$$

$$A'_0 \cdot A''_0 = 0\,, \tag{3b}$$

$$A'_0 \cdot A'''_0 = -A''^2_0\,, \tag{3c}$$

and likewise in $B$.

The acceleration felt by a point due to the gravitational effect of the string is, at first order, [14]

$$\bar{X}^\gamma_{,uv} = -\frac{1}{8}\eta^{\gamma\rho}\left(h_{\beta\rho,\alpha} + h_{\rho\alpha,\beta} - h_{\alpha\beta,\rho}\right)\bar{A}'^\alpha \bar{B}'^\beta\,. \tag{4}$$

Here $\eta_{\mu\nu}$ is the flat-space metric, and $h_{\alpha\beta}$ is the perturbation to that metric. We can compute the change of the tangent vectors due to gravitational back-reaction by integrating the acceleration induced by the unperturbed worldsheet[1] over a full oscillation,

$$\Delta A'^{\gamma} = 2\int_0^L X^{\gamma}_{,uv} du\,, \tag{5a}$$

$$\Delta B'^{\gamma} = 2\int_0^L X^{\gamma}_{,uv} dv\,. \tag{5b}$$

The metric depends on the choice of coordinates (i.e., the gauge) for the perturbed spacetime. Thus $\bar{X}^{\gamma}_{,uv}$ may contain gauge artifacts. However, $\Delta A'$ and $\Delta B'$ do not have this problem. The metric oscillates with the oscillation of the string, but $\Delta A'$ and $\Delta B'$ grow linearly with the number of oscillations (as long as we continue to use the approximation that the source worldsheet is unchanged). This provides a clean separation between effects that may and those that may not have gauge dependence.

Since the corrections to $A'$ leave $A'$ null, we will automatically have $\Delta A' \cdot A' = 0$. But because of the Lorentzian metric, adding $\Delta A'$ may change the length of $A'$, which represents a loss of energy from the string. Since we demand that $|\mathbf{A}'| = 1$, we must change the parameterization by redefining $v$. The same remarks apply to $B'(u)$.

This reparameterization raises the question of whether the divergences that we find below could be only artifacts of the parameterization. The answer is that so long as the divergent effect changes the direction of $A'$ or $B'$, it is not a parameterization artifact, because these changes of direction cannot be removed by reparameterization.

### B. The metric perturbation

We will now compute the metric perturbation at an observer position $\bar{X}$ due to some source point $X$. Let $\Delta X = X - \bar{X}$, the vector from the observer to the source, and let $\mathcal{I} = (\Delta X)^2$, the squared interval between source and observer.

Starting from the linearized Einstein equations,

$$\Box h_{\alpha\beta} = 16\pi G S_{\alpha\beta}\,, \tag{6}$$

where $G$ is Newton's constant and $S$ the trace-reversed stress-energy tensor, we solve by the method of Green's functions,

$$h_{\alpha\beta}(\bar{X}) = 8G\int d^4x S_{\alpha\beta}(x)\delta(\mathcal{I})\,, \tag{7}$$

where we take the integral only over source points $X$ in the past of $\bar{X}$. A string has a stress tensor of the form [14]

$$S_{\alpha\beta}(X) = \frac{\mu}{4}\int du\,dv\; s_{\alpha\beta}\;\delta^{(4)}(X - X(u,v))\,, \tag{8}$$

where $\mu$ denotes the energy per unit length of the string and we have defined[2] $s_{\alpha\beta} = \Sigma_{\alpha\beta}(A', B')$, with

$$\Sigma_{\alpha\beta}(P,Q) = P_\alpha Q_\beta + Q_\alpha P_\beta - \eta_{\alpha\beta}(P\cdot Q)\,. \tag{9}$$

[1] This is the approximation that was used in [11, 14, 15].
[2] The quantity $s_{\alpha\beta}$ here is twice the $\sigma_{\alpha\beta}$ of Ref. [11] and four times the $F_{\alpha\beta}$ of Ref. [14].

We pause here to note two important features of $\Sigma$. If $N$ is a null vector,

$$\Sigma_{\alpha\beta}(N,Q)N^\alpha = 0\,, \tag{10a}$$

$$\Sigma_{\alpha\beta}(P,Q)N^\alpha N^\beta = 2(N\cdot P)(N\cdot Q)\,. \tag{10b}$$

These features will lead to a number of useful simplifications further down the road.

Putting Eq. (8) into Eq. (7), we find

$$h_{\alpha\beta}(\bar{X}) = 2G\mu \int du\, dv\;\, s_{\alpha\beta}(X)\;\delta(\mathcal{I})\,. \tag{11}$$

The metric is thus determined by the effect of all places where the backward lightcone from the observation point intersects the string worldsheet, which we will call the *intersection line*.

We can eliminate one integral in Eq. (11) by changing variables in the $\delta$-function. For example, to eliminate $v$, we write

$$\delta(\mathcal{I}) = -\frac{\delta(v-v(u))}{\mathcal{I}_{,v}}\,, \tag{12}$$

where $v(u)$ denotes the (unique) value of $v$ for the given $u$ that puts the point $(u,v)$ on the past lightcone of the observer. (The negative sign in Eq. (12) appears because $\mathcal{I}_{,v}<0$). The result will be an integral giving the metric at $\bar{X}$ as a sum of the contributions due to the stress-energy at each source point. We could then differentiate $h_{\alpha\beta}$ and use Eq. (4) to find the acceleration. Indeed, this is the procedure used in Ref. [11].

In order to differentiate, though, we would need the metric not just on the worldsheet but nearby. It turns out to be easier to differentiate Eq. (11) first [14],

$$h_{\alpha\beta,\gamma}(\bar{X}) = 4G\mu \int du\, dv\;\, s_{\alpha\beta}(X)\;\delta'(\mathcal{I})X^\gamma\,. \tag{13}$$

Then we can write the derivative with respect to $\mathcal{I}$ in terms of a derivative with respect to $v$ (say),

$$h_{\alpha\beta,\gamma}(\bar{X}) = 4G\mu \int du\, dv \left(\frac{s_{\alpha\beta}(X)}{\mathcal{I}_{,v}}\right)\frac{\partial}{\partial v}\delta(\mathcal{I}),\,. \tag{14}$$

We integrate by parts and then proceed as above to get [14]

$$h_{\alpha\beta,\gamma}(\bar{X}) = 4G\mu \int du \left[\frac{1}{\mathcal{I}_{,v}}\frac{\partial}{\partial v}\left(\frac{s_{\alpha\beta}\Delta X_\gamma}{\mathcal{I}_{,v}}\right)\right]_{v=v(u)}\,. \tag{15}$$

Equation (15) gives the metric derivative at $\bar{X}$ as an integral over source points and allows us to consider $\bar{X}$ only on the worldsheet. We could also have chosen to convert $\delta'$ using $u$ instead of $v$, and (independently) to change variables in $\delta(\mathcal{I})$ to $u$ instead of $v$.

To apply Eq. (15), we proceed as follows. There are two branches to the intersection line near $\bar{X}$, one going mostly in the direction of decreasing $u$ and the other mostly in the direction of decreasing $v$. We will consider only the former, meaning source points where $\Delta u = u-\bar{u} < 0$ and $\Delta v = v-\bar{v} \geq 0$. The latter condition is necessary because if $\Delta u, \Delta v < 0$, the source point would be in chronological past of the observer, not on the lightcone.

Given the specific form of string, we can write an explicit expression for $\mathcal{I}(u,v)$. For a specific $u<0$, we can solve $\mathcal{I}=0$ for $v$. We then perform the operations in Eq. (15) to find $h_{\alpha\beta,\gamma}$.

We can write

$$\mathcal{I} = \left(\frac{\Delta A(v)+\Delta B(u)}{2}\right)^2 \tag{16}$$

so we have the derivative

$$\mathcal{I}_{,v} = \left(\frac{\Delta A(v)+\Delta B(u)}{2}\right)\cdot \Delta A' \tag{17}$$

### C. Coordinate system

We can simplify our calculations by using a coordinate system adapted to the worldsheet. For most purposes, we will use a pseudo-orthogonal coordinate system $(u,v,c,d)$ constructed around the observation point, with basis vectors $e_{(u)} = \bar{B}'/2$, $e_{(v)} = \bar{A}'/2$, and $e_{(c)}$ and $e_{(d)}$ any unit spacelike vectors perpendicular to $\bar{A}'$ and $\bar{B}'$ and to each other. Defining $Z = \bar{A}'\cdot\bar{B}'$, the corresponding covector basis is $e^{(u)} = 2\bar{A}'/Z$, $e^{(v)} = 2\bar{B}'/Z$, $e^{(c)} = e_{(c)}$, $e^{(d)} = e_{(d)}$, and the metric tensor is in $uvcd$ coordinates is

$$\eta_{\alpha\beta} = \begin{pmatrix} 0 & Z/4 & 0 & 0 \\ Z/4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \eta^{\alpha\beta} = \begin{pmatrix} 0 & 4/Z & 0 & 0 \\ 4/Z & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{18}$$

This basis allows us a number of simplifications in vector components. Namely,

- $\bar{A}'^v = 2$, $\bar{B}'^u = 2$, and all other components of both are zero.
- $\bar{A}'_u = Z/2$, $\bar{B}'_v = Z/2$, and all other components of both are zero.
- Because $\bar{A}'\cdot\bar{A}'' = 0$, we have $\bar{A}''^u = \bar{A}''_v = 0$, and similarly, $\bar{B}''^v = \bar{B}''_u = 0$.

There is a cancellation in $\Sigma_{uv}$, so that

$$\Sigma_{uv}(P,Q) = -\frac{Z}{4}(P\star Q)\,, \tag{19}$$

where we define $P\star Q = P_cQ_c + P_dQ_d$, which can be understood as the inner product in the subspace perpendicular to the worldsheet.

Finally, Eq. (4) becomes

$$\bar{X}^\gamma_{,uv} = -\frac{1}{2}\eta^{\gamma\rho}\left(h_{u\rho,v} + h_{\rho v,u} - h_{uv,\rho}\right)\,. \tag{20}$$

Then, making use of Eq. (18), we find the acceleration components in the $uvcd$ basis,

$$X^u_{,uv} = -\frac{2}{Z}h_{vv,u} \tag{21a}$$

$$X^v_{,uv} = -\frac{2}{Z}h_{uu,v} \tag{21b}$$

$$X^c_{,uv} = \frac{1}{2}(h_{uv,c} - h_{uc,v} - h_{vc,u}) \tag{21c}$$

$$X^d_{,uv} = \frac{1}{2}(h_{uv,d} - h_{ud,v} - h_{vd,u})\,. \tag{21d}$$

## III. NEAR A GENERIC POINT

We will now find the leading-order effect of back-reaction on the smooth string worldsheet, reproducing a result of Quashnock and Spergel [14]. We will choose the origin at the observation point. Then

$$\mathcal{I} = \left(\frac{A(v) + B(u)}{2}\right)^2 . \tag{22}$$

We will consider the branch of the intersection line going nearly in the $-u$ direction, so $|u| \gg |v|$. To find $v(u)$, we use Eqs. (2,3) and disregard terms of order $v^2$, $u^2 v$, $u^5$ and higher, to find

$$\mathcal{I} = \frac{Zuv}{2} - \frac{\bar{B}''^2 u^4}{48} . \tag{23}$$

Setting $\mathcal{I} = 0$ gives

$$v(u) = \frac{\bar{B}''^2 u^3}{24Z} . \tag{24}$$

Thus we have consistently disregarded terms higher than order $u^4$ in Eq. (23).

We will need to be more accurate in computing $\mathcal{I}_{,v}$. From Eq. (17), $\mathcal{I}_{,v} = A\cdot A'/2 + B\cdot A'/2$. But from Eqs. (2,3) the first term will be $\mathcal{O}(v^3)$. We will not be interested in effects at this level, and so we can write

$$\mathcal{I}_{,v} = \frac{B\cdot A'}{2} = uA'_u + \frac{(\bar{B}''\cdot A')u^2}{4} . \tag{25}$$

Higher orders in $u$ will not contribute. Outside the derivative in Eq. (15), we need only the first term of Eq. (25) and we can replace $A'$ with $\bar{A}'$. Thus we define

$$g(u,v) = \frac{s_{\alpha\beta}(u,v) X_\gamma}{A'_u(v) + u(\bar{B}''\cdot A')/4} \tag{26}$$

and using $A'_u = Z/2$ we can rewrite Eq. (15) as

$$h_{\alpha\beta,\gamma} = \frac{8G\mu}{Z}\int \frac{du}{u^2}\left(\frac{\partial g_{\alpha\beta\gamma}}{\partial v}\right) . \tag{27}$$

Because we would like to find contributions up to $\mathcal{O}(u)$ in the integrand, we will expand

$$\frac{\partial g}{\partial v} = \bar{g}_{,v} + u\bar{g}_{,uv} + \frac{u^2 \bar{g}_{,uuv}}{2} + \frac{u^3 \bar{g}_{,uuuv}}{6} + v\bar{g}_{,vv} . \tag{28}$$

We will not need higher orders.

Now $\bar{X} = 0$, so $X_\gamma$ must be differentiated. Furthermore, $X_{u,u} = \bar{B}'_u/2 = 0$ and $X_{u,uu} = \bar{B}''_u/2 = 0$. Thus in order to have a $u$ component in some differentiated $X$, we need to differentiate with respect to $v$, or 3 times with respect to $u$, and vice versa.

On the other hand, $\bar{s}_{u\beta} = \Sigma_{\alpha\beta}(\bar{A}', \bar{B}')\bar{B}'^{\alpha} = 0$, so $s_{\alpha\beta}$ must be differentiated. Differentiating with respect to $v$ just differentiates $A'$, so $\bar{s}_{u\beta,v\ldots v} = 0$ regardless of the number of derivatives. In order to have a $u$ component in $\bar{s}$, we much differentiate with respect to $u$, and the same for $v$.

Furthermore, $\bar{s}_{uu,u} = 0$ because of Eq. (10). Additional derivatives with respect to $v$ make no difference.

Now let us find the leading order term in Eq. (28). We need two derivatives, one for $s_{\alpha\beta}$ and one for $X_\gamma$. But among $\alpha, \beta, \gamma$ there must be $u$ and $v$. By the considerations above, we thus need to differentiate $s_{\alpha\beta}$ and $X_\gamma$ both with respect to $v$ or both with respect to $u$. Thus $\bar{g}_{,v}$ and $\bar{g}_{,uv}$ do not contribute, and the integral in Eq. (15) never diverges.

To go beyond this level, we need to consider the specific combinations of indices we need in Eq. (21). First consider $h_{vv,u}$. This involves $s_{vv}$. To get a term in Eq. (28) that doesn't vanish we need to go up to $s_{vv,vv}$. Thus we need the last term in Eq. (28), but both derivatives have been applied to $s$, leaving none for $X$, so $h_{vv,u} = 0$ at this order.

Now consider $h_{uu,v}$. Here we need to differentiate $s$ twice and $X$ once with respect to $u$. Thus we take the penultimate term of Eq. (28). There's one derivative with respect to $v$ left, and it acts on

$$\frac{s_{uu,uu}}{A'_u} = \frac{2A'_u B'''_u}{A'_u} = 2\bar{B}'''_u\,, \tag{29}$$

which has no $v$ dependence, so $h_{uu,v} = 0$.

So we are interested now only in $X^c_{,uv}$ and $X^d_{,uv}$. These have exactly the same form, so we will compute only the former.

There are 3 terms with the indices in different orders. First consider $h_{vc,u}$. To keep $s_{vc}$ from vanishing we need to differentiate with respect to $v$. Then we need to differentiate $X$ once with respect to $v$ or thrice with respect to $u$, using all the rest of the derivatives in either case. In the former case,

$$\bar{g}_{,vv} = \Sigma_{vc}(\bar{A}'', \bar{B}') = \frac{\bar{A}''_c Z}{2} \tag{30}$$

Differentiating $X_u$ with respect to $v$ gave $A'_u/2$, canceling the $A'_u$ in the denominator and a combinatoric factor of 2 from the placement of the derivatives. The other possibility gives

$$\bar{g}_{,uuuv} = \frac{\Sigma_{vc}(\bar{A}'', \bar{B}')\bar{B}'''_u}{2\bar{A}'_u} = -\frac{\bar{A}''_c \bar{B}''^2}{4}\,. \tag{31}$$

These terms give a contribution from each $u$ to $X^c_{,uv}$ of

$$\frac{G\mu \bar{A}''_c \bar{B}''^2 u}{12Z}\,. \tag{32}$$

Now we consider $h_{uc,v}$ and $h_{uv,c}$ together. We'll need to differentiate $s$ with respect to $u$, so $\bar{g}_{vv}$ does not contribute here. The other terms have one $v$ derivative. If we apply it to $X_\gamma$, we get $\bar{A}'_c = 0$ or $\bar{A}'_v = 0$, so we can take $B_\gamma/2$ for $X_\gamma$.

Thus we take

$$\frac{s_{uv}B_c - s_{uc}B_v}{2(A'_u + u\bar{B}'' \cdot A'/4)}\,, \tag{33}$$

differentiate with respect to $u$ 2 or 3 times, set $u = 0$, and differentiate with respect to $v$.

In the first term in the numerator, one derivative must act on $s$, and two on $B_c$ giving

$$\frac{3s_{uv,u}\bar{B}''_c}{2A'_u} = \frac{3(A'_u \bar{B}''_v - (Z/4)A' \cdot \bar{B}'')\bar{B}''_c}{2A'_u} \tag{34}$$

The first term has no $v$ dependence.

In the other term from Eq. (33), we need one derivative on $s$, and one on $B_v$. If we differentiate neither the denominator nor $s$ (again), the only possible $v$ dependence is in $s_{uc,u}/A'_u$, but this is just $B''_c$, because $\bar{B}''_u = 0$. So in these cases there's nothing to differentiate with respect to $v$.

The remaining terms are

$$\frac{3s_{uc,uu}Z}{4A'_u} - \frac{3s_{uc,u}(\bar{B}'' \cdot A')Z}{8(A'_u)^2} \,. \tag{35}$$

The second term is

$$\frac{3\bar{B}''_c(\bar{B}'' \cdot A')Z}{8A'_u} \tag{36}$$

and it cancels the second term in Eq. (34). We do not know any good explanation for this cancellation.

The first term in Eq. (35) is

$$\frac{3A'_c\bar{B}'''_u Z}{4A'_u} = -\frac{3A'_c\bar{B}''^2 Z}{8_u} \tag{37}$$

plus a term with no $v$ dependence. We must apply the $v$ derivative to $\bar{A}'_c$, so the contribution from $h_{uc,v}$ and $h_{uv,c}$ is

$$\bar{g}_{,uuuv} = \frac{3A''_c\bar{B}''^2}{4} \,, \tag{38}$$

and the contribution to $X^c_{,uv}$ is

$$\frac{G\mu\bar{A}''_c\bar{B}''^2 u}{2Z} \,. \tag{39}$$

Putting together Eqs. (32,39) gives the total contribution to $X^c_{,uv}$ from a sufficiently close source point,

$$\frac{7G\mu\bar{A}''_c\bar{B}''^2 u}{12Z} \tag{40}$$

The $d$ term is just the same, while from above $X^u_{,uv} = X^v_{,uv} = 0$. One can write a total contribution from all sources nearer than some small distance $u_{\max}$,

$$X^c_{,uv} = \frac{7G\mu\bar{A}''_c\bar{B}''^2}{12Z}\int_{-u_{\max}}^{0} u \, du = \frac{7G\mu\bar{A}''_c\bar{B}''^2}{12Z}\left(\frac{u^2_{\max}}{2}\right) . \tag{41}$$

Equation (41) reproduces the result of Appendix A in Ref. [14]. But since this effect grows as we get further from the observer, the total effect is dominated by distant places where this calculation does not apply.

The main importance of this result is that there is no divergent contribution from nearby points on a smooth worldsheet. When there are points where the worldsheet is not smooth, such as kinks and cusps, this result does not apply and the effect may diverge as one approaches these special points, as we now discuss.

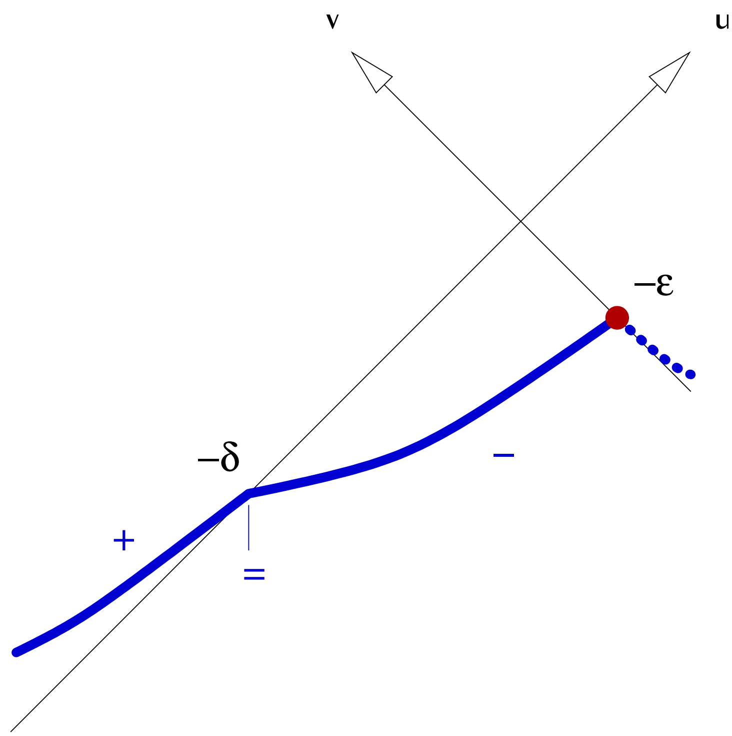


FIG. 1. A drawing of an observer point (red circle) at $(u, v) = (0, -\epsilon)$ near a kink in $A$ at $v = 0$. The intersection of the lightcone with the worldsheet is in blue, with the relevant branch solid and the other branch dotted. The region "below" the kink is labeled by $-$, above the kink by $+$, and at the kink by $=$. As the observer approaches the kink ($\epsilon \to 0$), the distance at which the intersection crosses the kink, $\delta$, will also go to zero.

## IV. CLOSE TO A KINK

We begin by introducing a kink in $A$ at $v = 0$. We will take the Taylor expansion of $B$ as before, but $A$ is no longer analytic, so let us instead consider a form which is straight on each side of the kink,

$$A(v) = \begin{cases} A'_- v & v < 0 \\ A'_+ v & v > 0 \end{cases} \tag{42}$$

Curved segments of $A$ would not affect the divergent behavior.

We will consider our observer to be at $\bar{u} = 0$ and $\bar{v} = -\epsilon < 0$. We will consider observers at $v > 0$ in Sec. IV C. The past lightcone in the mostly negative $v$ direction does not intersect the kink, so effect from such sources is the smooth result of the previous section. In the mostly negative $u$ direction it intersects the kink at some point we will call $u = -\delta$. The integral of Eq. (15) therefore covers three regimes: when $v < 0$ and $u > -\delta$, which we call *below* the kink, and denote related quantities with a subscript or superscript $-$; when $v > 0$ and $u < -\delta$, which we call *above* the kink, and which has subscript or superscript $+$; and finally when $v = 0$ and $u = -\delta$, which we call *at* the kink, and indicate by a subscript or superscript $=$. Fig. 1. shows an observer point and these three regions of its intersection line.

In the region below the kink, the existence of the kink has no effect, and the result is as in Sec. III, with no divergence. When the sources are above the kink, source quantities may no longer be similar to quantities at the observer, as assumed in Sec. III, so the calculation there are no longer applies and divergences are possible. In addition, at $u = -\delta$ there is a discontinuous change in both $s_{\alpha\beta}$ and $\mathcal{I}_{,v}$. Thus the integrand in Eq. (15) is a $\delta$-function in $u$, leading also to a divergent effect.

Before considering the regions individually, we wish to determine the relationship between $\delta$ and $\epsilon$. Let us start at the observer and move backwards along the lightcone, primarily in the $-u$ direction. We move first through the region below the kink, where $A = vA'_-$, and so to lowest order in $v$ we find

$$(\Delta A)^2 = \mathcal{O}(\epsilon^4)\,, \tag{43a}$$

$$(\Delta B)^2 = -\frac{u^4 \bar{B}''^2}{12}\,, \tag{43b}$$

$$\Delta A \cdot \Delta B = u(\epsilon + v) Z_-\,, \tag{43c}$$

where $Z_\pm = \bar{B} \cdot A'_\pm$, and thus $Z_-$ is the $Z$ of Eq. (18). Thus

$$\mathcal{I}_- = \frac{(\epsilon + v)uZ_-}{2} - \frac{u^4 \bar{B}''^2}{48}\,. \tag{44}$$

With the lightcone constraint $\mathcal{I} = 0$, this means that when we are at the kink and $(u, v) = (-\delta, 0)$, we find

$$\delta = \left(-\frac{24Z_-\epsilon}{\bar{B}''^2}\right)^{1/3}\,. \tag{45}$$

We next continue to the region above the kink. Now, $\bar{A} - A = vA'_+ + \epsilon A'_-$, and so, ignoring terms like $v^2$ and $u^4$ or higher, we find

$$\mathcal{I}_+ = \frac{\epsilon u Z_- + vuZ_+}{2} - \frac{u^4 \bar{B}''^2}{48}\,. \tag{46}$$

This allows us to write the general relationship

$$v(u)_\pm = \frac{(u^3 + \delta^3)\bar{B}''^2}{24 Z_\pm}\,. \tag{47}$$

Since we're concerned with $u$ of order $\delta$, $v(u)$ is of order $\delta^3$, and so we will be concerned only with terms at most linear in $v$.

Before moving on, we note that we can also write the general relationship

$$\mathcal{I}^\pm_{,v} = \frac{A'_\pm \cdot B}{2} = uA'_{u\pm} + \frac{u^2 A'_\pm \cdot \bar{B}''}{4}\,, \tag{48}$$

which is necessary for finding the denominator of the acceleration integrand.

Now, we can consider how divergences might arise as we integrate along the intersection line with respect to $u$, starting above the kink and crossing it.

### A. Divergent behavior above the kink

We begin on the side of the kink with $v > 0$, $u < -\delta$. Here, the only thing in Eq. (15) that can be differentiated with respect to $v$ is $\Delta X_\gamma$, and so find to lowest order that

$$h^+_{\alpha\beta,\gamma} = \frac{8G\mu}{Z_+^2} \int^{-\delta} du \frac{\bar{s}^+_{\alpha\beta} A'_{+\gamma}}{u^2} = \frac{8G\mu \bar{s}^+_{\alpha\beta} A'_{+\gamma}}{Z_+^2 \delta}\,. \tag{49}$$

We have included only the upper limit of integration, which would be the source of terms that diverge for small $\delta$. If we expand to one more order in $u$, we expect divergences of order $\ln\delta$, but we will not attempt to compute those.

Consulting Eq. (21), we see that all terms involve at least one $u$ index. But $\bar{s}^{+}_{u\beta} = 0$ from Eq. (10). Thus we must have $\gamma = u$, and so the only metric perturbation terms we need to consider are

$$h^{+}_{vv,u} = \frac{2G\mu(A'_{+}\cdot A'_{-})Z_{-}}{Z_{+}\delta}\,, \tag{50a}$$

$$h^{+}_{vc,u} = \frac{2G\mu A'_{+c}Z_{-}}{Z_{+}\delta}\,. \tag{50b}$$

The terms with $d$ instead of $c$ are analogous.

### B. Divergent behavior at the kink

Now we consider divergences as we integrate across the kink, where $u = -\delta$, $v = 0$. There is no jump in $\Delta X$ there, but $s_{\alpha\beta}$ and $\mathcal{I}_{,v}$ change discontinuously. So we define $F^{+}_{\alpha\beta}$ to be the value of $s_{\alpha\beta}/\mathcal{I}_{,v}$ immediately above the kink and $F^{-}_{\alpha\beta}$ to be the value immediately below,

$$F^{\pm}_{\alpha\beta} = -\frac{s^{\pm}_{\alpha\beta}}{\delta Z_{\pm} + \delta^{2}(A'_{\pm}\cdot\bar{B}')/4} = -\left(\frac{2}{\delta Z_{\pm}} - \frac{A'_{\pm}\cdot\bar{B}''}{Z^{2}_{\pm}}\right)s^{\pm}_{\alpha\beta}\,, \tag{51}$$

plus higher orders in $\delta$. For most of our purposes, we will only need the $1/\delta$ term, but the latter will be important later on. Now, we write

$$h^{=}_{\alpha\beta,\gamma} = 2G\mu\int du\frac{\delta(v)(F^{+}-F^{-})_{\alpha\beta}(\epsilon A'_{-} - B(-\delta))_{\gamma}}{\mathcal{I}_{,v}}\,. \tag{52}$$

We now substitute $v(u)$ given by $\mathcal{I} = 0$ and use the relation

$$\frac{\delta(v)}{\mathcal{I}_{,v}} = \frac{\delta(u+\delta)}{\mathcal{I}_{,u}}\,. \tag{53}$$

Now $\mathcal{I}_{,u} = \Delta X\cdot B'$, and at the kink crossing this becomes

$$\mathcal{I}_{,u} = \frac{\epsilon Z_{-}}{2} + \frac{\delta^{3}\bar{B}''^{2}}{12} = \frac{\delta^{3}\bar{B}''^{2}}{16}\,, \tag{54}$$

so

$$h^{=}_{\alpha\beta,\gamma} = \frac{32G\mu(F^{+}-F^{-})_{\alpha\beta}(\epsilon A'_{-} - B(-\delta))_{\gamma}}{\delta^{3}\bar{B}''^{2}}\,. \tag{55}$$

We will now consider specific indices of the metric perturbation derivatives in order to find the divergent behavior of the accelerations.

### 1. Divergences for $\gamma = u$

First, consider $\gamma = u$ and expand $\bar{B}(-\delta)$. The first nonvanishing term is $\delta^3 \bar{B}''^2/12$, which combines with $\epsilon A'_{-u}$ to give $\delta^3 \bar{B}''^2/16$, and so

$$h^{=}_{\alpha\beta,u} = 2G\mu(F^+ - F^-)_{\alpha\beta} \,. \tag{56}$$

We are interested only in $\alpha\beta = vv$ and $\alpha\beta = vc$. When we choose $vv$, $F^- = 0$ and

$$F^+_{vv} = -\frac{(A'_+ \cdot A'_-)Z_-}{Z_+\delta} \tag{57}$$

to leading order, and thus

$$h^{=}_{vv,u} = -\frac{2G\mu(A'_+ \cdot A'_-)Z_-}{Z_+\delta} \,. \tag{58}$$

This cancels the term in Eq. (50a). We have calculated all possibly divergent components of the $u$ direction acceleration, and as a consequence of this cancellation we find that $X^u_{,uv}$ has no $1/\delta$ divergence.

When we choose $vc$, we again have $F_- = 0$, but now to leading order we find

$$F^+_{vc} = -\frac{A'_{+c}Z_-}{Z_+\delta} \,, \tag{59}$$

and therefore

$$h^{=}_{vc,u} = -\frac{2G\mu A'_{+c}Z_-}{Z_+\delta} \,. \tag{60}$$

Once again, this cancels the above-kink region contribution, and so terms like $h_{vc,u}$ are not divergent. The reason for these cancellations can be seen by rewriting Eq. (15) using $\partial/\partial u$ instead of $\partial/\partial v$.

### 2. Divergences for $\gamma = v$

Now we consider terms with $\gamma = v$. Because $A'_{-v} = 0$, we need $B_v = \delta Z_-/2$, and therefore

$$h^{=}_{\alpha\beta,v} = -\frac{16G\mu Z_-(F^+ - F^-)_{\alpha\beta}}{\bar{B}''^2\delta^2} \,. \tag{61}$$

The only two choices of $\alpha\beta$ we need to consider are $uu$ and $uc$. For the former,

$$F^{\pm}_{uu} = \bar{B}''^2\delta \tag{62}$$

to first order, so $F^+_{uu} = F^-_{uu}$. Thus $h^{=}_{uu,v} = 0$, so $X^v_{,uv}$ has no $1/\delta$ divergence.

Now consider $uc$. Here we must take into account both terms of Eq. (51). Moreover, we will consider the two terms in

$$s^{\pm}_{uc} = A'_{\pm u}B'_c + A'_{\pm c}B'_u \tag{63}$$

individually.

Starting with the $A'_{\pm u}B'_c$ term, and with $B'_c = -\delta\bar{B}''_c$ when $u = -\delta$, we find that for this term

$$F^{\pm}_{uc} = \left(1 + \frac{(A'_\pm \cdot \bar{B}'')\delta}{2Z_\pm}\right)\bar{B}''_c \tag{64}$$

and therefore a contribution to the metric perturbation of

$$\frac{8G\mu Z_-\bar{B}''_c}{\delta\bar{B}''^2}\left[\frac{A'_+\cdot\bar{B}''}{Z_+}-\frac{A'_-\cdot\bar{B}''}{Z_-}\right]=\frac{8G\mu Z_-\bar{B}''_c}{\delta\bar{B}''^2}(A'_+\star\bar{B}'')\,. \tag{65}$$

Then taking the $A'_{\pm c}B'_u$ term, we must go to $B'_u=\delta^2\bar{B}'''_u/2=-\delta^2\bar{B}''^2/2$. Thus, for this term,

$$F^+_{uc}=\frac{\delta A'_{+c}\bar{B}''^2}{Z_+}\,. \tag{66}$$

Of course, $A'_{-c}=0$, and so $F^-_{uc}=0$ for this term. So in sum,

$$h^=_{uc,v}=\frac{8G\mu Z_-(A'_+\star\bar{B}'')\bar{B}''_c}{\delta\bar{B}''^2}-\frac{8G\mu Z_-A'_{+c}}{\delta Z_+}\,. \tag{67}$$

*3. Divergences for $\gamma=c$*

The remaining choice for $\gamma$ is $c$. Now the leading term comes from $B_c=-\delta^2\bar{B}''_c/2$, giving

$$h^=_{\alpha\beta,c}=\frac{16G\mu(F^+-F^-)_{\alpha\beta}\bar{B}''_c}{\delta\bar{B}''^2}\,. \tag{68}$$

But now, the only choice for $\alpha\beta$ that we can make is $uv$. At leading order,

$$s^+_{uv}=-\frac{(A'_+\star B')Z_-}{4}=\frac{\delta(A'_+\star\bar{B}'')Z_-}{4} \tag{69}$$

and $s^-_{uv}=0$, and thus $F^-_{uv}=0$ as well. Thus

$$h^=_{uv,c}=\frac{8G\mu Z_-(A'_+\star\bar{B}'')\bar{B}''_c}{\delta\bar{B}''^2Z_+}\,. \tag{70}$$

This is identical to the first term of Eq. (67), and contributes oppositely in Eq. (21). This cancellation is analogous to the one involving Eq. (36). The only remaining $1/\delta$ divergent term for the $c$ direction acceleration is the second half of Eq. (67), giving

$$X^c_{,uv}=\frac{4G\mu Z_-A'_{+c}}{\delta Z_+}=-\frac{2G\mu A'_{+c}}{Z_+}\left(\frac{\bar{B}''^2Z_-^2}{3\epsilon}\right)^{1/3}\,. \tag{71}$$

Thus the transverse accelerations diverge as an observer approaches a kink, but only as the inverse cube root of the distance. Equation (71) agrees with the acceleration reported in Ref. [10] for the loop discussed there.

## C. Observers above the kink

In the previous subsections, we considered observers below the kink, i.e., points that the kink is approaching. Here we will show that there are no divergences for observation points above the kink, i.e., where the kink has already passed by. We keep the forms of $A$ and $B$ above, but now we consider an observation point with $\bar{u}=0$, $\bar{v}=\epsilon>0$. The backward lightcone that intersects the kink is the one mostly in the negative $v$ direction. The intersection occurs at a point $v = 0$, $u = \delta > 0$, with $\delta = \mathcal{O}(v^3)$. This is the critical difference: because the lightcone now starts in the $-v$ direction, perpendicular to the kink motion, it quickly reaches the kink with little transverse motion.

We will use the $u$-$v$ exchanged version of Eq. (15),

$$h_{\alpha\beta,\gamma}(\bar{X}) = 4G\mu \int dv \left[\frac{1}{\mathcal{I}_{,u}} \frac{\partial}{\partial u} \left(\frac{s_{\alpha\beta}\Delta X_\gamma}{\mathcal{I}_{,u}}\right)\right]_{u=u(v)} . \tag{72}$$

Applying $\partial/\partial u$ does not lead to any $\delta$-functions, because the $u$ direction does not cross the kink.

Now

$$\mathcal{I}_{,u} = \Delta X \cdot B' = (A_\pm v - A_-\epsilon)/2 \cdot B' = (Z_\pm v - Z_-\epsilon)/2 \,, \tag{73}$$

where we ignore $\mathcal{O}(\delta)$. We will be concerned with $v$ of order $\epsilon$, in which case $\mathcal{I}_{,u} = \mathcal{O}(\epsilon)$, and $\mathcal{I}_{,u}$ does not vanish as $v \to 0$. (It does vanish as $v \to \epsilon$, but this is just the near-observer regime of Sec. III.) Furthermore, $\mathcal{I}_{,u}$ has no $u$ dependence. Thus in Eq. (72) we must differentiate either $s_{\alpha\beta}$ or $\Delta X_\gamma$. In the former case, we are left with $\Delta X_\gamma = O(\epsilon)$. Thus the integrand is $\mathcal{O}(\epsilon^{-1})$, and since the range of integration is $\mathcal{O}(\epsilon)$, the result is at most a constant in $\epsilon$.

The other possibility is that we apply $\partial/\partial u$ to $\Delta X_\gamma$, giving $B'_\gamma/2$, and leave $s_{\alpha\beta}$ undifferentiated. Since we are ignoring $\mathcal{O}(\delta)$, we can take $B'$ as $\bar{B}'$ both in $\Delta X_\gamma$ and in $s_{\alpha\beta}$. But $\bar{B}'$ has only one nonzero component, which is $v$. Thus $\gamma$ must be $v$ and also one of $\alpha$ and $\beta$ must be $v$ (or both must be $c$ or $d$), but no such term appears in Eq. (21). Thus there's no divergence for observers approaching the kink from above.

### D. Changes to the string near a kink

What does Eq. (71) tell us about how the worldsheet is modified around a kink? Because the kink we studied is at a fixed position in $v$, the effects on $A'$ and $B'$ are different. To find the correction to $B'$ at a certain fixed $u$, we integrate around the worldsheet in the $v$ direction, following Eq. (5). This line of integration will always pass across the kink and, since the divergent part of the acceleration near the kink is only like $v^{-1/3}$, there is no divergence after integration with respect to $v$. In fact, as discussed in Sec. II A, since no divergence appears in $\Delta B'$, we cannot say for sure that there is a divergent effect on $B'$ at all.

Conversely, we find the correction to $A'$ by fixing $v$ and integrating around the worldsheet in the $u$ direction. The kink always remains the same distance away, and the divergent $v^{-1/3}$ behavior remains in the correction to $A'$. This correction is always transverse to the worldsheet, but the worldsheet direction changes as we integrate the corrections to $A'$ at different observation points. Thus the divergent correction to $A'$ for a whole oscillation is quite general, except that it must be perpendicular to $A'$, so that $A'$ remains null. This divergence cannot be a gauge artifact.

The loss of length of the string is given by the change to the time component of $A'$, which generally diverges as $\bar{v}^{-1/3}$. The total loss of length gives the total energy emitted from the string. To compute this we integrate over $\bar{v}$, which gives a finite result as it should [12].

Now we will estimate the length scale at which a kink is rounded off. Define

$$K^\gamma = A'^\gamma_+ - A'^\gamma_- \tag{74}$$

for the tangent vectors at a pair of points of fixed $v$ above and below the kink. This is the kink's "turning vector" across that range in $v$, so decreases in $K$ constitute smoothing the kink out to that range. We will assume that the back-reaction is not affected by smoothing closer to the kink than the points of interest, so we can use Eq. (71), which we rewrite as

$$X^c_{,uv} = -\frac{2G\mu}{Z_+}\left(\frac{\bar{B}''^2 Z_-^2}{3L}\right)^{1/3}\left(\frac{L}{v}\right)^{1/3} A'_{+c}\,. \tag{75}$$

This modifies the vector $A'_-$, making it closer (because $Z_+ < 0$) to $A'_+$, and so decreasing the bending angle. However, the change in $A'_-$ is given by the projection of $K$ into directions transverse to the worldsheet,

$$K_\perp = A'_+ - \frac{YB' + Z_+ A'_-}{Z_-}\,, \tag{76}$$

with $Y = A'_+ \cdot A'_-$. The length of $K$ will be modified according to how much the transverse acceleration points in the direction of $K$, i.e., the magnitude of $K_\perp \cdot K/|K|$, introducing an overall factor

$$\frac{K_\perp \cdot K}{K^2} = \frac{-YZ_+/Z_- - YZ_+/Z_- - Y + Y}{-2Y} = \frac{Z_+}{Z_-}\,, \tag{77}$$

which may be more or less than one because of the Lorentzian metric. The instantaneous change to the length of $K$ at a particular point is thus

$$|K|' = -4G\mu\left(\frac{\bar{B}''^2}{3LZ_-}\right)^{1/3}\left(\frac{L}{v}\right)^{1/3}|K|\,, \tag{78}$$

where there is a factor of 2 from Eq. (5).

Now we integrate this projection with respect to $\bar{u}$ over one oscillation. This tells us about the rate of change of the length of $K$ per oscillation. Dividing by the loop oscillation time of $L/2$ converts this to an average rate of change,

$$\frac{d|K|}{dt} = -\frac{G\mu H}{L}\left(\frac{L}{v}\right)^{1/3}|K|\,, \tag{79}$$

where the dimensionless coefficient is given by

$$H = \int_0^L d\bar{u}\; 8\left(\frac{\bar{B}''^2}{3LZ_-}\right)^{1/3}\,. \tag{80}$$

Thus, $|K|$ decreases exponentially with time, with a time constant of $(G\mu H/L)(L/v)^{1/3}$, so the kink has been significantly rounded off to distance $v$ after a time

$$t_{\text{kink}} \approx \frac{L}{G\mu H}\left(\frac{v}{L}\right)^{1/3}\,. \tag{81}$$

The loop's lifetime is $t \approx L/(\Gamma G\mu)$, with $\Gamma$ the measure of the loop's power loss rate. At the end of the loop's lifetime, we can estimate that significant rounding extends to a distance

$$v_{\text{rounded}} \approx \left(\frac{H}{\Gamma}\right)^3 L\,. \tag{82}$$

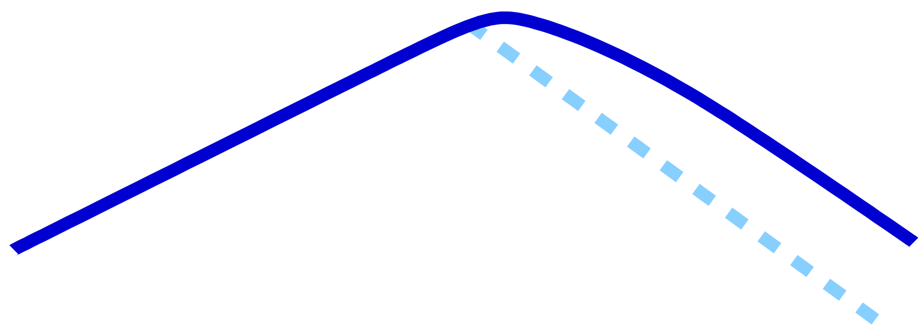

FIG. 2. How a kink is modified due to back-reaction. We show a segment of a worldsheet function, where the region above the kink (solid blue, on the left) does not change, but the region below the kink goes from being straight (dashed light blue) to having some curvature (solid blue, on the right). Note that the curvature dies out as one goes to the right, so there is some distance after which the $A'$ below the kink before and after back-reaction are effectively identical.

We show a drawing of this rounding process in Fig. 2.

Because $\Gamma$ is of order 50 for realistic loops, the rounding distance may be much less than $L$. Let us consider a "generic" loop, which has worldsheet functions which are mostly smooth circles except for a few large kinks. We take as typical values $|B''| = 2\pi/L$, $Z_\pm = -1$, so $H = 8(4\pi^2/3)^{1/3} \approx 20$ and $(H/\Gamma)^3 \approx 0.06$. This means that rounding process never has much effect on regions further from the kink than about $0.06L$; at such distances the kink mostly retains its original appearance.

If the kink is preventing the occurrence of a cusp, by jumping over what would otherwise be an intersection between $A'$ and $B'$, the smoothing process will reintroduce the cusp. However, the cusp will be weak, in the sense that little of the total string length will ever be involved in it. Of course this is a very simplified model. Strings taken from simulations have many kinks of various angles, and little smoothly bending parts of the string, so this analysis does not apply.

Our estimate of how the kink is rounded is only good if the change in one oscillation is small. This means that we require $v/L > (4G\mu H)^3$, but this is an incredibly tiny number, and so the preceding is valid until we are extremely close to the kink. For example, using roughly the current observational upper bound of $G\mu = 10^{-11}$ and our estimate of $H$ above, we find $v/L \gtrsim 10^{-30}$ as our requirement.

## V. CLOSE TO A CUSP

Now, we consider an observation point on a string with smooth $A$ and $B$, but place the observer very near to a cusp. As mentioned in Sec. II, a cusp is formed when $\mathbf{A}' = \mathbf{B}'$ or equivalently $A' = B'$, so points near a cusp have $Z = A' \cdot B' \ll 1$. Otherwise-well-behaved quantities such as Eq. (40) may thus diverge as the observation point approaches a cusp. We now analyze this situation.

## A. Coordinate system

While the *uvcd* coordinates greatly simplified our investigations of the kink (and the generic point), they are not well adapted to studying the cusp. If we define the *uvcd* basis at a point near the cusp, the vanishing of $Z$ leads to divergences in the metric and the lengths of the basis vectors, which make it difficult to distinguish actual divergences from coordinate divergences. Instead we will use a fixed basis for all points near the cusp, which we now define.

Let $e_{(w)} = A'/2$ (equivalently, $B'/2$) at the cusp, and let $e_{(m)}$ be $w$ with its spatial component reversed. Then let $e_{(p)}$ and $e_{(q)}$ be any unit spacelike vectors orthogonal to $e_{(w)}$, $e_{(m)}$, and to each other. In the *wmpq* basis, the metric tensor is

$$\eta_{\alpha\beta} = \begin{pmatrix} 0 & -1/2 & 0 & 0 \\ -1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \eta^{\alpha\beta} = \begin{pmatrix} 0 & -2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{83}$$

Like its *uvcd* cousin, this basis allows some simplifications in vector components. We expand about the cusp as in Eq. (2), getting

$$A_w = -\frac{A_0''^2}{12} v^3\,, \tag{84a}$$

$$A_m = -v + \frac{A_0''^2}{12} v^3\,, \tag{84b}$$

$$A_p = \frac{A_{0p}''}{2} v^2\,, \tag{84c}$$

with $A_q$ just $A_p$ with $p \to q$, and the $B$ dependencies the same under $A \to B$, $v \to u$. Then we find $v$ or $u$ dependence for any derivative of $A$ or $B$ by applying the appropriate number of derivatives and taking the lowest-order term.

We now take the cusp to be at the origin, and the observer to be at some point on the worldsheet $(\bar{u}, \bar{v})$ near the cusp. When we consider how different sources will affect the observer, we see that there are two regimes: one for when the sources are much closer to the observer than to the cusp, and one for when they are very far from either the observer or the cusp.

In the former case, the sources do not know about the cusp, and so the problem reduces to that of Sec. III, but the resulting effect may be quite large because $Z \ll 1$, i.e., the string is rapidly moving. But when the sources are far from the observer they cannot distinguish the observer from the cusp, and as a result their contributions to the acceleration integrand grow divergently.

Because the scale at which this growth is cut off is when the source is about as far from the observer as the observer is from the cusp, we may see divergent accelerations as the observer moves towards the cusp. Let's find such an effect now by finding the general form of the acceleration integrand, and thereby the leading-order divergent term in the acceleration.

## B. Sources far from the observer

Because we are now working with our origin at the cusp itself, we will make the replacement $B'_0 \to A'_0$ for the remainder of this section. We can now also write

$$A'_0 \cdot B''_0 = 0\,, \tag{85a}$$
$$A'_0 \cdot B'''_0 = -B''^2_0\,. \tag{85b}$$

We are considering sources close to the cusp, but much further from the cusp than the observer is. Thus we work in the regime $\bar{u}, \bar{v} \ll u, v \ll L$. Then the leading terms in $\mathcal{I}$ are those that which have a combined order in $u$ and $v$ of four, and the lightcone constraint becomes

$$0 = \mathcal{I} = \frac{A''_0 \cdot B''_0}{8} u^2 v^2 - \frac{B''^2_0}{12} u^3 v - \frac{A''^2_0}{12} u v^3 - \frac{A''^2_0}{48} v^4 - \frac{B''^2_0}{48} u^4\,. \tag{86}$$

Solving this homogeneous quartic gives $v(u) = \lambda_0 u$, with $\lambda_0$ some constant depending on the cusp parameters.

Rewriting Eq. (15) as

$$h_{\alpha\beta,\gamma} = -2G\mu \int du \frac{\mathcal{I}_{,v}\left[s_{\alpha\beta,v}(A+B)_\gamma + s_{\alpha\beta}A'_\gamma\right] - \mathcal{I}_{,vv}\left[s_{\alpha\beta}(A+B)_\gamma\right]}{\mathcal{I}^3_{,v}} \tag{87}$$

leads us to our next considerations: what are the lowest-order terms in $u$ once we have contracted the $\bar{A}'$ and $\bar{B}'$ vectors into the Christoffel symbol and made the replacement $v = v(u)$? To lowest order in $u$ and $v$,

$$A \cdot A''_0 = \frac{v^2}{2} A''^2_0\,, \tag{88a}$$
$$B \cdot A''_0 = \frac{u^2}{2}(A''_0 \cdot B''_0)\,, \tag{88b}$$

and the contractions with derivatives of $A$ and $B$ follow from there. From Eq. (9), we can write

$$s_{\sigma\alpha} A''^{\alpha}_0 = (u A''_0 \cdot B''_0) A'_{0\sigma} + (v A''^2_0) B'_{0\sigma} - (A' \cdot B') A''_{0\sigma}\,, \tag{89a}$$
$$s_{\sigma\alpha,v} A''^{\alpha}_0 = (u A''_0 \cdot B''_0) A''_{0\sigma} + (A''^2_0) B'_{0\sigma} - (A'' \cdot B') A''_{0\sigma}\,. \tag{89b}$$

As a final step before considering particular accelerations, we note that, to lowest order,

$$h_{\beta\sigma,\alpha} \bar{A}'^{\alpha} \bar{B}'^{\beta} = 4 h_{\sigma w,w}\,, \tag{90a}$$
$$h_{\sigma\alpha,\beta} \bar{A}'^{\alpha} \bar{B}'^{\beta} = 4 h_{\sigma w,w}\,, \tag{90b}$$
$$h_{\beta\sigma,\alpha} \bar{A}'^{\alpha} \bar{B}'^{\beta} = 4 h_{ww,\sigma}\,. \tag{90c}$$

So now we have all the ingredients necessary to begin calculating the orders of the metric perturbation (thus acceleration) integrands. While the integrand numerators depend critically on the acceleration direction, the denominators are always the same. We will always write

$$\mathcal{I}^3_{,v}|_{v=v(u)} = d_0 u^9 \tag{91}$$

where

$$d_0 = \left[ \frac{\lambda_0 \left( A_0'' \cdot B_0'' - \lambda_0 (\lambda_0 + 3) A_0''^2 \right) - B_0''^2}{12} \right]^3 . \tag{92}$$

The simple form of the denominator leads to the simple form of $d_0$. The numerator coefficients, which we will introduce in the following subsections, are generally far more complicated.

### C. $w$-direction acceleration

We know that $g^{w\alpha} = 0$ unless $\alpha = m$. Thus, in Eq. (4) with $\gamma = w$, it must be that $\rho = m$ everywhere. We turn to Eq. (90) to determine the orders of the terms involved.

Consider terms like $h_{ww,m}$ and $h_{wm,w}$. The terms in the numerator of Eq. (87) are generally of three types. The first two are (where each vector has its own index) $A'B'A'$, $A'B'B'$, $A''B'A$, or $A''B'B$ multiplied by $\mathcal{I}_{,v}$; the third is $A'B'A$ or $A'B'B$ multiplied by $\partial^2 \mathcal{I}/\partial v^2$. Thus, based on Eq. (84), we see that the lowest-order terms in the numerator are like $u^7$.

Thus, for accelerations in the $w$ direction, a source point $u$ away contributes

$$G\mu \, \frac{n_{0w}}{d_0} \left( \frac{1}{u^2} \right) , \tag{93}$$

where $n_{0w}$ is, like $d_0$, a constant which depends on the cusp parameters. It is more complicated than $d_0$, owing to the greater complexity of the numerator:

$$\begin{aligned} n_{0w} = {} & \frac{A_0''^4 (A_0'' \cdot B_0'')}{144} \left( \lambda_0^6 + 6\lambda_0^5 \right) - \frac{A_0''^2}{48} \left( A_0''^2 B_0''^2 + 4 (A_0'' \cdot B_0'')^2 \right) \lambda_0^4 + \frac{11 A_0''^2 B_0''^2 (A_0'' \cdot B_0'')}{144} \lambda_0^3 \\ & + \frac{A_0''^2 B_0''^2}{96} \left( A_0'' \cdot B_0'' - 3 B_0''^2 \right) \lambda_0^2 - \frac{A_0''^2 B_0''^4}{48} \lambda_0 + \frac{B_0''^2 (A_0'' \cdot B_0'')}{144} . \end{aligned} \tag{94}$$

Because the integrand has a divergence like $1/u^2$, the acceleration has a divergence like the inverse distance from the observer to the cusp[3].

### D. $m$-direction acceleration

Here, we use the same property of $g^{\alpha\beta}$ as above, but now replace in Eq. (4) all $\gamma$ by $w$. This leads to a number of cancellations when combining the terms in Eq. (90), meaning that we need only consider $h_{ww,w}$ and terms where the second derivative vectors are contracted onto the $(A + B)$ or $A'$ in Eq. (87).

Consulting the same equations as before, we see that these are perhaps the highest-order indices one could choose. The $h_{ww,w}$ has terms like $u^9$, and thus each source contributes

$$G\mu \frac{n_{0m}}{d_0} . \tag{95}$$

There are no divergences in this direction.

[3] And also like the logarithm of the same, if we continue to further orders.

### E. $p$-direction acceleration

Because the $p$ and $q$ directions are interchangeable, we only need to calculate the divergent behavior of one of them.

The only non-zero metric component involving $p$ is $g^{pp}$. Thus, for finding $X^p_{,uv}$, we set $\rho = p$ everywhere in Eq. (4). There are no cancellations.

We first consider terms like $h_{wp,w}$ and $h_{ww,p}$. They yield terms like $u^8$, and so the contribution for each source is

$$G\mu \, \frac{n_{0p}}{d_0} \left( \frac{1}{u} \right) . \tag{96}$$

This integrand has a divergence like $1/u$, and so the accelerations in the $p$ and $q$ directions diverge as the logarithm of the distance between the observer and the cusp.

### F. Total behavior of the cusp acceleration integrand

We now know how the acceleration for an observer near the cusp depends on the observer position for very distant sources. From Sec. III, we know that very near the observer, the integrand goes like $u$ only in the $c$ and $d$ directions. Now, we are interested to know how the cusp acceleration depends on the observer position when the sources are much closer to the observer than the observer is to the cusp, in order to compare the importance of the far and near regions of the integrand.

To do this, we express the contribution to the acceleration of a source point very near the observer as

$$\frac{7G\mu}{12} \frac{\bar{B}''^2}{\bar{A}' \cdot \bar{B}'} \left[ \bar{A}''^\gamma - \frac{\bar{A}'' \cdot \bar{B}'}{\bar{A}' \cdot \bar{B}'} \bar{A}'^\gamma \right] = -\frac{7G\mu}{6} \frac{B_0''^2}{(\bar{v}A_0'' - \bar{u}B_0'')^2} \left[ \bar{A}''^\gamma - \frac{2A_0'' \cdot (\bar{v}A_0'' - \bar{u}B_0'')}{(\bar{v}A_0'' - \bar{u}B_0'')^2} \bar{A}'^\gamma \right] u \,, \tag{97}$$

which is nothing but the expression for a regular point, Eq. (40), but now in four-vector form. We see that it might be possible for the coefficient to the $u$ to grow as $\bar{u}, \bar{v} \to 0$, depending on the orders of the components of $A'$ and $A''$. But finding the orders of those components via Eq. (84) shows that this will only be a concern for the $w$ direction.

To show this, consider a line of worldsheet points lying in some specific direction from the cusp, given by $\bar{v} = \chi \bar{u}$, with $\chi$ some constant. Making this substitution and using Eq. (84) to find $A'$ and $A''$ components[4], we find that the contribution per source in the $m$ direction goes as $u/\bar{u}$, in the $p$ and $q$ directions goes as $u/\bar{u}^2$, and in the $w$ direction is

$$\frac{7G\mu}{3} \frac{\chi A_0'' \cdot (\chi A_0'' - B_0'') B_0''^2}{(\chi A_0'' - B_0'')^4} \frac{u}{\bar{u}^3} \,. \tag{98}$$

Upon integration of $u$ up to something proportional to $\bar{u}$, the $m$, $p$, and $q$ directions do not increase as $\bar{u} \to 0$. But, something interesting has happened with the $w$ component. While the integrand itself is linear in $u$ very near the observer, the coefficient has a $1/\bar{u}^3$ dependence. As a consequence, the $w$-direction acceleration diverges as $1/\bar{u}$ in the near regime, just as it does in the far regime. Thus any estimate of the acceleration for a point near a cusp must account for the effect of both of those regimes.

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[4] Note that we now want the *upper* index vectors, as opposed to the *lower* index vectors as given in Eq. (84), and so we use e.g. $P^w = \eta^{wm} P_m$.

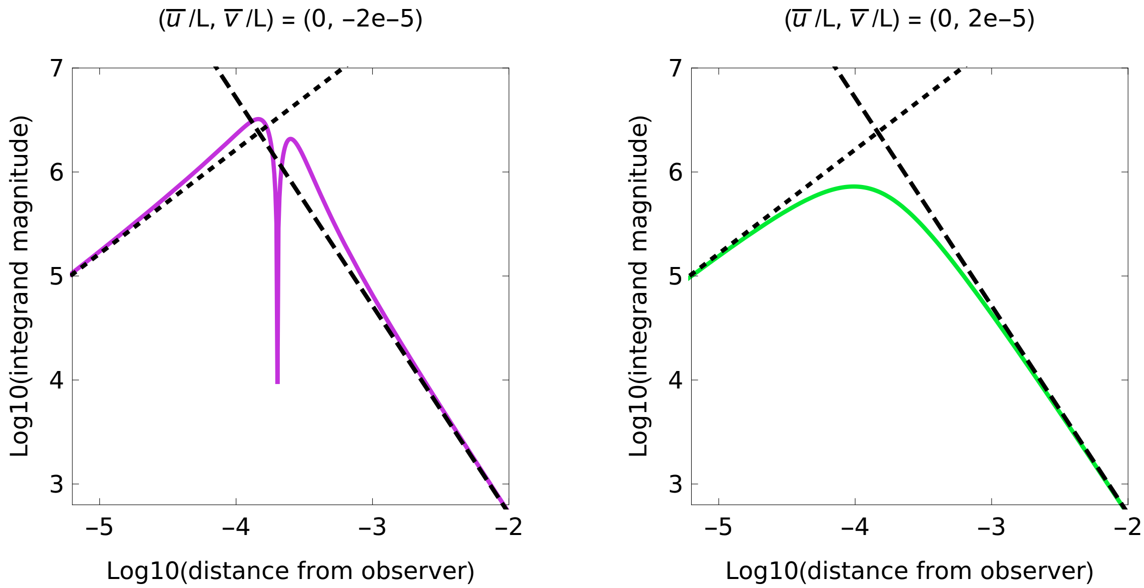


FIG. 3. The $w$ component of the cusp acceleration integrands for two observers with $\bar{u} = 0$, one located at $\bar{v}/L = -2 \cdot 10^{-5}$ in the past of the cusp (left panel, purple) and the other located at the same distance in the future of the cusp (right panel, green). In both plots, the short-dashed line indicates the predicted acceleration integrand when $u \ll \bar{u}$, while the long-dashed line is for when $u \gg \bar{u}$. Note that the left plot changes sign between regimes, while the right plot maintains the same sign throughout.

Moreover, the signs of these effects do not need to be the same. For sources very far away, all observers near the cusp see contributions from such distant points as having the same sign, as $n_{0w}$ and $d_0$ are independent of $\bar{u}$ and $\bar{v}$. But consider Eq. (98). Here, the overall sign depends on the sign of $\chi$, and the leading $1/\bar{u}$ means that sign will always be different for two points with the same $\chi$ on opposite sides of the cusp.

Plots of the acceleration integrands for two observers near a cusp, demonstrating the phenomena discussed in this section, may be found in Fig. 3. In order to obtain the solid lines from these plots, we carried out the calculation of the $w$-direction acceleration via Eqs. (4,87) for $A$ and $B$ Taylor-expanded about an observer near a cusp on the Kibble-Turok loop [17], keeping all terms up to fourth order in the lightcone constraint.

### G. Changes to the string near a cusp

We have concluded that the acceleration as we approach a cusp diverges like the inverse distance from the cusp to the observer (for the cusp direction) or like the logarithm of the same (for the transverse directions) and are only cut off by the near regime when $u$ is comparable to this distance. On the other hand, the cusp is a transient event which occurs at some precise $u$ and $v$ coordinates on the worldsheet. To find the total effect of back-reaction on a point near a cusp due to the combined contributions of the rest of the world sheet one should compute the change on the tangent vectors following Eq. (5). Upon integrating either of these expressions, we will find that the $w$ direction is still divergent, but only logarithmically, while the remaining directions are non-divergent, and so both $\Delta A'$ and $\Delta B'$ will be log-divergent in the $w$ direction. Since the divergences are seen in $\Delta A'$ and $\Delta B'$, they are not gauge artifacts. As in the kink case, integrating once again to determine the total loss of length will give a finite answer.

The corrections $\Delta A'$ and $\Delta B'$ for a single oscillation will never be large. For points very near the cusp, both corrections will be proportional to $G\mu$ times a logarithm. No logarithm appearing in cosmology is more than about 100, and $100G\mu$ is still tiny for any realistic $G\mu$.

The only divergent correction is in the $w$ direction, which is the direction of the cusp's motion, i.e., $A'_0 = B'_0$. Nearby points will have similar $A'$ and $B'$, so the correction acts mostly to decrease the energy of the string near the cusp without changing the directions of the tangent vectors. Reparameterization to return $\mathbf{A}'$ and $\mathbf{B}'$ to unit length will increase $A''$ and $B''$, because $A'$ and $B'$ change by the same amount over less parameter distance. This decreases the strength of the cusp by decreasing the area of the worldsheet in which $A'$ and $B'$ are nearly identical. The unit sphere looks more or less the same, but the $A'$ and $B'$ now move more quickly over the cusp point, resulting in weaker bursts of gravitational radiation in subsequent oscillations.

## VI. CONCLUSIONS

We have demonstrated that points on a string worldsheet near a kink or a cusp will feel a divergent acceleration due to those features. While points not located at the feature itself always have some small nearby region which looks smooth, divergent effects arise on a scale related to the distance from that point to the nearby feature.

That there is a divergent acceleration as an observer approaches a kink indicates that it is possible for the kink to be rounded off by gravitational back-reaction, in contrast to the claim of Ref. [15] that kinks are "opened", and may seem more similar to the "smoothing" of kinks used in Ref. [9]. However, this rounding happens on small distances at early times, and it takes a significant fraction of the loop lifetime until a large length of string has been bent across the kink. So while kinks are removed rapidly, the amount of string spread across the gaps on the unit sphere is small. Thus, cusps which form as a consequence of this will be very weak.

Our results on back-reaction at cusps suggest that they lose a significant amount of energy in the neighborhood of the cusp, making them weaker as time passes. The effect of back-reaction will also change the parameters that characterize the cusps, which could have important consequences for their observational signatures.

These results were found using the zero-thickness string approximation. Thus, once the observer approaches a kink or a cusp to a scale comparable to the string thickness $\delta$, we expect the expressions for the accelerations to change.[5] On the other hand, strings of cosmological and astrophysical significance always have length scales many orders of magnitude above their thicknesses,[6] so these results are applicable to all but an infinitesimal fraction of the string.

More importantly, the type of analysis done here is applicable only to isolated, simple features on strings, and we can accurately calculate only the initial effect. After a significant period of back-reaction, a string will have cusps that are partly depleted and look somewhat

[5] At that scale one would imagine that field theory effects of the type observed in simulations [18] would be the dominant contribution to back-reaction.

[6] For example: a Milky Way-scale string with $G\mu = 10^{-11}$ has $L/\delta \sim 10^{45}$.

like kinks, and kinks that are partly rounded and lead to weak cusps. To fully understand the evolution of loops under the influence of gravitational back-reaction, we need to numerically simulate back-reaction over the course of the loop lifetime. We will report on such simulations in future publications.

## VII. ACKNOWLEDGMENTS

Concurrently to the work described here, Chernoff, Flanagan, and Wardell [19] did related work on cosmic string back-reaction; that paper and this were submitted at the same time. As far as we know the results are in agreement where they overlap.

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