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Spacetime geometry fluctuations and geodesic deviation

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Abstract

The quantum fluctuations of the geodesic deviation equation in a flat background spacetime are discussed. We calculate the resulting mean squared fluctuations in the relative velocity and separation of test particles. The effect of these quantum fluctuations of the spacetime geometry is given in terms of the Riemann tensor correlation function. Three different sources of the Riemann tensor fluctuations are considered: a thermal bath of gravitons, gravitons in a squeezed state, and the graviton vacuum state.

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I. INTRODUCTION

In flat space, parallel lines maintain their separation forever. However, in curved spacetime, parallel geodesics do not remain parallel when are extended. The mathematical statement of this physical phenomena is given by the geodesic deviation equation, which shows that the tidal force of a gravitational field causes modification in the trajectories of neighboring particles [1]. Many studies concerning the behavior of the geodesic deviation equation in several background gravitational fields as well as their consequences can be found in [2–6]. In the general relativity theory, an important effect played by curvature is how it changes the relative separation between two geodesic particles. This is a manifestation of the gravitational field and hence the acceleration of the deviation vector between two nearby geodesics contains information about the curvature of the spacetime [7–10].

The properties of the curved spacetime which are reflected by physics in a gravitational field can be evaluated by analyzing the behavior of a set of neighboring geodesics, representing, for example, a bundle of photons or a distribution of massive test particles [11]. In order to study this phenomenon, many different approaches have been proposed [12–15].

On the other hand, the investigation of the Brownian motion, which can be described by the Langevin equation, played a very important role for the establishment of the atomic structure of matter. The discreteness character of matter (microscopic feature) causes fluctuations in the density of matter, which, in turn, causes observable effects on the motion of the Brownian particle (macroscopic feature) [16]. Recently, the solutions of Langevin-type equations in some astrophysical scenarios have been discussed in the literature [17–23].

The knowledge of the behavior of a Brownian particle immersed in a fluid of much smaller atoms, can give us, in principle, some relevant information about the physics of these objects [24]. Brownian motion of test particles coupled to quantized fields was studied in Refs. [25–27]. Similarly, we can study the Brownian motion of test particles in a fluctuating gravitational field to look for insights into quantum gravity [28] . In this way, we will use the geodesic deviation equation as a Langevin equation in which the Riemann tensor fluctuates. These quantum fluctuations of the curvature modify the motion of test particles and can be measured by the relative velocity dispersion after an interaction.

The quantum fluctuations of the spacetime geometry can be of two types: passive and active. The passive case is generated by fluctuations of the quantum matter fields, that is, from fluctuations in the source of the gravitational field which are described in terms of the stress and Ricci tensor correlation functions [29–35]. The active case is due to the quantum nature of gravity, that is, from fluctuations of the dynamical degrees of freedom of gravity itself, which are given in terms of the Riemann tensor correlation function [36–43].

This paper is organized as follows. In Section II we introduce the geodesic deviation equation and obtain an expression for the relative velocity dispersion. In Section III, we evaluate this expression and compute the relative distance fluctuations in the case of a thermal bath of gravitons. In Section IV, we do the same for gravitons in a squeezed state. In Section V, we sample the Riemann tensor correlation function for the case of the graviton vacuum state. Finally, in Section VI, the conclusions are given. We will use units in which $\hbar = c = 1$ throughout the paper. In Sections III and V, we use units in which $32\pi G = 32\pi\ell_{\rm Pl}^2 = 1$ where G is Newton's constant and $\ell_{\rm Pl}$, is the Planck length. However, in Section IV we will work in units where $G = \ell_{\rm Pl}^2 = 1$ for consistency with previous references. In all case, we will restore powers $\ell_{\rm Pl}$ in final results.

II. GEODESIC DEVIATION FLUCTUATIONS

Let us consider two test particles whose worldlines are nearby timelike geodesics, as illustrated in Fig. 1.

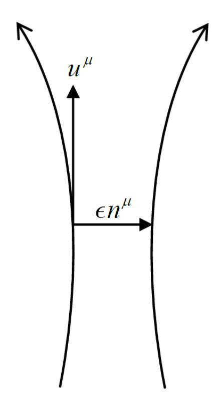

FIG. 1. Timelike geodesics for two nearby falling particles, with four-velocity u^μ, and separation $s^\mu = \epsilon n^\mu$, where n^μ is a unit spacelike vector.

The four-velocity, u^μ, and the separation vector, s^μ, are given by

$$u^\mu \equiv \frac{dx^\mu}{d\tau}, \tag{1}$$

and

$$s^\mu \equiv \frac{dx^\mu}{dn} = \epsilon n^\mu \,, \tag{2}$$

where $n^\mu n_\mu = 1$, and n is a parameter which labels nearby geodesics. Note that s^μ is a spacelike vector with magnitude $\epsilon(\tau)$ which connects two points on the two geodesics with the same values of the proper time, τ.

The variation of the separation vector between two neighboring geodesics is described by the geodesic deviation equation,

$$\frac{D^2 s^\mu}{d\tau^2} = -R^\mu{}_{\alpha\nu\lambda} u^\alpha s^\nu u^\lambda, \tag{3}$$

where $R^\mu_{\alpha\nu\lambda}$ is the Riemann tensor. For particles on the neighboring geodesics, their relative acceleration along the separation direction is given by [44]

$$\alpha \equiv n_\mu \frac{D^2 s^\mu}{d\tau^2} = -\epsilon\, R_{\mu\alpha\nu\lambda} n^\mu u^\alpha n^\nu u^\lambda. \tag{4}$$

Thus, if the particles start at rest at proper time $\tau = 0$, then we may approximately integrate the above expression to find their relative velocity at a later time $\tau = \tau_0$ as

$$v \equiv n_\mu \frac{Ds^\mu}{d\tau} \approx -\epsilon_0 \int_0^{\tau_0} d\tau\ R_{\mu\alpha\nu\lambda}(\tau)\ n^\mu u^\alpha n^\nu u^\lambda \,. \tag{5}$$

Here we assume that the separation change during this interval is small, so we may let $\epsilon \approx \epsilon(0) = \epsilon_0$, the initial separation. We also assume that n^μ is constant to leading order. Equation (5) defines a scalar velocity in the frame where $n^\mu = (0, \vec{n})$.

Now, let us suppose that the spacetime geometry is subject to quantum fluctuations. In fact, given an ensemble of geodesics, measurements of the relative velocity along the same line will give different results. Therefore, we must take the expectation value of these measurements as well as the standard deviation. To do this, we will assume that the Riemann tensor is subject to quantum fluctuations which can be, in principle, active, passive or both. We have to specify how the 4-vectors u^μ and s^μ behave under the fluctuations. The simplest assumption is that both u^μ and s^μ do not fluctuate to lowest order in the perturbations of spacetime. Physically, this is equivalent to assuming that both source and detector are located in a flat region, or both are rigidly attached to one another by non-gravitational forces. Finally, we assume that the perturbation is negligible at both source and detector.

The mean relative velocity of the particles, $\langle \upsilon \rangle$, is now obtained by averaging Eq. (5) as follows

$$\langle \upsilon \rangle = -\epsilon_0 \int_0^{\tau_0} d\tau \; \langle R_{\mu\alpha\nu\lambda}(\tau) \rangle \; n^\mu u^\alpha n^\nu u^\lambda \,. \tag{6}$$

The fluctuations around the mean trajectory in the direction of n^μ are described by

$$\begin{aligned} \Delta\upsilon &= \upsilon - \langle \upsilon \rangle \\ &= -\epsilon_0 \int_0^{\tau_0} d\tau \; [R_{\mu\alpha\nu\lambda}(\tau) - \langle R_{\mu\alpha\nu\lambda}(\tau) \rangle] \; n^\mu u^\alpha n^\nu u^\lambda. \end{aligned} \tag{7}$$

Therefore, the variance of the relative velocity, $\langle (\Delta\upsilon)^2 \rangle$, can be expressed as

$$\begin{aligned} \langle (\Delta\upsilon)^2 \rangle &= \langle \upsilon^2 \rangle - \langle \upsilon \rangle^2 \\ &= \epsilon_0^2 \int_0^{\tau_0} d\tau \int_0^{\tau_0} d\tau' \; C_{\alpha\lambda\mu\nu\gamma\delta\rho\sigma}(x, x') \; n^\alpha u^\lambda n^\mu u^\nu n^\gamma u^\delta n^\rho u^\sigma, \end{aligned} \tag{8}$$

where the Riemann tensor correlation function, $C_{\alpha\lambda\mu\nu\gamma\delta\rho\sigma}(x, x')$, is given by

$$C_{\alpha\lambda\mu\nu\gamma\delta\rho\sigma}(x, x') = \langle R_{\alpha\lambda\mu\nu}(x) R_{\gamma\delta\rho\sigma}(x') \rangle - \langle R_{\alpha\lambda\mu\nu}(x) \rangle \langle R_{\gamma\delta\rho\sigma}(x') \rangle. \tag{9}$$

This expression describes the fluctuations of the Riemann tensor. Here, the indices $\alpha\lambda\mu\nu$ refer to the spacetime point x (which corresponds to the point τ), while the indices $\gamma\delta\rho\sigma$ refer to the spacetime point x' (which corresponds to the point τ'). Equation (8) is our key result for the geodesic deviation fluctuations and it applies to both active and passive fluctuations of the spacetime geometry. In what follows, we will evaluate the relative velocity dispersion given by Eq. (8) for three different sources of active fluctuations.

III. THERMAL GRAVITON STATE

In this section, we will analyze the fluctuations produced by a thermal bath of gravitons, which may be created, for example, by the Hawking effect or cosmological particle production [45, 46]. In this case, let us suppose that the spacetime geometry fluctuates in such a way that [47]

$$\langle R^\alpha{}_{\lambda\mu\nu} \rangle = 0, \tag{10}$$

but

$$\langle R^\alpha{}_{\lambda\mu\nu} R^\gamma{}_{\delta\rho\sigma} \rangle \neq 0. \tag{11}$$

These two statements mean that we are neglecting the average spacetime curvature due to the bath of gravitons. Therefore, the average geometry corresponds to a flat Minkowski spacetime. Furthermore, since that we are dealing with a thermal quantum state at temperature T, the Riemann tensor correlation function can be written as

$$C_{\alpha\lambda\mu\nu\gamma\delta\rho\sigma} = \langle R_{\alpha\lambda\mu\nu}(x) R_{\gamma\delta\rho\sigma}(x')\rangle_\beta, \tag{12}$$

where $\langle R_{\alpha\lambda\mu\nu}(x) R_{\gamma\delta\rho\sigma}(x')\rangle_\beta$ is the thermal normal-ordered Riemann tensor two-point function, with $\beta = 1/T$. Therefore, Eq. (8) reduces to

$$\langle(\Delta v)^2\rangle = \epsilon_0^2 \int_0^{\tau_0} d\tau \int_0^{\tau_0} d\tau' \ \langle R_{\alpha\lambda\mu\nu}(x) R_{\gamma\delta\rho\sigma}(x')\rangle_\beta \ n^\alpha u^\lambda n^\mu u^\nu n^\gamma u^\delta n^\rho u^\sigma. \tag{13}$$

Let us choose the case where both source and detector are initially at rest with respect to one another and to the bath of gravitons. This choice is such that

$$u^\mu = (1,0,0,0), \tag{14}$$

$$n^\mu = (0,1,0,0), \tag{15}$$

where we have assumed that the particles are separated in the x-direction. Thus, substituting Eqs. (14)-(15) into Eq. (13), we obtain

$$\begin{aligned}\langle(\Delta v)^2\rangle &= \epsilon_0^2 \int_0^{\tau_0} d\tau \int_0^{\tau_0} d\tau' \ \langle R_{xtxt}(x) R_{xtxt}(x')\rangle_\beta \ n^x u^t n^x u^t n^x u^t n^x u^t \\ &= \epsilon_0^2 \int_0^{\tau_0} d\tau \int_0^{\tau_0} d\tau' \ \langle R_{txtx}(x) R_{txtx}(x')\rangle_\beta,\end{aligned} \tag{16}$$

where we have used the symmetry and cyclic properties of the Riemann tensor, namely,

$$R_{\alpha\lambda\mu\nu} = -R_{\lambda\alpha\mu\nu} = -R_{\alpha\lambda\nu\mu}. \tag{17}$$

Now, we introduce the thermal Riemann tensor two-point function which was constructed from the vacuum two-point function via the Matsubara method (see [44] and references therein). It is given by

$$\langle R_{txtx}(x) R_{txtx}(x')\rangle_\beta = \frac{1}{4}(\partial_t^4 - 2\partial_t^2\partial_x^2 + \partial_x^4) D_\beta, \tag{18}$$

with

$$D_\beta = \frac{1}{4\pi^2} \sum_{n=-\infty}^{+\infty}{}' \frac{1}{(\Delta\vec{x})^2 - (\Delta t + in\beta)^2}, \tag{19}$$

where

$$\partial_t \partial_{t'} = -\partial_t^2, \tag{20}$$

$$\Delta \vec{x} = \vec{x} - \vec{x}', \tag{21}$$

$$\Delta t = t - t'. \tag{22}$$

In the last summation, the prime denotes that we have removed the $n = 0$ term, which is the vacuum contribution.

Now we will examine the relative velocity dispersion in one space dimension, that is, we may choose $\Delta y = \Delta z = 0$. Then, we can write Eq. (19) as

$$D_\beta = \frac{1}{4\pi^2} \sum_{n=-\infty}^{+\infty}{}' \frac{1}{(\Delta x)^2 - (\Delta t + in\beta)^2} \,. \tag{23}$$

Next assume that the two particles both start at rest in our frame of reference , so we may use $d\tau = dt$ in Eq. (16), which becomes

$$\langle (\Delta \upsilon)^2 \rangle = I_t + I_{tx} + I_x, \tag{24}$$

with

$$I_t = \int_0^{t_0} dt \int_0^{t_0} dt' \left(\frac{1}{4} \partial_t^4 D_\beta \right), \tag{25}$$

$$I_{tx} = \int_0^{t_0} dt \int_0^{t_0} dt' \left(-\frac{1}{2} \partial_t^2 \partial_x^2 D_\beta \right), \tag{26}$$

$$I_x = \int_0^{t_0} dt \int_0^{t_0} dt' \left(\frac{1}{4} \partial_x^4 D_\beta \right), \tag{27}$$

where t_0 is the flight time, that is, the interaction time between the particles and the thermal bath.

We are interested in the real part of D_β. Thus, we may make the replacement

$$\sum_{n=-\infty}^{+\infty}{}' = 2 \sum_{n=1}^{+\infty}, \tag{28}$$

and take

$$\Re(D_\beta) = \frac{1}{2\pi^2} \sum_{n=1}^{+\infty} G, \tag{29}$$

with

$$G = \Re\left[\frac{1}{(\Delta x)^2 - (\Delta t + in\beta)^2} \right], \tag{30}$$

where $\Re$ denotes the real part. After that, we can evaluate Eq. (24) by using an algebraic manipulation program. However, the final expression is so long that no insight is gained by writing it out.

Next, we will assume that Δx is small compared to Δt and/or β, and hence can be ignored. We may compute the relative velocity between the two test particles taking the following limits in the derivatives:

$$\begin{aligned}\frac{1}{4}\,\partial_t^4 D_\beta\big|_{x\to x'} &= \frac{1}{2\pi^2}\sum_{n=1}^{+\infty}\left(\frac{1}{4}\,\partial_t^4 G\big|_{x\to x'}\right)\\ &= \frac{1}{2\pi^2}\sum_{n=1}^{+\infty}\Re\left[-\frac{30}{(\Delta t+in\beta)^6}\right],\end{aligned} \tag{31}$$

$$\begin{aligned}-\frac{1}{2}\,\partial_t^2\partial_x^2 D_\beta\big|_{x\to x'} &= \frac{1}{2\pi^2}\sum_{n=1}^{+\infty}\left(-\frac{1}{2}\,\partial_t^2\partial_x^2 G\big|_{x\to x'}\right)\\ &= \frac{1}{2\pi^2}\sum_{n=1}^{+\infty}\Re\left[\frac{20}{(\Delta t+in\beta)^6}\right],\end{aligned} \tag{32}$$

$$\begin{aligned}\frac{1}{4}\,\partial_x^4 D_\beta\big|_{x\to x'} &= \frac{1}{2\pi^2}\sum_{n=1}^{+\infty}\left(\frac{1}{4}\,\partial_x^4 G\big|_{x\to x'}\right)\\ &= \frac{1}{2\pi^2}\sum_{n=1}^{+\infty}\Re\left[-\frac{6}{(\Delta t+in\beta)^6}\right].\end{aligned} \tag{33}$$

Thus, we can write the thermal normal-ordered Riemann tensor two-point function as

$$\langle R_{txtx}(x)R_{txtx}(x')\rangle_\beta = \frac{1}{2\pi^2}\sum_{n=1}^{+\infty}\Re\left[-\frac{16}{(\Delta t+in\beta)^6}\right]. \tag{34}$$

Therefore, substituting Eqs. (25)-(33) into Eq. (24), we obtain

$$\langle(\Delta v)^2\rangle = \frac{\epsilon_0^2}{2\pi^2}\sum_{n=1}^{+\infty}\left[\frac{8}{5n^4\beta^4} - \frac{8}{5(t_0^2+n^2\beta^2)^2} + \frac{64t_0^2}{5(t_0^2+n^2\beta^2)^3} - \frac{64t_0^4}{5(t_0^2+n^2\beta^2)^4}\right]. \tag{35}$$

At this point, we can analyze the limits in which the time of observation t_0 is large compared to the thermal parameter β, and vice versa.

A. Case 1: $t_0 \ll \beta$ (short flight time or low temperature)

If $t_0 \ll \beta$, we have

$$\langle(\Delta v)^2\rangle \sim \frac{\epsilon_0^2}{2\pi^2}\sum_{n=1}^{+\infty}\frac{16t_0^2}{n^6\beta^6} = \frac{8\pi^4\epsilon_0^2t_0^2}{945\beta^6} = \frac{256\pi^5\ell_{\rm Pl}^2\epsilon_0^2\,t_0^2}{945\beta^6}. \tag{36}$$

Here the rms relative velocity is given by

$$(\Delta v)_{\rm rms} = \frac{16\pi^{5/2}\ell_{\rm Pl}\epsilon_0\, t_0}{3\sqrt{105}\beta^3}, \tag{37}$$

which grows linearly with the flight time. Recall that, following the convention in Ref. [44], we set $32\pi\ell_{\rm Pl}^2 = 1$ in this section.

B. Case 2: $t_0 \gg \beta$ (long flight time or high temperature)

In the limit when $t_0 \gg \beta$, that is, in the observationally reasonable limit where the wavelength of the gravitational waves is small compared to the flight time of the particles after the interaction with the thermal bath of gravitons, the expression reduces to

$$\langle(\Delta v)^2\rangle \sim \frac{\epsilon_0^2}{2\pi^2}\sum_{n=1}^{+\infty}\frac{8}{5n^4\beta^4} = \frac{2\pi^2\epsilon_0^2}{225\beta^4} = \frac{64\pi^3\ell_{\rm Pl}^2\epsilon_0^2}{225\beta^4}, \tag{38}$$

where $\ell_{\rm Pl}$ is the Planck length. In this case, the rms relative velocity approaches a constant, namely,

$$(\Delta v)_{\rm rms} = \frac{8\pi^{3/2}\ell_{\rm Pl}\epsilon_0}{15\beta^2}. \tag{39}$$

There are several features of this result which require comment. First, it is proportional to the Planck length, $\ell_{\rm Pl}$. This arises from the fact that the graviton Riemann tensor correlation function, Eq. (9), is proportional to $G = \ell_{\rm Pl}^2$, and reflects the smallness of quantum gravity effects. Second, the relative velocity of the test particles is proportional to their initial separation, ϵ_0. This comes because the tidal acceleration, Eq. (3), is proportional to the particle separation. Recall that we have assumed that this separation does not change dramatically, so $\epsilon(\tau) \approx \epsilon_0$. In addition, we assumed in Eq. (30) that $\Delta x \lesssim \beta$, which implies that $\epsilon_0 \lesssim \beta$. Finally, note that $(\Delta v)_{\rm rms} \propto \beta^{-2} = T^2$, which can be understood on dimensional grounds, given the factors of $\ell_{\rm Pl}$ and ϵ_0 in the numerator. However, in thermal equilibrium one expects $(\Delta v)_{\rm rms} \propto T^{1/2}$. The reason for this difference is that Eq. (39) describes the state of the system after a time long compared to β, but well before thermal equilibrium has been reached. Equilibrium requires a balance between graviton absorption and emission processes, the latter of which is not included in our analysis. In addition, the equilibrium value of the mean speed should be independent of the value of the coupling constant, G, which Eq. (39) is not. The weakness of gravity insures that the time required to reach thermal equilibrium is much longer that the scales which we consider.

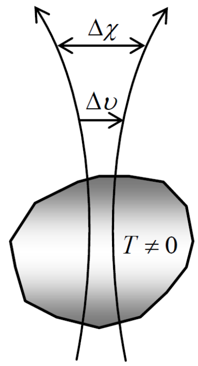

FIG. 2. Distance and velocity between two nearby timelike geodesics after interaction with the thermal bath of gravitons.

C. Position fluctuations

From the previous calculations, note that the relative velocity dispersion is not zero, that is, the nearby timelike geodesics are affected by the gravitons. In this way, we are interested in computing the relative distance dispersion between the two test particles after their interaction with the thermal bath of gravitons. The mean squared distance fluctuation in the x-direction, as represented in Fig. 2, can be calculated as follows:

$$\langle(\Delta\chi)^2\rangle = \epsilon_0^2 \int_0^{\mathcal{T}} dt_1 \int_0^{t_1} dt \int_0^{\mathcal{T}} dt_1' \int_0^{t_1'} dt' \ \langle R_{txtx}(x) R_{txtx}(x')\rangle_\beta . \tag{40}$$

Following the same procedure used to obtain Eq. (35), namely, setting $\Delta x, \Delta y, \Delta z \approx 0$ in the denominator of the Riemann tensor correlation function, which means $|\Delta\vec{x}| \lesssim \Delta t$, the relative distance dispersion is given by

$$\langle(\Delta\chi)^2\rangle = \frac{\epsilon_0^2}{2\pi^2} \sum_{n=1}^{+\infty} \left[\frac{4}{15n^2\beta^2} + \frac{4t^2}{5n^4\beta^4} + \frac{4(3t^4 - 6t^2n^2\beta^2 - n^4\beta^4)}{15(t^2 + n^2\beta^2)^3} \right], \tag{41}$$

where we have set $t = \mathcal{T} > t_0$, which means that t is the measurement time, i.e, the total time given by the sum of the flight time and the time elapsed after the interaction.

In the $t \ll \beta$ limit, the relative distance dispersion and its root-mean-square value are given, respectively, by

$$\langle(\Delta\chi)^2\rangle \sim \frac{\epsilon_0^2}{2\pi^2} \sum_{n=1}^{+\infty} \frac{4t^4}{n^6\beta^6} = \frac{64\pi^5 \ell_{\rm Pl}^2 \epsilon_0^2\, t^4}{945\beta^6}, \tag{42}$$

$$(\Delta\chi)_{\rm rms} = \frac{8\pi^{5/2}\ell_{\rm Pl}\,\epsilon_0 t^2}{3\sqrt{105}\beta^3}. \tag{43}$$

On the other hand, for the $t \gg \beta$ limit, we have

$$\langle(\Delta\chi)^2\rangle \sim \frac{\epsilon_0^2}{2\pi^2}\sum_{n=1}^{+\infty}\frac{4t^2}{5n^4\beta^4} = \frac{32\pi^3\ell_{\rm Pl}^2\epsilon_0^2\,t^2}{225\beta^4}, \tag{44}$$

$$(\Delta\chi)_{\rm rms} = \frac{4\sqrt{2}\pi^{3/2}\ell_{\rm Pl}\epsilon_0\,t}{15\beta^2}. \tag{45}$$

From Eqs. (39), (37), (45), and (43), we have

$$(\Delta\chi)_{\rm rms} = \frac{(\Delta v)_{\rm rms}}{2t_0}t^2 \qquad (\text{for } t \ll \beta). \tag{46}$$

$$(\Delta\chi)_{\rm rms} = \frac{(\Delta v)_{\rm rms}}{\sqrt{2}}t \qquad (\text{for } t \gg \beta), \tag{47}$$

Therefore, we find in both cases a form of gravitational wave memory effect (see Ref. [2] and references therein), such that

$$(\Delta\chi)_{\rm rms} \sim (\Delta v)_{\rm rms}\,t. \tag{48}$$

Note that while the test particles are in the thermal graviton bath, our assumption that $\Delta x \lesssim \beta$ in Eq. (40) requires that $(\Delta\chi)_{\rm rms} \lesssim \beta$. However, the timelike geodesics are subsequently modified due to their passage through the thermal bath. This modification is reflected in the growing distance between the pair of particles and can in principle become large as t grows despite the smallness of the Planck length.

IV. GRAVITONS IN A SQUEEZED QUANTUM STATE

In this section we will consider a spacetime region filled with gravitons in a squeezed state which produce quantum fluctuations on test particle geodesics. In principle, this squeezed state could be due to quantum creation of gravitons in a background gravitational field, as for example, in the course of a cosmological expansion or in the Hawking process of black hole evaporation [48–51]. The squeezed state is represented by $|\alpha,\zeta\rangle$, where α and ζ are the displacement and squeezed parameters, respectively. Consider a gravitational plane wave mode in a squeezed state. The normal-ordered Riemann tensor correlation function can be expressed as [47]

$$:C_{\alpha\lambda\mu\nu\gamma\delta\rho\sigma}(x,x'):= 4(\ell_{[\alpha}A_{\lambda][\mu}\ell_{\nu]})(\ell_{[\gamma}A_{\delta][\rho}\ell_{\sigma]})F(x,x'), \tag{49}$$

with

$$F(x,x') = [\cosh(2r) - 1]\cos[\ell_\epsilon(x^\epsilon - x'^\epsilon)] - \sinh(2r)\cos[\ell_\epsilon(x^\epsilon + x'^\epsilon) + \theta], \tag{50}$$

where $\ell^\epsilon = (\omega_g, \ell^x, \ell^y, \ell^z)$ is the specific wave vector of the excited mode, ω_g is the angular frequency, $A_{\mu\nu}$ is the polarization tensor, and the parameters r, θ are defined such that $\zeta = r\mathrm{e}^{i\theta}$. We have used the following convention for the antisymmetrized tensor,

$$T_{[\mu\nu]} = \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu})\,. \tag{51}$$

In this section only, we follow the convention of Ref. [47], and use units in which $\ell_{\mathrm{Pl}} = 1$. Substituting Eqs. (49) and (50) into Eq. (8), the relative velocity dispersion for test particles subjected to gravitons in a squeezed state is given by

$$\langle(\Delta\upsilon)^2\rangle = 4\epsilon_0^2(\ell_{[\alpha}A_{\lambda][\mu}\ell_{\nu]})(\ell_{[\gamma}A_{\delta][\rho}\ell_{\sigma]})n^\alpha u^\lambda n^\mu u^\nu n^\gamma u^\delta n^\rho u^\sigma f_1(\omega_g, t_0), \tag{52}$$

where the function $f_1(\omega_g, t_0)$ is calculated from Eq. (16), again with $d\tau = dt$, leading to

$$\begin{aligned} f_1(\omega_g, t_0) &= \int_0^{\tau_0} d\tau \int_0^{\tau_0} d\tau'\ F(x,x') \\ &= \int_0^{t_0} dt \int_0^{t_0} dt'\ \{[\cosh(2r) - 1]\cos[\omega_g(t - t') + \ell_x(x - x')] \\ &\quad - \sinh(2r)\cos[\omega_g(t + t') + \ell_x(x + x') + \ell_y(y + y') + \ell_z(z + z') + \theta]\} \\ &= \frac{4}{\omega_g^2}\sin^2\left(\frac{\omega_g t_0}{2}\right)\{2\sinh^2(r)\cos[\ell_x(x - x')] \\ &\quad - \sinh(2r)\cos[\omega_g t_0 + \ell_x(x + x') + \ell_y(y + y') + \ell_z(z + z') + \theta]\}, \end{aligned} \tag{53}$$

where t_0 is the flight time, the interaction time between the geodesic particles and the quantum state under consideration.

In this last result, we have chosen $\Delta y = \Delta z = 0$, that is, we are again assuming that Δy and Δz are small. It is worth calling attention to the fact that $f_1(\omega_g, t_0)$, and hence $\langle(\Delta\upsilon)^2\rangle$, are independent of the displacement parameter α. Therefore, the fluctuations depend only on the squeezing parameter ζ in such a way that $\zeta = 0$ (a coherent state) induces no fluctuations:

$$r = 0 \Rightarrow \zeta = 0 \Rightarrow f_1 = 0 \Rightarrow \langle(\Delta\upsilon)^2\rangle = 0 \qquad \text{(classical wave)}.$$

We will assume that the gravitational wave mode is in the transverse tracefree (TT) gauge, in which the gravitational perturbations have only spatial components h_{ij}, satisfying

$\partial^i h_{ij} = 0$ and $h^i_i = 0$. In fact, this choice is only a matter of convenience, since our results were obtained from the linearized Riemann tensor, which is gauge invariant. Thus, the first vector product of Eq. (52) is given by

$$
\begin{aligned}
(\ell_{[\alpha}A_{\lambda][\mu}\ell_{\nu]})n^\alpha u^\lambda n^\mu u^\nu &= (\ell_{[x}A_{t][x}\ell_{t]})n^x u^t n^x u^t \\
&= \frac{1}{4}(\ell_x A_{tx}\ell_t - \ell_x A_{tt}\ell_x - \ell_t A_{xx}\ell_t + \ell_t A_{xt}\ell_x) \\
&= -\frac{1}{4}(\ell_t)^2 A_{xx} \\
&= -\frac{1}{4}\omega_g^2 A_+,
\end{aligned}
\tag{54}
$$

where the polarization tensor $A_{\mu\nu}$ is given by

$$
A_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{xx} & A_{xy} & 0 \\ 0 & A_{xy} & -A_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_+ & A_\times & 0 \\ 0 & A_\times & -A_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{55}
$$

which is obviously traceless and purely spatial:

$$
A_{0\nu} = 0, \quad \eta^{\mu\nu}A_{\mu\nu} = 0, \quad A_{3\nu} = 0. \tag{56}
$$

Therefore, there exists a nonzero effect on the relative velocity dispersion due to gravitons in a squeezed state, which depends on the (+) polarization as well as upon position. It is given by

$$
\begin{aligned}
\langle(\Delta v)^2\rangle &= 4\epsilon_0^2\left(-\frac{1}{4}\omega_g^2 A_+\right)\left(-\frac{1}{4}\omega_g^2 A_+\right) f_1(\omega_g, t_0) \\
&= \frac{\epsilon_0^2}{4}\omega_g^4 A_+^2 f_1(\omega_g, t_0).
\end{aligned}
\tag{57}
$$

In the $t_0 \to 0$ limit, we can expand Eq. (57) for fixed r in order to find

$$
\begin{aligned}
\langle(\Delta v)^2\rangle \sim \frac{\epsilon_0^2}{4}\omega_g^4 A_+^2 \{ & 2\sinh^2(r)\cos[\ell_x(x-x')] \\
& - \sinh(2r)\cos[\ell_x(x+x') + \ell_y(y+y') + \ell_z(z+z') + \theta]\} t_0^2.
\end{aligned}
\tag{58}
$$

A. Classical time-dependence

In this subsection ,we examine the expectation value of the relative velocity, $\langle v\rangle$, which is given in terms of the first order contribution in the Riemann tensor fluctuations, $\langle R_{\alpha\lambda\mu\nu}(x)\rangle$.

This quantity gives the classical time dependent variation since it depends only upon the displacement parameter, α. If $\zeta = 0$ the squeezed state becomes a coherent state, which can describe a classical wave.

In order to evaluate $\langle \upsilon \rangle$, we can do a single integration of the expectation value of the Riemann tensor over the proper time $d\tau$ by using Eq. (6). However, we want to calculate $\langle \upsilon \rangle^2$ directly from Eq. (8). Then, the squared mean change in relative velocity can be written as

$$\langle \upsilon \rangle^2 = \epsilon_0^2 \int_0^{\tau_0} d\tau \int_0^{\tau_0} d\tau' \ \langle R_{\alpha\lambda\mu\nu}(x)\rangle \langle R_{\gamma\delta\rho\sigma}(x')\rangle n^\alpha u^\lambda n^\mu u^\nu n^\gamma u^\delta n^\rho u^\sigma. \tag{59}$$

The right-hand side of the Riemann tensor correlation function for gravitons in a squeezed state is given by [47]

$$\langle R_{\alpha\lambda\mu\nu}(x)\rangle \langle R_{\gamma\delta\rho\sigma}(x')\rangle = 4 \sum_\ell (\ell_{[\alpha} A_{\lambda][\mu} \ell_{\nu]})(\ell_{[\gamma} A_{\delta][\rho} \ell_{\sigma]}) G(x, x'), \tag{60}$$

where

$$G(x, x') = \alpha^2 \mathrm{e}^{i\ell_\epsilon (x^\epsilon + x'^\epsilon)} + (\alpha^*)^2 \mathrm{e}^{-i\ell_\epsilon (x^\epsilon + x'^\epsilon)} + 2|\alpha|^2 \cos[\ell_\epsilon (x^\epsilon - x'^\epsilon)]. \tag{61}$$

Thus, following the same procedure used to obtain Eq. (53), and performing the integration of $G(x, x')$, we get

$$\begin{aligned}
g_1(\omega_g, t_0) &= \int_0^{\tau_0} d\tau \int_0^{\tau_0} d\tau' \ G(x, x') \\
&= \int_0^{t_0} dt \int_0^{t_0} dt' \ \{2|\alpha|^2 \cos[\omega_g (t - t') + \ell_x (x - x')] \\
&\quad + (\alpha^*)^2 \mathrm{e}^{-i[\omega_g (t+t') + \ell_x (x+x') + \ell_y (y+y') + \ell_z (z+z')]} + \alpha^2 \mathrm{e}^{i[\omega_g (t+t') + \ell_x (x+x') + \ell_y (y+y') + \ell_z (z+z')]}\} \\
&= -\frac{\mathrm{e}^{-i[2\omega_g t_0 + \ell_x (x+x') + \ell_y (y+y') + \ell_z (z+z')]} (\mathrm{e}^{i\omega_g t_0} - 1)^2}{\omega_g^2} \\
&\quad \times \{(\alpha^*)^2 + \alpha^2 \mathrm{e}^{2i[\omega_g t_0 + \ell_x (x+x') + \ell_y (y+y') + \ell_z (z+z')]} \\
&\quad + 2|\alpha|^2 \mathrm{e}^{i[\omega_g t_0 + \ell_x (x+x') + \ell_y (y+y') + \ell_z (z+z')]} \cos[\ell_x (x - x')]\}.
\end{aligned} \tag{62}$$

Therefore, the classical time dependent variation of the relative velocity for a single mode is characterized by

$$\begin{aligned}
\langle \upsilon \rangle^2 &= 4\epsilon_0^2 (\ell_{[\alpha} A_{\lambda][\mu} \ell_{\nu]})(\ell_{[\gamma} A_{\delta][\rho} \ell_{\sigma]}) n^\alpha u^\lambda n^\mu u^\nu n^\gamma u^\delta n^\rho u^\sigma g_1(\omega_g, t_0) \\
&= 4\epsilon_0^2 \left(-\frac{1}{4}\omega_g^2 A_+\right)\left(-\frac{1}{4}\omega_g^2 A_+\right) g_1(\omega_g, t_0) \\
&= \frac{\epsilon_0^2}{4} \omega_g^4 A_+^2 g_1(\omega_g, t_0).
\end{aligned} \tag{63}$$

B. Special case: Transverse gravitational waves

Here, we will analyze the special case of transversely propagating gravity waves. These waves propagate with wave vector given by $\ell^\mu = \omega_g(1,0,0,1)$, while the test particles continue to have 4-vectors given by $u^\mu = (1,0,0,0)$ and $n^\mu = (0,1,0,0)$. Then, for a gravitational wave propagating in the z-direction, we just need to set $\ell_x = \ell_y = 0$ into Eq. (53) in order to define the function $f_2(\omega_g, t_0)$ as

$$f_2(\omega_g, t_0) = \frac{4}{\omega_g^2}\sin^2\left(\frac{\omega_g t_0}{2}\right)\{2\sinh^2(r) - \sinh(2r)\cos[\omega_g t_0 + \omega_g(z+z') + \theta]\}. \tag{64}$$

Therefore, in the special case of transverse gravitational waves, the relative velocity dispersion is given by

$$\langle(\Delta v)^2\rangle = \frac{\epsilon_0^2}{4}\omega_g^4 A_+^2 f_2(\omega_g, t_0). \tag{65}$$

From Eqs. (57) and (65), we conclude that the fluctuations in the relative velocity depend on the degree of squeezing, measured by the parameter ζ.

On the other hand, the classical time dependent variaton of the relative velocity is given by

$$\langle v\rangle^2 = \frac{\epsilon_0^2}{4}\omega_g^4 A_+^2 g_2(\omega_g, t_0), \tag{66}$$

where the function $g_2(\omega_g, t_0)$ is defined in a similar way as $g_1(\omega_g, t_0)$, when $\ell_x = \ell_y = 0$, namely,

$$\begin{aligned} g_2(\omega_g, t_0) = &-\frac{\mathrm{e}^{-i\omega_g[2t_0+(z+z')]}(\mathrm{e}^{i\omega_g t_0}-1)^2}{\omega_g^2} \\ &\times\{(\alpha^*)^2 + 2|\alpha|^2\mathrm{e}^{i\omega_g[t_0+(z+z')]} + \alpha^2\mathrm{e}^{2i\omega_g[t_0+(z+z')]}\}. \end{aligned} \tag{67}$$

Note that both functions $g_1(\omega_g, t_0)$ and $g_2(\omega_g, t_0)$ depend on the displacement parameter, α, but are independent of the squeeze parameter, r. Therefore, we can say the same for Eqs. (63) and (66). Furthermore, in the $\alpha = 0$ limit, we have that $g_1(\omega_g, t_0) = g_2(\omega_g, t_0) = 0$, which means a coherent state ($r = 0$ and $\alpha \neq 0$) exhibits regular time variation but does not fluctuate. In fact, from Eq. (53) we can see that $\langle(\Delta v)^2\rangle = 0$ for $r = 0$.

C. Estimating $\langle(\Delta v)^2\rangle$ from the value of the stress tensor

In this subsection, we will estimate the order of magnitude of $\langle(\Delta v)^2\rangle$ in the squeezed vacuum state when $\alpha = 0$ and $r \gg 1$. In order to do this, we will assume the (+) polarization for the gravitational waves, which implies $A_\times = 0$. These assumptions lead to $\langle \upsilon \rangle^2 = 0$ and, therefore, we have $\langle (\Delta\upsilon)^2 \rangle = \langle \upsilon^2 \rangle$. Thus, from Eqs. (53) and (57), with a suitable choice of θ, in a situation where many modes are excited, we find the following asymptotic behavior for large r:

$$\begin{aligned}\langle (\Delta\upsilon)^2 \rangle &\approx \frac{\epsilon_0^2}{4}\omega_g^4 \frac{8\pi}{\omega_g V}\frac{V}{(2\pi)^3}(\Delta\ell_x)(\Delta\ell_y)(\Delta\ell_z)\frac{2\mathrm{e}^{2r}}{\omega_g^2} \\ &= \epsilon_0^2 \frac{\omega_g \mathrm{e}^{2r}}{2\pi^2}(\Delta\ell_x)(\Delta\ell_y)(\Delta\ell_z), \end{aligned} \tag{68}$$

where the contribution from the (+) polarization is given by $A_+ = \sqrt{8\pi/\omega_g V}$. In the latter result, we have summed the modes when the density of states is large, namely,

$$\sum_\ell \to \frac{V}{(2\pi)^3}\int d^3\ell = \frac{V}{(2\pi)^3}(\Delta\ell_x)(\Delta\ell_y)(\Delta\ell_z), \tag{69}$$

where V is the quantization volume. Now, from the effective stress tensor in the linearized theory, the vacuum energy density for large r is given by [47]

$$: T_{00} : \approx \frac{\omega_g \mathrm{e}^{2r}}{32\pi^3}(\Delta\ell_x)(\Delta\ell_y)(\Delta\ell_z). \tag{70}$$

Then, substituting Eq. (70) into Eq. (68), the relative velocity dispersion can be expressed as

$$\langle (\Delta\upsilon)^2 \rangle \approx \frac{\omega_g \mathrm{e}^{2r}}{2\pi^2}\frac{32\pi^3\epsilon_0^2}{\omega_g \mathrm{e}^{2r}} : T_{00} : = 16\pi\epsilon_0^2\,\ell_{\mathrm{Pl}}^2 \,: T_{00} : . \tag{71}$$

In this case, the rms relative velocity dispersion is given by

$$(\Delta\upsilon)_{\mathrm{rms}} \approx 4\ell_{\mathrm{Pl}}\,\epsilon_0\sqrt{\pi\,\rho_g}\,, \tag{72}$$

where $\rho_g =: T_{00} :$ is the graviton energy density. Recall that Eq. (72) is valid only when $\epsilon_0 \lesssim \lambda_g$, with λ_g being the characteristic wavelength of the gravitons in the bath. Unlike the case of a thermal bath, where both energy density and characteristic wavelength are determined by the temperature, here ρ_g and λ_g can be independent of one another.

It is of interest to make an estimate of $(\Delta\upsilon)_{\mathrm{rms}}$ in the context of a cosmological model in which gravitons contribute a non-negligible fraction of the total energy density. Consider a spatially flat Robertson-Walker universe in which the present value of the Hubble parameter is H_0, and the present total energy density is ρ_T. The Friedmann equation gives

$$H_0^2 = \frac{8\pi}{3}\,G\,\rho_T\,. \tag{73}$$

Assume that gravitons contribute a fraction r of the total energy density, so $\rho_g = r\,\rho_T$. Observational data on the expansion rate of the universe are consistent with $r = 0$, and lead to an upper bound on $r \lesssim 0.04$ [51]. We may combine Eqs. (72) and (73) to write

$$\frac{(\Delta v)_{\rm rms}}{\epsilon_0} = \sqrt{6\,r}\,H_0 \lesssim 0.5\,H_0\,. \tag{74}$$

This relation gives an upper bound on and potentially an estimate of the fractional relative speed $(\Delta v)_{\rm rms}/\epsilon_0$ in terms of the Hubble parameter, whose value observed by the Planck satellite is $H_0 \approx 67$km/s/Mpc [52]. If r is near its upper bound, and ϵ_0 is of the order of a few Mpc, then Eq. (74) gives the estimate $(\Delta v)_{\rm rms} \approx 10^2$ km/s, which is of the order of the peculiar motions of individual galaxies. This estimate would require a present day graviton bath with $\lambda_g \gtrsim 1$Mpc, for which there is no obvious physical origin. Smaller values of ϵ_0 lead to smaller relative speeds from Eq. (74), However, in the early universe where graviton density could be much larger, the effect could increase.

D. Position fluctuation

In order to compute the relative distance dispersion for two geodesic particles subject to the gravitons in a squeezed state, we follow the same procedure used in the thermal case, and perform two more integrals of the function $F(x, x')$:

$$\langle(\Delta\chi)^2\rangle = 4\epsilon_0^2(\ell_{[\alpha}A_{\lambda][\mu}\ell_{\nu]})(\ell_{[\gamma}A_{\delta][\rho}\ell_{\sigma]})n^\alpha u^\lambda n^\mu u^\nu n^\gamma u^\delta n^\rho u^\sigma f_3(\omega_g, t_0), \tag{75}$$

where the function $f_3(\omega_g, t_0)$ is given by

$$\begin{aligned} f_3(\omega_g, \mathcal{T}) &= \int_0^{\mathcal{T}} d\tau_1 \int_0^{\tau_1} d\tau \int_0^{\mathcal{T}} d\tau_1' \int_0^{\tau_1'} d\tau'\; F(x, x') \\ &= \int_0^{\mathcal{T}} dt_1 \int_0^{t_1} dt \int_0^{\mathcal{T}} dt_1' \int_0^{t_1'} dt'\; \{[\cosh(2r) - 1]\cos[\omega_g(t-t') + \ell_x(x-x')] \\ &\quad - \sinh(2r)\cos[\omega_g(t+t') + \ell_x(x+x') + \ell_y(y+y') + \ell_z(z+z') + \theta]\}, \end{aligned} \tag{76}$$

such that

$$
\begin{aligned}
f_3(\omega_g, t) = \frac{1}{\omega_g^4} & \{2\sinh^2(r)\cos[l_x(x-x')][t^2\omega_g^2 - 2t\omega_g\sin(t\omega_g) - 2\cos(t\omega_g) + 2] \\
& + \sinh(2r)\{t^2\omega_g^2\cos[\theta + l_y(y+y') + l_z(z+z') + l_x(x+x')] \\
& +2t\omega_g\sin[\theta + l_y(y+y') + l_z(z+z') + l_x(x+x')] \\
& -2t\omega_g\sin[\theta + l_y(y+y') + l_z(z+z') + l_x(x+x') + t\omega_g] \\
& +2\cos[\theta + l_y(y+y') + l_z(z+z') + l_x(x+x') + t\omega_g] \\
& -\cos[\theta + l_y(y+y') + l_z(z+z') + l_x(x+x') + 2t\omega_g] \\
& -\cos[\theta + l_y(y+y') + l_z(z+z') + l_x(x+x')]\}\},
\end{aligned} \tag{77}
$$

where we have set $t = \mathcal{T} > t_0$, which means that t is the total time.

Therefore, the relative distance dispersion is given by

$$
\langle(\Delta\chi)^2\rangle = \frac{\epsilon_0^2}{4}\omega_g^4 A_+^2 f_3(\omega_g, t). \tag{78}
$$

In the $t \to 0$ limit, we can expand Eq. (78) for fixed r in order to get

$$
\begin{aligned}
\langle(\Delta\chi)^2\rangle \sim \frac{\epsilon_0^2}{4}\omega_g^4 A_+^2 & \left\{\frac{1}{2}\sinh^2(r)\cos[\ell_x(x-x')]\right. \\
& \left. -\frac{1}{4}\sinh(2r)\cos[\ell_x(x+x') + \ell_y(y+y') + \ell_z(z+z') + \theta]\right\} t^4.
\end{aligned} \tag{79}
$$

From Eqs. (58) and (79), we have

$$
\langle(\Delta\chi)^2\rangle = \frac{\langle(\Delta\upsilon)^2\rangle}{4} t^2. \tag{80}
$$

The root-mean-square value is given by

$$
(\Delta\chi)_{\rm rms} = \frac{(\Delta\upsilon)_{\rm rms}}{2} t. \tag{81}
$$

Therefore, we conclude that

$$
(\Delta\chi)_{\rm rms} \sim (\Delta\upsilon)_{\rm rms}\ t. \tag{82}
$$

This is the same behavior as for the thermal case, given by Eq. (48). As before, we need to require that $(\Delta\chi)_{\rm rms} \lesssim \lambda_g$ while the test particles are in the graviton bath, However, after leaving the bath, $(\Delta\chi)_{\rm rms}$ can become arbitrarily large due to the gravitational memory effect.

V. GRAVITON VACUUM STATE

In this section, we will deal with fluctuations of the Riemann tensor in the graviton vacuum state in linearized quantum gravity. This state must approximate a corresponding state in full quantum gravity in a suitable limit, and exhibits nontrivial fluctuation effects. From Eq. (8), the relative velocity dispersion can be expressed as

$$\langle(\Delta v)^2\rangle = \epsilon_0^2 \int_0^{\tau_0} d\tau \int_0^{\tau_0} d\tau' \; C_{txtxtxtx}(x, x'). \tag{83}$$

where the Riemann tensor correlation function, $C_{txtxtxtx}(x, x')$, is given in terms of the vacuum two-point function, namely,

$$C_{txtxtxtx}(x, x') = \langle R_{txtx}(x) R_{txtx}(x')\rangle = \frac{1}{4}(\partial_t^4 - 2\partial_t^2\partial_x^2 + \partial_x^4)D, \tag{84}$$

with

$$D = \frac{1}{4\pi^2[(\Delta\vec{x})^2 - (\Delta t)^2]}. \tag{85}$$

Substituting Eqs. (20)-(22) into Eq. (84), we can write the full expression of the Riemann tensor correlation function in the graviton vacuum state as

$$\begin{aligned} C_{txtxtxtx}(\Delta t, \Delta x, \Delta y, \Delta z) = \frac{4}{\pi^2}\Bigg[& \frac{(\Delta x^2 - \Delta t^2)^2 + (\Delta y^2 + \Delta z^2)^2}{(\Delta x^2 + \Delta y^2 + \Delta z^2 - \Delta t^2)^5} \\ & - \frac{4(\Delta x^2 - \Delta t^2)(\Delta y^2 + \Delta z^2)}{(\Delta x^2 + \Delta y^2 + \Delta z^2 - \Delta t^2)^5}\Bigg], \end{aligned} \tag{86}$$

where $\Delta t = t - t'$, $\Delta x = x - x'$, $\Delta y = y - y'$, and $\Delta z = z - z'$. Note that this correlation function is singular in the limit of coincident spacetime points, so that the integral in Eq. (83) diverges. We adopt the viewpoint that the correct resolution of this problem involves averaging over space and time.

A. Space and Time Averaging of Quantum Field Operators

It is well known that while expectation values of products of local field operators diverge, averaging the operators over a finite spacetime region produces finite expectation values. This technique of smearing a field operator with a smooth test function is used as a formal device in rigorous approaches to quantum field theory [53]. However, there is evidence that this averaging can have a physical meaning in specific contexts. For example, the one-loop QED correction to potential scattering can be estimated by averaging the quantum electric field operator of a time of the order that spent by an electron in the vicinity of the potential barrier [54]. The averaging of quadratic operators, such as the stress tensor, in time or space and time is essential to define a probability distribution for the fluctuations of such operators [55–57]. Furthermore, the results are very sensitive to the choice of the averaging function. The more rapidly the averaging function switches on and off, the greater is the probability of a large fluctuation. The physical origin of this sensitivity is as follows: averaging suppresses the contributions of the high frequency modes of the theory and renders the fluctuations finite. However, more rapid switching leaves a larger contribution from high frequency modes and hence both a larger variance for the fluctuations and a greater probability for a large fluctuation.

We adopt the viewpoint that the details of the averaging function are to be determined by the specific physical situation in question. In the present context of the fluctuations in the Riemann tensor in a bundle of geodesics, this involves averaging over the world tube defined by the bundle. This approach was used in Refs. [58, 59]. This averaging will produce finite results which depend upon the details of the averaging, as these details provide a physical cutoff on the high frequency mode contributions. In this view, there is no need for a formal regularization and renormalization procedure. We replace the integrations on the proper time in Eq. (83) by four-dimensional spacetime integrations:

$$\langle(\Delta v)^2\rangle = \epsilon_0^2 \int_{-\infty}^{+\infty} d^4x\ f(x) \int_{-\infty}^{+\infty} d^4x'\ f(x')\ C_{txtxtxtx}(x,x'), \tag{87}$$

where $f(x)$ is the sampling function, which is normalized so that $\int d^3x\ f(x) = 1$ and has dimensions of $1/length^3$. It describes the history of a wave packet, and involves integrating in time and averaging in space.

Here we will first average in space, and then integrate in time. For the spatial averaging, we consider a Lorentzian sampling function of width ϕ in each of the rectangular coordinates x, y, z and x', y', z',

$$g_{\rm L}(u,\phi) = \frac{\phi}{\pi(u^2+\phi^2)}, \tag{88}$$

so that

$$\int_{-\infty}^{+\infty} du\ g_{\rm L}(u,\phi) = 1. \tag{89}$$

This has the effect of averaging over a spatial scale of order ϕ. Therefore, the Riemann tensor correlation function, averaged over the spatial directions, may be defined by

$$
\begin{aligned}
\hat{C}(t-t',b) &= \int_{-\infty}^{+\infty} d^3x\ f(x) \int_{-\infty}^{+\infty} d^3x'\ f(x')\ C_{txtxtxtx}(x,x') \\
&= \int_{-\infty}^{+\infty} dx\ g_{\mathrm{L}}(x,\phi) \int_{-\infty}^{+\infty} dy\ g_{\mathrm{L}}(y,\phi) \int_{-\infty}^{+\infty} dz\ g_{\mathrm{L}}(z,\phi) \\
&\quad \times \int_{-\infty}^{+\infty} dx'\ g_{\mathrm{L}}(x',\phi) \int_{-\infty}^{+\infty} dy'\ g_{\mathrm{L}}(y',\phi) \int_{-\infty}^{+\infty} dz'\ g_{\mathrm{L}}(z',\phi) \\
&\quad \times C_{txtxtxtx}(\Delta t, \Delta x, \Delta y, \Delta z) \\
&= \frac{4[3b^4 + 6b^2(t-t')^2 - (t-t')^4]}{\pi^2[3b^2 + (t-t')^2]^5}\,. \qquad (90)
\end{aligned}
$$

Here we have used the following identity

$$
\int_{-\infty}^{+\infty} dx\ g_{\mathrm{L}}(x,\phi) \int_{-\infty}^{+\infty} dx'\ g_{\mathrm{L}}(x',\phi)\ F(x-x') = \int_{-\infty}^{+\infty} d\Delta x\ g_{\mathrm{L}}(\Delta x, b)\ F(\Delta x), \qquad (91)
$$

with $b = 2\phi$. Note that the lightcone singularity present in Eq. (86), is no longer present in $\hat{C}(t-t',b)$. We may interpret the latter quantity as an acceleration correlation function which has been averaged in space, but not in time.

B. Direct time integration

Because $\hat{C}(t-t',b)$ is finite for all values of its arguments, so long as $b \neq 0$, one option seems to be to integrate it directly in time to find the associated velocity and position fluctuations. Define a velocity correlation function obtained by direct time integration by

$$
\langle v(t_1)v(t_2)\rangle_{\mathrm{DTI}} = \epsilon_0^2 \int_0^{t_1} dt \int_0^{t_2} dt' \hat{C}(t-t')\,, \qquad (92)
$$

and the associated velocity variance at time t_0 by

$$
\langle(\Delta v(t))^2\rangle_{\mathrm{DTI}} = \langle v(t_0)v(t_0)\rangle_{\mathrm{DTI}}\,. \qquad (93)
$$

The latter quantity is found to be

$$
\langle(\Delta v)^2\rangle_{\mathrm{DTI}} = \frac{16\epsilon_0^2\, t_0\, \ell_{\mathrm{Pl}}^2[27b^5 t_0 + 60b^3 t_0^3 + 9bt_0^5 + 7\sqrt{3}(3b^2+t_0^2)^3 \arctan(t_0/\sqrt{3}b)]}{81\pi\, b^5(3b^2+t_0^2)^3}\,. \qquad (94)
$$

In the limit that t_0 becomes large for fixed b, we find

$$
\langle(\Delta v)^2\rangle_{\mathrm{DTI}} \sim \frac{56\,\sqrt{3}\epsilon_0^2\,\ell_{\mathrm{Pl}}^2}{81\,b^5} t_0 - \frac{64\ell_{\mathrm{Pl}}^2\epsilon_0^2}{27\pi b^4} + \mathcal{O}\left(\frac{1}{t_0}\right)^2. \qquad (95)
$$

One may also find the associated position fluctuations by further time integrations:

$$\langle(\Delta\chi)^2\rangle_{\rm DTI} = \int_0^{t_0} dt_1 \int_0^{t_0} dt_2 \langle v(t_1) v(t_2)\rangle_{\rm DTI}\,, \tag{96}$$

and find

$$\langle(\Delta\chi)^2\rangle_{\rm DTI} \sim \frac{56\epsilon_0^2 \ell_{\rm Pl}^2\, t_0^3}{81\sqrt{3} b^5} \tag{97}$$

in the limit of large t_0.

These results are puzzling, because they imply that the mean squared velocity of the particle grows linearly in time. This is only possible if there is an external energy source. It is useful to examine a somewhat different limit. Let $b = c\,t_0$, where $c > 0$ is a constant. Now Eq. (94) take the form

$$\langle(\Delta v)^2\rangle_{\rm DTI} = \frac{K}{t_0^4}\,, \tag{98}$$

where K is a constant. Now $\langle(\Delta v)^2\rangle_{\rm DTI} \to 0$ as $t_0 \to \infty$. Thus if both b and t_0 become large together, then the velocity variance vanishes. At this point, it is unclear whether the linear growth found in Eq. (94) is due to holding b fixed, or to the sudden time switching used in the direct time integration approach.

.

C. Lorentzian Time Integration

We next adopt an indirect way of integrating in time, which we call Lorentzian time integration. It involves a dimensionless Lorentzian function given by

$$\bar{g}_{\rm L}(u,\varphi) = \frac{\varphi^2}{\pi(u^2+\varphi^2)}\,. \tag{99}$$

This function has the following property

$$\int_{-\infty}^{+\infty} du\; \bar{g}_{\rm L}(u,\varphi) = \varphi, \tag{100}$$

so φ is the effective interval of integration. That is, $\int_{-\infty}^{+\infty} du\; \bar{g}_{\rm L}(u,\varphi)\, F(u)$ is an integral of $F(u)$ over an interval of order φ centered about $u = 0$. The advantages of this approach are that the integral can be finite even if $F(u)$ has a singularity somewhere in the range of integration, and it avoids sudden switching.

In this subsection, we will use Lorentzian time integration to study velocity fluctuations. Thus, we integrate $\hat{C}(t-t', b)$ using two functions of the form of Eq. (99), and define the velocity variance as

$$
\begin{aligned}
\langle (\Delta v(t))^2 \rangle_{\text{LTI}} &= \epsilon_0^2 \int_{-\infty}^{+\infty} dt\; \bar{g}_{\text{L}}(t,\varphi) \int_{-\infty}^{+\infty} dt'\; \bar{g}_{\text{L}}(t',\varphi)\; \hat{C}(t-t',b) \\
&= \frac{a\epsilon_0^2}{4} \int_{-\infty}^{+\infty} d\tau\; \bar{g}_{\text{L}}(\tau,a)\; \hat{C}(\tau,b)\,, \qquad (101)
\end{aligned}
$$

where $a = 2\varphi$. In the last step, we used the fact that

$$
\int_{-\infty}^{+\infty} dt\; \bar{g}_{\text{L}}(t,\varphi) \int_{-\infty}^{+\infty} dt'\; \bar{g}_{\text{L}}(t',\varphi)\; F(t-t') = \frac{a}{4} \int_{-\infty}^{+\infty} d\tau\; \bar{g}_{\text{L}}(\tau,a)\; F(\tau)\,. \qquad (102)
$$

The integrand in the second line of Eq. (101) has first order poles at $\tau = \pm ia$ and fifth order poles at $\tau = \pm i\sqrt{3}\,b$. The integral may be performed by contour integration, with the result

$$
\langle (\Delta v(t))^2 \rangle_{\text{LTI}} = \frac{2\ell_{\text{Pl}}^2\, a^2 \epsilon_0^2}{81\pi\, b^5\, (a^2-3b^2)^5}\, (7\sqrt{3}a^9 - 108\sqrt{3}a^7 b^2 + 594\sqrt{3}a^5 b^4 \qquad (103)
$$

$$
+ 1296 a^4 b^5 - 4860\sqrt{3}a^3 b^6 + 7776 a^2 b^7 + 1215\sqrt{3} a b^8 - 3888 b^9)\,. \qquad (104)
$$

In the limit that a becomes large for fixed b, we have

$$
\langle (\Delta v(t))^2 \rangle_{\text{LTI}} \sim \frac{14\sqrt{3}\, a\epsilon_0^2\, \ell_{\text{Pl}}^2}{81\, b^5}\,. \qquad (105)
$$

Given that the duration of the time integration is proportional to a, this is essentially the same result as in Eq. (95), with the velocity variance growing linearly in the flight time. In fact, if we set $a = 4\pi t_0$, the two asymptotic forms are identical. We can also consider the limit where a and b are proportional to one another: set $b = c\,a$, so Eq. (104) takes the form

$$
\langle (\Delta v)^2 \rangle_{\text{LTI}} = \frac{K'}{a^4}\,, \qquad (106)
$$

for some constant K'. Now $\langle (\Delta v)^2 \rangle_{\text{LTI}} \to 0$ as $a \to \infty$.

Both Eqs. (105) and (106) are in qualitative agreement with the corresponding results, Eqs. (95) and (98), found using direct time integration. This indicates that the linear growth of $\langle (\Delta v(t))^2 \rangle$ in time is not an artifact of sudden temporal switching. However, the velocity variance does not grow when both the flight time and spatial scale increase together. This result suggests that we should examine more general space and time averagings.

D. Averaging over World Tubes of increasing Width

In both of the previous subsections, the spatial scale b was a constant, which means that we were averaging over the history of a bundle of rays with a fixed spatial cross section. However, more realistic beams tend to spread in width as they propagate. Now we explore an averaging method which describes this spreading. Return to the Riemann tensor correlation function, Eq. (86). Now we average it with Lorentzians of width b in (x, y, z), but width b' in (x', y', z'). However, this is equivalent to averaging with Lorentzians of width $b + b'$ in each of Δx, Δy, and Δz, because of the identity

$$\int_{-\infty}^{+\infty} dx\ g_{\rm L}(x, b) \int_{-\infty}^{+\infty} dx'\ g_{\rm L}(x', b')\ F(x - x') = \int_{-\infty}^{+\infty} d\Delta x\ g_{\rm L}(\Delta x, b + b')\ F(\Delta x)\,. \quad (107)$$

Thus, we may define

$$\begin{aligned}\hat{C}(t - t', b, b') &= \int_{-\infty}^{+\infty} dx\, dy\, dz\, g_{\rm L}(x, b)\, g_{\rm L}(y, b)\, g_{\rm L}(z, b) \\ &\times \int_{-\infty}^{+\infty} dx'\, dy'\, dz'\, g_{\rm L}(x', b')\, g_{\rm L}(y', b')\, g_{\rm L}(z', b')\ C_{txtxtxtx}(\Delta t, \Delta x, \Delta y, \Delta z) \\ &= \int_{-\infty}^{+\infty} d\Delta x\, d\Delta y\, d\Delta z\, g_{\rm L}(\Delta x, b + b')\, g_{\rm L}(\Delta y, b + b')\, g_{\rm L}(\Delta z, b + b') \\ &\times C_{txtxtxtx}(\Delta t, \Delta x, \Delta y, \Delta z)\,. \end{aligned} \quad (108)$$

This may be evaluated with the result

$$\hat{C}(t - t', b, b') = \frac{4[3(b + b')^4 + 6(b + b')^2(t - t')^2 - (t - t')^4]}{\pi^2[3(b + b')^2 + (t - t')^2]^5}\,. \quad (109)$$

So far, b and b' have been constants, but they may be functions of time without changing any of the above analysis. Let them be linear functions given by

$$b = b(t) = c\,t + b_0 \quad \text{and} \quad b' = b'(t') = c\,t' + b_0\,, \quad (110)$$

where $c > 0$ and $b_0 > 0$ are constants. These functions describe a bundle of rays which starts with a nonzero width, which then grows linearly in time as the rays propagate. Now the variable width averaged acceleration correlation function becomes

$$\hat{C}_{vw}(t, t') = \hat{C}(t - t', b(t), b'(t')) = \frac{4\{3[c(t + t') + 2b_0]^4 + 6[c(t + t') + 2b_0]^2(t - t')^2 - (t - t')^4\}}{\pi^2\{3[c(t + t') + 2b_0]^2 + (t - t')^2\}^5}\,. \quad (111)$$

The velocity variance of obtained from an integral upon t and t' of $\hat{C}_{vw}(t,t')$. In the limit of a long flight time, this variance becomes

$$\langle(\Delta v)^2\rangle = \epsilon_0^2 \int_0^\infty dt \int_0^\infty dt'\, \hat{C}_{vw}(t,t')\,. \tag{112}$$

This integral is finite so long as both c and b_0 are nonzero. This is most easily seen by transforming to polar coordinates, defined by $t = \tau\,\sin\theta$ and $t' = \tau\,\cos\theta$, so

$$\langle(\Delta v)^2\rangle = \epsilon_0^2 \int_0^\infty d\tau \int_0^{\pi/2} d\theta\,\tau\,\hat{C}_{vw}(\tau,\theta)\,. \tag{113}$$

The integrand is finite as $\tau \to 0$ so long as $b_0 > 0$. As $\tau \to \infty$, the integrand falls as $1/\tau^5$ for all θ if $c > 0$, and hence the integral converges at the upper limit. We can also now understand why we found $\langle(\Delta v)^2\rangle$ growing with increasing flight time in the two previous subsections. Both of those cases correspond to $c = 0$ in the present notation. If $c = 0$, the integrand in Eq. (113) grows for large τ if $\theta = \pi/4$, which is the $t = t'$ line. Note that on dimensional grounds, $\langle(\Delta v)^2\rangle \propto {b_0}^{-4}$. The integral in Eq. (112) may be evaluated numerically as a function of the parameter c, and the result is plotted in Fig. 3.

In summary, we have found that averaging over a geodesic bundle with a fixed spatial cross section leads to a mean squared velocity which grows linearly in time. This requires an external energy source to supply the added kinetic energy to the particles. However, if the cross section grows linearly in time, as would be the case for a diverging beam of particles, then the mean squared velocity approaches a constant value. Furthermore, this asymptotic value is very small unless the initial cross section is close to the Planck scale. In other contexts, the lack of secular growth of vacuum fluctuation effects can be linked to anti-correlations [59]. It is of interest to explore whether similar anti-correlations exist here as well. This is a topic for future research.

VI. CONCLUSIONS

In this work, we have analyzed the effects of fluctuations of the spacetime geometry on the motion of test particles using the geodesic deviation equation. Just as a classical gravitational field leads to tidal acceleration and changes in the relative velocities of test particles, a fluctuating gravitational field leads to fluctuations in these relative velocities and consequently fluctuations in the relative separations of the particles. We treat the geodesic

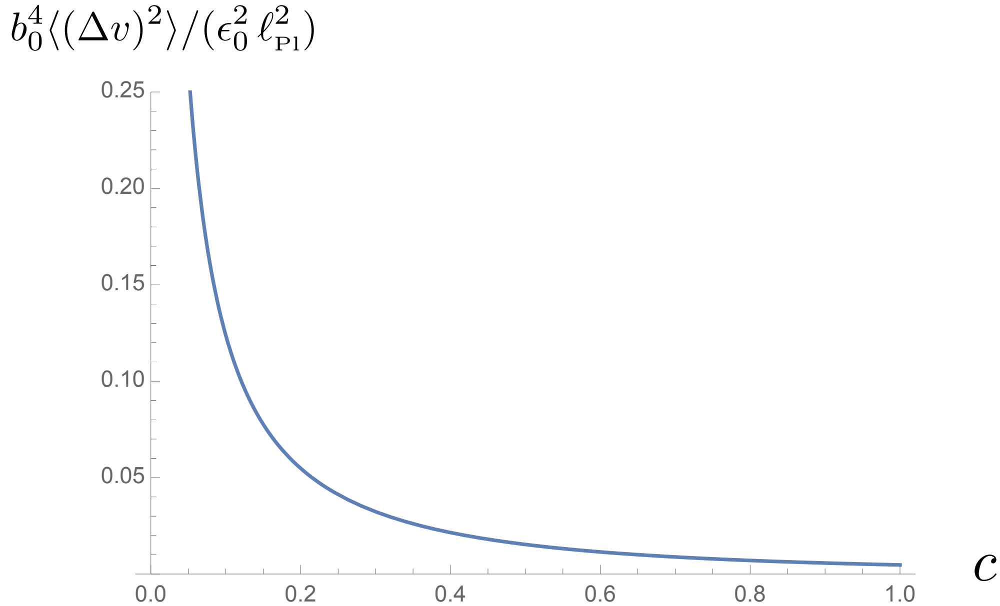

FIG. 3. The velocity variance for the case of variable width spatial sampling is plotted as a function of the parameter c.

deviation equation as a Langevin equation, which may be integrated to express the relative velocity and position variances as integrals of a Riemann tensor correlation function. Here we have considered fluctuations around an average flat spacetime background produced by linear quantum gravity effects. Thus we are dealing with active fluctuations of the dynamical degrees of freedom of gravity, as opposed to passive fluctuations driven by a matter stress tensor. The source of the spacetime geometry fluctuations could be either a bath of gravitons, or the graviton vacuum fluctuations. We have consider both a thermal bath of gravitons, and a bath of gravitons in a squeezed vacuum state. As expected, the velocity and position variances tend to be very small, and are suppressed by the square of the ratio of the Planck length to a characteristic length scale of the system. In the case of a thermal graviton bath, the variance of the relative velocity approaches a constant at late time, but root-mean squared position fluctuation grows linearly in time. This can be interpreted as a version of the gravitational memory effect [2].

The discussion of graviton vacuum fluctuation effects, given in Section V, requires averaging over both space and time to produce finite results. We view this as averaging over a world tube which describes the history of a set of test particles. We find results which can be very sensitive to the details of the averaging. This is to be expected, because the details of the rate of switch-on or switch-off determine the frequencies of the vacuum graviton modes which contribute to the final results. In this viewpoint, the details of the world tube of the test particles provides a physical cutoff which renders the theory finite, so there is no need for formal regularization or renormalization. A test of this viewpoint could in principle come from experiments which measure the dependence of the observable fluctuation effects upon the shape of this world tube. Although quantum gravity effects are far from testability, the same issues could arise in analog systems with electromagnetic field fluctuations which might be more accessible to experiment. This is a topic for future study, to determine whether large effects from rapid switching are real, or artificial effects which need to be subtracted by renormalization. In this paper, we have worked only to lowest order in the Riemann tensor. Whether the viewpoint which we adopt can be generalized to higher orders remains to be determined.

In the case where the spatial width of the bundle of geodesics is held constant, we find that the mean squared relative velocity grows linearly with the flight time. This seems to require an external energy source to maintain the constant width. However, it also raises the interesting possibility of enhanced quantum gravity effects for long flight times. However, we also find that if the spatial width is allowed to grow, even if very slowly, as the particles propagate, then the mean squared relative velocity approaches a constant.

There seem to be some subtle effects of spacetime geometry fluctuations in linearized quantum gravity which may elucidate the effects to be expected in a complete quantum gravity theory.

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