Effective line elements and black-hole models in canonical loop quantum gravity

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Effective line elements and black-hole models in canonical (loop) quantum gravity

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Canonical quantization is often used to suggest new effects in quantum gravity, in the dynamics as well as the structure of space-time. Usually, possible phenomena are first seen in a modified version of the classical dynamics, for instance in an effective Friedmann equation, but there should also be implications for a modified space-time structure. Quantum space-time effects, however, are often ignored in this setting because they are not obvious: they require a careful analysis of gauge transformations and the anomaly problem. It is shown here how modified space-time structures and effective line elements can be derived unambiguously, provided an off-shell anomaly-free system of modified constraints exists. The resulting effective line elements reveal signature change as an inescapable consequence of non-classical gauge transformations in the presence of holonomy modifications. The general framework is then specialized to black-hole models in loop quantum gravity. In contrast to previous studies, a self-consistent space-time structure is taken into account, leading to a new picture of black-hole interiors.

I. INTRODUCTION

The covariance problem is highly non-trivial in modified or quantum canonical gravity. Not just the dynamics but also the structure of space-time may change, and the correct space-time structure is to be derived from the theory. A self-consistent space-time structure within a given theory is important not only for an interpretation of the theory but also for explicit derivations based on effective line elements. A line element $\mathrm{d}s^2$, by definition, provides a distance measure independent of the choice of coordinates. In any theory with classical covariance, based on Riemannian geometry, the metric tensor $g_{\mu\nu}$ has the correct transformation properties such that the familiar equation

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \tag{1}$$

results in a coordinate-independent expression.

In canonical gravity, the components of $g_{\mu\nu}$ are split into different types of variables: the spatial metric q_{ab}, the shift vector N^a, and the lapse function N. They now define the line element in the form [1]

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + q_{ab}\left(\mathrm{d}x^a + N^a\mathrm{d}t\right)\left(\mathrm{d}x^b + N^b\mathrm{d}t\right) \tag{2}$$

with a time coordinate t and spatial coordinates x^a, $a = 1, 2, 3$ in $x^\mu = (t, x^a)$. There is a one-to-one correspondence between the components of $g_{\mu\nu}$ and the components of q_{ab} together with N^a and N. However, in the canonical formulation, the coefficients in the space-time line element play rather different roles: The spatial metric q_{ab} provides the phase-space coordinates, along with momenta p^{cd} related to extrinsic curvature. Lapse and shift, on the other hand, are multipliers of first-class constraints $H[N]$ and $D[N^a]$, and therefore determine a gauge but do not have non-zero momenta.

Gauge transformations, generated by $H[N]$ and $D[N^a]$ via Poisson brackets, seem to change only phase-space variables, that is, the spatial components q_{ab} of the metric (2). A generic coordinate transformation, on the other hand, should change all the coefficients in the space-time line element. There should therefore be gauge transformations of lapse and shift as well, in addition to those of q_{ab}. In classical canonical gravity, it is known that such transformations are indeed implied by the constrained system given by $H[N]$ and $D[N^a]$, in particular by the consistent interplay of equations of motion and gauge transformations generated by the constraints. (See [2] for details.) Such an interplay relies on the *off-shell* form of Poisson brackets of the constraints, or the hypersurface-deformation brackets

$$\{D[N_1^a], D[N_2^b]\} = D[\mathcal{L}_{N_1}N_2^a] \tag{3}$$

$$\{H[N], D[N_1^a]\} = -H[\mathcal{L}_{N_1}N] \tag{4}$$

$$\{H[N_1], H[N_2]\} = \pm D[q^{ab}(N_1\partial_b N_2 - N_2\partial_b N_1)] \tag{5}$$

where the $\pm$ in the last equation indicates space-time signature. (An equivalent formulation [3] may use an extended phase space in which lapse and shift do have momenta, subject to additional constraints that they vanish. These additional constraints are not relevant for space-time transformations and therefore do not affect considerations of space-time structure.)

However, it seems that these general properties have not always been taken into account in modified or quantum canonical gravity. In this setting, motivated by different kinds of formal aspects of quantization, one modifies some of the constraints, in most cases the Hamiltonian constraint $H[N]$. A large class of examples includes homogeneous models of quantum cosmology, in which only a single constraint, $H[N]$ with spatially constant N, is non-trivial and implies a modified Friedmann equation when quantized. The modified Friedmann equation then leads to modified evolution equations for the scale factor $a(t)$, which may show interesting new effects. One could then insert a modified scale factor in an "effective" line element $\mathrm{d}s^2 = -N(t)^2\mathrm{d}t^2 + a(t)^2\mathrm{d}s_k^2$, with the appropriate spatially isotropic line element $\mathrm{d}s_k^2$ and a lapse function $N(t)$ corresponding to one's choice of time. But without an analysis of space-time structure, such a line element is coordinate dependent and therefore meaningless if $H[N]$ is modified such that the constraints no longer obey the classical hypersurface-deformation brackets. In such a case, gauge transformations of a according to modified constraints are not consistent with coordinate transformations of t and x^a. In some cases, the constraint brackets may still be closed but modified, in which case a suitable effective line element requires additional modifications. And if the constraint brackets no longer close, gauge transformations are broken and no effective line element exists. These statements underline the importance of the off-shell brackets of constraints, the derivation of which presents one of the most challenging tasks in approaches to canonical quantum gravity.

Fortunately, there are some models of canonical quantum gravity in which closed modified constraint brackets can be derived in their off-shell form. These include some versions of cosmological perturbations [4–6], and spherically symmetric models with non-perturbative inhomogeneity [7–10]. Here, we will use the latter models because their equations are less lengthy and show the relevant features more clearly. After a review of covariance in classical canonical gravity, we will show how all the required ingredients of covariance can be derived in modified canonical models, provided one has access to the off-shell brackets of constraints. As an application, we will obtain effective line elements and black-hole models with crucial new ingredients not considered before.

In particular, we shall show that given a deformation of the bracket between two modified Hamiltonian constraints, the same deformation function appears in the time-time component of the effective line-element. If the bracket is deformed such that its structure function changes sign, the time-time component of the effective line element turns positive, while the remaining contributions remain positive definite. This result unequivocally demonstrates the change in signature of the space-time metric which had earlier been deduced more indirectly from the change in sign of the hypersurface-deformation brackets, or from field equations they imply [11, 12]. Thus, signature-change is a necessary effect of having deformed hypersurface-deformations such that their structure function changes sign, as long as one is careful in deriving an invariant line-element that is consistent with the modified gauge transformations of the effective theory.

II. EFFECTIVE EQUATIONS IN SPHERICALLY SYMMETRIC MODELS OF LOOP QUANTUM GRAVITY

In spherically symmetric models, the canonical line element (2) takes the form

$$\mathrm{d}s^2 = -N(t,x)^2\mathrm{d}t^2 + q_{xx}(t,x)\left(\mathrm{d}x + N^x(t,x)\mathrm{d}t\right)^2 + q_{\varphi\varphi}(t,x)\mathrm{d}\Omega^2 \tag{6}$$

with two independent metric components q_{xx} and $q_{\varphi\varphi}$, and just one non-zero component N^x of the shift vector field, all depending on t and just the radial spatial coordinate x.

In order to be closer to models of loop quantum gravity, we express the metric components in terms of a densitized spatial triad with components E^x and E^φ, such that

$$q_{xx} = \frac{(E^\varphi)^2}{E^x} \quad , \quad q_{\varphi\varphi} = E^x \,. \tag{7}$$

For our purposes, it is sufficient to view these equations as part of a canonical transformation. (For details of the derivation of these and some of the following relations, see [13–15]. In general, E^x could take positive or negative values, indicating the orientation of space. We will assume $E^x > 0$ without loss of generality.) The triad components then acquire momenta K_x and K_φ such that

$$\{K_x(x), E^x(y)\} = 2G\delta(x,y) \quad \text{and} \quad \{K_\varphi(x), E^\varphi(y)\} = G\delta(x,y)\,. \tag{8}$$

The relationship between the momenta and partial derivatives of triad variables (or extrinsic curvature) is determined by the constraints, solving $\dot{E}^x = \{E^x, H[N] + D[N^a]\}$ and $\dot{E}^\varphi = \{E^\varphi, H[N] + D[N^a]\}$ for K_x and K_φ. In classical gravity, the constraints are

$$H[N] = -\frac{1}{G}\int \mathrm{d}x\, N\left(\frac{E^\phi}{2\sqrt{E^x}}K_\varphi^2 + K_\varphi\sqrt{E^x}K_x + \frac{E^\phi}{2\sqrt{E^x}} - \frac{((E^x)')^2}{8\sqrt{E^x}E^\phi}\right.$$
$$\left. + \frac{\sqrt{E^x}(E^x)'(E^\phi)'}{2(E^\phi)^2} - \frac{\sqrt{E^x}(E^x)''}{2E^\phi}\right) \tag{9}$$

and

$$D[N^x] = \frac{1}{2G}\int \mathrm{d}x\, N^x\left(2K_\varphi'\, E^\varphi - K_x(E^x)'\right)\,. \tag{10}$$

Their off-shell brackets mimick the full ones, with

$$q^{xx} = \frac{E^x}{(E^\varphi)^2} \tag{11}$$

replacing q^{ab} in (5).

A. Holonomy-modified effective constraints

Holonomy modifications in spherically symmetric models change the dependence of the Hamiltonian constraint on K_φ. Similarly to the full theory of loop quantum gravity [16, 17], spherically symmetric models in a loop quantization have operators only for exponentials $\exp(i\delta K_\varphi)$ of K_φ, which is part of the components of a U(1)-connection [14]. Prior to quantization, the Hamiltonian constraint therefore is modified by correction terms with higher powers of K_φ, such that the classical polynomial form is extended to a periodic function [15], again mimicking the construction in the full theory [18]. The diffeomorphism constraint, by contrast, is represented directly via its finite spatial transformations, without modifications. In order to analyze potential consequences of loop quantizations, one can therefore consider holonomy modifications in the Hamiltonian constraint, with the important condition that they not break gauge transformations. This condition is implemented by making sure that the constraints remain first class, that is, their Poisson brackets are still closed after holonomy modification.

The constraint brackets remain closed if we use a Hamiltonian constraint of the form

$$H[N] = -\frac{1}{G}\int \mathrm{d}x\, N \left(\frac{E^\phi}{2\sqrt{E^x}} f_1(K_\varphi) + f_2(K_\varphi)\sqrt{E^x} K_x + \frac{E^\phi}{2\sqrt{E^x}} - \frac{((E^x)')^2}{8\sqrt{E^x}E^\phi} \right. \\ \left. + \frac{\sqrt{E^x}(E^x)'(E^\phi)'}{2(E^\phi)^2} - \frac{\sqrt{E^x}(E^x)''}{2E^\phi}\right) \tag{12}$$

with two functions $f_1(K_\varphi)$ and $f_2(K_\varphi)$, such that [7, 10]

$$f_2(K_\varphi) = \frac{1}{2}\frac{\mathrm{d}f_1(K_\varphi)}{\mathrm{d}K_\varphi}\,. \tag{13}$$

The structure function in the bracket of two modified Hamiltonian constraints then acquires a factor of

$$\beta(K_\varphi) = \frac{1}{2}\frac{\mathrm{d}^2 f_1(K_\varphi)}{\mathrm{d}K_\varphi^2}\,. \tag{14}$$

In these derivations, the diffeomorphism constraint is left unmodified: In most constructions of the kinematics of loop quantum gravity, one does not use holonomies to quantize the diffeomorphism constraint but rather represents its finite action [17]. This procedure does not suggest holonomy modifications in the diffeomorphism constraint. Moreover, in the Hamiltonian constraint, only the K_φ-dependence is modified while the K_x-dependence remains linear. This form is motivated by differences in the roles played by these two extrinsic-curvature components in holonomies. While K_x appears in extended, non-local holonomies, exponentiating the radially integrated K_x, K_φ appears in "point" holonomies [19] which just exponentiate a multiple of iK_φ at a given point x. (See [14] for the corresponding quantum representation.) Holonomy effects on K_x can therefore be made small by choosing short curves for holonomies, while K_φ-modifications remain unchanged. The same arguments suggest that any non-linear modifications of the K_x-dependence should also introduce higher spatial derivatives of K_x from a derivative expansion of extended holonomies. However, no anomaly-free set of modified constraints is known in this form [10].

The function $\beta(K_\varphi)$ is close to one only when $f_1(K_\varphi)$ is close to the classical behavior, $f_1(K_\varphi) \approx K_\varphi^2$, which is usually the case for small K_φ. For other values, the constraint brackets do not have the classical form, and therefore the coefficients of a canonical line element (2) no longer transform in the classical way consistent with coordinate transformations. In such regimes, we have a modified space-time structure which requires a careful analysis before a coordinate-independent effective line element can be defined. We will pursue this question in the next section, but first indicate how strongly the space-time structure can be modified: Near a local maximum of f_1, β is negative, and the structure function in the bracket of two Hamiltonian constraints changes sign, indicating signature change according to (5). An analysis taking this new effect into account will, in general, lead to drastically different results, compared with one that assumes a Lorentzian "effective" line element in which one just inserts metric components subject to a modified dynamics. (See for instance [20].)

The remaining function, $f_1(K_\varphi)$, is not restricted by the condition that constraint brackets close. It is usually chosen such as to mimic holonomy components, for instance

$$f_1(K_\varphi) = \frac{\sin^2(\delta K_\varphi)}{\delta^2} \tag{15}$$

with a real parameter δ. In this case,

$$f_2(K_\varphi) = \frac{\sin(2\delta K_\varphi)}{2\delta} \tag{16}$$

and

$$\beta(K_\varphi) = \cos(2\delta K_\varphi)\,. \tag{17}$$

The function (15) is chosen as a simple example of a function that corresponds to a U(1)-holonomy in spherically symmetric models (replacing SU(2)-holonomies in the full theory). The precise choice depends on quantization ambiguities [21], as almost any periodic function could be obtained from U(1) characters.

B. Evolution, gauge and coordinates

The functions f_1 and f_2 appear in modified field equations generated by $H[N]+D[N^a]$:

$$\dot{F}=\{F,H[N]+D[N^a]\} \tag{18}$$

for any phase-space function F. For the basic variables, we obtain

$$\dot{E}^x = 2N\sqrt{E^x}\, f_2(K_\varphi)+N^x(E^x)' \tag{19}$$

$$\dot{E}^\varphi = N\sqrt{E^x}K_x\frac{\mathrm{d}f_2(K_\varphi)}{\mathrm{d}K_\varphi}+\frac{NE^\varphi}{2\sqrt{E^x}}\frac{\mathrm{d}f_1(K_\varphi)}{\mathrm{d}K_\varphi}+(N^xE^\varphi)' \tag{20}$$

and

$$\begin{aligned}\dot{K}_x =& -N\left(-\frac{E^\varphi}{2(E^x)^{3/2}}f_1(K_\varphi)+\frac{1}{\sqrt{E^x}}f_2(K_\varphi)K_x-\frac{E^\varphi}{2(E^x)^{3/2}}+\frac{((E^x)')^2}{8E^\varphi(E^x)^{3/2}}+\frac{(E^x)'(E^\varphi)'}{2\sqrt{E^x}(E^\varphi)^2}\right.\\ &\left.-\frac{(E^x)''}{2\sqrt{E^x}E^\varphi}\right)-\left(\frac{N(E^x)'}{2\sqrt{E^x}E^\varphi}\right)'+\left(\frac{N\sqrt{E^x}(E^\varphi)'}{(E^\varphi)^2}\right)'+\left(\frac{N\sqrt{E^x}}{E^\varphi}\right)''+(N^xK_x)' \end{aligned}\tag{21}$$

$$\begin{aligned}\dot{K}_\varphi =& N\left(-\frac{1}{2\sqrt{E^x}}(1+f_1(K_\varphi))-\frac{((E^x)')^2}{8\sqrt{E^x}(E^\varphi)^2}+\frac{\sqrt{E^x}(E^x)'(E^\varphi)'}{(E^\varphi)^3}-\frac{\sqrt{E^x}(E^x)''}{2(E^\varphi)^2}\right)\\ &+\left(\frac{N\sqrt{E^x}(E^x)'}{2(E^\varphi)^2}\right)'+N^xK_\varphi'\,. \end{aligned}\tag{22}$$

For a given choice of N and N^x, solutions for the phase-space variables are obtained in a fixed gauge. Because the modified constraints remain first class, they still generate gauge transformations

$$\delta F=\{F,H[\epsilon^0]+D[\epsilon^x]\} \tag{23}$$

consistent with the dynamics: The evolution of gauge-transformed initial values is equal to a gauge transformation of evolved initial values. For the phase-space variables, gauge transformations

$$\delta E^x = 2\epsilon^0\sqrt{E^x}\, f_2(K_\varphi)+\epsilon^x(E^x)' \tag{24}$$

$$\delta E^\varphi = \epsilon^0\sqrt{E^x}K_x\frac{\mathrm{d}f_2(K_\varphi)}{\mathrm{d}K_\varphi}+\frac{\epsilon^0E^\varphi}{2\sqrt{E^x}}\frac{\mathrm{d}f_1(K_\varphi)}{\mathrm{d}K_\varphi}+(\epsilon^xE^\varphi)' \tag{25}$$

and

$$\begin{aligned}\delta K_x =& -\epsilon^0\left(-\frac{E^\varphi}{2(E^x)^{3/2}}f_1(K_\varphi)+\frac{1}{\sqrt{E^x}}f_2(K_\varphi)K_x-\frac{E^\varphi}{2(E^x)^{3/2}}+\frac{((E^x)')^2}{8E^\varphi(E^x)^{3/2}}+\frac{(E^x)'(E^\varphi)'}{2\sqrt{E^x}(E^\varphi)^2}\right.\\ &\left.-\frac{(E^x)''}{2\sqrt{E^x}E^\varphi}\right)-\left(\frac{\epsilon^0(E^x)'}{2\sqrt{E^x}E^\varphi}\right)'+\left(\frac{\epsilon^0\sqrt{E^x}(E^\varphi)'}{(E^\varphi)^2}\right)'+\left(\frac{\epsilon^0\sqrt{E^x}}{E^\varphi}\right)''+(\epsilon^xK_x)' \end{aligned}\tag{26}$$

$$\begin{aligned}\delta K_\varphi =& \epsilon^0\left(-\frac{1}{2\sqrt{E^x}}(1+f_1(K_\varphi))-\frac{((E^x)')^2}{8\sqrt{E^x}(E^\varphi)^2}+\frac{\sqrt{E^x}(E^x)'(E^\varphi)'}{(E^\varphi)^3}-\frac{\sqrt{E^x}(E^x)''}{2(E^\varphi)^2}\right)\\ &+\left(\frac{\epsilon^0\sqrt{E^x}(E^x)'}{2(E^\varphi)^2}\right)'+\epsilon^xK_\varphi' \end{aligned}\tag{27}$$

have a form similar to the evolution equations because they are generated by the same constraints, but their meaning is quite different. For effective line elements, it is important to consider evolution equations as well as gauge transformations, and their interplay.

We will now approach the question of how modified gauge transformations could be related to coordinate transformations. The components of the spatial metric are subject to modified gauge transformations according to (24) and (25). Gauge transformations of the remaining components of an effective line element, N and N^x, are not as obvious, but they are uniquely determined by the constrained system. In the next section we review these aspects in general form, and then apply them to our modified spherically symmetric systems.

III. CANONICAL COVARIANCE

Covariance is not realized in a manifest way in canonical gravity, but it must still be present, given that the Hamiltonian formulation is equivalent to the more common Lagrangian one. Because covariance does not directly meet the eye in canonical versions of modified or quantum gravity, it is easy to overlook its importance when one works in a particular gauge without worrying whether other gauges would produce the same physical effects. If covariance is implemented properly, on the other hand, consistency conditions are imposed on possible quantum modifications, eliminating some (but perhaps not all) of the usual ambiguities. There are therefore important reasons to study covariance in canonical gravity, which remains one of the major open problems in approaches such as loop quantum gravity.

In this section, we briefly review the general features of covariance in classical canonical gravity, and then make them more explicit in generic spherically symmetric models. These details will allow us to see how covariance can be implemented in spherically symmetric models which are no longer classical but modified by holonomy effects motivated by loop quantum gravity. We will notice an important new term which is implied by holonomy modifications and necessary to obtain well-defined and gauge invariant effective line elements. This new term has not been included in any one of the previously existing black-hole models of loop quantum gravity.

A. Constraints, structure functions, and line elements

In canonical gravity, space-time coordinate transformations are replaced by gauge transformations of phase-space functions $F(q_{ab}, p^{cd})$ generated by the diffeomorphism and Hamiltonian constraints, $D[\epsilon^i]$ and $H[\epsilon^0]$. The diffeomorphism constraint generates deformations within a spatial slice used for the canonical decomposition, while the Hamiltonian constraint generates deformations of a spatial slice along its normal direction n^a in space-time. A combination of both transformations, generated by $H[\epsilon^0] + D[\epsilon^i]$, therefore implies a deformation along a space-time vector field $\epsilon^a = \epsilon^0 n^a + \epsilon^i s_i^a$, using a basis s_i^a, $i = 1, 2, 3$ of the spatial tangent bundle. For a generic phase-space function F, we obtain the gauge transformation

$$\delta F = \{F, H[\epsilon^0] + D[\epsilon^i]\} . \tag{28}$$

In order to compare these space-time deformations with infinitesimal coordinate changes, one should first translate the tangential-normal decomposition of a space-time vector field into a space-time decomposition, with reference to the time-evolution vector field $t^a = Nn^a + N^a$ of an ADM formulation [1]. Here, N and N^a are the lapse function and shift vector field of the space-time metric which is being transformed. A space-time vector field ξ^a, referring to the time direction and the previous spatial basis s_i^a, is decomposed as $\xi^a = \xi^0 t^a + \xi^i s_i^a$. If we want to find a gauge transformation which implements a coordinate transformation along ξ^a, we should choose the components ϵ^0 and ϵ^i such that ϵ^a and ξ^a are the same vector field, just written in different bases (see also [22]):

$$\epsilon^0 n^a + \epsilon^i s_i^a = \frac{\epsilon^0}{N} t^a + \left(\epsilon^i - \frac{N^i}{N}\epsilon^0\right) s_i^a = \xi^0 t^a + \xi^i s_i^a . \tag{29}$$

Comparing coefficients, a gauge transformation by (ϵ^0, ϵ^i) should therefore correspond to a coordinate transformation in a direction given by

$$\xi^0 = \frac{\epsilon^0}{N} \quad , \quad \xi^i = \epsilon^i - \frac{N^i}{N}\epsilon^0 . \tag{30}$$

We then have the identity

$$\mathcal{L}_\xi F = \{F, H[\epsilon^0] + D[\epsilon^i]\} \tag{31}$$

for phase-space functions $F(q_{ab}, p^{cd})$, provided the identities (30) hold for the components of ξ^a and ϵ^a.

As mentioned briefly in the introduction, in canonical gravity, gauge transformations generated by H and D act only on the spatial metric q_{ab} and its momentum p^{cd}. These constraints do not directly generate transformations of lapse and shift, which form the time-time and time-space components of the space-time line element

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + q_{ab}(\mathrm{d}x^a + N^a\mathrm{d}t)(\mathrm{d}x^b + N^b\mathrm{d}t) . \tag{32}$$

However, a generic coordinate transformation clearly changes not just q_{ab} but also the components g_{0a} of the space-time metric. In order to discuss covariance and line elements in canonical gravity, one could consider the extended phase space in which also lapse and shift have momenta, that is, the phase space on which primary constraints have not yet been solved [3]. Alternatively, the transformations follow from the requirement that the canonical equations of motion are gauge covariant [2, 23].

The latter viewpoint is instructive because it makes it clear that the off-shell constraint brackets, forming Dirac's hypersurface-deformation algebroid, are important in this context: Canonical equations of motion for the phase-space variables, formulated in a specific gauge, depend on the corresponding lapse and shift. A gauge transformation (31) changes the phase-space variables, and the transformed variables will obey equations of motion consistent with the original ones only if lapse and shift are properly transformed too. We are looking for a commutation property: The evolution of gauge-transformed initial data must be equal to a gauge transformation of the evolved initial data. Since equations of motion as well as gauge transformations are generated by the same constraints H and D, the former by $H[N]+D[N^a]$ and the latter by $H[\epsilon]+D[\epsilon^a]$, the commutation of gauge and evolution depends on the Poisson brackets of the constraints. An explicit calculation [2] shows that a constrained system with structure functions F_{AB}^D, such that the constraints obey

$$\{C_A,C_B\}=F_{AB}^D C_D\,, \tag{33}$$

leads to covariant equations of motion generated by $C[N^A]$, provided the multipliers N^A are subject to a gauge transformation

$$\delta_\epsilon N^A=\dot{\epsilon}^A+N^B\epsilon^C F_{BC}^A\,. \tag{34}$$

(Summed-over indices such as D may be continuous, implying integrations.)

For canonical gravity, the structure functions can be read off from the hypersurface-deformation brackets (3)–(5). They imply that the lapse function N should be subject to a gauge transformation

$$\delta_\epsilon N=\dot{\epsilon}^0+\epsilon^i\partial_i N-N^i\partial_i\epsilon^0 \tag{35}$$

and the shift vector field to a gauge transformation

$$\delta_\epsilon N^i=\dot{\epsilon}^i+\epsilon^j\partial_j N^i-N^j\partial_j\epsilon^i-q^{ij}(N\partial_j\epsilon^0-\epsilon^0\partial_j N)\,. \tag{36}$$

If these transformations are combined with a gauge transformation of the spatial metric, the full space-time metric is transformed according to a space-time coordinate change.

The explicit result shows that the off-shell brackets and their structure functions are relevant for the correct transformation of lapse and shift. This result underlines the importance of the full anomaly problem of modified or quantum gravity, making sure that the modified constraints obey not just some closed or partially closed system, but a modified version of the hypersurface-deformation brackets. In general, it is not enough to have closure on a phase-space on which some of the constraints have been solved (as in [24]), or of a reformulated constraint system (such as partial Abelianizations in [25]). Only if one has access to the full off-shell brackets, in such a way that the usual hypersurface-deformation brackets are obtained in the classical limit, can one realize modified versions of the transformations (35) and (36), a crucial ingredient of effective line elements. This conclusion will be made much more explicit by our analysis of spherically symmetric modls in what follows. (Examples of systems which formally have closed brackets of modified constraints but are not covariant exist, see [26, 27].)

B. Spherically symmetric line elements

By definition, the variables E^x and E^φ used in spherically symmetric models are such that the spatial line element is given by

$$\mathrm{d}s^2=\frac{(E^\varphi)^2}{E^x}\mathrm{d}x^2+E^x(\mathrm{d}\vartheta^2+\sin^2\vartheta\mathrm{d}\varphi^2)\,. \tag{37}$$

Therefore, the spatial metric has components

$$q_{xx}=\frac{(E^\varphi)^2}{E^x}\quad\text{and}\quad q_{\varphi\varphi}=E^x\,. \tag{38}$$

A corresponding space-time line element is

$$\mathrm{d}s^2=-N(x,t)^2\mathrm{d}t^2+q_{xx}(x,t)(\mathrm{d}x+N^x(x,t)\mathrm{d}t)^2+q_{\varphi\varphi}(x,t)(\mathrm{d}\vartheta^2+\sin^2\vartheta\mathrm{d}\varphi^2)\,. \tag{39}$$

We can now apply a spherically symmetric coordinate transformation along the vector field $(\xi^0,\xi^x)=(\epsilon^0/N,\epsilon^x-(N^x/N)\epsilon^0)$, using (30), directly to the line element by inserting $t'=t+\xi^0$ and $x'=x+\xi^x$ for t and x. The calculations are lengthy, but not as long as without symmetry assumptions, and they will ultimately be instructive when we discuss modified constraints.

After inserting transformed coordinates $x^a+\xi^a$ in the line element, we collect coefficients of $\mathrm{d}x^2$, $\mathrm{d}x\mathrm{d}t$ and $\mathrm{d}t^2$ in the new line element, in this order. The new coefficient of $\mathrm{d}x^2$ receives a contribution from a first-order expansion of $q_{xx}(x',t')$, one from expanding

$$\mathrm{d}x'=\mathrm{d}(x+\epsilon^x-(N^x/N)\epsilon^0)=\mathrm{d}x+(\epsilon^x-(N^x/N)\epsilon^0)^\bullet\mathrm{d}t+(\epsilon^x-(N^x/N)\epsilon^0)'\mathrm{d}x \tag{40}$$

in $\mathrm{d}x^2$, as well as one from the old $\mathrm{d}x\mathrm{d}t$ term in which we include, for now, only the $\mathrm{d}x$-term in

$$\mathrm{d}t'=\mathrm{d}(t+\epsilon^0/N)=\mathrm{d}t+(\epsilon^0/N)^\bullet\mathrm{d}t+(\epsilon^0/N)'\mathrm{d}x\,. \tag{41}$$

The latter two originate from cross-terms in squares, and therefore count twice. We obtain

$$\delta q_{xx}=\frac{\epsilon^0}{N}\dot{q}_{xx}+\left(\epsilon^x-\frac{N^x}{N}\epsilon^0\right)q'_{xx}+2q_{xx}\left(\epsilon^x-\frac{N^x}{N}\epsilon^0\right)'+2q_{xx}N^x\left(\frac{\epsilon^0}{N}\right)'\,. \tag{42}$$

Similarly, the new $\mathrm{d}x\mathrm{d}t$-term receives several contributions from which, using δq_{xx}, we have

$$\delta N^x=\dot{\epsilon}^x+\epsilon^x(N^x)'-N^x(\epsilon^x)'-\frac{1}{q_{xx}}\left(N(\epsilon^0)'-\epsilon^0N'\right)\,. \tag{43}$$

Finally, we derive

$$\delta N=\dot{\epsilon}^0+N'\epsilon^x-N^x(\epsilon^0)' \tag{44}$$

using both δq_{xx} and δN^x. (See App. A for details.) These results confirm the general expressions (35) and (36); note in particular a component of the inverse metric in (43). (There is no inverse $q_{\varphi\varphi}$ because the shift vector cannot have a φ-dependent φ-component if it preserves spherical symmetry.)

C. Holonomy modifications

As a result of the preceding calculations, we see that a line element of the form (39) is meaningful, that is, coordinate independent, only if lapse and shift are transformed according to (44) and (43). For classical gravity, these transformations are implied by the general result (34) for constrained systems. However, in the presence of holonomy modifications, some of the structure functions are modified. The same general result, (34), then implies that δN^x has to be modified to

$$\delta N^x=\dot{\epsilon}^x+\epsilon^x(N^x)'-N^x(\epsilon^x)'-\frac{\beta}{q_{xx}}\left(N(\epsilon^0)'-\epsilon^0N'\right) \tag{45}$$

where β/q_{xx} is the new structure function in the bracket of two Hamiltonian constraints.

Similarly, even though this structure function does not appear explicitly in (35), the classical (43) is used in a derivation of (44) for spherically symmetric models, as shown in App. A. This derivation can no longer proceed in the classical form if structure functions are modified such that $\beta\neq\pm1$. In particular, details of the derivation of (44) show that the q_{xx}-independent contributions to the new $\mathrm{d}t^2$-term are given by

$$-2N\dot{N}\frac{\epsilon^0}{N}-2NN'\left(\epsilon^x-\frac{N^x}{N}\epsilon^0\right)-2N^2\left(\frac{\epsilon^0}{N}\right)^\bullet=-2N\dot{\epsilon}^0-2NN'\epsilon^x+2N^xN'\epsilon^0\,. \tag{46}$$

Upon using δN^x, a new q_{xx}-independent term $2NN^x(\epsilon^0)'-2N^xN'\epsilon^0$ is generated by the contribution $q_{xx}^{-1}(N(\epsilon^0)'-\epsilon^0N')$ to δN^x, and the result (44) follows. If δN^x is modified to (45), the correct δN as required by (34) no longer follows unless further modifications are implemented in just the right way.

Since the term in δN^x relevant for a derivation of δN is multiplied with β, the classical cancellations leading to (44) still happen if all the terms in (46) are multiplied with β. This is achieved if one performs a coordinate transformation not of the classical line element (39), but of an effective line element

$$\mathrm{d}s^2=-\beta(x,t)N(x,t)^2\mathrm{d}t^2+q_{xx}(x,t)(\mathrm{d}x+N^x(x,t)\mathrm{d}t)^2+q_{\varphi\varphi}(x,t)(\mathrm{d}\vartheta^2+\sin^2\vartheta\mathrm{d}\varphi^2)\,. \tag{47}$$

Holonomy modification not only imply corrections to the spatial metric by modified equations of motion generated by the Hamiltonian constraint, they also require a new factor of β of N^2 in the time-time component of the space-time line element. Signature change for $\beta < 0$ is an immediate consequence.

In the derivation of δN from a coordinate transformation we also made use of δq_{xx} in addition to δN^x. Since the constraints are modified by holonomy terms, gauge transformations and equations of motion for q_{xx} are modified as well. However, a direct calculation shows that the modified equations of motion for E^x and E^φ imply that the classical relation (42) between a gauge transformation of $q_{xx} = (E^\varphi)^2/E^x$ and its space and time derivatives remains unchanged, for any choice of f_1 and f_2. (See App. B.)

The new factor of β in the time-time component produces the correct modification (45) as required by the modified constraint brackets, and it ensures that the derivation of δN still results in (44), also required by the constraint brackets. The new time-time component, transformed by a coordinate transformation, now has an expansion given by

$$\delta(-\beta N^2) = -2\beta N\delta N - \dot{\beta}\epsilon^0 N - \beta'\left(\epsilon^x - \frac{N^x}{N}\epsilon^0\right)N^2 = -2\beta N\left(\delta N + \frac{N}{2\beta}\delta\beta\right) \tag{48}$$

with

$$\delta N + \frac{\delta\beta}{2\beta}N = \dot{\epsilon}^0 + N'\epsilon^x - N^x(\epsilon^0)' + \frac{1}{2}\frac{\dot{\beta}}{\beta}\epsilon^0 + \frac{1}{2}\frac{\beta'}{\beta}(N\epsilon^x - N^x\epsilon^0) = \frac{(\sqrt{\beta}\epsilon^0)^\bullet + (\sqrt{\beta}N)'\epsilon^x - N^x(\sqrt{\beta}\epsilon^0)'}{\sqrt{\beta}}\,, \tag{49}$$

assuming $\beta > 0$ in the last step, as well as scalar transformation. Therefore, for $\beta > 0$ we have

$$\delta(-\beta N^2) = -2\sqrt{\beta}N\left((\sqrt{\beta}\epsilon^0)^\bullet + (\sqrt{\beta}N)'\epsilon^x - N^x(\sqrt{\beta}\epsilon^0)'\right)\,, \tag{50}$$

and transformations of the effective line element can be mapped to those of the classical line element by absorbing a factor of $\sqrt{\beta}$ in the lapse function N and in the time component ϵ^0 of a gauge transformation. Alternatively, one can view this as absorbing $\sqrt{\beta}$ in the normal vector n^a, which is then no longer a unit vector. The latter viewpoint is an example for the general result of [28] based on properties of Lie algebroids. If $\beta < 0$, $\sqrt{|\beta|}$ can be absorbed in the same way, provided one maps to a Euclidean 4-dimensional line element.

IV. SPECIFIC SOLUTIONS OF EFFECTIVE LINE ELEMENTS

The equations of motion (19)–(22) form a set of coupled, non-linear, partial differential equations. An important class of classical solutions which are easier to derive are stationary ones, which we can try to look for also in the modified case. In Eq. (19), the left-hand side must then be zero, as must be the shift vector: We are looking for the zeros of one of the holonomy-modification functions, $f_2(K_\varphi) = 0$. Potential candidates of stationary solutions are therefore such that

$$K_\varphi = 0 \quad , \quad K_\varphi = \frac{\pi}{2\delta} \quad , \quad \ldots \tag{51}$$

(assuming a modification function of the form (15).) The value of $K_\varphi = 0$ corresponds to the classical solution, as shall be demonstrated in more detail in the next section. The value of $K_\varphi = \pi/(2\delta)$ may be considered a stationary solution in the deep quantum regime of strongly modified constraints, corresponding to a Euclidean phase in an effective line element. The presence of Euclidean solutions means that we have to generalize the meaning of "stationary." If there are four spatial dimensions but no time, the intuitive understanding of stationary solutions does not apply. The formal definition as solutions with a timelike Killing vector can, however, be generalized to a definition in which any solution with a Killing vector transversal to the hypersurfaces of a canonical $3+1$ decomposition is considered stationary (or static if the Killing vector is normal to the hypersurfaces).

Another space(-time) property requires more care. It is not always obvious how new stationary solutions for large K_φ should be interpreted. We can see this difficulty if we first describe what happens in the well-understood case of the classical Schwarzschild solution in canonical form. We know that in this case a classical stationary solution of the canonical equations of motion corresponds to the entire space-time region outside the horizon. In this region, $K_\varphi = 0$ does not identify a unique spatial slice but rather specifies a gauge condition on the slicing of space-time. In other regions of the same space-time, however, an equation such as $K_\varphi = 0$ can have a different meaning. For instance, in the interior, formulated as a Kantowski-Sachs model, K_φ is no longer identically zero and a fixed value then corresponds to a unique spatial hypersurface within a homogeneous slicing. The existence of different possible meanings of $K_\varphi = 0$ depending on the position in space-time and its gauge shows that, without a further detailed analysis of slicings and gauges, we cannot uniquely decide which version new stationary solutions with $\delta K_\varphi = \pi/2$ should correspond to.

Additional solutions, with even larger values of $K_\varphi = n\pi/(2\delta)$ for integer n, could indicate the existence of further alternating cycles of Euclidean and Lorentzian phases. However, before embarking on a detailed analysis of the quantum space(-time) dynamics, it is mute to speculate about potential transitions or phase-changes between the two classical signatures. Here, we are mainly interested in developing an understanding of the approach to larger values of K_φ and eventually the first Euclidean region.

A. The classical Schwarzschild solution

We begin with a simpler task: recovering the classical solution in an effective line element. In order to solve the full set of spherically symmetric equations, we first choose the gauge $E^x = x^2$ and then examine its flow as generated by the diffeomorphism constraint:

$$\delta E^x = \{E^x(x), D[N^x]\} = N^x(x)\,(E^x(x))' \approx 2N^x x \tag{52}$$

vanishes, and therefore the gauge condition is conserved, provided $N^x = 0$. With this first gauge choice, $E^x = x^2$, we solve the diffeomorphism constraint $D[N^x] = 0$ and obtain

$$K_x = \frac{E^\varphi}{x} K_\varphi' \,. \tag{53}$$

We can now rewrite the Hamiltonian constraint with this partial gauge fixing

$$H[N] = -\frac{1}{2G}\int \mathrm{d}x\, N \left((1 + f_1(K_\varphi)) \frac{E^\varphi}{x} + 2E^\varphi f_2(K_\varphi) K_\varphi' - \frac{3x}{E^\varphi} + \frac{2x^2 (E^\varphi)'}{(E^\varphi)^2} \right) . \tag{54}$$

It generates a flow given by

$$\delta_N E^\varphi := \{E^\varphi, H[N]\} = \left(\frac{NE^\varphi}{x} - \frac{\mathrm{d}}{\mathrm{d}x}(NE^\varphi) \right) f_2(K_\varphi) + NE^\varphi K_\varphi' \frac{\mathrm{d}f_2}{\mathrm{d}K_\varphi} \,, \tag{55}$$

$$\delta_N K_\varphi := \{K_\varphi, H[N]\} = -\frac{N}{2x}(1 + f_1(K_\varphi)) - \frac{N}{2}\frac{\mathrm{d}f_1}{\mathrm{d}x} + \frac{Nx}{2(E^\varphi)^2} + \frac{N' x^2}{(E^\varphi)^2} \,. \tag{56}$$

For a stationary solution, E^φ is conserved. Since K_φ is constant, $\delta_N K_\varphi$ implies an equation for N using (56), depending on E^φ restricted by $H[N] = 0$ in (54).

We have already stated that the classical stationary solution should correspond to $K_\varphi = 0$. Since K_φ is fixed, its flow generated by $H[N]$, (56), must vanish:

$$-\frac{N}{2x} + \frac{Nx}{2(E^\varphi)^2} + \frac{N'x^2}{(E^\varphi)^2} = 0 \tag{57}$$

or

$$(E^\varphi)^2 = x^2 \left(1 + 2x\frac{N'}{N} \right) . \tag{58}$$

(One may view $K_\varphi = 0$ as a gauge-fixing condition.)

Finally, the Hamiltonian constraint (54) implies a differential equation for E^φ,

$$(E^\varphi)^3 - 3x^2 E^\varphi + 2x^3 \frac{\mathrm{d}E^\varphi}{\mathrm{d}x} = 0 \,, \tag{59}$$

with the general solution

$$E^\varphi = \frac{x}{\sqrt{1 + c/x}} \,. \tag{60}$$

The integration constant $c = -2M$ can be fixed, as usual, by referring to the mass for large x. Using this solution in (58) then implies a differential equation

$$2\frac{\mathrm{d}N}{\mathrm{d}x} = -\frac{Nc}{x(x+c)} \tag{61}$$

for the lapse function, with the solution $N(x) = \sqrt{1+c/x}$. (An additional integration constant in this solution would simply rescale the lapse function.)

We can now formulate a line element. Since $\beta(K_\varphi) = 1$ for $K_\varphi = 0$, the effective line element, obtained by inserting our solutions in

$$\mathrm{d}s^2 = -\beta N^2 \mathrm{d}t^2 + \frac{(E^\varphi)^2}{E^x} (\mathrm{d}x + N^x \mathrm{d}t)^2 + E^x \mathrm{d}\Omega^2 \,, \tag{62}$$

is identical with the classical Schwarzschild line element

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{x}\right) \mathrm{d}t^2 + \frac{1}{1-2M/x} \mathrm{d}x^2 + x^2 \mathrm{d}\Omega^2 \,. \tag{63}$$

B. Holonomy-modified interior

The same arguments as used in classical general relativity show that the values of $x < 2M$ can no longer correspond to a stationary solution because, according to (63), x then plays the role of a time coordinate and the metric depends on x. The gauge choice $K_\varphi = 0$ is therefore unavailable in the interior. Moreover, $E^x = x^2$ is not a good gauge choice if x becomes a time coordinate (unless one would like to use $\sqrt{E^x}$ as internal time).

The interior, classically, is no longer stationary but spatially homogeneous because the metric does not depend on the new spatial coordinate t, and it remains spherically symmetric. The homogeneity condition $(E^x)' = 0$ can be used as a new gauge choice, also in the modified constraints. (Here and in what follows, a prime always denotes a spatial derivative, while a dot is a time derivative. In the interior of (63), a prime therefore refers to a derivative by t, while a dot refers to a derivative by x. This convention is used here because it is consistent with the dots and primes in the general equations of motion (19)–(22).) The diffeomorphism constraint then implies $K'_\varphi = 0$ for non-degenerate triads. The Hamiltonian constraint, with spatially constant E^x and K_φ, implies that K_x/E^φ is spatially constant. And with this condition $\dot{E}^\varphi/E^\varphi$, according to (20), is homogeneous if we assume homogeneous N and $N^x = 0$. If we choose homogeneous initial data, they remain homogeneous. Spatially homogeneous solutions therefore exist for any choice of holonomy-modification functions. Moreover, any such solution is connected to the stationary exterior through a horizon: The classical homogeneous interior, with $N(T) = 1/\sqrt{2M/T-1}$, $E^x(T) = T^2$ and $E^\varphi(T) = T\sqrt{2M/T-1}$ where $T = x$ in terms of the coordinates used in (63), has a vanishing $K_\varphi(T) = \dot{E}^x/(2N\sqrt{E^x}) = \sqrt{2M/T-1} = 0$ at the horizon. Therefore, holonomy modifications are expected to be small around the horizon, and the classical matching surface is unmodified.

By these arguments, we may consider modified homogeneous solutions as a black-hole interior. Since the classical relationship $T = x$ with an exterior coordinate may no longer be realized, we call the new time coordinate in the interior η. (We will also choose a different lapse function for our modified interior solutions.) The function $\sqrt{E^x(\eta)}$ would then be one of the scale factors of a modified Kantowski–Sachs model, accompanied by $E^\varphi(\eta)/\sqrt{E^x(\eta)}$ as an independent one in an anisotropic homogeneous model. Their time dependence is given by the previous equations of motion, (19)–(22), setting all spatial derivatives equal to zero and using a spatially constant lapse function. The Hamiltonian constraint, also with zero spatial derivatives, then implies a Friedmann-type equation.

It is convenient to choose the lapse function to be $N = \sqrt{E^x}$, an anisotropic version of conformal time. (For solutions in a different choice, see [29].) In the equations of motion

$$\dot{E}^x = 2E^x f_2(K_\varphi) \tag{64}$$

$$\dot{E}^\varphi = E^x K_x \frac{\mathrm{d}f_2}{\mathrm{d}K_\varphi} + \frac{1}{2} E^\varphi \frac{\mathrm{d}f_1}{\mathrm{d}K_\varphi} \tag{65}$$

$$= E^x K_x \frac{\mathrm{d}f_2}{\mathrm{d}K_\varphi} + E^\varphi f_2(K_\varphi) \tag{66}$$

$$\dot{K}_x = \frac{E^\varphi}{2E^x} f_1(K_\varphi) - K_x f_2(K_\varphi) + \frac{E^\varphi}{2E^x} \tag{67}$$

$$\dot{K}_\varphi = -\frac{1}{2}(1 + f_1(K_\varphi)) \,, \tag{68}$$

the last one decouples from the rest. For f_1 as in (15), it can be solved by

$$K_\varphi(\eta) = \frac{1}{\delta} \arctan\left(\frac{\delta}{\sqrt{1+\delta^2}} \tan\left(-\frac{1}{2}\sqrt{1+\delta^2}(\eta - \eta_0) \right) \right) \,. \tag{69}$$

(The form of this function resembles generalizations of Chebyshev polynomials considered in [30], but we are not aware of a simplified form in our case.)

This function is of interest because it appears in the new coefficient βN^2 in an effective line element, derived from the deformation function $\beta(K_\varphi) = \cos(2\delta K_\varphi)$ in the bracket of two spherically symmetric Hamiltonian constraints. In particular, signature change is indicated by sgnβ. Inserting our solution $K_\varphi(\eta)$, we obtain

$$\beta(\eta) = \frac{1 - \frac{\delta^2}{1+\delta^2}\tan^2(-\frac{1}{2}\sqrt{1+\delta^2}(\eta-\eta_0))}{1 + \frac{\delta^2}{1+\delta^2}\tan^2(-\frac{1}{2}\sqrt{1+\delta^2}(\eta-\eta_0))}\,. \tag{70}$$

Signature change happens when η reaches a value such that

$$\frac{\delta}{\sqrt{1+\delta^2}}\tan\left(-\frac{1}{2}\sqrt{1+\delta^2}(\eta-\eta_0)\right) = 1\,. \tag{71}$$

"After" this time, η is no longer a time coordinate but can still be used as the fourth coordinate in the Euclidean region. Changing η is then no longer time evolution but can still be interpreted as a transition between different slices in the Euclidean core, determined by the homogeneous gauge. For increasing η, the value $\beta(\eta) = -1$ of Euclidean space is approached for $\tan(-\frac{1}{2}\sqrt{1+\delta^2}(\eta-\eta_0)) \to \infty$, or

$$\eta = \eta_0 - \frac{\pi}{\sqrt{1+\delta^2}}\,. \tag{72}$$

At this point, $\delta K_\varphi = \frac{1}{2}\pi$, a value which will be discussed in more detail in the following subsection.

With our $K_\varphi(\eta)$, we obtain

$$E^x(\eta) = \frac{M^2}{(1+\delta^2)^2}\left(1 + 2\delta^2 + \cos\left(\sqrt{1+\delta^2}(\eta-\eta_0)\right)\right)^2 \tag{73}$$

from (64). Using the Hamiltonian constraint,

$$\frac{E^\varphi}{2\sqrt{E^x}}f_1(K_\varphi) + \sqrt{E^x}K_x f_2(K_\varphi) + \frac{E^\varphi}{2\sqrt{E^x}} = 0 \tag{74}$$

for spatially homogeneous variables, we can decouple the remaining variables E^φ and K_x: We first obtain

$$\frac{K_x}{E^\varphi} = -\frac{1+f_1(K_\varphi)}{2E^x f_2(K_\varphi)} \tag{75}$$

from the Hamiltonian constraint, and insert this result in

$$\frac{\dot{E}^\varphi}{E^\varphi} = E^x\frac{K_x}{E^\varphi}\frac{\mathrm{d}f_2}{\mathrm{d}K_\varphi} + f_2(K_\varphi) = -\frac{1+f_1(K_\varphi)}{2f_2(K_\varphi)}\frac{\mathrm{d}f_2}{\mathrm{d}K_\varphi} + f_2(K_\varphi)\,. \tag{76}$$

This equation is slightly simpler if we write it directly for the metric component $q_{xx} = (E^\varphi)^2/E^x$:

$$\frac{\dot{q}_{xx}}{q_{xx}} = -\frac{1+f_1(K_\varphi)}{2f_2(K_\varphi)}\frac{\mathrm{d}f_2}{\mathrm{d}K_\varphi}\,, \tag{77}$$

using the relation between f_1 and f_2. The solution for E^φ is

$$E^\varphi(\eta) = M\sin\left(\sqrt{1+\delta^2}(\eta-\eta_0)\right)\,, \tag{78}$$

which we can finally use to obtain

$$K_x(\eta) = \frac{1+\delta^2}{2M}\,\frac{1+2\delta^2+\sqrt{1+\delta^2} - \left(\sqrt{1+\delta^2}-1\right)\cos\left(\sqrt{1+\delta^2}(\eta-\eta_0)\right)}{\left(1+2\delta^2+\cos\left(\sqrt{1+\delta^2}(\eta-\eta)\right)\right)^2}\,. \tag{79}$$

The usual choice of f_1, (15), also allows us to express the K-dependent terms in the Hamiltonian constraint in terms of time derivatives of triad components, using the equations of motion: Trigonometric identities result in

$$\frac{\mathrm{d}f_2}{\mathrm{d}K_\varphi} = \sqrt{1-4\delta^2 f_2(K_\varphi)^2} = \sqrt{1-\delta^2\left(\frac{\dot{E}^x}{E^x}\right)^2} \tag{80}$$

such that

$$K_x = \frac{\dot{E}^\varphi - E^\varphi f_2(K_\varphi)}{2E^x \mathrm{d}f_2/\mathrm{d}K_\varphi} = \frac{\dot{E}^\varphi - \frac{1}{2}\frac{E^\varphi}{E^x}\dot{E}^x}{2E^x\sqrt{1-\delta^2(\dot{E}^x/E^x)^2}}\,. \tag{81}$$

Moreover,

$$f_1(K_\varphi) = \frac{1-\mathrm{d}f_2/\mathrm{d}K_\varphi}{2\delta^2} = \frac{1-\sqrt{1-\delta^2(\dot{E}^x/E^x)^2}}{2\delta^2}\,. \tag{82}$$

Eliminating K-components in the Hamiltonian constraint, we therefore obtain the "effective" Hamiltonian

$$\begin{aligned} H[N] &= -L_0\frac{N}{\sqrt{E^x}}\left(E^\phi\frac{1-\sqrt{1-\delta^2(\dot{E}^x/E^x)^2}}{4\delta^2} + \frac{\left(\dot{E}^\varphi - \frac{1}{2}\frac{E^\varphi}{E^x}\dot{E}^x\right)\dot{E}^x}{2E^x\sqrt{1-\delta^2(\dot{E}^x/E^x)^2}} + \frac{1}{2}E^\phi\right) \\ &= -\frac{L_0N}{2\sqrt{E^x}\sqrt{1-\delta^2(\dot{E}^x/E^x)^2}}\left(\dot{E}^\varphi\frac{\dot{E}^x}{E^x} + \frac{1}{2\delta^2}E^\phi\left((1+2\delta^2)\sqrt{1-\delta^2(\dot{E}^x/E^x)^2}-1\right)\right) \end{aligned} \tag{83}$$

for a finite region of coordinate length $L_0 = \int \mathrm{d}x$.

C. Effective line element in the deep quantum regime

The homogeneous interior evolves such that K_φ grows as we move away from the horizon. (Note that η decreases from η_0 at the horizon to $\eta_0 - \pi/\sqrt{1+\delta^2}$ at the transition of signature change. Similarly, the time coordinate t in a classical interior, corresponding to r in the usual Schwarzschild line element, decreases from $t = 2M$ at the horizon to $t = 0$ at the singularity. Formally, therefore, $K_\varphi(\eta)$ given in (69) is a decreasing function.) It eventually reaches a value where $\beta = 0$, and the signature changes to 4-dimensional Euclidean space. If we just follow formal solutions in this regime, K_φ still grows in the direction normal to hypersurfaces of the canonical decomposition, which is no longer a time direction. We should therefore switch to a 4-dimensional boundary-value problem, or a 2-dimensional one for t and x in the spherically symmetric reduction. There is not much known about possible boundary conditions in the deep quantum regime, and therefore the interior solutions becomes uncertain at this point. However, we use the fact that large K_φ will be reached as a motivation to take a closer look at the new stationary solutions realized for $K_\varphi = \pi/(2\delta)$.

We choose to fix the gauge in the same way as before, using $E^x = x^2$, but assigning the value $\pi/(2\delta)$ to K_φ. (After a transition to Euclidean signature, $E^x = x^2$ is again a good gauge choice even though we may remain in the interior.) For this choice, $f_1(K_\varphi) = 1/\delta^2$ and $f_2(K_\varphi) = 0$. (Note that the resulting solutions will not have a classical limit because f_1 diverges for $\delta \to 0$.) We then proceed as before: Compatibility of the flow of K_φ with the gauge choice can be deduced from (56) and implies

$$(E^\varphi)^2 = \frac{x^2}{1+1/\delta^2}\left(1+2x\frac{N'}{N}\right)\,. \tag{84}$$

Solving for the Hamiltonian constraint then gives

$$E^\varphi = \frac{x}{\sqrt{1+1/\delta^2-c/x}} \tag{85}$$

with an integration constant c To simplify notation, we introduce the parameter $\bar{\delta} = (1+1/\delta^2)^{-1}$ from hereon. We solve for the lapse

$$N(x) = \sqrt{1-\frac{c\bar{\delta}}{x}}\,. \tag{86}$$

All the consistency conditions, such as preservation of E^φ, still hold.

The effective line element, with $\beta(K_\varphi) = -1$ for $K_\varphi = \pi/(2\delta)$, is

$$\mathrm{d}s^2 = \left(1-\frac{c\bar{\delta}}{x}\right)\mathrm{d}\tau^2 + \frac{\bar{\delta}}{1-c\bar{\delta}/x}\,\mathrm{d}x^2 + x^2\mathrm{d}\Omega^2\,. \tag{87}$$

A Newtonian limit in Euclidean space would suggest $c\bar{\delta} = 2M$, such that

$$\mathrm{d}s^2 = \left(1 - \frac{2M}{x}\right)\mathrm{d}\tau^2 + \frac{\bar{\delta}}{1-2M/x}\,\mathrm{d}x^2 + x^2\mathrm{d}\Omega^2\,. \tag{88}$$

(The Newtonian limit is, of course, not a straightforward notion in Euclidean space. We are merely redefining an integration constant at this point, but the suggestive identification of M as a potential mass parameter might be useful.)

The Euclidean solution is valid for $x > 2M$. There is no legitimate solution for $x < 2M$ because it would imply two negative components of the metric, but no such signature is allowed by the effective line elements derived here. The value $x = 2M$ must therefore be interpreted as a boundary of Euclidean space, rather than a horizon as in the classical Schwarzschild solution. Analyzing the effective line element for x close to this value then tells us whether the boundary is regular or forms a conical singularity, just as in applications of Euclidean gravity to thermodynamic properties of black holes.

D. Black-hole thermodynamics

Our new stationary solution suggests interesting applications to black-hole thermodynamics because it automatically appears as a Euclidean effective line element; no Wick rotation is necessary to derive it.

In (88), the boundary at $x = 2M$ is a conical singularity unless the metric coefficients $g_{\tau\tau}$ and g_{xx} in an expansion around this value agree with the 2-dimensional plane in polar coordinates. Defining $x = 2M + \xi$ with small ξ, we obtain

$$1 - \frac{2M}{x} \approx \frac{1}{2M}\xi \tag{89}$$

and write the 2-dimensional effective line element in τ and ξ as

$$\mathrm{d}s^2 \approx \frac{1}{2M}\xi\mathrm{d}\tau^2 + 2M\bar{\delta}\frac{\mathrm{d}\xi^2}{\xi}\,. \tag{90}$$

Defining $r = \sqrt{8M\xi\bar{\delta}}$ and $\phi = \tau/(4M\sqrt{\bar{\delta}})$ then implies the 2-plane line element $\mathrm{d}s^2 \approx \mathrm{d}r^2 + r^2\mathrm{d}\phi^2$. We have the full 2-plane rather than a conical singularity provided that ϕ is periodic with period 2π, which implies a period of $\tau_{\mathrm{T}} = 8\pi M\sqrt{\bar{\delta}}$ for τ. In black-hole thermodynamics, this value is usually regarded as the inverse of the Hawking temperature

$$T_{\mathrm{E}} = \frac{1}{\tau_{\mathrm{T}}} = \frac{1}{8\pi M\sqrt{\bar{\delta}}} \tag{91}$$

for our Euclidean black hole. The corresponding entropy, using the area law, is

$$S = \frac{A}{4} = 4\pi M^2 = \frac{1}{16\pi T_{\mathrm{E}}^2\bar{\delta}}\,, \tag{92}$$

and the specific heat

$$C = \frac{\partial S}{\partial T_{\mathrm{E}}} = -\frac{1}{8\pi T_{\mathrm{E}}^3\bar{\delta}} < 0 \tag{93}$$

indicates that this black hole is thermodynamically unstable. If $\delta \ll 1$, $\bar{\delta} \approx \delta^2 \ll 1$ implies that $T_{\mathrm{E}} \gg T_{\mathrm{L}}$ where T_{L} is the temperature of a (classical) Lorentzian black hole with the same mass M as used in (91). The limit of $\delta \to 0$ of such a Euclidean black hole is therefore not a classical black hole, but rather an object that evaporates very quickly if it is in any way connected to a Lorentzian space-time.

Properties of higher stationary solutions with $K_\varphi = n\pi/(2\delta)$ with integer n can be obtained by substituting δ/n for δ in the preceding equations, or $\bar{\delta} = (1 + n^2/\delta^2)^{-1}$ for $\bar{\delta}$. For small δ, $\bar{\delta} \approx \delta^2/n^2$ is even smaller than for $n = 1$, and evaporation would proceed even more quickly. We note, however, that a physical interpretation of parameters such as M and T, obtained here exclusively in a Euclidean framework without the physical basis of a Wick-rotated Lorentzian space-time, is not clear at this point and needs further study.

V. DISCUSSION

We have studied effective line elements derived from anomaly-free spherically symmetric models of modified canonical gravity. We used canonical effective methods which are not new, but had not yet been applied in this context. Our results highlight the importance of off-shell constraint brackets and related closure conditions. Moreover, the specific form of hypersurface-deformation brackets in the classical limit is crucial; it cannot be replaced by a reformulated constrained system (such as a partial Abelianization) even if it remains first class. These detailed relationships between basic properties of canonical gravity reveal a direct road from modified structure functions to coefficients in effective line elements and related geometrical effects. In particular, if hypersurface-deformation brackets are modified such that the bracket of two normal deformations changes sign, the corresponding effective line element changes signature.

Applied to models of loop quantum gravity, therefore, signature change in effective line elements has been found to be an inevitable consequence if covariance is preserved by holonomy effects. It is not possible to interpret changing signs of structure functions as a dynamical instability of matter or metric perturbations on an otherwise Lorentzian space-time with a well-posed initial-value problem, as it is sometimes attempted in early-universe models. Since the only effective line element consistent with modified hypersurface-deformation brackets becomes Euclidean when holonomy effects are strong, only 4-dimensional boundary-value problems can be well-posed in this regime.

We have focused here on spherically symmetric solutions of modified canonical gravity. New black-hole models with a Euclidean core have been obtained, correcting the proposal of [31, 32] and several more recent follow-up studies. Our effective line elements also give rise to new non-classical stationary solutions, but their thermodynamical behavior indicates that they may not be relevant for a long-term analysis of quantum space-time.

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Appendix A: Coordinate transformations and lapse and shift

In the main text, we have shown some steps in the derivation of δq_{xx} from a coordinate transformation of the canonical spherically symmetric line element; see (42). Here, we provide further details of a similar derivation of δN^x and δN, given in (43) and (44). We will highlight steps in the derivation that will change if the hypersurface-deformation brackets are deformed.

In order to compute δN^x, we collect all terms that receive differentials $\mathrm{d}x\mathrm{d}t$ after substituting $t' = t + \epsilon^0/N$ and $x' = x + \epsilon^x - (N^x/N)\epsilon^0$ in the spherically symmetric line element

$$\mathrm{d}s^2 = -N(x,t)^2\mathrm{d}t^2 + q_{xx}(x,t)(\mathrm{d}x + N^x(x,t)\mathrm{d}t)^2 + q_{\varphi\varphi}(x,t)(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2)\,. \tag{A1}$$

Such contributions results from all terms except those that have angle differentials. In the final expression,

$$\begin{aligned}\frac{1}{2}\delta\mathrm{d}s^2|_{\mathrm{d}x\mathrm{d}t} = {}& \frac{\epsilon^0}{N}\dot{q}_{xx}N^x + \left(\epsilon^x - \frac{N^x}{N}\epsilon^0\right)q'_{xx}N^x + \frac{\epsilon^0}{N}q_{xx}\dot{N}^x + \left(\epsilon^x - \frac{N^x}{N}\epsilon^0\right)q_{xx}(N^x)' \\ &+ q_{xx}N^x\left(\left(\epsilon^x - \frac{N^x}{N}\epsilon^0\right)' + \left(\frac{\epsilon^0}{N}\right)^\bullet\right) \\ &+ q_{xx}\left(\epsilon^x - \frac{N^x}{N}\epsilon^0\right)^\bullet + q_{xx}(N^x)^2\left(\frac{\epsilon^0}{N}\right)' - N^2\left(\frac{\epsilon^0}{N}\right)'\,,\end{aligned} \tag{A2}$$

the first line results from expanding q_{xx} and N^x in $q_{xx}N^x\mathrm{d}x'\mathrm{d}t'$, while the second line results from expanding $\mathrm{d}x'\mathrm{d}t'$ in the same term. The last line contains those terms that produce $\mathrm{d}x\mathrm{d}t$ in an expansion of $q_{xx}\mathrm{d}x^2$, $q_{xx}(N^x)^2\mathrm{d}t^2$, and $-N^2\mathrm{d}t^2$, respectively. In an expansion, all these terms appear twice, which we have taken into account by dividing by two on the left-hand side of the equation.

The first two terms in (A2) can be recognized as similar contributions to δq_{xx}, given in (42). The two remaining terms in (42) then change (A2) to

$$\frac{1}{2}\delta\mathrm{d}s^2|_{\mathrm{d}x\mathrm{d}t} = N^x\delta q_{xx} - 2q_{xx}N^x\left(\epsilon^x - \frac{N^x}{N}\epsilon^0\right)' - 2q_{xx}(N^x)^2\left(\frac{\epsilon^0}{N}\right)'$$

$$+\frac{\epsilon^0}{N}q_{xx}\dot{N}^x+\left(\epsilon^x-\frac{N^x}{N}\epsilon^0\right)q_{xx}(N^x)'$$
$$+q_{xx}N^x\left(\left(\epsilon^x-\frac{N^x}{N}\epsilon^0\right)'+\left(\frac{\epsilon^0}{N}\right)^\bullet\right)$$
$$+q_{xx}\left(\epsilon^x-\frac{N^x}{N}\epsilon^0\right)^\bullet+q_{xx}(N^x)^2\left(\frac{\epsilon^0}{N}\right)'-N^2\left(\frac{\epsilon^0}{N}\right)'\,. \tag{A3}$$

Since this result should equal the first-order contribution to $(q+\delta q_{xx})(N+\delta N^x)$, the terms on the right other than $N^x\delta q_{xx}$ give us

$$\delta N^x=-2N^x\left(\epsilon^x-\frac{N^x}{N}\epsilon^0\right)'-2(N^x)^2\left(\frac{\epsilon^0}{N}\right)'$$
$$+\frac{\epsilon^0}{N}\dot{N}^x+\left(\epsilon^x-\frac{N^x}{N}\epsilon^0\right)(N^x)'+N^x\left(\left(\epsilon^x-\frac{N^x}{N}\epsilon^0\right)'+\left(\frac{\epsilon^0}{N}\right)^\bullet\right)$$
$$+\left(\epsilon^x-\frac{N^x}{N}\epsilon^0\right)^\bullet+(N^x)^2\left(\frac{\epsilon^0}{N}\right)'-\frac{1}{q_{xx}}N^2\left(\frac{\epsilon^0}{N}\right)'\,. \tag{A4}$$

Notice that we now obtained an inverse metric component in the last term, which will be sensitive to modified constraint brackets. If we take all the derivatives, the result can be simplified to

$$\delta N^x=\dot{\epsilon}^x+\epsilon^x(N^x)'-N^x(\epsilon^x)'-\frac{1}{q_{xx}}\left(N(\epsilon^0)'-\epsilon^0N'\right)\,, \tag{A5}$$

which is (43).

The transformation δN of the lapse function is derived from an expansion of all terms in $\mathrm{d}s^2$ with at least one $\mathrm{d}t$. We first compute

$$\delta\mathrm{d}s^2|_{\mathrm{d}t^2}=-2N\dot{N}\frac{\epsilon^0}{N}-2NN'\left(\epsilon^x-\frac{N^x}{N}\epsilon^0\right)-2N^2\left(\frac{\epsilon^0}{N}\right)^\bullet \tag{A6}$$
$$+\left(\dot{q}_{xx}(N^x)^2+2q_{xx}N^x\dot{N}^x\right)\frac{\epsilon^0}{N}+\left(q_{xx}'(N^x)^2+2q_{xx}N^x(N^x)'\right)\left(\epsilon^x-\frac{N^x}{N}\epsilon^0\right)$$
$$+2q_{xx}(N^x)^2\left(\frac{\epsilon^0}{N}\right)^\bullet$$
$$+2q_{xx}N^x\left(\epsilon^x-\frac{N^x}{N}\epsilon^0\right)^\bullet\,,$$

where the first line results from an expansion of $-N^2\mathrm{d}t^2$, the next two lines from $q_{xx}(N^x)^2\mathrm{d}t^2$, and the last from $2q_{xx}N^x\mathrm{d}x\mathrm{d}t$. This result should be equal to the first-order contribution in

$$-(N+\delta N)^2+(q_{xx}+\delta q_{xx})(N^x+\delta N^x)^2=-N^2+q_{xx}(N^x)^2-2N\delta N+(N^x)^2\delta q_{xx}+2q_{xx}\delta N^x+\cdots \tag{A7}$$

Using δq_{xx} in (42), which does not change for holonomy-modified constraints, we can first rewrite

$$\delta\mathrm{d}s^2|_{\mathrm{d}t^2}=(N^x)^2\delta q_{xx}-2N\dot{N}\frac{\epsilon^0}{N}-2NN'\left(\epsilon^x-\frac{N^x}{N}\epsilon^0\right)-2N^2\left(\frac{\epsilon^0}{N}\right)^\bullet \tag{A8}$$
$$+2q_{xx}N^x\dot{N}^x\frac{\epsilon^0}{N}+2q_{xx}N^x(N^x)'\left(\epsilon^x-\frac{N^x}{N}\epsilon^0\right)$$
$$+2q_{xx}(N^x)^2\left(\left(\frac{\epsilon^0}{N}\right)^\bullet-N^x\left(\frac{\epsilon^0}{N}\right)'\right)$$
$$+2q_{xx}N^x\left(\left(\epsilon^x-\frac{N^x}{N}\epsilon^0\right)^\bullet-N^x\left(\epsilon^x-\frac{N^x}{N}\epsilon^0\right)'\right)\,,$$

We next have to collect terms that provide δN^x in (A7) before we can read off δN. We will do so for a holonomy-modified expression of the form (45), or

$$\delta N^x=\dot{\epsilon}^x+\epsilon^x(N^x)'-N^x(\epsilon^x)'-\frac{\beta}{q_{xx}}\left(N(\epsilon^0)'-\epsilon^0N'\right)\,, \tag{A9}$$

in order to highlight the lack of cancellations if an unmodified effective line element were used. Since the δN^x-term in (A7) is the only first-order contribution in this expression with a coefficient of q_{xx}, it must produce all terms in the second, third and fourth line in (A8). By taking derivatives, it is straightforward, if a bit tedious, to see that this is indeed the case. However, the β-dependent term in δN^x contains the inverse metric component $1/q_{xx}$, which produces a term in $q_{xx}\delta N^x$ independent of q_{xx}. This remaining term should be combined with the first line in (A8) such that $-2N\delta N$ results. We find

$$\begin{aligned} \not{\delta} N &= \dot{N}\frac{\epsilon^0}{N} + N'\left(\epsilon^x - \frac{N^x}{N}\epsilon^0\right) + N\left(\frac{\epsilon^0}{N}\right)^{\bullet} + \beta\frac{N^x}{N}\left(N'\epsilon^0 - N(\epsilon^0)'\right) \\ &= \dot{\epsilon}^0 + N'\epsilon^x + (\beta-1)N^x\frac{N'}{N}\epsilon^0 - \beta N^x(\epsilon^0)'\,. \end{aligned} \tag{A10}$$

For $\beta = 1$, this result is identical with the lapse transformation (35) required for consistency of evolution and gauge. However, for $\beta \neq 1$, (A10) and (A9) are not consistent with gauge transformations of multipliers of holonomy-modified constraints: While (A9) is implied by a deformed bracket of two Hamiltonian constraints, the lapse transformation should not depend on β because, according to (34) it is sensitive only to the unmodified bracket $\{H[N], D[N^a]\}$.

We can obtain consistent results by modifying the original line element (A1) such that expansion coefficients change in the right way. This modification then implies the effective line element. We first note that all contributions to (A10) other than the β-dependent term originate from an expansion of $-N^2\mathrm{d}t^2$. If we modify this contribution to the line element to $-\beta N^2\mathrm{d}t^2$, all terms in the first line of (A10) are proportional to β, and the classical cancellations are revived:

$$\begin{aligned} \delta_\beta N &= \beta\dot{N}\frac{\epsilon^0}{N} + \beta N'\left(\epsilon^x - \frac{N^x}{N}\epsilon^0\right) + \beta N\left(\frac{\epsilon^0}{N}\right)^{\bullet} + \beta\frac{N^x}{N}\left(N'\epsilon^0 - N(\epsilon^0)'\right) \\ &= \beta\left(\dot{\epsilon}^0 + N'\epsilon^x - N^x(\epsilon^0)'\right) = \beta\delta N\,, \end{aligned} \tag{A11}$$

where δN is the classically expected transformation. If β depends on t or x, the expansion of $-\beta N^2$ will recieve additional terms, which can be combined to

$$\begin{aligned} & -2\beta N\delta N - \dot{\beta}\epsilon^0 N - \beta'\left(\epsilon^x - \frac{N^x}{N}\epsilon^0\right)N^2 \\ &= -2N\beta\left(\delta N\frac{1}{2}\frac{\dot{\beta}}{\beta}\epsilon^0 + \frac{1}{2}\frac{\beta'}{\beta}(N\epsilon^x - N^x\epsilon^0)\right) \\ &= -2\sqrt{\beta}N\left((\sqrt{\beta}\epsilon^0)^{\bullet} + (\sqrt{\beta}N)' - N^x(\sqrt{\beta}\epsilon^0)'\right)\,. \end{aligned} \tag{A12}$$

Gauge transformations of lapse and shift for holonomy-modified constraints are therefore consistent with the effective line element (47), but not with the classical form of the line element. Equivalently, as shown by (A12), one can work with the usual line element after a redefinition of the lapse function N and gauge parameter ϵ^0 absorbing $\sqrt{\beta}$, provided $\beta > 0$. If $\beta < 0$, one can absorb $\sqrt{-\beta}$ in the lapse function and gauge parameter of Euclidean space.

Appendix B: Gauge transformation of the spatial metric

The spatial metric has two independent components in spherically symmetric models, q_{xx} and $q_{\varphi\varphi}$. In this appendix, we show that their gauge transformations, expressed in terms of their time and space derivatives, are not modified for non-classical f_1 and f_2. The result for q_{xx} is important in our derivation of lapse and shift transformations, but the example of $q_{\varphi\varphi}$ is simpler. We present it first as a warm-up.

Since $q_{\varphi\varphi} = E^x$, we can directly use (24) and write

$$\delta q_{\varphi\varphi} = \delta E^x = 2\epsilon^0\sqrt{E^x}f_2(K_\varphi) + \epsilon^x(E^x)'\,. \tag{B1}$$

This expression is modified for non-classical f_2. However, if we use the modified equation of motion (19), solved for

$$2\sqrt{E^x}f_2(K_\varphi) = \frac{1}{N}\left(\dot{E}^x - N^x(E^x)'\right)\,, \tag{B2}$$

we obtain

$$\begin{aligned} \delta q_{\varphi\varphi} &= \frac{\epsilon^0}{N}\left(\dot{E}^x - N^x(E^x)'\right) + \epsilon^x(E^x)' \\ &= \frac{\epsilon^0}{N}\dot{q}_{\varphi\varphi} + \left(\epsilon^x - \frac{N^x}{N}\epsilon^0\right)q'_{\varphi\varphi}\,. \end{aligned}$$

There are no modifications in this equation.

For q_{xx}, we proceed in a similar way and write

$$\begin{aligned}\delta q_{xx} &= \delta\left(\frac{(E^\varphi)^2}{E^x}\right) = -\frac{(E^\varphi)^2}{(E^x)^2}\delta E^x + 2\frac{E^\varphi}{E^x}\delta E^\varphi \\ &= -\frac{(E^\varphi)^2}{(E^x)^2}\left(2\epsilon^0\sqrt{E^x}\; f_2(K_\varphi) + \epsilon^x (E^x)'\right) \\ &\quad +2\frac{E^\varphi}{E^x}\left(\epsilon^0\sqrt{E^x}K_x\frac{\mathrm{d}f_2(K_\varphi)}{\mathrm{d}K_\varphi} + \frac{\epsilon^0 E^\varphi}{2\sqrt{E^x}}\frac{\mathrm{d}f_1(K_\varphi)}{\mathrm{d}K_\varphi} + (\epsilon^x E^\varphi)'\right) \\ &= 2\frac{E^\varphi}{\sqrt{E^x}}K_x\frac{\mathrm{d}f_2(K_\varphi)}{\mathrm{d}K_\varphi}\epsilon^0 + \left(\frac{\epsilon^x (E^\varphi)^2}{E^x}\right)' + \frac{(E^\varphi)^2}{E^x}(\epsilon^x)' + \frac{(E^\varphi)^2}{(E^x)^{3/2}}\epsilon^0\left(\frac{\mathrm{d}f_1(K_\varphi)}{\mathrm{d}K_\varphi} - 2f_2(K_\varphi)\right) .\end{aligned}$$

If we now use the equations of motion to eliminate K_x, we have

$$\begin{aligned}\delta q_{xx} &= \frac{2\epsilon^0 E^\varphi}{E^x N}\left(\dot{E}^\varphi - \frac{E^\varphi}{2E^x}\left(\dot{E}^x + N\sqrt{E^x}\left(\frac{\mathrm{d}f_1(K_\varphi)}{\mathrm{d}K_\varphi} - 2f_2(K_\varphi)\right) - N^x (E^x)'\right) - (N^x E^\varphi)'\right) \\ &\quad + \left(\frac{\epsilon^x (E^\varphi)^2}{E^x}\right)' + \frac{(E^\varphi)^2}{E^x}(\epsilon^x)' + \frac{(E^\varphi)^2}{(E^x)^{3/2}}\epsilon^0\left(\frac{\mathrm{d}f_1(K_\varphi)}{\mathrm{d}K_\varphi} - 2f_2(K_\varphi)\right) \\ &= \frac{\epsilon^0}{N}\left(\frac{(E^\varphi)^2}{E^x}\right)^\bullet + \left(\epsilon^x - \frac{N^x}{N}\epsilon^0\right)\left(\frac{(E^\varphi)^2}{E^x}\right)' - 2\frac{(E^\varphi)^2}{E^x}\frac{\epsilon^0}{N}(N^x)' + 2\frac{(E^\varphi)^2}{E^x}(\epsilon^x)' \\ &= \frac{\epsilon^0}{N}\dot{q}_{xx} + \left(\epsilon^x - \frac{N^x}{N}\epsilon^0\right)q'_{xx} - 2q_{xx}\frac{\epsilon^0}{N}(N^x)' + 2q_{xx}(\epsilon^x)' .\end{aligned}$$

Also this equation is unmodified by holonomy terms. Note that the condition (13) for anomaly-free constraints would simplify the derivation, but it is not required for δq_{xx}, expressed in terms of its time and space derivatives, to be of the classical form.

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