Hydrodynamics with parametric slowing down and fluctuations near the critical point

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Hydro+: hydrodynamics with parametric slowing down and fluctuations near the critical point

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Abstract

The search for the QCD critical point in heavy-ion collision experiments requires dynamical simulations of the bulk evolution of QCD matter as well as of fluctuations. We consider two essential ingredients of such a simulation: a generic extension of hydrodynamics by a *parametrically* slow mode or modes ("Hydro+") and a description of fluctuations out of equilibrium. By combining the two ingredients we are able to describe the bulk evolution and the fluctutations within the same framework. Critical slowing down means that equilibration of fluctuations could be as slow as hydrodynamic evolution and thus fluctuations could significantly deviate from equilibrium near the critical point. We generalize hydrodynamics to partial-equilibrium conditions where the state of the system is characterized by the off-equilibrium magnitude of fluctuations in addition to the usual hydrodynamic variables – conserved densities. We find that the key element of the new formalism – the extended entropy taking into account the off-equilibrium fluctuations – is remarkably similar to the 2PI action in quantum field theory. We show how the new Hydro+ formalism reproduces two major effects of critical fluctuations on the bulk evolution: the strong frequency dependence of the anomalously large bulk viscosity as well as the stiffening of the equation of state with increasing frequency or wave-number. While the agreement with known results confirms its validity, the fact that Hydro+ achieves this within a local and deterministic framework gives it significant advantages for dynamical simulations.

I. INTRODUCTION AND OUTLINE

Hydrodynamics is a universal and versatile theory which describes (as the name implies) the dynamics of fluids or, more generally, systems whose evolution at long distance and time scales essentially consists of redistribution of conserved quantities (energy, momentum and charges) towards achieving global thermodynamic equilibrium throughout the system. Although many results of this paper are general, its primary focus is the application of relativistic hydrodynamics to the evolution of the fireball of hot and dense QCD matter created in heavy-ion collisions. In particular, this work is motivated by one of the major goals of the heavy-ion collision experiments – the discovery of the QCD critical point through the beam energy scan of the QCD phase diagram [1, 2]. Hydrodynamic description has shown remarkable and non-trivial agreement with many results of heavy-ion collision experiments (see, e.g., [3] for a concise review and further references). So far, however, such applications of hydrodynamics have been limited to the regime where the equation of state does not contain a critical point, e.g., at negligible baryon number densities. In order to facilitate the search for the QCD critical point it is essential to extend hydrodynamic description of heavy-ion collisions into the regime of finite baryon densities and, specifically, into the vicinity of the critical point.

There is, however, a major problem with applying hydrodynamics near a critical point. Applicability of hydrodynamics rests on the possibility of a certain scale separation. The time scales for achieving *local* equilibrium are usually much shorter than the time scales needed to reach global equilibrium throughout the system. This scale separation exists because the conserved densities relax to equilibrium by diffusion and the relaxation time is proportional to the square of the size of the inhomogeneity involved. The local equilibrium is achieved at time scales necessary to smooth out inhomogeneities on the scale of the correlation length ξ which is, typically, microscopically small. The global equilibrium may require times which are arbitrarily long for arbitrarily large systems. This separation of scales disappears at the critical point as the correlation length ξ diverges (see, e.g., Ref. [4] for more discussion).

At first sight, it might seem that hydrodynamics works as long as the correlation length ξ, while becoming large, remains much shorter than the scale of the inhomogeneities, ℓ. For heavy-ion collisions where the relevant size of the system is of order 10 fm while the correlation length is most likely not to exceed $2-3$ fm [5, 6] one would then expect a reasonable scale separation. However, such an argument would miss an important point: the time for relaxation of inhomogeneities of size ξ, essential for establishing local thermodynamic equilibrium, grows faster than their size and become very long near the critical point (critical slowing down). This time grows with ξ as ξ^z, where $z \approx 3$ [7]. Since $z > 1$, hydrodynamics breaks down even earlier, at much longer time and distance scales than the naive argument would suggest. For example, a sound wave with period shorter than the time of relaxation to local equilibrium cannot be described hydrodynamically, which happens when the wavelength $\ell \sim \xi^3$ (in units of temperature) or shorter, which is much larger than just ξ. For heavy-ion collisions the time scales needed to reach local equilibrium could become comparable to typical evolution times, invalidating hydrodynamic description near the critical point well before ξ is as large as ℓ.

One way this breakdown is manifested is in the growth of gradient corrections to the constitutive equation for the stress tensor, i.e., non-equilibrium corrections to pressure, proportional to bulk viscosity ζ and expansion rate (gradient of velocity) $\theta = \boldsymbol{\nabla} \cdot \boldsymbol{v} \sim 1/\ell$. The bulk viscosity diverges at the critical point as $\zeta \sim \xi^{z-\alpha/\nu} \sim \xi^3$ (for simplicity, we round exponents to integer values). This leads to the breakdown of the gradient expansion, i.e., of the locality of hydrodynamic description, when $\zeta\theta \sim \xi^3/\ell \gtrsim 1$, i.e., already for $\ell \lesssim \xi^3$, i.e., earlier, at much larger ℓ than just ξ suggested by the naive argument.

The goal of this paper is to develop an extended hydrodynamic description which overcomes this shortcoming of ordinary hydrodynamics. One way to understand the predicament here is to compare this situation with the breakdown of an effective theory for low-energy modes at scales comparable to the energy scales at which the mode which is next-to-lowest can get excited. The solution in this case is well-known – include the latter mode into the effective theory description. Therefore it is logical to consider extending hydrodynamics by adding an additional mode or, rather, as we shall see, a set of modes, describing relaxation processes responsible for the critical slowing down.

Of course, this only makes sense if the mode we add is still *parametrically* slower than the remaining (infinitely many) microscopic modes which are not included explicitly.[1] We

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[1] Hydrodynamics is a truncation of the full theory, justified by the slowness of hydrodynamic degrees of freedom. Due to conservation, the relaxation rates of these modes are controlled by the scale of inhomogeneity ℓ, and are proportional to ℓ^{-2}. The rate for the slow mode we add should be also controlled by some parameter independently of ℓ.

shall consider the role of such parametric separation in general and in the vicinity of the critical point in particular, where it is controlled by the divergence of the correlation length.

A. Outline

Broadly speaking, our paper contains two major ingredients. They are independent but, obviously, related. The first is presented in Section II and the second in Sections III and IV. They are combined in Section V.

First, in Section II, we consider a generic extension of the relativistic hydrodynamics by a mode which is slow not because it is a conserved density, but because a parameter controlling its relaxation rate can be independently tuned to make the rate arbitrarily small. Such a model could describe many different systems where due to *microscopic* dynamics some relaxation processes are slow. For example, if an additional charge exists which is only approximately conserved, so that its relaxation is not diffusive, but nevertheless slow. In Appendix A we discuss a specific example with partially conserved axial charge playing this role. Another example could be a system where some channels of chemical equilibration (of relative particle abundances) are slow. Such situations arise in cosmological and astrophysical contexts due to slowness of electroweak processes.

We use this simple model, which we shall call "Hydro+" to illustrate how the competition between the two relaxation scales (hydrodynamic and non-hydrodynamic) produces two regimes, one where ordinary hydrodynamics is sufficient and another where Hydro+ is necessary. This allows us to illustrate the basic mechanisms and in particular show how the divergent bulk viscosity and corresponding breakdown of the hydrodynamic gradient expansion is related to the slow mode. This mechanism is known [8–10], and our purpose here is to review and present it in *relativistically covariant* form, setting the stage for the generalization necessary near the QCD critical point.

To introduce the second ingredient we need to address the question of what is the physical origin of the critically slow processes near the critical point. It is known that the critical slowing down affects relaxation of fluctuations at scales of order ξ, essential for establishing local equilibrium. In order to find an appropriate description of these processes, for our purposes, we develop a more general approach to evolution of fluctuations in hydrodynamics in Sections III and IV.

In Section III we introduce the notion of the partial-equilibrium entropy S_2 which is a functional not only of the *average values* of the conserved densities, but also of the *magnitude of their fluctuations* (and correlations), i.e., one- and two-point functions of these densities. We show that this entropy has a remarkable (mostly mathematical) similarity to the 2-particle irreducible (2PI) action in quantum field theory.

Having derived the "2PI entropy" S_2 we then use it in Section IV to write evolution/relaxation equations for the conserved densities (which are ordinary hydrodynamic equations) together with the relaxation equations for the 2-point correlators. The equations for 2-point correlators show a remarkable similarity to kinetic (Boltzmann) equations, as has been observed long time ago in a related work by Andreev *et al.* [11–14], or more recently, in the context of heavy-ion collisions away from the critical point, in Ref.[15, 16]. Kinetic equations for correlations functions have also appeared in the work of Kawasaki [17] and others [18–20] in the calculations of kinetic coefficients and higher-order correlation functions near critical points in non-relativistic systems. In this paper we use a similar "kinetic" approach together with the 2PI entropy formalism we introduced to extend applicability of hydrodynamics for simulations near the QCD the critical point.

To this end, we defer further development of the 2PI entropy formalism to future work, and focus on its application to Hydro+ near the critical point in Section V. We identify the slowest mode of fluctuations and propose the equation which describes its evolution as an extension/generalization of ordinary hydrodynamics. To demonstrate how this extension works near the critical point and to verify its validity we compare the frequency-dependent bulk response in Hydro+ to the existing result due to Onuki [21, 22] obtained using a different approach based on a stochastic hydrodynamic model of critical fluctuations. We find that the new formalism simplifies this analytical calculation, while also providing a different intuitive perspective. The main advantage of the new formalism is that it is *local* and *deterministic*, while still capturing the dynamics of fluctuations. These properties make it much easier to use for simulating dynamics of the heavy-ion collision fireball expanding in the vicinity of the QCD critical point.

II. HYDRODYNAMICS WITH AN ADDITIONAL SLOW MODE

A. The general framework

In this section, we will formulate a local description of the evolution of hydrodynamical variables such as energy density ε, a conserved charge (e.g., baryon number) density n and fluid velocity u^μ coupled to *one* additional non-hydrodynamic, but nevertheless slow, scalar mode which will be denoted by ϕ (see also Ref. [23]).

If the mode ϕ is *parametrically* slower than other microscopic modes we can, in a window of time scales, consider *partial-equilibrium* state in which the entropy reaches its maximum under an additional constraint that the expectation value of that slow mode is ϕ. We can then introduce partial-equilibrium entropy $s_{(+)}(\varepsilon, n, \phi)$ as the logarithm of the number of states satisfying this constraint. The generalized thermodynamic potentials are defined as usual via derivatives of entropy

$$ds_{(+)} = \beta_{(+)}\, d\varepsilon - \alpha_{(+)}\, dn - \pi d\,\phi\,. \tag{1}$$

Here $\beta_{(+)}(\varepsilon, n, \phi)$ and $\alpha_{(+)}(\varepsilon, n, \phi)$ are generalized inverse temperature and chemical potential to temperature ratio, respectively, in the partial-equilibrium state. The variable $\pi(\varepsilon, n, \phi)$ is the generalized thermodynamic potential (or "force") corresponding to ϕ. In a *complete* equilibrium at given ε and n the variable ϕ must relax to its equilibrium value $\bar{\phi}(\varepsilon, n)$ which maximizes the generalized entropy, i.e.,

$$s(\varepsilon, n) = \max_{\phi} s_{(+)}(\varepsilon, n, \phi) = s_{(+)}(\varepsilon, n, \bar{\phi}(\varepsilon, n)), \tag{2}$$

and thus

$$\pi(\varepsilon, n, \bar{\phi}(\varepsilon, n)) = 0. \tag{3}$$

Hydro+ equations are usual energy-momentum conservation $\partial_\mu\, T^{\mu\nu} = 0$, charge conservation $\partial_\mu\, J^\mu = 0$ and an additional equation which describes the relaxation of ϕ towards equilibrium that we will specify shortly.

The components of $T^{\mu\nu}$ must be local functionals of the variables u^μ, ε, n, ϕ. One can expand in powers of derivatives, as usual:

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu + p_{(+)}\, g_\perp^{\mu\nu} + \Delta T^{\mu\nu}\,, \tag{4}$$

where

$$g_\perp^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu \tag{5}$$

is the transverse (spatial in the local rest frame) part of $g^{\mu\nu}$ [2] and the function $p_{(+)}(\varepsilon, n, \phi)$ is the generalized partial-equilibrium pressure. Here $\Delta T^{\mu\nu}$ denotes contributions to the stress-energy tensor due to the gradients of u^μ, ε, n, ϕ, which vanish in a static homogeneous system. Throughout this paper, we will use Landau frame choice to define ε and 4-velocity u^μ, i.e., $u_\mu \, \Delta T^{\mu\nu} = 0$. Similarly, the definition $n = u_\mu \, J^\mu$ implies:

$$J^\mu = n \, u^\mu + \Delta J^\mu \tag{6}$$

with $u \cdot \Delta J = 0$. Again ΔJ^μ will vanish in a static and homogeneous system and can be expanded in powers of derivatives.

The five equations for hydrodynamic variables can now be written explicitly:

$$D\epsilon = -w_{(+)}\theta - (\partial_\mu u_\nu) \, \Delta T_\perp^{\mu\nu} \,, \tag{7a}$$

$$Dn = -n \, \theta - \partial \cdot \Delta J \,, \tag{7b}$$

$$w_{(+)} Du^\nu = -\partial_\perp^\nu p - \delta_{\perp}{}^{\nu}_{\lambda} \partial_\mu \, \Delta T^{\mu\lambda} \,. \tag{7c}$$

The equation for ϕ must describe relaxation of this quantity to equilibrium value $\bar{\phi}(\varepsilon, n)$ and can be written as

$$D \, \phi = -F_\phi - A_\phi \, \theta \,, \tag{7d}$$

Here $F_\phi(\varepsilon, n, \phi)$ is the "returning force" which, at given ε and n, drives ϕ back to its equilibrium value. I.e., $F_\phi = 0$ when $\pi = 0$ (see Eq. (2)). The coefficient $A_\phi(\varepsilon, n, \phi)$ describes the susceptibility of the quantity ϕ to isotropic compression/expansion. In (7), we also introduced notations

$$w_{(+)} \equiv \varepsilon + p_{(+)}, \quad D \equiv u \cdot \partial \quad \text{and} \quad \theta \equiv \partial \cdot u \,. \tag{8}$$

For conventional hydrodynamics, the system of equations is closed once we supply the equation of state and constitutive relations for ΔJ_μ and $\Delta T^{\mu\nu}$. Similarly, Hydro+ will be closed if we, in addition, supply $F_\phi(\varepsilon, n, \phi)$ and $A_\phi(\varepsilon, n, \phi)$.

Second law of thermodynamics imposes constraints on the form of the constituitive equations. For ordinary hydrodynamics it requires $\beta p = s - \beta\varepsilon + \alpha n$ and positivity of kinetic

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[2] We use the mostly positive convention for the metric, therefore, $u \cdot u = -1$.

coefficients in ΔJ^μ and $\Delta T^{\mu\nu}$. Similarly, there are constraints on $p_{(+)}(\varepsilon, n, \phi)$, $F_\phi(\varepsilon, n, \phi)$ and $A_\phi(\varepsilon, n, \phi)$ from the generalized second law of thermodynamics which requires $\partial^\mu s_\mu^{(+)} \geq 0$. The generalized (partial-equilibrium) entropy current is given by

$$s^\mu_{(+)} = s_{(+)}\, u^\mu + \Delta s^\mu \,, \tag{9}$$

where Δs^μ denotes contribution to entropy current from gradients. The divergence of the entropy current then becomes:

$$\partial \cdot s_{(+)} = \left(s_{(+)} - \beta_{(+)}\, w_{(+)} + \alpha_{(+)}\, n + \pi A_\phi\right)\, \theta \\ + \alpha_{(+)}\, (\partial \cdot \Delta J) + \pi\, F_\phi + \beta_{(+)}\, (\partial_\mu u_\nu)\, \Delta T^{\mu\nu}_\perp + \partial \cdot \Delta s \,, \tag{10}$$

where we used (1) and Eq. (7c), while neglecting terms cubic in gradients. In order to guarantee $\partial \cdot s_{(+)} \geq 0$, we need the first term on the R.H.S of (10) to vanish. This condition relates pressure $p_{(+)}$ and "compressibility" A_ϕ and other generalized thermodynamic functions such as $s_{(+)}$, $\alpha_{(+)}$, $\beta_{(+)}$:

$$\beta_{(+)} p_{(+)} = \, s_{(+)} - \beta_{(+)}\varepsilon + \alpha_{(+)}\, n + \pi\, A_\phi \,. \tag{11}$$

Therefore:

$$\beta_{(+)}\, dp_{(+)} = -w_{(+)}\, d\beta_{(+)} + n\, d\alpha_{(+)} - \pi\, d\phi + d\left(\pi A_\phi\right)\,, \tag{12}$$

where we used (1). The last term, $\partial\cdot\Delta s$, in Eq. (10) must cancel total derivatives arising from integration by parts in the preceeding three terms, leaving only positive definite contributions to $\partial \cdot s_{(+)}$. This condition determines the form of Δs^μ.

The dissipative terms $\Delta T^{\mu\nu}$, ΔJ^μ and F_ϕ are also constrained by the second law of thermodynamics similarly to ordinary hydrodynamics.

To the first order in gradients, we still find the usual form for the gradient corrections to stress energy tensor:

$$\Delta T^{\mu\nu} = -\eta_{(+)} \left(\partial^\mu_\perp u^\nu + \partial^\nu_\perp u^\mu - \frac{2}{3} g^{\mu\nu}_\perp \theta\right) - \zeta_{(+)} g^{\mu\nu}_\perp\, \theta \,. \tag{13}$$

The second law of thermodynamics requires that $\zeta_{(+)}, \eta_{(+)} \geq 0$.

To first order in gradients, we now have additional term in ΔJ^μ

$$\Delta J^\mu = -\lambda_{\alpha\alpha}\, \partial^\mu_\perp\, \alpha - \lambda_{\alpha\pi}\, \partial^\mu_\perp\, \pi \,, \tag{14}$$

and

$$F_\phi = \gamma_\pi \pi - \partial_\perp \cdot \left(\lambda_{\pi\pi}\, \partial\pi + \lambda_{\alpha\pi}\, \partial\alpha_{(+)} \right) , \tag{15}$$

with $\gamma_\pi \geq 0$ and a semi-positive definite matrix λ_{ab} $(a, b = \alpha, \pi)$. Eqs. (14) and (15), take into account Onsager reciprocity. The first term in F_ϕ is allowed because $D\phi$ can remain finite even if the system is homogeneous since ϕ is *not* a conserved quantity.

Equation (7d) for ϕ reads, upon substituting Eq. (15):

$$D\phi = -\gamma_\pi\, \pi - \partial_\perp \cdot \left(\lambda_{\pi\pi}\, \partial\pi + \lambda_{\alpha\pi}\, \partial\alpha_{(+)} \right) - A_\phi\, \theta \,. \tag{16}$$

At sufficiently long times, i.e., in the limit $\pi = 0$, Eq. (14) reproduces the conventional constitutive relation for the dissipative part of J^μ with $\lambda_{\alpha\alpha}$ giving the conventional conductivity (times temperature), σT, i.e. $\Delta J^\mu = -\sigma T \partial_\perp^\mu \alpha$. In Appendix A we use hydrodynamics with partially conserved axial charge as an example of a theory with nonzero γ_ϕ, $\lambda_{\alpha\alpha}$, $\lambda_{\alpha\pi}$, and $\lambda_{\pi\pi}$.

A note on the power counting in the gradient expansion of constituitive equations is in order here. Unlike ordinary hydrodynamics, where the small parameter controlling the expansion such as Eq. (14) is the typical wave vector magnitude, k, Hydro+ has an additional *independent* small parameter – the relaxation rate, Γ_π, of the slow non-hydrodynamic mode. Given $\gamma_\pi \sim \Gamma_\pi$ (see Eq. (23)), the two terms in Eq. (15) are the lowest-order terms in the expansions in Γ_π and k respectively, allowed by isotropy and the second law of thermodynamics.[3]

Furthermore, the need for retaining terms of order k^2 in Eqs. (15) depends on the relative magnitude of k^2 and Γ_π. For the purpose of the power counting, in this paragraph, we measure parameters, such as k or Γ_π, in microscopic units, e.g., units set by the mean free path or temperature. Note that Hydro+ regime, where ordinary hydrodynamics fails, begins already at $k \sim \Gamma_\pi$ (more precisely, Γ_π / c_s [23]). The terms of order k^2 in Eq. (15) need to be kept if we want to apply Hydro+ to even higher k of order $\sqrt{\Gamma_\pi}$. On the other hand, in the regime $k \ll \sqrt{\Gamma_\pi}$ (including the Hydro+ regime $k \gtrsim \Gamma_\pi$) the k^2 terms in Eq. (15) are negligible.

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[3] In general, k^2 terms which are *not total derivatives* could also appear in Eq. (15), e.g., $\gamma_1 \pi (\partial_\perp \alpha_{(+)})^2$, etc. (obeying second law constraints, i.e., $\gamma_1 \geq 0$). In the most notable example, which we present in Appendix A, where ϕ is an approximately conserved density whose conservation is violated by a controllably small parameter (e.g., quark mass, m_q), such terms must vanish, i.e., $\gamma_1 \to 0$, in the limit $\Gamma_\pi \to 0$ (i.e., $m_q \to 0$) due to the conservation of ϕ in this limit. That means such terms are parametrically smaller (by a power of Γ_π, or m_q) than the k^2 terms in Eq. (15) in such a case.

It is instructive to compare and contrast the single-mode Hydro+ and the model of chiral fluid dynamics (CFD) (see, e.g., [24, 25]) considered recently in the context of the QCD critical point. The equation (linearized for simplicity) of the non-hydrodynamic mode σ, $\partial \cdot \partial\sigma + \eta_\sigma \partial_t \sigma + m_\sigma^2 \sigma = 0$, is different from the (correspondingly linearized) equation (16). Most notably, the mode σ is propagating, not relaxational, unless, or course, one considers $\omega \ll \eta_\sigma$. In this case the relaxation rate $\Gamma_\pi = m_\sigma^2/\eta_\sigma$ is vanishing when $m_\sigma \to 0$, provided η_σ does not vanish in this limit, which, however, it does in the model. If one ignores the physics of the model and considers η_σ as a free phenomenological parameter then the model will become an example of a single-mode Hydro+, provided the Lorentz invariance is also restored by replacing $\partial_t \sigma \to D\sigma$. This should be expected since Hydro+ is a general effective theory which should match any model in the appropriate limit, the specifics of the model being reflected in the values of phenomenological parameters such as γ_π, A_ϕ, etc. It should be also noted that, due to the mixing of the scalar field with baryon density, addition of such a field to hydrodynamics, generally, will not produce an independently slow mode even when $m_\sigma \to 0$ [26] unless an additional parameter is tuned.

Summarizing this section, we have considered a generalization of hydrodynamics, or "Hydro+", which describes the coupled evolution of hydrodynamic degrees of freedom and an additional parametrically slow scalar mode. As in ordinary hydrodynamics, the second law of thermodynamics imposes constraints on the form and parameters of the theory. The inputs of Hydro+ include the generalized entropy $s_{(+)}(\varepsilon, n, \phi)$, the "compressibility coefficient" A_ϕ and transport coefficients such as $\eta_{(+)}, \zeta_{(+)}$ and $\gamma_\pi, \lambda_{\alpha\alpha}, \lambda_{\alpha\pi}, \lambda_{\pi\pi}$ which appear in constitutive equations (13), (14) and (15).

For very slow processes, i.e., processes slower than the relaxation time of the slow mode (equivalently, at $\pi = 0$), Hydro+ reduces to conventional hydrodynamics. While the relationship between $\eta_{(+)}$ and $\lambda_{\alpha\alpha}$ to the conventional hydrodynamic coefficients η and λ (given by Kubo formulas at $\omega \to 0$) is trivial, this is not the case for the bulk viscosity ζ because it receives contribution from the slow mode proportional to its large relaxation time, as pointed out long ago by Leontovich and Mandelstam [8–10]. In the next section we discuss this effect in more detail using Hydro+ with a single slow non-hydrodynamic mode. Generalization of this effect to the case of the critical point leads to the critical divergence of the bulk viscosity which we discussed already in the introduction.

B. Bulk viscosity and sound in Hydro+

The presence of the slow mode has a profound effect on the response of the system to expansion or compression. In hydrodynamics the fluid's expansion/compression leads to the corresponding change in the densities of the conserved quantities. If the size of the system (or its part) undergoing expansion/compression is large enough that the contribution of diffusive processes are negligible, the conserved quantities, energy and charge, remain the same and the densities, ε and n, simply scale with the volume. We can describe this process by the linearized Eqs. (7a), (7a), where $\theta = \boldsymbol{\nabla}\cdot\boldsymbol{v}$ is the expansion rate:

$$D\varepsilon = -w\theta + \dots \qquad \text{and} \quad Dn = -n\theta + \dots \tag{17}$$

where the ellipsis denotes the terms of higher order in gradients.

The pressure, on the other hand, is a function of the variables ε and n only in equilibrium (at $\pi = 0$). In ordinary hydrodynamics the pressure adjusts to its equilibrium value $p(\varepsilon, n)$ on a microscopically short time scale, negligible compared to the timescale of expansion. In Hydro+, in contrast, the pressure $p_{(+)}$ depends also on the variable ϕ or, π, whose relaxation rate to equilibrium, $\pi = 0$, can be arbitrarily slow. As a result, if we write the linearized deviation of pressure from equilibrium due to infinitesimal expansion/compression θ we find an additional term proportional to π:

$$p_{(+)}(\varepsilon, n, \pi) = p(\varepsilon, n) + p_\pi(\varepsilon, n)\pi + \dots \tag{18}$$

The deviation of π from equilibrium is due to expansion, and the amount of this deviation is proportional to θ. To express this explicitly, we can substitute $\phi(\varepsilon, n, \pi)$ into Eq. (7d) to rewrite it as an equation for π:

$$\phi_\pi D\pi = -\gamma_\pi \pi + \left[w \left(\frac{\partial\phi}{\partial\varepsilon}\right)_{n\pi} + n \left(\frac{\partial\phi}{\partial n}\right)_{\varepsilon\pi} - A_\phi \right] \theta + \dots \tag{19}$$

where we used Eqs. (17) and defined

$$\phi_\pi \equiv \left(\frac{\partial\phi}{\partial\pi}\right)_{\varepsilon n} . \tag{20}$$

Using Maxwell relations (see Appendix B) one can express the quantity in the square brackets in terms of p_π:

$$\beta p_\pi = -\left[w \left(\frac{\partial\phi}{\partial\varepsilon}\right)_{n\pi} + n \left(\frac{\partial\phi}{\partial n}\right)_{\varepsilon\pi} - A_\phi \right] \tag{21}$$

and rewrite Eq. (19) as

$$D\pi = -\Gamma_\pi \pi - \frac{\beta p_\pi}{\phi_\pi}\theta + \dots \tag{22}$$

where we defined the relaxation rate

$$\Gamma_\pi \equiv \frac{\gamma_\pi}{\phi_\pi} \, . \tag{23}$$

This linearized equation can be solved for π as

$$\pi = \frac{\beta p_\pi}{\phi_\pi}\frac{1}{i\omega - \Gamma_\pi}\theta \tag{24}$$

where ω is the frequency of the oscillation of the variables around equilibrium. Substituting into Eq. (18) we find that pressure deviates from its equilibrium value by the amount proportional to the expansion rate θ:

$$p_{(+)} = p - \frac{\beta p_\pi^2}{\phi_\pi}\frac{1}{\Gamma_\pi - i\omega}\theta \, . \tag{25}$$

By definition, the coefficient of θ at $\omega = 0$ is the contribution of the slow mode to the bulk viscosity:

$$\Delta\zeta(0) = \frac{\beta p_\pi^2}{\phi_\pi \Gamma_\pi}. \tag{26}$$

This contribution diverges when Γ_π vanishes.

At non-zero ω the coefficient of θ can be related to the Green's function G_R of the operator $T_i^i/3$, which is natural since it describes the response to compression:

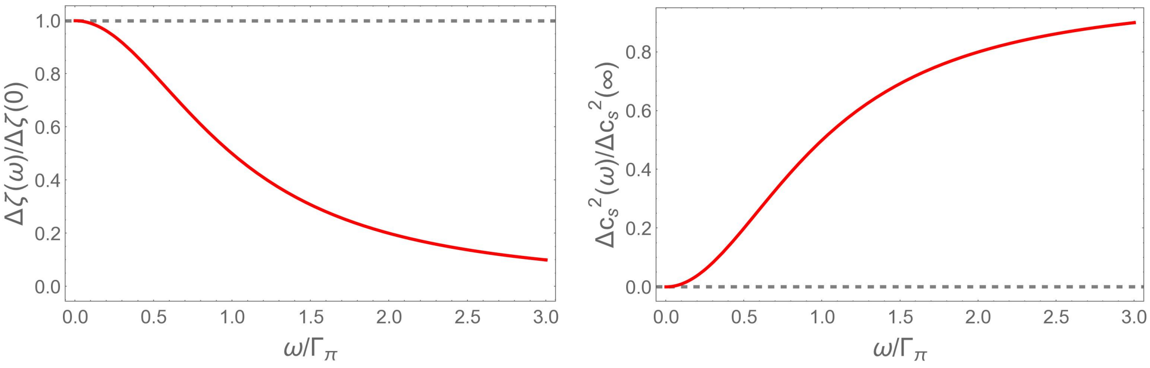
$$\frac{\beta p_\pi^2}{\phi_\pi}\frac{1}{\Gamma_\pi - i\omega} \equiv \frac{i\Delta G_R(\omega)}{\omega} \, . \tag{27}$$

The frequency dependent bulk viscosity can be defined as the real part of that coefficient, or $-\mathrm{Im}\, G_R/\omega$, in accordance with the Kubo formula,

$$\Delta\zeta(\omega) = \frac{-\mathrm{Im}\,\Delta G_R(\omega)}{\omega} = \Delta\zeta(0)\,\frac{\Gamma_\pi^2}{\Gamma_\pi^2 + \omega^2} \, . \tag{28}$$

This quantity describes dissipation during expansion/compression at frequency ω. Note that $\Delta\zeta(\omega)$ drops off when $\omega \gtrsim \Gamma_\pi$. This means that if we were to naively extend the conventional hydrodynamics with frequency-independent ζ to $\omega \gtrsim \Gamma_\pi$ we would overestimate the amount of dissipation (see Fig. 1(a)).

The imaginary part of the coefficient of θ in Eq. (25), i.e., the real part of ΔG_R in Eq. (27), is related to the contribution of the slow mode to the sound speed. This can be seen from



(a) (b)

FIG. 1. (color online) The frequency dependence of the contribution of the slow mode to the bulk viscosity, or to $-\mathrm{Im}\, G_R/\omega$, (left), as well as to the speed of sound, or $\mathrm{Re}\, G_R/\omega$, (right). Dashed horizontal lines represent the (lack of) frequency dependence of this quantities in ordinary hydrodynamics.

Eq. (25) by expressing oscillation δp of pressure in terms of oscillations $\delta\varepsilon$ of energy density and also using Eq. (17) to express θ in terms of $\delta\varepsilon$ ($\theta = i\omega\, \delta\varepsilon/w$):

$$\delta p_{(+)} = \left(c_s^2 + \Delta G_R/w\right) \delta\varepsilon \,. \tag{29}$$

The expression $c_s^2 + \Delta G_R/w$ in Eq. (29) can be viewed as the speed of sound (squared), or equation of state stiffness, in two limits when it becomes almost real, with the imaginary part related to the attenuation of the sound.

Let us define the frequency-dependent contribution to the speed of sound (corresponding to phase velocity)

$$\Delta c_s^2(\omega) = \frac{\mathrm{Re}\, \Delta G_R(\omega)}{w} = \frac{\beta p_\pi}{\phi_\pi w} \frac{\omega^2}{\Gamma^2 + \omega^2} \tag{30}$$

(see also Fig. 1(b)). The imaginary part (attenuation) becomes negligible in the limit $\omega \ll \Gamma_\pi$, when the sound speed is given by c_s^2 – the usual hydrodynamic sound speed, and also in the limit $\omega \gg \Gamma_\pi$ when the sound speed is given by a larger value

$$c_{s(+)}^2 = c_s^2 + \Delta c_s^2(\infty), \qquad \text{where} \quad \Delta c_s^2(\infty) = \frac{\beta p_\pi^2}{w\phi_\pi} \,. \tag{31}$$

Comparing Eq. (26) and (31) we find:

$$\zeta(0) = w\, \frac{\Delta c_s^2(\infty)}{\Gamma_\pi} \tag{32}$$

– the Landau-Khalatnikov formula (cf. Ref.[10]).

Note that $\Delta c_s^2 > 0$ is a consequence of thermodynamic stability. The fact that $c_{(+)}^2 > c_s^2$, i.e., the Hydro+ equation of state is stiffer, is natural since some (slow) degrees of freedom are effectively "frozen" at high frequencies. Thus naively extending ordinary hydrodynamics with equilibrium equation of state to higher frequencies would underestimate the stiffness (see Fig. 1(b)). [4]

Substituting the pressure oscillation given by Eq. (29) into the linearized hydrodynamic equations we find the dispersion relation for the sound as well as the non-hydrodynamic slow mode given by the three solutions of

$$\omega^2 = k^2 \left(c_s^2 + \frac{\omega}{\omega + i\Gamma_\pi} \, \Delta c_s^2(\infty) \right) . \tag{33}$$

In Fig. 2 the real and imaginary parts of the sound dispersion relation given by Eq. (33) are plotted for illustration. Note that the behavior of sound attenuation rate changes from quadratic in the regime $\omega \ll \Gamma_\pi$ to a constant for $\omega \gg \Gamma_\pi$:

$$\omega \ll \Gamma_\pi : \qquad \mathrm{Im}\, \omega = -\frac{k^2}{2} \frac{\Delta c_s^2(\infty)}{\Gamma_\pi} = -\frac{k^2}{2} \frac{\Delta\zeta(0)}{w} \, ; \tag{34}$$

$$\omega \gg \Gamma_\pi : \qquad \mathrm{Im}\, \omega = -\frac{\Gamma_\pi}{2} \frac{\Delta c_s^2(\infty)}{c_{s(+)}^2} \, . \tag{35}$$

For even larger ω one also has to take into account the usual $\mathcal{O}(k^2)$ contribution unrelated to the slow mode coming from $\zeta_{(+)}$ (as well as as from $\eta_{(+)}$ and $\lambda_{(+)}$).

To summarize this section we have considered the response of a system with a parametrically slow mode, described by Hydro+, to bulk expansion/compression. Most notably, for frequencies $\omega \gtrsim \Gamma_\pi$ the effective stiffness $\delta p/\delta\varepsilon$, or the sound speed, increases (see Figs. 1(b) and 2(b)), while the frequency-dependent bulk viscosity drops (see Fig. 1(a)). Note that without this drop the sound attenuation rate in Eq. (34) would have overcome the sound frequency (compare dashed lines on Figs. 2(a) and 2(b)). Instead, the sound attenuation rate (rather than growing as k^2) saturates at a constant (Fig. 2(a)). Such frequency and wave-vector dependence is beyond the reach of conventional hydrodynamics.

We want to emphasize again that these results, for $\omega \gtrsim \Gamma_\pi$, are reliable if the slow mode ϕ is *parametrically* slower than all the other non-hydrodynamic modes. I.e., if Γ_π

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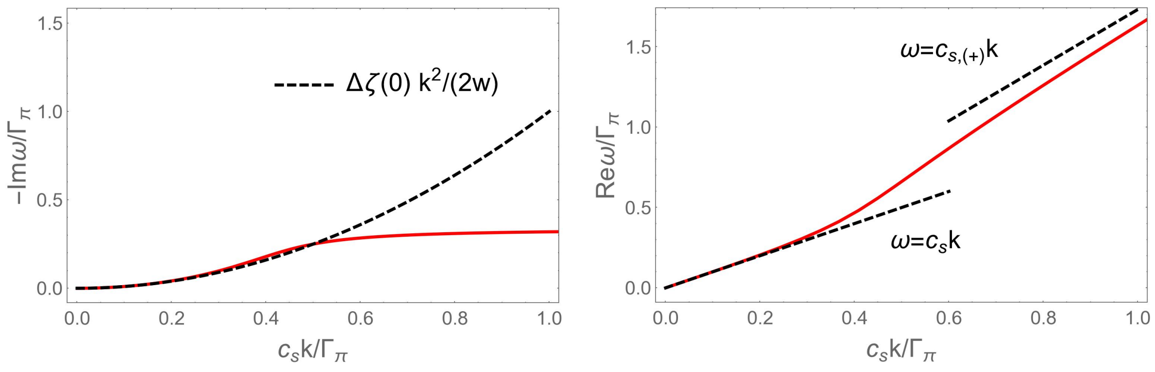
[4] Also note that, while the hydrodynamic speed of sound c_s^2 is given by the usual derivative $(\partial p/\partial\varepsilon)$ at $d\varepsilon/w = dn/n$ and $\pi = 0$, the speed $c_{s(+)}^2$ is given by the derivative where instead of $\pi = 0$ the condition $d\phi/A_\phi = d\varepsilon/w = dn/n$ holds (see Appendix B). This is different from the $\phi = \mathrm{const}$ condition in Ref.[10] because even if the relaxation term in Eq. (7d) can be neglected in the Hydro+ regime ($\omega \gg \Gamma_\pi$), the variable ϕ oscillates with θ if its compressibility A_ϕ is nonzero.



(a) (b)

FIG. 2. (color online) Sound dispersion relation, i.e., real (right) and imaginary (left) part of sound frequency as a function of k, in single-mode Hydro+ determined by solving (33), compared to ordinary hydrodynamics (dashed lines for $\omega \ll \Gamma_\pi$). The quantities are normalized to make the plots scale-independent. The dimensionless ratio $\Delta c_s^2/c_s^2$ is set to 2 for concreteness.

is much smaller than the microscopic (non-hydrodynamic) relaxation rates. This is an essential condition which distinguishes Hydro+ from other descriptions which add degrees of freedom not parametrically separated from other microscopic modes, e.g., conventional Israel-Stewart hydrodynamics [27] (see, e.g., Ref. [28] for a discussion of the (in)applicability of Israel-Stewart theory.) If, however, one treats the relaxation time τ_Π of the trace of the stress tensor as a parameter which can be made arbitrarily large, then an Israel-Stewart theory becomes an example of a single-mode Hydro+ (with $p_\pi = 1$). As expected, bulk viscosity diverges as $\tau_\Pi \to \infty$ in such a theory [29].

III. ENTROPY OF FLUCTUATIONS AND 2PI ACTION

A. Introduction

The purpose of this section is to introduce a particular new set of degrees of freedom into hydrodynamics. Unlike the usual hydrodynamic degrees of freedom, which describe local averages of conserved densities, these additional degrees of freedom describe fluctuations (and their correlations). This section explains the conceptual framework which makes this possible, i.e., explains how one needs to think about these degrees of freedom and the type of states of the system that they describe. Our limited goal here is to make this understanding quantitative enough that we can determine the entropy associated with these additional degrees of freedom, i.e., $s_{(+)}$. However, the conceptual framework we lay out is general and broad and could be extended to other interesting problems (e.g., non-gaussian fluctuations).

We shall attempt to answer the question: what is the physical meaning of a state characterized by given magnitudes of fluctuations and correlations which are not equal to equilibrium values? What we need to know is how much entropy the state with non-equilibrium fluctuations/correlations is missing (because we "know" more about it) compared to complete-equilibrium state. The answer to that question will be given by Eq. (57) at the end of this section. The result and the formalism leading to it is remarkably similar to the formalism of the 2-particle irreducible (2PI) effective action in quantum field theory [30–34]. The similarity is mathematical, while the physical origin and meaning is different.[5]

The purpose of this section is to derive Eq. (57) in such a way as to elucidate its conceptual meaning in the present context. A reader who finds Eq. (57) and its physical meaning evident following the intuitive explanation in the paragraph below it may skip directly to Eq. (57) on first reading. Another important result in this section is Eq. (56) which describes "renormalization" of the equation of state by fluctuations.

The discussion will be pedagogical and self-contained. We shall begin with a standard textbook introduction (e.g., Ref. [36]) to statistical mechanics and the concept of local thermodynamic equilibrium in order to emphasize the points which will be essential for our purpose.

The most important point to keep in mind is that a *state* in statistical physics describes an *ensemble* of microscopic quantum states of the system. The concept reflects the impracticality of describing the evolution of a macroscopic system by specifying and evolving its (highly excited) microscopic state. Not only is such a description unworkable, it is also unrealistic when we have no practical way to prepare such a pure microscopic state for a macroscopic system.

Instead, the statistical description deals with the ensemble of "similar" microscopic states. The ensemble is characterized by the set of *probabilities* of each state to be in the ensemble (the density matrix), or alternatively, the set of correlation functions, or expectation values of operators in the ensemble. The space of all possible statistical states is thus greater than

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[5] The 2PI action formalism has been used before to describe non-equilibrium evolution of quantum systems [35]. Although still different, this application of 2PI bears the closest similarity to our approach. We hope that our discussion will provide an intuitive insight into the meaning of this formalism as well.

the Hilbert space of microscopic (pure) states.

B. Equilibrium fluctuations

Equilibrium states form a special class of the statistical states in which the probability of a microscopic state to appear in the ensemble is a function *only* of conserved quantities (quantum numbers) of this state, such as energy, charge, momentum, etc. An equilibrium state is a very good approximation to a macroscopic system (i.e., a system with many degrees of freedom) averaged over time intervals sufficiently long compared to microscopic time scales (e.g., mean free time between collisions in a gas). Below we shall denote the set of the conserved quantities by a vector Ψ.

As a warm-up let us first consider the simplest equilibrium statistical state – the microcanonical ensemble – where the only microscopic quantum states in the ensemble are those with Ψ within a small interval $\Delta\Psi$ around a given value $\bar{\Psi}$. The size of the interval $\Delta\Psi$ is assumed to be small compared to characteristic scale of the variation of the density of states, but much larger than the level spacing. For a macroscopic system, with very large density of states, this is easily satisfied. The probability of each state in this interval is the same. Then entropy, which we shall denote $S_0(\bar{\Psi})$, is given simply by the logarithm of the number of the states in the ensemble.

Next let us consider canonical ensemble – an open system in contact with a much larger (infinite) reservoir of conserved quantities Ψ. All states of the system can now appear, but the probability of each state is now weighed by the exponential $\exp(J\Psi)$, where J is a set (a vector) of thermodynamic variables conjugate to Ψ (e.g., μ/T for charge, or $-1/T$ for energy): $J = -\partial S_{\text{reservoir}}/\partial\Psi$, and we use symbolic notation $J\Psi = \sum_n J_n\Psi_n$ for the sum of the products of each variable and its conjugate. This extra weight reflects the dependence of the number of states of the reservoir when it exchanges energy with our system.

The entropy of the canonical ensemble can be calculated by using the standard formula:

$$S = -\sum_i p_i \log p_i \tag{36}$$

where the sum runs over all states and p_i is the probability of a state labeled i. Since this is an equilibrium state (by definition) the value of p_i depends only on the value Ψ for this state. The probability of each state is the same up to a factor $e^{J\Psi}$. Since the density of states is large we can replace the sum over i with the integral over Ψ, taking into account the density of states:

$$\sum_i \to \int (d\Psi/\Delta\Psi)e^{S_0}. \tag{37}$$

The *normalized* probability for each microscopic state is given by

$$p_i = \exp(J\Psi - W(J)) \tag{38}$$

where

$$e^{W(J)} = \int (d\Psi/\Delta\Psi)e^{S_0(\Psi)+J\Psi} \tag{39}$$

The entropy of the canonical ensemble in Eq. (36) is then given by

$$S = -\int (d\Psi/\Delta\Psi)e^{S_0(\Psi)+J\Psi-W(J)}(J\Psi - W(J)) = W(J) - J\langle\Psi\rangle \tag{40}$$

Since $\langle\Psi\rangle = dW/dJ$, $S(\langle\Psi\rangle)$ is a Legendre transform of $W(J)$.

C. Partial-equilibrium fluctuations

As the next step let us consider so-called *partial* (or, incomplete) equilibrium states [36], i.e., the states where the equilibrium has been achieved locally, for regions of size of order, say, ℓ, much larger than the microscopic scale (such as mean free path), but the global (complete) equilibrium has not yet been achieved. This is a common situation when the system is observed over *finite* periods of time because the relaxation time grows with the size of the system. The evolution of these states is described by hydrodynamics. We can generalize the calculation of entropy for these states by using additivity of entropy. Each of the subsystems of size ℓ can be treated as an open system in equilibrium with its local environment and its entropy calculated using the formulas for a canonical ensemble above. The total entropy is then the sum of the entropies of the parts. More formally, this means considering microscopic state probabilities p_i as functionals of the (slowly varying on the scale ℓ) fields Ψ, and replacing the integrals in Eq. (40) with the path integrals. We thus find the entropy of a partial equilibrium state as a functional of slowly varying $\bar{\Psi} \equiv \langle\Psi\rangle$:

$$S_1[\bar{\Psi}] = -\int \mathcal{D}\Psi e^{S_0[\Psi]+J\Psi-W[J]}(J\Psi - W[J]) = W[J] - J\langle\Psi\rangle \tag{41}$$

where $\mathcal{D}\Psi$ is the functional integral measure and $J\Psi$ now denotes the sum over the variables as before as well as the integral over space (i.e., sum over the locally equilibrated subsystems):

$J\Psi = \sum_n \int_{\boldsymbol{x}} J_n(\boldsymbol{x})\Psi_n(\boldsymbol{x})$. The entropy of a partial equilibrium state is thus given by the Legendre transform of the "one-particle irreducible" (1PI) generating functional $W[J]$ of the correlation functions of Ψ.

Finally, we must realize that the partial-equilibrium state we described above is still a special case and in many important situations is inadequate for describing a real system out of equilibrium even on the time scales sufficient for local equilibration. Since only local equilibration had enough time to occur, the profile of Ψ in the system on scales longer than ℓ is still different from equilibrium (constant in space) value. Also, each local value of $\bar{\Psi}$ is different in different members of the ensemble. In the states we just described these fluctuations within the ensemble are, however, completely determined by $\bar{\Psi}$ (or, alternatively, J) and the density of states. This property of the ensemble is, however, unrealistic in many important cases when not only $\bar{\Psi}$, but also its variations in space on scales longer than ℓ between different members of the ensemble are not what they should be in equilibrium. The relaxation time of such long-wavelength fluctuations is of the same order as the relaxation time for long-wavelength inhomogeneities of $\bar{\Psi}$ itself and thus these fluctuations would typically be out of equilibrium at these time scales.

In other words, we must consider states where not only one-point function $\bar{\Psi} = \langle\Psi\rangle$, but also the two-point functions $\langle\Psi_n(\boldsymbol{x})\Psi_m(\boldsymbol{y})\rangle$, which we shall denote symbolically as $\langle\Psi\Psi\rangle$, and, in general, higher order correlation functions are still away from equilibrium values. In other words, the state should be characterized by the probability *functional* $p_i[\Psi]$, which is not completely determined by the average value $\bar{\Psi}$, but can be an arbitrary functional, which will evolve in time as it approaches the equilibrium form.

To formalize this observation, we recall that the equilibrium probability of the microscopic state we considered is given by equation (38). To allow for a state with arbitrary value of a two-point function we can consider the probability in the form:

$$p_i = \exp(J\Psi + \frac{1}{2}\Psi K\Psi - W_2[J,K]) \tag{42}$$

where we introduced an arbitrary quadratic form matrix/operator K to parameterize the deviation of the probability distribution from equilibrium. The normalization of the probability is given by

$$e^{W_2[J,K]} = \int \mathcal{D}\Psi e^{S_0[\Psi]+J\Psi+\frac{1}{2}\Psi K\Psi} \tag{43}$$

where $\Psi K \Psi$ denotes $\int_{\boldsymbol{xy}} \sum_{nm} \Psi_n(\boldsymbol{x}) K_{nm}(\boldsymbol{x}, \boldsymbol{y}) \Psi_m(\boldsymbol{y})$. The entropy of such a partial equilibrium state is given by the standard formula Eq. (36):

$$S_2 = -\sum_i p_i \log p_i = \int \mathcal{D}\Psi e^{S_0[\Psi] + J\Psi + \frac{1}{2}\Psi K \Psi - W_2[J,K]} (J\Psi + \frac{1}{2}\Psi K \Psi - W_2[J, K])$$
$$= W_2[J, K] - J\langle\Psi\rangle - \frac{1}{2}\langle\Psi K \Psi\rangle \quad (44)$$

We find that the entropy is the Legendre transform of the 2PI generating functional $W_2[J, K]$. It should be possible to generalize this to higher point correlators, and this should be important to do in order to study higher moments of fluctuations near the critical point. We defer this to further work.

Here we focus on states described by one- and two-point functions. It is convenient to introduce a correlator

$$G = \langle\Psi\Psi\rangle - \bar{\Psi}\bar{\Psi} \quad (45)$$

in terms of which

$$\langle\Psi K \Psi\rangle = \operatorname{tr} KG + \bar{\Psi} K \bar{\Psi} \quad (46)$$

It is convenient to substitute Eq. (46) into Eq. (44):

$$S_2 = W_2[J, K] - J\langle\Psi\rangle - \frac{1}{2}\bar{\Psi} K \bar{\Psi} - \frac{1}{2} \operatorname{tr} KG \quad (47)$$

and express S_2 as a functional of $\bar{\Psi}$ and G using implicit equations

$$\bar{\Psi} = \frac{\delta W_2}{\delta J} \qquad \text{and} \qquad G + \bar{\Psi}\bar{\Psi} = 2\frac{\delta W_2}{\delta K} \quad (48)$$

It is useful to note that the derivatives of $S_2[\bar{\Psi}, G]$ are given by

$$\frac{\delta S_2}{\delta \bar{\Psi}} = -J \qquad \text{and} \qquad 2\frac{\delta S_2}{\delta G} = -K \quad (49)$$

Therefore, one can think of K as a thermodynamic restoring force bringing the system back to equilibrium.

To be more explicit, we shall evaluate W_2 in Eq. (43) in the saddle point approximation (which should be appropriate in the regime $\ell \gg \xi$ due to central limit theorem):

$$W_2[J, K] \approx S_0[\bar{\Psi}] + J\bar{\Psi} + \frac{1}{2}\bar{\Psi} K \bar{\Psi} - \frac{1}{2} \log \det(\mathrm{C} - \mathrm{K}), \quad (50)$$

where we introduced quadratic form matrix/operator

$$C = -\frac{\delta^2 S_0}{\delta\bar{\Psi}\delta\bar{\Psi}}. \quad (51)$$

The value of $\bar\Psi$ is determined by J and K via the saddle-point equation

$$\frac{\delta S_0}{\delta\bar\Psi} + J + K\bar\Psi = 0. \tag{52}$$

Substituting Eq. (50) into Eq. (44) and using (46) we find

$$S_2[\bar\Psi, G] \approx S_0[\bar\Psi] - \frac{1}{2}\operatorname{tr} KG - \frac{1}{2}\log\det(\mathrm{C}-\mathrm{K}) \tag{53}$$

To eliminate K in favor of G we apply Eq. (48) to Eq. (47) to find

$$G = (C-K)^{-1} \tag{54}$$

which we substitute into Eq. (54)

$$S_2[\bar\Psi, G] \approx S_0[\bar\Psi] - \frac{1}{2}\operatorname{tr}(CG-1) + \frac{1}{2}\log\det\mathrm{G}. \tag{55}$$

It is more convenient to express microscopic action S_0 in terms of the 1PI effective action S_1 defined in Eq. (41). It is easy to see that $S_1 = S_2|_{K=0}$, i.e.

$$S_1 \approx S_0 + \frac{1}{2}\log\det\mathrm{C} \tag{56}$$

and thus

$$S_2[\bar\Psi, G] \approx S_1[\bar\Psi] - \frac{1}{2}\operatorname{tr}(CG-1) + \frac{1}{2}\log\det\mathrm{CG}. \tag{57}$$

One can see that $S_2 \le S_1$. The entropy is maximized when $G = C^{-1}$, i.e., when the fluctuations (characterized by the 2-point function G) are in equilibrium ($K = 0$).

To understand this result physically it is useful to keep in mind that entropy is a measure of the uncertainty of the system. Since larger fluctuations mean larger uncertainty, the entropy should increase with G. The last, logarithmic, term in Eq. (57) describes this. One can think of $\sqrt{\det\mathrm{G}}$ as a measure of the “spread” of the thermodynamic state probability distribution over the microscopic states, and the logarithm of it is the entropy. However, the increase in the magnitude of fluctuations around equilibrium comes at an expense: due to the convexity of the entropy the average entropy of a state decreases when the fluctuations get larger, i.e., $\langle S\rangle = S_0 + \delta^2 S/(\delta\Psi\delta\Psi)\langle\delta\Psi\delta\Psi\rangle < S_0$. This effect is described by the second term in Eq. (57). The balance of these two effects leads to the maximum at the equilibrium value of the fluctuations given by $G = C^{-1}$.

D. Scale separation and mode distribution function

Let us consider a system in a partial-equilibrium state where the equilibrium is "complete" only for subsystems of a macroscopic size ℓ (or smaller). If $\ell \gg \xi$, in such a state the contribution of fluctuations is suppressed according to the central limit theorem, where ξ is the correlation length of fluctuations. This separation of scales $\ell \gg \xi$ is also reflected in the dependence of the fluctuation correlator $\bar{G}(\boldsymbol{x}_1, \boldsymbol{x}_2) = C^{-1}(\boldsymbol{x}_1, \boldsymbol{x}_2)$ on $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. Indeed, in the fully equilibrated state ($\ell = \infty$), which is homogeneous, the local value of Ψ is position-independent and the two-point correlator, $\bar{G}(\boldsymbol{x}_1, \boldsymbol{x}_2) = C^{-1}(\boldsymbol{x}_1, \boldsymbol{x}_2)$, depends only on the difference $\boldsymbol{x}_1 - \boldsymbol{x}_2$. The typical scale for this dependence is $|\boldsymbol{x}_1 - \boldsymbol{x}_2| \sim \xi$. In the partial equilibrium states $\bar{\Psi}(\boldsymbol{x})$ depends on $\boldsymbol{x}$ very slowly and similarly the dependence of $\bar{G}(\boldsymbol{x}_1, \boldsymbol{x}_2)$ on $(\boldsymbol{x}_1 + \boldsymbol{x}_2)/2$ is slow, and is associated with the scales longer than ℓ which is much longer than the scale ξ of $|\boldsymbol{x}_1 - \boldsymbol{x}_2|$ dependence.

This separation of scales is most conveniently exploited by performing Wigner transform of G, i.e., Fourier transform w.r.t. $\boldsymbol{x}_1 - \boldsymbol{x}_2$.

$$G_{\boldsymbol{Q}}(\boldsymbol{x}) = \int_{\Delta \boldsymbol{x}} G\left(\boldsymbol{x} + \frac{\Delta \boldsymbol{x}}{2}, \boldsymbol{x} - \frac{\Delta \boldsymbol{x}}{2}\right) e^{i\boldsymbol{Q}\Delta \boldsymbol{x}}. \tag{58}$$

The states we consider are characterized by $G_{\boldsymbol{Q}}$ which vary slowly with $\boldsymbol{x}$, compared to the scale set by characteristic value of Q.

For such states the 2PI action simplifies. The functional trace in Eq. (57) becomes an integral over $\boldsymbol{x}$ and over $\boldsymbol{Q}$ of a matrix function of $G_{\boldsymbol{Q}}(\boldsymbol{x})$, i.e,

$$S_2[\bar{\Psi}, G] \approx S_1[\bar{\Psi}] + \frac{1}{2} \int_{\boldsymbol{x}} \int_{\boldsymbol{Q}} \mathrm{Tr}\, \left(1 - C_{\boldsymbol{Q}} G_{\boldsymbol{Q}} + \log C_{\boldsymbol{Q}} G_{\boldsymbol{Q}}\right). \tag{59}$$

where $C_{\boldsymbol{Q}} = \bar{G}_{\boldsymbol{Q}}^{-1}$ is the Wigner transform of C from Eq. (51). We use intuitive short-hand notations for spatial and wave-vector integrations respectively:

$$\int_{\boldsymbol{x}} \ldots = \int d^3\boldsymbol{x} \ldots \quad \text{and} \quad \int_{\boldsymbol{Q}} \ldots \equiv \int d^3\left(\boldsymbol{Q}/2\pi\right) \ldots . \tag{60}$$

It is notable that a similar scale separation occurs in kinetic theory where the particle distribution function is also a Wigner transform of a two-point correlator. Mathematical similarity notwithstanding, the physical origin of the separation is different in that case – the slowness of the collision rate compared to typical particle momenta. In our case such a Wigner transform would be more appropriately called a *mode* distribution function, since the variable $\boldsymbol{Q}$ is a wave-vector of the mode and not a particle momentum. The integral over variables $\boldsymbol{x}$ and $\boldsymbol{Q}$ is the phase-space integral. The evolution equation for the mode distribution function is similar to a kinetic equation, and one could use this term to describe it, with the understanding that it does not describe particle kinetics, but rather the mode kinetics.[6]

IV. KINETICS OF FLUCTUATIONS

A. Relaxation equations

The evolution of the system we are describing is governed by the second law of thermodynamics, i.e., the evolution proceeds towards the maximum of the entropy under (energy-momentum, charge, etc) conservation constraints. Hydrodynamic equations are essentially these constraints (supplemented with constitutive equations). We can use the same approach to describe the evolution of the 2-point (as well as 1-point) functions towards the maximum of the 2PI entropy S_2. In order to do that, we need to supplement the usual set of hydrodynamic equations for the conserved densities $\bar{\Psi}$, with the equations which describe the time evolution of the 2-point functions G. Following the same logic, we can write these equations as relaxation equations. Since we focus on the quadratic fluctuations, we would need linearized form of the hydrodynamic equations. Writing the hydrodynamic equations (in matrix notations) in the Onsager form and linearizing them we find

$$D\Psi = -\gamma J = -L\delta\Psi + \mathcal{O}(\delta\Psi^2)\,, \tag{61}$$

where γ is the Onsager matrix and L is the linear evolution matrix which are related via

$$\gamma = LC^{-1}\,. \tag{62}$$

The explicit form of these matrices is presented in Section IV D.

We can now write the linearized evolution equation for the 2-point functions in matrix notations as

$$\partial_t G = -L(G-\bar{G}) - (G-\bar{G})L^\dagger + \mathcal{O}(G-\bar{G})^2\,, \tag{63}$$

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[6] For the sound channel the similarity is not just mathematical, but also physical: The corresponding matrix element of $G_{\boldsymbol{Q}}$ can be identified with the distribution function of phonon quasiparticles, and the corresponding equation with the kinetic (Boltzmann) equation for the phonon quasiparticles as in Ref. [11].

where $\bar{G} = C^{-1}$ is the equilibrium value of G. This equation is easy to derive from equation (61) with a noise term.

In order to determine the full non-linear form of Eq. (63) we need to use the expression for the entropy Eq. (57) and the second law of thermodynamics. In order for the law to hold we need the relaxation equation to have the Onsager form, i.e., the relaxation rate should be proportional to the thermodynamic force

$$K = -2\frac{\delta S_2}{\delta G} = -\left(G^{-1} - \bar{G}^{-1}\right) = C\left(G - \bar{G}\right)C + \mathcal{O}(G - \bar{G})^2\,, \tag{64}$$

where we used $C = \bar{G}^{-1}$ (factor of 2 is due to symmetry of G). It is easy to see that the equation which obeys this condition and agrees with Eq. (63) has the form:

$$\partial_t G = -\gamma K C^{-1} - C^{-1} K \gamma = \gamma\left(G^{-1}\bar{G} - 1\right) + \left(\bar{G}G^{-1} - 1\right)\gamma\,. \tag{65}$$

So far the discussion of the kinetics was general and did not assume separation of scales, $Q \gg 1/\ell$, discussed in the previous subsection. Such scale separation simplifies equations in terms of the Wigner transformed functions. Symbolically the form of equations in terms of $G_{\boldsymbol{Q}}$ remains the same, but we can replace all matrix/operators (G, C, L, γ) by their Wigner transforms evaluated at the common wave-vector $\boldsymbol{Q}$.

B. Mode decomposition

One can solve the linear equation (61) by decomposing the set of variables Ψ into the right eigenmodes of the operator L. Then the solution is given by the sum of projectors

$$L = \sum_n \lambda_n P_n, \qquad \text{where } P_n \equiv \psi_n \theta_n^\dagger \tag{66}$$

and ψ and θ are the right an left eigenmodes of L respectively:

$$L\psi_m = \lambda_m \psi_m, \qquad \theta_n^\dagger L = \lambda_n \theta_n^\dagger \tag{67}$$

Although the right and left eigen*values* are the same (roots of the characteristic polynomial), the right and left eigen*vectors* are, in general, different (i.e., $P_n^\dagger \neq P_n$) and not orthogonal among themselves. Instead they form a set of dual bases which are *mutually* orthogonal, i.e.,

$$\theta_n^\dagger \psi_m = \delta_{mn}, \qquad \text{or } P_n P_m = \delta_{mn} P_n\,. \tag{68}$$

One can show this by multiplying the first and the second of equations (67) by $\theta_n^\dagger$ on the left and by ψ_n on the right respectively.

Although L is not hermitian, the Onsager matrix/operator in Eq. (62) is, which means

$$LC^{-1} = C^{-1}L^\dagger \,. \tag{69}$$

Therefore matrices C and C^{-1} (which are hermitian) can be written as

$$C = \sum_n c_n \theta_n \theta_n^\dagger, \qquad C^{-1} = \sum_n c_n^{-1} \psi_n \psi_n^\dagger \,. \tag{70}$$

The result for C^{-1} can be derived by "sandwiching" Eq. (69) between $\theta_m^\dagger$ and θ_n and using Eq. (67) and Eq. (68), while the result for C – by doing the same to equation $CL = L^\dagger C$ using $\psi_m^\dagger$ and ψ_n. [7]

If the matrix L were hermitian, this would reduce to a familiar result that C can be diagonalized in the same basis as L since Eq. (69) would become commutativity condition.

C. Projection onto the slowest mode

Our discussion of fluctuations in this section is general. In Section V we shall consider a special case where a *parametric* separation of scales appears between the relaxation rates of different modes of fluctuations, such as the case near a critical point. The slowest relaxing mode in this case is the heat diffusion at constant pressure (see Section IV D), whose relaxation time ℓ^2/D_p is longest because the diffusion coefficient D_p vanishes at the critical point. This is due to the divergence of heat capacity $c_p \sim \xi^2$ and the relation $D_p = \kappa/c_p$, where $\kappa = \lambda\,(\beta w/n)^2$ is the heat conductivity. As a result $D_p \sim \xi^{-1}$, even despite the divergence of $\lambda \sim \xi$ (we round all powers of ξ to integer values for simplicity).

In such a situation one may consider a partial-equilibrium state where the "complete" (local) equilibrium of all modes is achieved on length scales ℓ, except for the slowest mode, whose equilibrium still needs more time to be reached. In this case we can neglect the fluctuations of all equilibrated modes since their contribution is typically suppressed by

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[7] The coefficients c_n can be chosen arbitrarily by adjusting the normalization of ψ_n and θ_n, while still preserving Eqs. (68). One could, for example, choose $c_n \equiv 1$. However, there are other considerations which make certain other choices preferable.

central limit theorem by a factor $(\xi/\ell)^3 \ll 1$.[8] The unequlibrated fluctuation mode can then be treated using the formalism we introduce in the next section.

The slowest mode of equation (61), ψ_1, corresponds to the smallest (in terms of its real part) eigenvalue of L, λ_1.[9] I.e., the slowest mode is the projection $P_1\Psi = \psi_1(\theta_1^\dagger\Psi)$. It is easy to see that the relaxation rates in Eq. (63) are given by $\lambda_n + \lambda_m$, i.e., the slowest relaxation rate corresponds to $n = m = 1$ and the slowest mode is given by the projection $P_1(G - C^{-1})P_1^\dagger$. If we neglect all other (faster) modes, the matrix G will take the form:

$$G = C^{-1} + P_1(G - C^{-1})P_1^\dagger = C^{-1} + (\phi - c_1^{-1})\psi_1\psi_1^\dagger = (1 + (\phi c_1 - 1)P_1)\, C^{-1} \tag{71}$$

where we introduced

$$\phi \equiv \theta_1^\dagger G \theta_1 \tag{72}$$

and used Eq. (70).

Since G is not just a discrete matrix, but an operator whose kernel $G(\boldsymbol{x},\boldsymbol{y})$ is a 5×5 matrix, the spectrum of modes is not discrete, but a continuous spectrum of hydrodynamic modes. For the partial equilibrium states we consider, where the rate of variation w.r.t. $\boldsymbol{x}+\boldsymbol{y}$ is much slower than w.r.t. $\boldsymbol{x}-\boldsymbol{y}$, i.e., $1/\ell \ll Q$, the problem simplifies, as we have seen in Eq. (59), where the action can be written as a local functional of the Wigner transform $G_{\boldsymbol{Q}}(\boldsymbol{x})$. We can then consider the lowest eigenmode of $G_{\boldsymbol{Q}}$ locally and define the corresponding projection

$$\phi_{\boldsymbol{Q}}(\boldsymbol{x}) \equiv \theta_1^\dagger G_{\boldsymbol{Q}}(\boldsymbol{x})\theta_1 \tag{73}$$

which is related to $G_{\boldsymbol{Q}}$ as ϕ is related to G in Eq. (71). Substituting this expression for $G_{\boldsymbol{Q}}$ into the 2PI entropy in Eq. (59) we find

$$S_2[\bar\Psi, G] \approx S_1[\bar\Psi] + \frac{1}{2}\int_{\boldsymbol{x}}\int_{\boldsymbol{Q}} \left(1 - \phi_{\boldsymbol{Q}}/\bar\phi_{\boldsymbol{Q}} + \log(\phi_{\boldsymbol{Q}}/\bar\phi_{\boldsymbol{Q}})\right) \tag{74}$$

where

$$\bar\phi_{\boldsymbol{Q}}(\boldsymbol{x}) \equiv c_1^{-1} = \theta_1^\dagger \bar G_{\boldsymbol{Q}}\theta_1\,. \tag{75}$$

Here $\phi_{\boldsymbol{Q}}(\boldsymbol{x})$ is an additional degree of freedom whose local equilibrium $\bar\phi_{\boldsymbol{Q}}$ depends on $\boldsymbol{x}$ via the dependence on local equilibrium value of hydrodynamic variables (1-point functions) $\bar\Psi$.

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[8] There are, of course, special measurements where such fluctuations give leading contributions, e.g., in long-time tails of correlators [11, 37]. In this case, since $\ell^2 \sim t$, the suppression factor $(\xi/\ell)^3 \sim t^{-3/2}$ leads to the characteristic half-integer power tail.

[9] It is also useful to note that due to $L = \gamma C$ that mode is also the flattest direction of the quadratic form C. A simple explicit example of this could be found in Ref.[26].

In order to write the kinetic equation of the slowest mode $\phi_{\boldsymbol{Q}}$ we would need to eliminate faster modes using the equations of motion such as Eq. (65). For linear equations this would simply amount to the projection we described above. However due to non-linearities this procedure is more complicated and should essentially capture the known physics of the "mode-coupling" [21, 38, 39]. Here we shall use the result of the mode-coupling calculations in [21, 39] to write the resulting equation as

$$D\phi_{\boldsymbol{Q}} = -\gamma_\pi(\boldsymbol{Q})\pi_{\boldsymbol{Q}}\,, \tag{76}$$

where

$$\pi_{\boldsymbol{Q}} \equiv -\frac{\delta S_2}{\delta \phi_{\boldsymbol{Q}}} = \frac{1}{2}\left(\bar{\phi}_{\boldsymbol{Q}}^{-1} - \phi_{\boldsymbol{Q}}^{-1}\right)\,. \tag{77}$$

The coefficient $\gamma_\pi(\boldsymbol{Q})$ can be related to the relaxation rate $\Gamma(\boldsymbol{Q})$ which is known from the mode-coupling calculations and we shall discuss it below (see Eq. (95)). We have also omitted the $A_\phi(\boldsymbol{Q})$ term because we shall choose the slowest mode to be s/n (see Section IV D), which has zero compressibility, i.e., $D(s/n) = 0 \cdot \theta + \ldots$. We remind the reader that the choice is arbitrary (see Appendix C), and a more detailed calculation would be needed to determine what the optimal choice of the slow mode should be, and if that choice has nonzero A_ϕ. In addition, one should also consider Hydro+ terms with $\lambda_{\pi\pi}$ and $\lambda_{\alpha\pi}$, but we shall defer this as well as a more nuanced choice of $\phi_{\boldsymbol{Q}}$ and the derivation of Eq. (76) to future work.

D. The slowest mode

For completeness, we present here an explicit form of the matrix operators L, C and γ and identify the slowest mode (or more precisely, the branch of modes).

Writing linearized hydrodynamics in coordinates $\delta\Psi = (w\delta\boldsymbol{v}(\boldsymbol{k}), \delta\varepsilon(\boldsymbol{k}), \delta n(\boldsymbol{k}))$ we find:

$$L = \begin{pmatrix} \mathbb{V} & i\boldsymbol{k}p_\varepsilon & i\boldsymbol{k}p_n \\ 1 & 0 & 0 \\ n/w & \boldsymbol{k}^2\lambda\alpha_\varepsilon & \boldsymbol{k}^2\lambda\alpha_n \end{pmatrix} \tag{78}$$

where subscripted thermodynamic variables p_ε, p_n, α_ε, α_n denote derivatives of the variable with respect to the subscript variable (ε or n) while the other variable is held fixed (e.g.,

$p_\varepsilon \equiv (\partial p/\partial \varepsilon)_n)$ and $\mathbb{V} = \eta \boldsymbol{k}^2 + \left(\zeta + \frac{1}{3}\eta\right) \boldsymbol{k} \otimes \boldsymbol{k}$ is a matrix of viscous relaxation.

$$C = \begin{pmatrix} \beta/w & 0 & 0 \\ 0 & \beta_\varepsilon & \beta_n \\ 0 & -\alpha_\varepsilon & -\alpha_n \end{pmatrix} \tag{79}$$

which is symmetric by virtue of the Maxwell relation $\beta_n = -\alpha_\varepsilon$. Therefore the Onsager matrix is given by

$$\gamma = LC^{-1} = \frac{1}{\beta} \begin{pmatrix} w\mathbb{V} & i\boldsymbol{k}w & i\boldsymbol{k}n \\ -i\boldsymbol{k}w & 0 & 0 \\ -i\boldsymbol{k}n & 0 & \boldsymbol{k}^2\beta\lambda \end{pmatrix} \tag{80}$$

where we performed some standard thermodynamic Jacobian calculus to simplify the result.

To lowest order in k (ideal hydrodynamics), the smallest eigenvalue of matrix L is 0 and the corresponding (unnormalized) right and left eigenvectors are:

$$\psi_1 \sim (\boldsymbol{0}, 1, -p_\varepsilon/p_n) \ , \quad \theta_1 \sim (\boldsymbol{0}, 1, -w/n) \ . \tag{81}$$

Since $-p_\varepsilon/p_n = (\partial n/\partial \varepsilon)_p$, this simply means that in the mode $\delta\Psi \sim \psi_1$ the $\delta\varepsilon$ and δn fluctuations are such that $\delta p = 0$, i.e., pressure does not fluctuate. The projection on that mode from an arbitrary fluctuation, i.e, $\theta_1 \delta\Psi \sim \delta\varepsilon - (w/n)\delta n$ is proportional to the fluctuation of s/n since:

$$\delta\left(\frac{s}{n}\right) = \frac{\beta}{n}\left(\delta\varepsilon - \frac{w}{n}\delta n\right) \ . \tag{82}$$

Thus we can describe the slowest mode as the diffusion of entropy per baryon at fixed pressure.

Therefore, the variable $\phi_{\boldsymbol{Q}}$ defined by projection on the slowest mode in Eq. (73) can be identified with the Wigner transform of the correlator of s/n:

$$\phi_{\boldsymbol{Q}}(\boldsymbol{x}) \sim \int_{\Delta\boldsymbol{x}} \left\langle \delta m \left(\boldsymbol{x} + \frac{\Delta\boldsymbol{x}}{2}\right) \delta m \left(\boldsymbol{x} - \frac{\Delta\boldsymbol{x}}{2}\right) \right\rangle e^{i\boldsymbol{Q}\Delta\boldsymbol{x}} \ . \tag{83}$$

where we defined

$$m \equiv \frac{s}{n}. \tag{84}$$

The normalization of $\phi_{\boldsymbol{Q}}$ is arbitrary, as can be seen from the expression for the 2PI action in Eq. (74), where it cancels. Moreover, it is worth noting that any function of $\phi_{\boldsymbol{Q}}$ can be chosen to represent slow relaxation, due to the reparameterization invariance of Hydro+. However, one needs to be aware that the value of compressibility A_ϕ may depend on that choice (see Appendix C).

V. HYDRO+ NEAR THE QCD CRITICAL POINT

A. Formulation

In Section IV we identified slow degree(s) of freedom near the QCD critical point $\phi_{\boldsymbol{Q}}$ and derived partial-equilibrium entropy density $s_{(+)}(\varepsilon, n, \phi)$:

$$s_{(+)}(\varepsilon, n, \phi_{\boldsymbol{Q}}) = s\,(\varepsilon, n) + \frac{1}{2}\int_{\boldsymbol{Q}} \left\{ \log\frac{\phi_{\boldsymbol{Q}}}{\bar{\phi}_{\boldsymbol{Q}}(\varepsilon, n)} - \frac{\phi_{\boldsymbol{Q}}}{\bar{\phi}_{\boldsymbol{Q}}(\varepsilon, n)} + 1 \right\}, \tag{85}$$

where $s(\epsilon, n)$ is the ordinary equilibrium entropy and $\bar{\phi}_{\boldsymbol{Q}}(\varepsilon, n)$ is the local equilibrium value of the non-hydrodynamics slow mode $\phi_{\boldsymbol{Q}}$ determined by local values of ε and n. Similarly to ε and n being energy and charge density in the local rest frame defined by 4-velocity u^μ, $\boldsymbol{Q}$ is also a wave vector in the same frame, thus ensuring Lorentz invariance of $s_{(+)}$. We are now ready to write the Hydro+ equations which we propose to describe the evolution near the critical point. The equations are usual conservation laws $\partial_\mu T^{\mu\nu} = 0$, $\partial_\mu J^\mu = 0$ and a relaxation rate equation (76) for $\phi_{\boldsymbol{Q}}$.

The constitutive relation for $T^{\mu\nu}$ and J^μ now read:

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu + p_{(+)}\, g_\perp^{\mu\nu} - \eta_{(+)} \left(\partial_\perp^\mu u^\nu + \partial_\perp^\nu u^\mu - \frac{2}{3} g_\perp^{\mu\nu}\theta \right) - \zeta_{(+)} g_\perp^{\mu\nu}\,\theta\,, \tag{86}$$

$$J^\mu = n\, u^\mu + \lambda\, \partial_\perp^\mu\, \alpha_{(+)}\,. \tag{87}$$

Here $p_{(+)}$ is related to $s_{(+)}$ by

$$\beta_{(+)} p_{(+)} = s_{(+)} - \beta_{(+)}\epsilon + \alpha_{(+)}\, n\,, \tag{88}$$

In this subsection we shall discuss the necessary ingredients for Hydro+. Our central ingredient is the partial-equilibrium equation of state, which depends on the equilibrium entropy $s(\epsilon, n)$ and the equilibrium value of $\bar{\phi}_{\boldsymbol{Q}}(\varepsilon, n)$. Here, $s(\varepsilon, n)$ is the complete-equilibrium equation of state which includes the thermodynamic behavior near the critical point. For QCD, the equation of state in the relevant region, i.e., at finite baryon density, is not reliably known from the first-principle lattice simulation. An approach which is being pursued is to use an efficient paramaterization of equation of state which, on the one hand matches the reliable lattice data at small density (chemical potential) and on the other hand incorporates correct universal critical behavior, in order to minimize the number of free parameters to be determined by comparing the simulation with experimental data [40].

In this work we shall take $\phi_{\boldsymbol{Q}}$ to be the Wigner transform of the correlator $\langle\delta m(\boldsymbol{x})\delta m(\boldsymbol{y})\rangle$. We shall assume the separation of scales $1/\ell \ll Q \sim \xi^{-1}$, where ℓ is the scale of spatial variation of local values of ε, n, u^μ as well as $\phi_{\boldsymbol{Q}}$. The local equilibrium value, $\bar{\phi}_{\boldsymbol{Q}}$ is given by the Wigner transform of the equilibrium correlator $\langle\delta m(\boldsymbol{x})\delta m(\boldsymbol{0})\rangle$ at given ε and n. In the scaling regime (ξ much larger than microscopic scale, such as $1/T$) the dependence of this quantity on Q and ξ must enter through a universal scaling function, i.e.,

$$\bar{\phi}_{\boldsymbol{Q}} = \int_{\Delta\boldsymbol{x}} e^{i\boldsymbol{Q}\cdot\Delta\boldsymbol{x}}\langle\delta m(\Delta\boldsymbol{x})\delta m(\boldsymbol{0})\rangle = \bar{\phi}_{\boldsymbol{0}} f_2(Q\xi,\Theta) \tag{89}$$

where f_2 is a universal function of two scaling variables with $\Theta = \Theta(\varepsilon, n)$ denoting the variable which, similarly to ξ, depends on the local values of ε and n, but in contrast to ξ is invariant under scaling (e.g., parameter θ in $R\theta$ parameterization of the universal scaling equation of state [41]).

The value of $\bar{\phi}_{\boldsymbol{0}}$ is given by the magnitude of the fluctuation of $m = \int_{\boldsymbol{x}} m(\boldsymbol{x})/V$. This can be found from the standard text-book analysis of the equilibrium fluctuations and is given by

$$\bar{\phi}_{\boldsymbol{0}} = V\langle(\delta m)^2\rangle = \frac{c_p}{n^2} \tag{90}$$

(see Appendix D for derivation).

The universal function f_2 is normalized as $f_2(0,\Theta) = 1$ and can be determined, in principle, using methods described in, e.g., Ref.[42], or by a lattice computation. Similarly to the equation of state $s(\varepsilon, n)$ it is a *static* thermodynamic quantity. The asymptotic behavior of $f(x,\Theta)$ at large and small x is given by

$$f_2 \to 1 + a_1 x^2 + \ldots, \quad x \ll 1, \tag{91}$$

$$f_2 \to x^{-2+\eta}(b_1 + b_2 x^{-(1-\alpha)/\nu} + b_3 x^{-1/\nu}\ldots), \quad x \gg 1, \tag{92}$$

where coefficients a_i, b_i are functions of Θ (see Ref. [43]).

For practical purposes a simplified expression independent of Θ

$$f_2(x) \approx (1+x^2)^{-1} \tag{93}$$

often referred to as Ornstein-Zernike (OZ) form [7, 17] could provide a reasonable approximation in limited applications. However, even though it gives correct small-x asymptotics, the incorrect large x asymptotics leads to incorrect large-ω behavior of frequency-dependent bulk viscosity which we discuss in Section V B.

The coefficient $\gamma_\pi(\boldsymbol{Q})$ in Eq. (76) is given, similarly to Eq. (23), by

$$\gamma_\pi(\boldsymbol{Q}) = \phi_\pi(\boldsymbol{Q})\Gamma(\boldsymbol{Q}) = 2\bar{\phi}^2_{\boldsymbol{Q}}\Gamma(\boldsymbol{Q})\,, \tag{94}$$

where we defined $\phi_\pi(\boldsymbol{Q}) = (\partial\phi_{\boldsymbol{Q}}/\partial\pi_{\boldsymbol{Q}})_{\varepsilon n}$ and used Eq. (77).

The momentum-dependent relaxation rate of the critical slow mode $\Gamma(\boldsymbol{Q})$ for model H has been computed by Kawasaki [17] (see also Appendix 6B in Ref.[22] or Appendix B of Ref. [44] for derivations):

$$\Gamma(\boldsymbol{Q}) = 2\Gamma_\xi\, K(Q\xi)\,, \tag{95}$$

where

$$K(x) = (3/4)\left[1 + x^2 + \left(x^3 - x^{-1}\right)\,\arctan(x)\right] \tag{96}$$

is sometimes referred to as Kawasaki function.[10] This form describes the experimental data for various fluids over a range of temperatures near the critical point remarkably well (see for example Fig. 4 of Ref. [7]).[11] The characteristic critical slowing down rate is defined as the diffusion rate at wave number ξ^{-1}

$$\Gamma_\xi = D_p\,\xi^{-2} = \frac{T}{6\pi\eta\xi^3} \qquad (z \approx 3)\,, \tag{97}$$

where D_p is thermal (more precisely, entropy per baryon at constant pressure) diffusion constant due to the convection of critical density fluctuations, the physical mechanism of which is described in, e.g., Ref.[7].

Finally, let us discuss the input values for Hydro+ kinetic coefficients $\zeta_{(+)}$, $\eta_{(+)}$ and λ. Since the critical behavior of bulk viscosity ζ (at zero frequency) is now due to the dynamics of additional slow mode $\phi_{\boldsymbol{Q}}$, the input value $\zeta_{(+)}$ will not sensitively depend on the correlation length and should match with the smooth behavior of bulk viscosity away from the critical point. Since shear viscosity η has very weak divergence at the critical point, it is reasonable to neglect this divergence and use $\eta_{(+)}$ interpolated from the smooth behavior away from the critical point.

The implementation of conductivity λ is more subtle. λ diverges as ξ (rounding the exponent to integer) due to the convection of enhanced density fluctuations [7, 17]. Somewhat similar to divergence of ζ, the divergence of λ is due to slowness of relaxation of a

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[10] The factor of 2 relative to Ref.[22] is due to the fact that $\Gamma(\boldsymbol{Q})$ is the relaxation rate of a 2-point function.

[11] The calculation of the Kawasaki function involves function f_2, and it is an example when the Ornstein-Zernike ansatz (93) is adequate.

non-hydrodynamic mode involving transverse momentum and charge density fluctuations. In principle, this effect could be described by an extension of our approach using 2PI formalism by introducing an additional mode. We defer this to future work. As a provisional recipe, consistent with other choices, one could use, near the critical point,

$$\lambda = \left(\frac{nT}{w}\right)^2 c_p D_p = \left(\frac{nT}{w}\right)^2 \frac{Tc_p}{6\pi\eta\xi} \,. \tag{98}$$

B. Frequency dependence of bulk response

In order to illustrate how the Hydro+ formalism described in Section V A works, we shall study the bulk response in this theory. It is useful to keep in mind that the formalism in Section V A is essentially a multi-mode generalization of the Hydro+ theory we discussed in Section II and the results here are easily obtained by generalizing calculation in Section II B. The purpose of this section is to demonstrate that this simpler formalism reproduces known properties derived earlier using a different approach by Kawasaki and Onuki [17, 21], which we briefly review in the next section.

As in Sec. II B, we consider expansion/compression fluctuations around the static and homogeneous background. A straightforward generalization of Eq. (27) leads to the expression for the bulk response function

$$\frac{i\Delta G_R(\omega)}{\omega} = \beta \int_{\boldsymbol{Q}} \frac{p_\pi^2(\boldsymbol{Q})}{\phi_\pi(\boldsymbol{Q})} \frac{1}{\Gamma(\boldsymbol{Q}) - i\omega} \,, \tag{99}$$

where we generalized the definition of p_π, ϕ_π for one slow mode Eqs. (18), (20) to a branch of modes labeled by $\boldsymbol{Q}$:

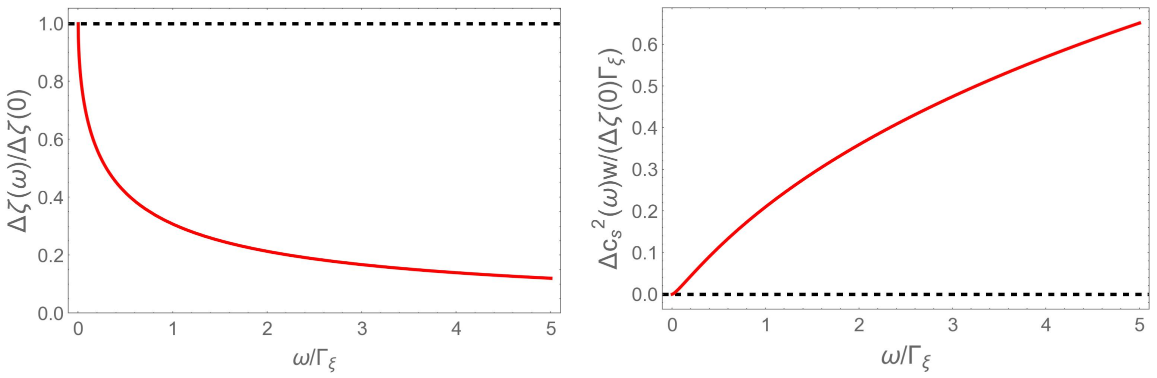
$$p_\pi(\boldsymbol{Q}) \equiv \left(\frac{\delta p_{(+)}}{\delta \pi_{\boldsymbol{Q}}}\right)_{\varepsilon,n} = -\frac{w}{\beta}\left(\frac{\partial \bar{\phi}_{\boldsymbol{Q}}}{\partial \varepsilon}\right)_m \,, \tag{100}$$

$$\phi_\pi(\boldsymbol{Q}) = \left(\frac{\partial \phi_{\boldsymbol{Q}}}{\partial \pi_{\boldsymbol{Q}}}\right)_{\varepsilon,n} = 2\,\bar{\phi}_{\boldsymbol{Q}}^2 \,. \tag{101}$$

In Eq. (100) we have also used a natural generalization of Eq. (B1). Consequently,

$$\frac{i\Delta G_R(\omega)}{\omega} = \frac{w^2}{2\beta} \int_{\boldsymbol{Q}} \left(\frac{\partial \log \bar{\phi}_{\boldsymbol{Q}}}{\partial \varepsilon}\right)_m^2 \frac{1}{\Gamma(\boldsymbol{Q}) - i\omega} \,. \tag{102}$$

Substituting the expression for the equilibrium mode distribution $\bar{\phi}_{\boldsymbol{Q}}$ we can determine the contribution of critical fluctuations to frequency-dependent to bulk viscosity and the



(a) (b)

FIG. 3. (color online) The frequency dependence of the (critical contribution to) bulk viscosity, or $\Delta\zeta = -\mathrm{Im}\, G_R/\omega$ (left); and the speed of sound (stiffness of equation of state), or $\Delta c_s^2 = \mathrm{Re}\, G_R/w$, (right) near the critical point from Eq. (102). The quantities and the frequency are normalized to make plots scale-independent. Dashed horizontal lines illustrate the results from ordinary hydrodynamics extrapolated beyond its range of validity.

frequency dependent equation of state stiffness (sound group velocity) which are related to G_R as

$$\Delta\zeta(\omega) = -\mathrm{Im}\, \Delta G_R/\omega; \quad \Delta c_s^2(\omega) = \mathrm{Re}\, \Delta G_R/w. \tag{103}$$

Using the OZ form in Eq. (93) one obtains the results in agreement with Kawasaki Ref.[39], which do not have the correct large ω behavior. If one uses an ansatz satisfying the asymptotic behavior in Eqs. (91) the correct asymptotic behavior is reproduced, as in Ref. [22].

To illustrate the frequency dependence of the bulk viscosity and stiffness (sound speed) we have chosen the ansatz for $\bar\phi_{\boldsymbol{Q}}$ similar to the one used by Onuki in Ref. [21, 22]. Rather than choosing the function $\bar\phi_{\boldsymbol{Q}}$, we chose ansatz for its derivative $(\partial \log \bar\phi_{\boldsymbol{Q}}/\partial\varepsilon)_m^2 \sim (1+x^2)^{-(1-\alpha)/\nu}$, where $x = Q\xi$, which satisfies the necessary asymptotics following from Eqs. (91). In particular, $(\partial \log \bar\phi_{\boldsymbol{Q}}/\partial\varepsilon)_m \sim Q^{-(1-\alpha)/\nu}$ at large Q, which translates into the large ω asymptotics $G_R \sim \omega^{\alpha/(z\nu)}$ in accordance with Ref.[22], where z is the dynamical critical scaling exponent, $\Gamma_\xi \sim \xi^{-z}$, as in Eq. (97).

In Fig. 3 we show the resulting frequency dependence for $\Delta\zeta$ and Δc_s^2. Note that, as in Section II B, the ordinary hydrodynamics extrapolated beyond its region of validity overpredicts bulk viscosity and underpredicts the stiffness. [12]

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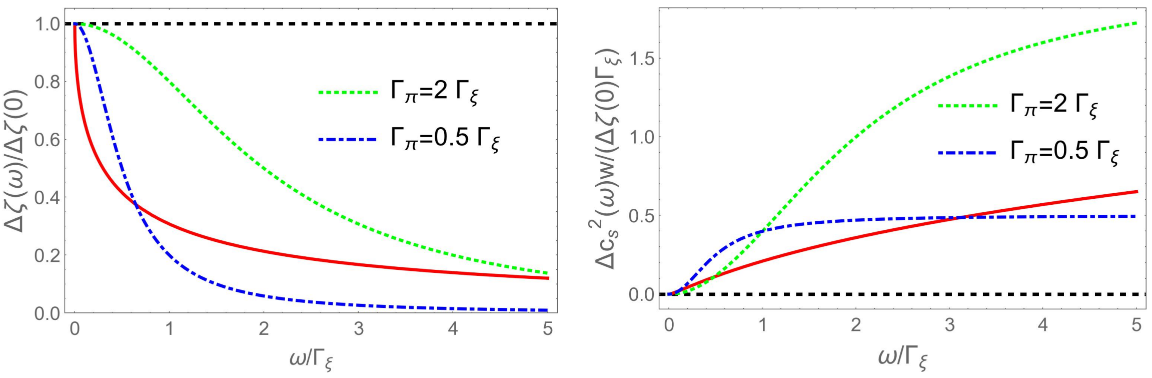
[12] The stiffening at higher frequencies is a counterpart of the well-known effect of softening of the equation



(a) (b)

FIG. 4. (color online) Comparison of the frequency dependence of the bulk viscosity, or $\Delta\zeta$ (left); and the speed of sound (stiffness of equation of state), or Δc_s^2, (right) between the two theories: "kinetic" Hydro+ in Section V A and single-mode Hydro+ in Section II for two choices of the matching scale ratio Γ_π/Γ_ξ. Dashed horizontal lines illustrate the results of ordinary hydrodynamics extrapolated beyond its range of validity.

Although these results are similar to the single-mode Hydro+ theory in Section II B, it is instructive to compare and emphasize the differences. For this purpose we combine the plots from Figs. 1 and Figs. 3 on the same graph. To make the comparison we need to choose what scales in two theories to match. The relevant scale in the single-mode theory is the rate Γ_π of the relaxation of the single mode ϕ. In the "kinetic" Hydro+ theory there is a spectrum of modes, with a characteristic scale given by Γ_ξ. We show two choices of Γ_π-to-Γ_ξ matching corresponding to $\Gamma_\pi/\Gamma_\xi = 0.5$ and 2 in Fig. 4. One can view this comparison as an answer to the question: How well could a single-mode theory match the results of a full "kinetic" Hydro+ approach.

One can see that it is hard to match both large and small frequency behavior of the bulk viscosity, due to completely different asymptotics of $\Delta\zeta(\omega)$ in two theories.[13] The same is true for the stiffness, i.e., Δc_s^2. It is also notable that the choice of Γ_π/Γ_ξ which makes Δc_s^2 agree better, could lead to worse agreement for $\Delta\zeta$. These differences notwithstanding, it is also clear that even a single-mode Hydro+ theory gives better description than naive extrapolation of the ordinary hydrodynamics beyond its range of validity. We can conclude

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of state at $\omega \to 0$, characterized by vanishing of the hydrodynamic sound speed c_s^2. More quantitatively, since $c_s^2 \sim \xi^{-\alpha/\nu}$ the dynamical scaling translates this behavior into high frequency scaling $\omega^{\alpha/(\nu z)}$.

[13] The small-ω behavior $\Delta\zeta \sim -\omega^{-1/2}$ evident in Fig. 3(a) is the half-integer long-time hydrodynamic tail [11, 37, 45]. It cannot be matched by a single-mode theory where $\Delta\zeta$ is analytic at zero frequency.

that a single-mode theory could be used as a rough illustration of some features of the critical slowing down, but it cannot describe this phenomenon fully.

C. Comparison to a loop calculation

In the previous section, we have evaluated critical mode contribution to the retarded Green's function using Hydro+ formalism. To elucidate the correspondence between this formalism and an earlier calculation by Onuki [21, 22], and to make the paper self-contained, we shall provide here a sketch of the computation of the same quantity using the approach of Refs. [21, 22] (see also Ref. [16, 46]) in this subsection.

The starting point of Ref. [21] is the relation between $\Delta G_R(\omega)$ and non-linear non-equilibrium pressure Δp:

$$\frac{i\Delta G_R(\omega)}{\omega} = \beta \int_0^\infty dt\, e^{i\omega\, t} \int_{\boldsymbol{x}} \langle\, \Delta p\,(t,\boldsymbol{x})\;\Delta p(0,\boldsymbol{0})\,\rangle\,, \tag{104}$$

where we used the fluctuation-dissipation relation. The non-linear non-equilibrium pressure can be related to corresponding contribution to entropy density, Δs:

$$\beta\,\Delta p = -w\,\Delta\beta + n\Delta\alpha = -w\left(\frac{\partial \Delta s}{\partial \varepsilon}\right)_m\,, \tag{105}$$

where we used

$$\Delta\beta = \left(\frac{\partial \Delta s}{\partial \varepsilon}\right)_n\,, \qquad \Delta\alpha = -\left(\frac{\partial \Delta s}{\partial n}\right)_\varepsilon \tag{106}$$

and thermodynamic relation (B2). Near the critical point, it is sufficient to keep only the contribution of the slowest mode δm to Δs:

$$\int_{\boldsymbol{x}} \Delta s = -\frac{1}{2}\int_{\boldsymbol{Q}} \bar{\phi}^{-1}_{\boldsymbol{Q}}\,|\delta m_{\boldsymbol{Q}}(t)|^2\,. \tag{107}$$

where $\bar{\phi}_{\boldsymbol{Q}}$ is the equal-time correlator of δm as in Eq. (89) and $\delta m_{\boldsymbol{Q}}$ is the Fourier transform of $\delta m(\boldsymbol{x})$. Substituting Eqs. (105) and (107) into Eq. (104) we find (using the scale separation $1/\ell \ll Q$)

$$\frac{i\Delta G_R(\omega)}{\omega} = \frac{w^2}{2\beta}\int_0^\infty dt\, e^{i\omega\, t}\left(\frac{\bar{\phi}^{-1}_{\boldsymbol{Q}}}{\partial\varepsilon}\right)^2_m \langle \delta m_{\boldsymbol{Q}}(t)\,\delta m_{-\boldsymbol{Q}}(0)\rangle\,\langle \delta m_{-\boldsymbol{Q}}(t)\,\delta m_{\boldsymbol{Q}}(0)\rangle \tag{108}$$

This result is best illustrated by a simple one-loop diagram shown in Fig. 5. The vertex factor, $(\partial\bar{\phi}^{-1}_{\boldsymbol{Q}}/\partial\varepsilon)$, is a third derivative of the entropy Δs (see Eq. (107)), which is intuitively

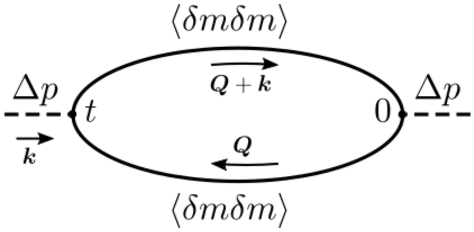

FIG. 5. The one-loop diagram representing critical mode contribution to bulk response in Eq. (108). Since $k \sim 1/\ell$, and $Q \sim 1/\xi$, the separation of scales $\ell \gg \xi$ means $Q \gg k$ — a typical hierarchy of scales in an HTL calculation.

natural considering analogy between the entropy of fluctuations and the action in field theory.

Using the expression for the unequal-time correlator

$$\langle\, \delta m_{\boldsymbol{Q}}(t)\, \delta m_{-\boldsymbol{Q}}(0)\, \rangle = \bar{\phi}_{\boldsymbol{Q}}\, e^{-\Gamma(\boldsymbol{Q})t/2} \tag{109}$$

and performing the time integration in Eq. (108) we obtain precisely Eq. (102). The factor 1/2 in the exponent in Eq. (109) is due to the fact that $\Gamma(\boldsymbol{Q})$ is the relaxation rate of a *two*-point function (essentially, of $(\delta m)^2$), while Eq. (109) represents the correlation function of the one-point function δm (see also Eq. (95) and footnote 10).

The comparison between Hydro+ and the loop calculation in this Section sheds additional light on the analogy between Hydro+ and the kinetic (Boltzmann) description in thermal field theory. The response in thermal field theory, which can be calculated in the hard-thermal loop (HTL) approach, is non-local, as manifested by the well-known Landau damping. This response is due to almost on-shell propagation of weakly coupled (quasi)particles. However, one can replace HTL approach by an equivalent kinetic description of these particles using a Boltzmann equation (or, in the case of gauge-field dynamics, Vlasov equations, coupling particles to classical fields). The advantage of the kinetic approach is that it is local and thus more intuitive and conceptually satisfying. Locality is also an indispensable property for numerical simulations of real-time dynamics. Hydro+ is similar to kinetic theory in this respect. While non-instantaneous bulk response (e.g., frequency-dependent bulk viscosity) is hard to implement in a simulation directly, Hydro+ reproduces this phenomenon using local (instantaneous) dynamics of additional modes very similar to kinetic description.

VI. SUMMARY AND DISCUSSION

We considered an extended hydrodynamic theory, or Hydro+, which describes evolution of partial-equilibrium states characterized by off-equilibrium values of non-hydrodynamic but slow variables. In general, such an extension can be justified if the additional non-hydrodynamic variable is still much slower than the remaining microscopic variables which are left (integrated) out. This condition distinguishes our approach from other extended hydrodynamic proposals, such as, e.g., the well-known Israel-Stewart second-order hydrodynamics [27], where additional variables (components of stress tensor), in general, relax to equilibrium as fast as the other (infinitely many) microscopic modes. As a result, applicability of Israel-Stewart theory (beyond ordinary hydrodynamic regime) is questionable [28]. Unlike Israel-Stuart hydrodynamics, we wish to consider a systematic limit in which the variables we keep are slow either because they are conserved (i.e., hydrodynamic), or because there exists another parameter, independent of the scale of inhomogeneity ℓ, controlling the slowness of hydrodynamic variables.

In Section II we describe a general formulation of Hydro+ and discuss the chiral (anomalous) fluid as a simple example in Appendix A. We show how a competition between scales of hydrodynamic evolution and non-hydrodynamic slow mode relaxation gives rise to two distinct regimes of frequencies. For $\omega \ll \Gamma_\pi$ the slow mode it completely equilibrated and simply tracks the hydrodynamic variables. Ordinary hydrodynamics apply in this regime, but the effect of the slow mode is manifested in a large contribution to bulk viscosity proportional to $1/\Gamma_\pi$ – a phenomenon already known in the context of non-relativistic fluids [8–10]. For $\omega \gg \Gamma_\pi$ the slow mode is effectively "frozen", which leads to a different, stiffer equation of state and the drop of the bulk viscosity.

The phenomenon of critical slowing down near a critical point is very similar to the situation where Hydro+ is applicable. The slowest non-hydrodynamic relaxation rate is controlled by the value of the correlation length ξ i.e., $\Gamma_\xi \sim \xi^{-3}$, independently of the scale of inhomogeneity ℓ which controls the slowness of the conserved hydrodynamic modes, $\Gamma_{\text{hydro}} \sim \ell^{-2}$ for relaxation or ℓ^{-1} for propagation. This sets the stage for the scale competition characteristic of Hydro+.

Ordinary hydrodynamics breaks down when $\Gamma_{\text{hydro}} \gtrsim \Gamma_\xi$, since the condition of the separation of hydrodynamic and non-hydrodynamic relaxation rates is violated leading to non-locality of the theory. Our goal in this paper is to determine how to add the slow mode, or modes, to hydrodynamics in order to extend its validity to the regime where the evolution rate of the hydrodynamic modes is comparable to, or larger than, the relaxation rate Γ_ξ.

The central element of Hydro+ is the extended equation of state, given by entropy, $s_{(+)}$, of the partial-equilibrium state characterized by the values of the non-hydrodynamic mode as well as the hydrodynamic variables. One of the major results of this paper is the expression for the entropy as a function(al) of the off-equilibrium values of *fluctuations*. In Section III we derive a general expression which we find to bear a natural mathematical resemblance to the 2PI action in quantum field theory. In addition to one-point functions – the local values of the hydrodynamic variables – the 2PI entropy depends on the 2-point functions – the off-equilibrium values of the fluctuations. In Section IV we use the extended entropy to write Hydro+ equations generalizing the single-mode theory discussed in Section II.

In Section V we focus on the slowest mode near the critical point – diffusion of entropy per (baryon) charge, and write down a set of equations for the coupled evolution of critically slow magnitude of fluctuations and the hydrodynamic modes. Since the additional slow variable is a measure of fluctuations, i.e., a 2-point function, the corresponding variable has an index, $\boldsymbol{Q}$, which is the Fourier transform of the spatial separation of the points in the 2-point function. In that respect the slow variable $\phi_{\boldsymbol{Q}}$ is similar to the phase-space particle distribution function in kinetic theory and can be called mode distribution function.

We show that the Hydro+ theory defined in Section V A reproduces known phenomena associated with critical slowing down. In particular, we show that the bulk viscosity receives critically enhanced contribution $\Delta\zeta \sim \xi^3$. As a result, ordinary hydrodynamics breaks down when the bulk relaxation (sound attenuation) rate Γ_{hydro} is of order $\Gamma_\xi \sim \xi^{-3}$. Naively extending hydrodynamics beyond that (to larger frequencies or wave numbers) one overpredicts the actual amount of dissipation because the frequency-dependent bulk viscosity drops for larger frequencies. This phenomenon is captured by Hydro+ as illustrated by Fig. 3(a).

We also note that as a consequence of the critical slowing down the stiffness of the equation of state increases for frequencies above Γ_ξ. Again, the ordinary hydrodynamics will underpredict the stiffness (measured by the frequency dependent sound velocity) as shown in Fig. 3(b).

The purpose of our paper is to introduce the approach and discuss its advantages as well as to point out limitations which could be addressed in future work.

One of the advantages of the approach to fluctuations encoded in Hydro+ over another popular approach being discussed in the literature based on the stochastic hydrodynamics [47, 48] is that Hydro+ eliminates the need for solving stochastic equations. Even though two approaches lead to similar results, deterministic description of fluctuations may prove advantageous in numerical simulations. One of the reasons is that stochastic fluctuations introduce strong cutoff dependence of equation of state as well as kinetic coefficients as has been observed and discussed, e.g., in Refs.[12, 16]. This dependence need to be canceled, which creates a numerically ill-conditioned problem. On the other hand, the 2PI equation of state we introduce in Section III is already renormalized (see Eq. (56)), i.e., the UV divergences due to fluctuations are included into it.

This paper would be incomplete without a discussion of the domain of applicability of Hydro+. As we already pointed out, ordinary hydrodynamics breaks down for frequencies (or rates) larger than the rate Γ_ξ which becomes critically slow as $\Gamma_\xi \sim \xi^{-3}$. Hydro+ extends the range of applicability to higher frequencies. Unlike the simple single-mode Hydro+ discussed in Section II, which could be applicable all the way to the microscopic scale (collision rates, or T), Hydro+ near the critical point has another limitation – the rate of the relaxation of the next-to-slowest mode. This rate is *parametrically* faster than Γ_ξ, but still slower than microscopic scale, e.g., T. The corresponding mode is the relaxation of the transverse velocity (shear) fluctuations on the characteristic scale ξ of density fluctuations, with the rate $\Gamma_\xi^T = (\eta/w)\xi^{-2} \sim \xi^{-2}$.[14] Comparing Γ_ξ^T to Γ_ξ we see that Hydro+ extends the range of applicability by a factor $\Gamma_\xi^T/\Gamma_\xi \sim T\xi \gg 1$.

The emergence of the scale Γ_ξ^T is due to the important role played by the transverse modes in the Model H dynamic universality class [7]. Indeed, the characteristic rate Γ_ξ in Eq. (97) (and the corresponding value of the critical exponent $z \approx 3$) depends on the divergence of conductivity $\lambda \sim \xi$. This critical behavior of λ is driven by the enhanced fluctuations of charge density and relies on transverse velocity (shear) modes relaxing faster than ω. Therefore, λ will only reach its critical behavior $\lambda \sim \xi$ for processes much slower than Γ_ξ^T and the use of Eq. (97) (and $z \approx 3$) is only meaningful for $\omega \ll \Gamma_\xi^T$. To extend Hydro+ to time scales shorter than $1/\Gamma_\xi^T$ on would need to add the fluctuations of shear modes as additional non-hydrodynamic variables, which should be possible to do along the

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[14] Here, for simplicity, we neglect ξ-dependence of η, a common approximation, since it is very weak [7].

lines similar to Section IV.

The 2PI formalism we introduce in Section III is suited for treating the off-equilibrium evolution of Gaussian fluctuations. The sensitivity of *non*-Gaussian measures of fluctuations makes them important signatures of the QCD critical point [49, 50]. In order to incorporate the evolution of non-Gaussianity into Hydro+ formalism one needs to extend 2PI formalism to non-Gaussian fluctuations. This would lead to a generalization of 2PI entropy to, e.g., 3PI and 4PI, and a hierarchy of kinetic equations similar to the hierarchy of cumulant equations in Ref. [51]. We defer these and other developments of Hydro+ to future work.

Appendix A: Application of Hydro+ to chiral (anomalous) fluid

We consider a chiral fluid, the constituents of which include (approximately) massless fermions. In this system, the axial current $J^\mu_A \equiv \bar{\psi}\gamma^\mu\gamma^5\psi$ is conserved only approximately. The conservation is violated by a small fermion mass (and/or by quantum anomaly, as in non-Abelian gauge theories, where topological sphaleron fluctuations induce fermion chirality flips). Therefore:

$$\partial_\mu\, J^\mu_A = -\gamma_A\, \alpha_A\,. \tag{A1}$$

Here γ_A is an Onsager coefficient describing chirality violating processes and $\alpha_A = \beta\mu_A$ where μ_A is the axial chemical potential. In the absence of a background magnetic field B^μ, the currents ΔJ^μ_V and ΔJ^μ_A are given by:

$$\Delta J^\mu_V = \lambda_{VV}\, \partial^\perp_\mu \alpha_V + \lambda_{VA}\, \partial^\perp_\mu \alpha_A\,, \qquad \Delta J^\mu_A = \lambda_{VA}\, \partial^\perp_\mu \alpha_V + \lambda_{AA}\, \partial^\perp_\mu \alpha_A\,. \tag{A2}$$

Parameters λ_{VV}, etc. are vector and axial conductivity coefficients (conductivity times temperature). Substituting (A2) into (A1), we then have:

$$D\, n_V = -n_V\, \theta - \partial_\perp \cdot (\, \lambda_{VV}\, \partial\alpha_V + \lambda_{VA}\, \partial\alpha_A\,) \tag{A3}$$

$$D\, n_A = -n_A\, \theta - \gamma_A\, \alpha_A - \partial_\perp \cdot (\, \lambda_{AA}\, \partial\alpha_A + \lambda_{VA}\, \partial\alpha_V\,)\,. \tag{A4}$$

We now identify axial charge density n_A with ϕ. Comparing (A4) with (7d) and (15), we have $A_\phi \equiv n_A$, and $\gamma_\pi \equiv \gamma_A$. In general, with n_A finite, λ_{AV} is non-zero. Moreover, λ_{AA} and λ_{VV} can be different from each other. Such expectation has been confirmed in an explicit perturbative computation [52].

In this theory the analog of p_π is zero due to parity and thus bulk viscosity and sound propagation velocity are not affected by the slow mode as it is in a more general case described in Section II. However, a similar enhancement was found in Ref. [53] for the conductivity along the magnetic field, which receives anomalous contribution from the slow non-hydrodynamics mode:

$$\Delta\lambda = \chi_V T \,\frac{\Delta v_{\rm cmw}^2}{\Gamma_\pi}\,, \tag{A5}$$

where χ_V is the charge susceptibility. Similarly to Eq. (32), $\Delta v_{\rm cmw}^2$ denotes the increase of the speed of chiral magnetic wave between hydrodynamic regime $\omega \ll \Gamma_\pi$ and Hydro+ regime $\omega \gg \Gamma_\pi$, and Γ_π is the rate of the slow mode (axial charge) relaxation.

Appendix B: A thermodynamic relation for p_π

This section supplies derivation of Eq. (21), which can be also written as

$$\left(\frac{\partial p_{(+)}}{\partial \pi}\right)_{\varepsilon n} = -\frac{w}{\beta}\left[\left(\frac{\partial \bar\phi}{\partial \varepsilon}\right)_m - \frac{A_\phi}{w}\right] \qquad \text{at } \pi = 0 \tag{B1}$$

due to Eq. (82), which means

$$\left(\frac{\partial n}{\partial \varepsilon}\right)_m = \frac{n}{w}\,. \tag{B2}$$

We begin from (12) evaluated at $\pi = 0$:

$$\beta\, dp_{(+)} = -w\, d\beta_{(+)} + n\, d\alpha_{(+)} + A_\phi d\pi\,. \tag{B3}$$

Considering variation of π at ε and n fixed we can write

$$\beta\, p_\pi = -w\beta_\pi + n\alpha_\pi + A_\phi \tag{B4}$$

where index π denotes derivatives with respect to π at ε and n fixed evaluated at $\pi = 0$, e.g.,

$$p_\pi \equiv \left(\frac{\partial p_{(+)}(\varepsilon, n, \pi)}{\partial \pi}\right)\Bigg|_{\pi=0} \equiv \left(\frac{\partial p_{(+)}}{\partial \pi}\right)_{\varepsilon,n}\Bigg|_{\pi=0}\,. \tag{B5}$$

We can then use Maxwell relations applied to the differential

$$d\left(s_{(+)} + \pi\phi\right) = \beta_{(+)} d\varepsilon - \alpha_{(+)} dn + \phi d\pi \tag{B6}$$

to relate derivative w.r.t. π to the derivatives of ϕ (at $\pi = 0$, i.e., in equilibrium):

$$\beta_\pi \equiv \left(\frac{\partial \beta_{(+)}}{\partial \pi}\right)_{\varepsilon n} = \left(\frac{\partial \phi}{\partial \varepsilon}\right)_{\pi n}, \qquad \alpha_\pi \equiv \left(\frac{\partial \alpha_{(+)}}{\partial \pi}\right)_{\varepsilon n} = -\left(\frac{\partial \phi}{\partial n}\right)_{\pi \varepsilon}. \tag{B7}$$

Substituting Maxwell relations (B7) into Eq. (B4) we obtain Eq. (21) or (with Eq. (B2)) Eq. (B1).

Appendix C: Reparameterization covariance in Hydro+

The choice of the slow mode is not unique, but since the physics cannot depend on that choice the equations of Hydro+ must possess reparameterization invariance which we describe here.

Let us consider another choice of the slow variable, ϕ', which is a function of the original choice ϕ and, possibly, of ε and n, i.e.,

$$\phi' = f(\phi, \varepsilon, n). \tag{C1}$$

The equation governing evolution of ϕ' must have a similar form to Eq. (7d), i.e.,

$$D\phi' = -F'_\phi - A'_\phi \theta, \tag{C2}$$

where the relationship between the new parameters F'_ϕ and A'_ϕ and the original ones can be found by substituting Eq. (C1) into Eq. (C2) and matching to Eq. (7d):

$$F'_\phi = f_\phi F_\phi, \qquad A'_\phi = f_\phi A_\phi + w f_\varepsilon + n f_n \tag{C3}$$

where

$$f_\phi = \left(\frac{\partial f}{\partial \phi}\right)_{\varepsilon n} \quad f_\varepsilon \equiv \left(\frac{\partial f}{\partial \varepsilon}\right)_{n\phi}; \quad f_n \equiv \left(\frac{\partial f}{\partial n}\right)_{\varepsilon\phi} \tag{C4}$$

One can easily check that

$$\pi' = f_\phi \pi; \quad \phi'_\pi = f_\phi^2 \phi_\pi; \quad \gamma'_\pi = f_\phi^2 \gamma_\pi; \quad p'_\pi = f_\phi p_\pi. \tag{C5}$$

These transformations leave the relationship Eq. (B1) between p_π and $\partial\bar\phi/\partial\varepsilon$ invariant (note that transformation of A_ϕ and the role it plays is reminiscent of a gauge potential). Also, the combination p_π^2/ϕ_π in the definition of Δc_s^2 is invariant as it should be expected.

Appendix D: Fluctuations of m and p

The fluctuations of the thermodynamic quantities Ψ are described by the probability distribution $P \sim e^{S(\Psi)+\bar J \Psi}$ (see Section III and Ref. [36]). The change of P under a fluctuation $\Psi - \bar\Psi \equiv \delta\Psi$ (of arbitrary, not necessarily small, magnitude) is given by

$$\Delta(\log P) \equiv \log \frac{P(\Psi)}{P(\bar\Psi)} = S(\bar\Psi + \delta\Psi) - S(\Psi) + \bar J \delta\Psi \equiv \Delta S + \bar J \delta\Psi\,, \tag{D1}$$

where ΔS is the change of the entropy of the system and $\bar{J}\delta\Psi$ is the change of the entropy of the environment as the amounts $\delta\Psi$ of *conserved* quantities are exchanged between the system and the environment characterized by thermodynamic potentials $\bar{J}$. Since $\bar{\Psi}$ is the equilibrium value, the $\mathcal{O}(\delta\Psi)$ terms in Eq. (D1) cancel. For the case we consider $\Psi = (\varepsilon, n)$ and $J = (-\beta, \alpha)$, Taylor expanding the entropy to second order in $\delta\Psi$ we can write

$$\Delta(\log P)/V = \frac{1}{2}\left(\delta\beta\delta\varepsilon - \delta\alpha\delta n\right) + \mathcal{O}(\delta^3)\,, \tag{D2}$$

where the factor of volume V comes from the space integration. Expressing the variables p and $m = s/n$ in terms of β, α, ε, n one finds to linear order:

$$\beta\delta p = n\delta\alpha - w\delta\beta + \mathcal{O}(\delta^2) \tag{D3}$$

$$n^2\delta m = \beta(n\delta\varepsilon - w\delta n) + \mathcal{O}(\delta^2) \tag{D4}$$

Solving for $\delta\varepsilon$ and $\delta\alpha$ and substituting into Eq. (D1) one finds (upon cancellation of $\delta\beta\delta n$ terms):

$$\Delta(\log P)/V = \frac{\beta n}{2}\left(\delta\left(\frac{1}{n}\right)\delta p - \delta\left(\frac{1}{\beta}\right)\delta m\right) + \mathcal{O}(\delta^3) \tag{D5}$$

Expressing $\delta(1/n)$ and $\delta(1/\beta)$ in terms δm and δp and using the Maxwell relations stemming from

$$d\left(\frac{w}{n}\right) = \frac{1}{\beta}\,dm + \frac{1}{n}\,dp \tag{D6}$$

one arrives at

$$\Delta(\log P)/V = -\frac{1}{2}\left(\frac{\beta}{wc_s^2}(\delta p)^2 + \frac{n^2}{c_p}(\delta m)^2\right) + \mathcal{O}(\delta^3)\,. \tag{D7}$$

In the thermodynamic limit, i.e., for large $V \sim k^{-3} \gg \xi^3$ (or $\ell \gg \xi$), fluctuations are small, their probability distribution is approximately Gaussian, and we obtain Eq. (90) for their variance.

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