Scalar enhancement of the photon electric field by the tail of the graviton propagator

S. P. Miao, T. Prokopec, and R. P. Woodard

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The Graviton Tail almost Completely Wags the Dog

S. P. Miao[1*], T. Prokopec[2⋆] and R. P. Woodard[3†]

[1] *Department of Physics, National Cheng Kung University*
No. 1, University Road, Tainan City 70101, TAIWAN

[2] *Institute for Theoretical Physics, Spinoza Institute & EMMEΦ*
Utrecht University, Postbus 80.195, 3508 TD Utrecht, NETHERLANDS

[3] *Department of Physics, University of Florida,*
Gainesville, FL 32611, UNITED STATES

ABSTRACT

One graviton loop corrections to the vacuum polarization on de Sitter show two interesting infrared effects: a secular enhancement of the photon electric field strength and a long range running of the Coulomb potential. We show that the first effect derives solely from the "tail" term of the graviton propagator, but that the second effect does not. Our result agrees with the earlier observation that the secular enhancement of massless fermion mode functions derives from solely from the tail term. We discuss the implications this has for the important project of generalizing to quantum gravity the Starobinsky technique for summing the series of leading infrared effects from inflationary quantum field theory.

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* e-mail: spmiao5@mail.ncku.edu.tw
⋆ email: T.Prokpec@uu.nl
† e-mail: woodard@phys.ufl.edu

1 Introduction

It has long been clear that there is something peculiar about long wavelength gravitons on cosmological backgrounds [1]. Unlike photons, which are precluded by conformal invariance from locally perceiving the expansion of the Universe, inflationary expansion leads to the production of gravitons [2, 3]. This process is the source of the tensor power spectrum predicted by primordial inflation [4].

Long wavelength gravitons also make a peculiar contribution to the retarded propagator, which DeWitt and Brehme famously denoted as the "tail term" [5]. Unlike the usual delta function *on* the past light-cone, the tail contribution is nonzero *inside* the past light-cone [6]. This fact has great relevance to computations of gravitational radiation reaction in binary mergers [7, 8, 9]. It is also responsible for the curious infrared "running" of the Newtonian potential induced by the one loop gravitational vacuum polarization of conformal matter on de Sitter background [10, 11],

$$\Psi = -\frac{GM}{ar}\left\{1 + \frac{4G}{15\pi a^2r^2} + \frac{2GH^2}{5\pi}\ln(aHr) + O(G^2)\right\}. \tag{1}$$

Here H is the Hubble constant, $a = e^{Ht}$ is the de Sitter scale factor and r is the co-moving position. The fractional correction of $\frac{4G}{15\pi a^2r^2}$ is just the de Sitter descendant of the flat space effect which has long been known [12, 13]. The new term proportional to GH^2 is specific to nonzero Hubble constant and causes perturbation theory to break down, both for large r and at late times. Even though conformal matter induces almost the same vacuum polarization, in de Sitter conformal coordinates, as in flat space, the gravitational *response* to that source is very different on account of the strong de Sitter tail term.

Analytic continuation carries the tail term of the retarded propagator into the tail part of the Feynman propagator which can mediate quantum graviton effects to other particles [14, 15]. An important example is the one graviton contribution to the electromagnetic vacuum polarization [16]. This induces an infrared running of the Coulomb potential similar to (1) [17],

$$\Phi = \frac{Q}{4\pi r}\left\{1 + \frac{2G}{3\pi a^2r^2} + \frac{2GH^2}{\pi}\ln(aHr) + O(G^2)\right\}. \tag{2}$$

As with the Newtonian potential (1), the fractional correction $\frac{2G}{3\pi a^2r^2}$ is just the de Sitter analogue of what happens in flat space [18], while the new term proportional to GH^2 causes perturbation theory to break down at large r and at late times. The gravitational vacuum polarization on de Sitter also causes a secular enhancement of the electric field of a plane wave photon [19],

$$F_{0i}^{1\ \rm loop} \longrightarrow \frac{2GH^2}{\pi}\ \ln(a) \times F_{0i}^{\rm tree}\ . \tag{3}$$

Like (2), this result signals a late time breakdown of perturbation theory.

A common feature in all three results (1), (2) and (3) is the breakdown of perturbation theory when $\ln(a) \sim \frac{1}{GH^2}$. Uncovering what happens after this time requires going beyond perturbation theory. For the very similar infrared logarithms of scalar potential models Starobinsky has developed a stochastic formalism [20] which exactly reproduces the leading infrared logarithms at each loop order [21, 22], and can be summed to elucidate the nonperturbative regime [23]. The same technique can be applied to a Yukawa-coupled scalar [24], and to scalar quantum electrodynamics [25]. However, it has not yet been generalized to quantum gravity.

The obstacle to applying Starobinsky's formalism has been the derivative interactions of quantum gravity. These frustrate the proof [21, 22] that works for scalar potential models. Derivative interactions also mean that the lowest order renormalization counterterms contribute at leading logarithm order, which means that dimensional regularization must be retained until a fully renormalized result is obtained [26]. The problem remains, despite notable progress understanding the simpler derivative interactions of nonlinear sigma models [27, 28].

A notable advance was the discovery [26] that only the tail part of the graviton propagator is responsible for the secular enhancement of massless fermions on de Sitter background [29, 30]. The purpose of this paper is to see if the tail term alone also explains the secular enhancement of dynamical photons (3) and the logarithmic running of the Coulomb potential (2). In section 2 we review the relevant Feynman rules and identify precisely those parts of the vacuum polarization which are responsible for the two effects. Section 3 computes the tail contribution to the vacuum polarization. Our results are discussed in section 4.

2 Notation

The purpose of this section is to review notation. We begin with the Feynman rules which were used to compute the vacuum polarization [16]. This is where we define the "tail" part of the graviton propagator which plays a central work in this study. We also describe how the tensor structure of the vacuum polarization is represented using two structure functions, and we give the order GH^2 contributions to these structure functions which are responsible for the enhancement of dynamical photons (3) and the logarithmic running of the Coulomb potential (2).

2.1 Feynman Rules

The Lagrangian relevant to our study is,

$$\mathcal{L} = \frac{[R-(D-2)(D-1)H^2]\sqrt{-g}}{16\pi G} - \frac{1}{4} F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} \sqrt{-g} + \Delta\mathcal{L} + \mathcal{L}_{GF} \; . \quad (4)$$

Here D is the spacetime dimension, H is the de Sitter Hubble constant and G is Newton's constant. The two counterterms we require are,

$$\Delta\mathcal{L} = \overline{C} H^2 F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} \sqrt{-g} + \Delta C H^2 F_{ij} F_{k\ell} g^{ik} g^{j\ell} \sqrt{-g} \; . \quad (5)$$

The noninvariant term (Roman indices are purely spatial) proportional to ΔC is required because of de Sitter breaking in the graviton sector [16, 31]. Our electromagnetic and gravitational gauge fixing terms are [14, 15],

$$\mathcal{L}_{GF} = -\frac{1}{2} a^{D-4} \Big[\eta^{\mu\nu} A_{\mu,\nu} - (D-4) H a A_0\Big]^2 - \frac{1}{2} a^{D-2} \eta^{\mu\nu} F_\mu F_\nu \; , \quad (6)$$

where $a \equiv -\frac{1}{H\eta}$ is the de Sitter scale factor (at conformal time η) and the gravitational term is,

$$F_\mu \equiv \eta^{\rho\sigma} \Big[h_{\mu\rho,\sigma} - \frac{1}{2} h_{\rho\sigma,\mu} + (D-2) H a h_{\mu\rho} \delta^0_{\ \sigma}\Big] \; . \quad (7)$$

Here and henceforth $h_{\mu\nu}$ is the conformally transformed graviton field whose indices are raised and lowered with the (spacelike) Minkowski metric,

$$g_{\mu\nu} \equiv a^2 \widetilde{g}_{\mu\nu} \equiv a^2 \Big[\eta_{\mu\nu} + \kappa h_{\mu\nu}\Big] \qquad , \qquad \kappa^2 \equiv 16\pi G \; . \quad (8)$$

Our gauge breaks de Sitter invariance but it does provide the simplest possible expressions for the photon and graviton propagators. They each take the form of a sum of constant tensor factors times scalar propagators,

$$i\Big[{}_\mu\Delta_\rho\Big](x;x') = \overline{\eta}_{\mu\rho} \times a a' i\Delta_B(x;x') - \delta^0_{\ \mu} \delta^0_{\ \rho} \times a a' i\Delta_C(x;x') \; , \quad (9)$$

$$i\Big[{}_{\mu\nu}\Delta_{\rho\sigma}\Big](x;x') = \sum_{I=A,B,C} \Big[{}_{\mu\nu} T^I_{\rho\sigma}\Big] \times i\Delta_I(x;x') \; , \quad (10)$$

where $\overline{\eta}_{\mu\nu} \equiv \eta_{\mu\nu} + \delta^0{}_\mu \delta^0{}_\nu$ is the spatial part of the Minkowski metric. The gravitational tensor factors are,

$$\Big[{}_{\mu\nu}T^A_{\rho\sigma}\Big] = 2\overline{\eta}_{\mu(\rho}\overline{\eta}_{\sigma)\nu} - \frac{2}{D-3}\,\overline{\eta}_{\mu\nu}\overline{\eta}_{\rho\sigma} \; , \qquad (11)$$

$$\Big[{}_{\mu\nu}T^B_{\rho\sigma}\Big] = -4\delta^0{}_{(\mu}\overline{\eta}_{\nu)(\rho}\delta^0{}_{\sigma)} \; , \qquad (12)$$

$$\Big[{}_{\mu\nu}T^C_{\rho\sigma}\Big] = \frac{2}{(D-2)(D-3)}\Big[(D-3)\delta^0{}_\mu\delta^0{}_\nu + \overline{\eta}_{\mu\nu}\Big]\Big[(D-3)\delta^0{}_\rho\delta^0{}_\sigma + \overline{\eta}_{\rho\sigma}\Big] \; . \qquad (13)$$

Here and henceforth parenthesized indices are symmetrized.

It is useful to expand the three scalar propagators in progressively less and less singular terms,

$$i\Delta_I(x;x') = \frac{i\Delta(x;x')}{(aa')^{\frac{D}{2}-1}} + i\delta\Delta_I(x;x') + i\Delta_{\Sigma I}(x;x') \quad , \quad I = A, B, C \; . \qquad (14)$$

Here the massless scalar propagator in flat space is

$$i\Delta(x;x') = \frac{\Gamma(\frac{D}{2}-1)}{4\pi^{\frac{D}{2}}\Delta x^{D-2}} \quad , \quad \Delta x^2(x;x') \equiv \Big\Vert \vec{x}-\vec{x}'\Big\Vert^2 - \Big(|\eta-\eta'|-i\epsilon\Big)^2 \; . \qquad (15)$$

Note that $i\Delta(x;x')$ has the leading, $1/\Delta x^{D-2}$ singularity. The three $1/\Delta x^{D-4}$ terms are,

$$(aa')^{\frac{D}{2}-2} i\delta\Delta_A(x;x') = \frac{H^2}{4\pi^{\frac{D}{2}}}\Bigg\{\frac{\Gamma(\frac{D}{2}+1)}{2(D-4)}\frac{1}{\Delta x^{D-4}} - \frac{\pi\cot(\frac{\pi D}{2})\Gamma(D-1)}{4\Gamma(\frac{D}{2})}\Big(\frac{aa'H^2}{4}\Big)^{\frac{D}{2}-2} + \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\Big(\frac{aa'H^2}{4}\Big)^{\frac{D}{2}-2}\ln(aa')\Bigg\}, \qquad (16)$$

$$(aa')^{\frac{D}{2}-2} i\delta\Delta_B(x;x') = \frac{H^2}{4\pi^{\frac{D}{2}}}\Bigg\{\frac{\Gamma(\frac{D}{2})}{\Delta x^{D-4}} - \frac{\Gamma(D-2)}{\Gamma(\frac{D}{2})}\Big(\frac{aa'H^2}{4}\Big)^{\frac{D}{2}-2}\Bigg\}, \qquad (17)$$

$$(aa')^{\frac{D}{2}-2} i\delta\Delta_C(x;x') = \frac{H^2}{4\pi^{\frac{D}{2}}}\Bigg\{\frac{(\frac{D}{2}-3)\Gamma(\frac{D}{2}-1)}{\Delta x^{D-4}} + \frac{\Gamma(D-3)}{\Gamma(\frac{D}{2})}\Big(\frac{aa'H^2}{4}\Big)^{\frac{D}{2}-2}\Bigg\}. \qquad (18)$$

The $i\delta\Delta_I(x;x')$ determine the coincidence limits in dimensional regularization, but only $i\delta\Delta_A(x;x')$ produces a nonzero tail term when $D=4$. The three $i\Delta_{\Sigma I}(x;x')$ terms are each infinite series of less singular powers, which vanish for $D=4$. They play no role in our analysis, but their expansions are given in Appendix A for completeness.

We can now identify the "tail" part of the graviton propagator,

$$i\Big[{}_{\mu\nu}\Delta^{\rm tail}_{\rho\sigma}\Big](x;x') \equiv \Big[{}_{\mu\nu}T^A_{\rho\sigma}\Big] \times i\delta\Delta_A(x;x') \; . \tag{19}$$

The purpose of this paper is to check whether or not replacing the full graviton propagator by (19) gives those parts of the vacuum polarization which are responsible for the secular enhancement of dynamical photons (3) and the logarithmic running of the Coulomb potential (2).

2.2 Representing Vacuum Polarization

The one graviton loop contribution to the vacuum polarization can be expressed in terms of expectation values of variations of the action,

$$\begin{aligned} i\Big[{}^{\mu}\Pi^{\nu}\Big](x;x') = \Big\langle \Omega \Big| \Big[\frac{i\delta S}{\delta A_{\mu}(x)}\Big]_{hA} \times \Big[\frac{i\delta S}{\delta A_{\nu}(x')}\Big]_{hA} \Big| \Omega \Big\rangle \\ + \Big\langle \Omega \Big| \Big[\frac{i\delta^2 S}{\delta A_{\mu}(x) \delta A_{\nu}(x')}\Big]_{hh} \Big| \Omega \Big\rangle \; . \end{aligned} \tag{20}$$

The subscripts hA and hh indicate that the operator in square brackets is to be expanded to that order in the weak fields $h_{\mu\nu}$ and A_{μ}. Expression (20) is ideal for our study because each of these two expectation values is *separately* transverse, and for *any* graviton field.

The tensor structure of the de Sitter background vacuum polarization can be represented using two structure functions [32, 33, 34],

$$i\Big[{}^{\mu}\Pi^{\nu}\Big](x;x') = \Big(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}\Big)\partial_{\rho}\partial'_{\sigma}F(x;x') + \Big(\overline{\eta}^{\mu\nu}\overline{\eta}^{\rho\sigma} - \overline{\eta}^{\mu\sigma}\overline{\eta}^{\nu\rho}\Big)\partial_{\rho}\partial'_{\sigma}G(x;x'). \tag{21}$$

Each of the two terms on the right hand side of (21) is transverse so we can work out contributions to $F(x;x')$ and $G(x;x')$ separately, from each of the two expectation values in (20), and from any part of the graviton propagator such as (19). Given a transverse contribution to $i[{}^{\mu}\Pi^{\nu}](x;x')$, the corresponding contributions to the structure functions can be inferred from selected components [34],

$$i\Big[{}^{0}\Pi^{0}\Big](x;x') = -\vec{\nabla} \cdot \vec{\nabla}' F(x;x') \; , \tag{22}$$

$$\eta_{\mu\nu} \times i\Big[{}^{\mu}\Pi^{\nu}\Big](x;x') = (D-1)\partial \cdot \partial' F(x;x') + (D-2)\vec{\nabla} \cdot \vec{\nabla}' G(x;x') \; . \tag{23}$$

The same considerations imply that the two relevant counterterms (5) make the following contributions [16],

$$\Delta F(x;x') = 4\overline{C}H^2 a^{D-4} i\delta^D(x-x') \ \ , \ \ \Delta G(x;x') = 4\Delta C H^2 a^{D-4} i\delta^D(x-x') \ . \tag{24}$$

The full one loop vacuum polarization [16] contains some parts which are de Sitter-ized versions of the flat space result [18]. However, the secular enhancement of dynamical photons (3) and the logarithmic running of the Coulomb potential (2) originate in the intrisically de Sitter portions of the structure functions,

$$F_{\rm dS}(x;x') \;=\; \frac{\kappa^2 H^2}{(2\pi)^4}\Bigg\{ 2\pi^2 \ln(a) i\delta^4(x-x') + \frac{1}{4}\partial^2 \Bigg[\frac{\ln(\frac14 H^2 \Delta x^2)}{\Delta x^2}\Bigg] + \partial_0^2 \Bigg[\frac{\ln(\frac14 H^2 \Delta x^2) + 2}{\Delta x^2}\Bigg]\Bigg\}, \tag{25}$$

$$G_{\rm dS}(x;x') \;=\; \frac{\kappa^2 H^2}{(2\pi)^4}\Bigg\{ -\frac{8}{3}\pi^2 \ln(a) i\delta^4(x-x') - \frac{1}{3}\partial^2 \Bigg[\frac{\ln(\frac14 H^2 \Delta x^2)}{\Delta x^2}\Bigg]\Bigg\}. \tag{26}$$

The enhancement of dynamical photons actually derives entirely from just the $\ln(a)$ part of $F_{dS}(x;x')$ [19]. In contrast, all terms on the first lines of (25-26) contribute to the logarithmic running of the Coulomb potential [17]. The terms on the second line of expression (25) do not contribute to either the enhancement of photons or the running of the Coulomb potential.

3 Vacuum Polarization from the Tail

This section presents the key computation of the tail contribution to the two structure functions of the vacuum polarization. Because each of the terms in the operator expression (20) is separately transverse, as is the contribution from the counter-action, we derive separate results for each of the three diagrams in Figure 1. Because the counterterms contribute at leading logarithm order it is necessary to retain dimensional regularization until the end. (The same thing was found in deriving the tail contribution to the fermion wave function [26].) However, extensive simplifications result from anticipating terms which must vanish in the renormalized, unregulated limit. We begin with the simple 4-point contribution, then proceed to the more complicated contribution from two 3-point vertices, and finally add the appropriate counterterms.

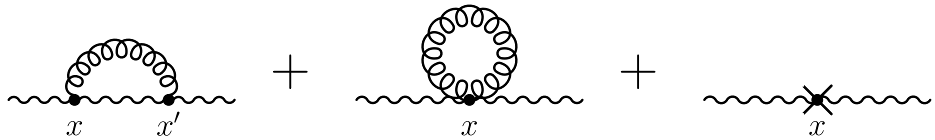

Figure 1: Feynman diagrams relevant to the one loop vacuum polarization from gravitons. Wavy lines are photons, curly lines are gravitons and the cross represents counterterms.

3.1 The 4-point contribution

The primitive 4-point contribution is the middle diagram of Fig. 1 and has the operator representation,

$$i\Big[{}^{\mu}\Pi^{\nu}_{\rm 4pt}\Big](x;x') = \partial_\rho \partial'_\sigma \Big\langle \Omega \Big| a^{D-4}\sqrt{-\widetilde{g}}\,\Big(\widetilde{g}^{\mu\sigma}\widetilde{g}^{\nu\rho} - \widetilde{g}^{\mu\nu}\widetilde{g}^{\rho\sigma}\Big) i\delta^D(x-x') \Big| \Omega \Big\rangle_{hh} . \quad (27)$$

This expression is exact. Because the tail contribution comes from the purely spatial components of the graviton field we can use relation (22) to write a simple relation for the tail part of the structure function $F(x;x')$,

$$-\vec{\nabla}\cdot\vec{\nabla}' F_{\rm 4t}(x;x') = \partial_i \partial'_j \Big\langle \Omega \Big| a^{D-4}\sqrt{-\widetilde{g}}\,\widetilde{g}^{ij} i\delta^D(x-x') \Big| \Omega \Big\rangle_{\rm tail} . \quad (28)$$

Isotropy implies,

$$F_{\rm 4t}(x;x') = -\frac{1}{D-1} \Big\langle \Omega \Big| a^{D-4}\sqrt{-\widetilde{g}}\,\widetilde{g}^{kk} i\delta^D(x-x') \Big| \Omega \Big\rangle_{\rm tail} , \quad (29)$$

$$= \frac{1}{4} D(D-5)\kappa^2 a^{D-4} i\delta\Delta_A(x;x) i\delta^D(x-x') . \quad (30)$$

Expression (30) agrees with the result (66) reported in [16].

Relation (23) determines the structure function $G(x;x')$,

$$(D-1)\partial\cdot\partial' F_{\rm 4t} + (D-2)\vec{\nabla}\cdot\vec{\nabla}' G_{\rm 4t} = \partial_0 \partial'_0 \Big\langle \Omega \Big| a^{D-4}\sqrt{-\widetilde{g}}\,\widetilde{g}^{kk} i\delta^D(x-x') \Big| \Omega \Big\rangle_{\rm tail}$$
$$+ \partial_i \partial'_j \Big\langle \Omega \Big| a^{D-4}\sqrt{-\widetilde{g}}\,\Big(\widetilde{g}^{ik}\widetilde{g}^{jk} + \widetilde{g}^{ij}(1-\widetilde{g}^{kk})\Big) i\delta^D(x-x') \Big| \Omega \Big\rangle_{\rm tail} . \quad (31)$$

Using relation (29) and exploiting isotropy implies,

$$G_{\rm 4t} = \Big\langle \Omega \Big| \frac{a^{D-4}\sqrt{-\widetilde{g}}}{(D-1)(D-2)} \Big(\widetilde{g}^{k\ell}\widetilde{g}^{k\ell} + \widetilde{g}^{kk}[(D-2) - \widetilde{g}^{\ell\ell}]\Big) i\delta^D(x-x') \Big| \Omega \Big\rangle_{tail} , \quad (32)$$

$$= -\Big[D - \Big(\frac{D-1}{D-3}\Big)\Big] \kappa^2 a^{D-4} i\delta\Delta_A(x;x) i\delta^D(x-x') . \quad (33)$$

Expression (33) agrees with the result (67) reported in [16].

3.2 The 3-point contribution

The primitive 3-point contribution is the left hand diagram of Fig. 1. From the first term of the operator expression (20) we can infer a simpler operator expression for it,

$$i\Big[{}^{\mu}\Pi^{\nu}_{\rm 3pt}\Big](x;x') = -\partial_{\rho}\partial'_{\sigma}\Bigg\{\Big\langle\Omega\Big|\Big[\sqrt{-\widetilde{g}}\,\widetilde{g}^{\rho[\alpha}\widetilde{g}^{\beta]\mu}\Big]_{h(x)}\times\Big[\sqrt{-\widetilde{g}}\,\widetilde{g}^{\sigma[\gamma}\widetilde{g}^{\delta]\nu}\Big]_{h(x')}\Big|\Omega\Big\rangle \times 4(aa')^{D-4}\partial_{\alpha}\partial'_{\gamma} i\Big[{}_{\beta}\Delta_{\delta}\Big](x;x')\Bigg\}\;, \tag{34}$$

where square bracketed indices are anti-symmetrized. If we specialize to just the tail contribution then the expectation value on the first line of (34) goes like $1/\Delta x^{D-4}$. Hence the entire curly-bracketed term is at most logarithmically divergent, and that only when both of the derivatives on the second line of (34) act on the most singular part of the photon propagator (9). Because the less singular parts vanish for $D=4$ we can make the simplification,

$$4(aa')^{D-4}\partial_{\alpha}\partial'_{\gamma} i\Big[{}_{\beta}\Delta_{\delta}\Big](x;x') \longrightarrow 4(aa')^{\frac{D}{2}-2}\eta_{\beta\delta}\partial_{\alpha}\partial'_{\gamma} i\Delta(x;x')\;. \tag{35}$$

Substituting (35) in (34), and exploiting relation (22), gives an operator expression for the tail contribution to the $F(x;x')$ structure function,

$$\begin{aligned}-\vec{\nabla}\cdot\vec{\nabla}' F_{3t}(x;x') &= -\partial_i\partial'_j\Bigg\{\Big\langle\Omega\Big|\Big[\sqrt{-\widetilde{g}}\,\widetilde{g}^{ik}\Big]_{h(x)}\times\Big[\sqrt{-\widetilde{g}}\,\widetilde{g}^{i\ell}\Big]_{h(x')}\Big|\Omega\Big\rangle_{\rm tail} \\ &\qquad\times(aa')^{\frac{D}{2}-2}\Big[\delta_{k\ell}\partial_0\partial'_0-\partial_k\partial'_\ell\Big] i\Delta(x;x')\Bigg\}\;, \qquad (36)\\ &= -\kappa^2\partial_i\partial'_j\Bigg\{\Big\langle\Omega\Big|\frac{1}{4}h^2\delta^{ik}\delta^{j\ell}-\frac{1}{2}h^{ik}h\delta^{j\ell}-\frac{1}{2}h\delta^{ik}h^{j\ell}+h^{ik}h^{j\ell}\Big|\Omega\Big\rangle_{\rm tail} \\ &\qquad\times(aa')^{\frac{D}{2}-2}\Big[\delta_{k\ell}\partial_0\partial'_0-\partial_k\partial'_\ell\Big] i\Delta(x;x')\Bigg\}\;. \qquad (37)\end{aligned}$$

Substituting the tail part of the propagator (19) and performing the simple contractions implies,

$$F_{3t}(x;x') = \kappa^2 i\delta\Delta_A(x;x')(aa')^{\frac{D}{2}-2}\Big[(D-1)\partial_0\partial'_0-\vec{\nabla}\cdot\vec{\nabla}'\Big] i\Delta(x;x')\;. \tag{38}$$

The final step is extracting the derivatives from inside the square brackets of (38), which is done generically in Appendix B. From relation (58) we infer,

$$F_{3t}(x;x') = -\frac{\kappa^2 H^2\partial\cdot\partial'}{64\pi^4}\left[\frac{\ln(\frac{1}{4}H^2\Delta x^2)-4}{\Delta x^2}\right] - \frac{\kappa^2 H^2\partial_0\partial'_0}{16\pi^4}\left[\frac{\ln(\frac{1}{4}H^2\Delta x^2)+2}{\Delta x^2}\right]$$

$$-\frac{\kappa^2 H^{D-2}(D-1)\Gamma(\frac{D}{2}+1)\, i\delta^D(x-x')}{(4\pi)^{\frac{D}{2}}(D-3)(D-4)} \; . \quad (39)$$

Both the divergence and the $\ln(\frac14 H^2\Delta x^2)$ terms agree with the results reported in equations (129) and (130) of [16].

Relations (22-23) provide an operator expression for the $G(x;x')$ structure function,

$$(D-1)\partial\cdot\partial' F(x;x') + (D-2)\vec{\nabla}\cdot\vec{\nabla}' G(x;x') = \vec{\nabla}\cdot\vec{\nabla}' F(x;x') + i\Big[{}^k\Pi^k\Big](x;x') \; . \quad (40)$$

Specializing (40) to the 3-point tail contribution gives,

$$(D-2)\vec{\nabla}\cdot\vec{\nabla}' G_{3\mathrm{t}}(x;x') = -(D-2)\vec{\nabla}\cdot\vec{\nabla}' F_{3\mathrm{t}}(x;x') + (D-1)\partial_0\partial_0' F_{3\mathrm{t}}(x;x')$$
$$-\partial_\rho\partial_\sigma'\Bigg\{\Big\langle\Omega\Big|\Big[\sqrt{-\widetilde{g}}\Big(\widetilde{g}^{\rho\alpha}\widetilde{g}^{\beta\mu}-\widetilde{g}^{\rho\beta}\widetilde{g}^{\alpha\mu}\Big)\Big]_{h(x)}\times\Big[\sqrt{-\widetilde{g}}\Big(\widetilde{g}^{\sigma\gamma}\widetilde{g}^{\delta\nu}-\widetilde{g}^{\sigma\delta}\widetilde{g}^{\gamma\nu}\Big)\Big]_{h(x')}\Big|\Omega\Big\rangle_{\mathrm{tail}}$$
$$\times(aa')^{D-4}\eta_{\beta\delta}\partial_\alpha\partial_\gamma' i\Delta(x;x')\Bigg\} \; . \quad (41)$$

The $\rho=\sigma=0$ component of the contraction in (41) cancels the factor of $(D-1)\partial_0\partial_0' F_{3t}(x;x')$. Expanding out the remaining terms gives,

$$(D-2)\vec{\nabla}\cdot\vec{\nabla}' G_{3\mathrm{t}}(x;x') = -(D-2)\vec{\nabla}\cdot\vec{\nabla}' F_{3\mathrm{t}}(x;x')$$
$$+\partial_0\partial_i'\Bigg\{\Big\langle\Omega\Big|\sqrt{-\widetilde{g}}\,\widetilde{g}^{k\ell}\times\sqrt{-\widetilde{g}}\Big(\widetilde{g}^{ij}\widetilde{g}^{k\ell}-\widetilde{g}^{i\ell}\widetilde{g}^{jk}\Big)\Big|\Omega\Big\rangle_{\mathrm{tail}}(aa')^{\frac{D}{2}-2}\partial_0\partial_j' i\Delta(x;x')\Bigg\}$$
$$+\partial_i\partial_0'\Bigg\{\Big\langle\Omega\Big|\sqrt{-\widetilde{g}}\Big(\widetilde{g}^{ij}\widetilde{g}^{k\ell}-\widetilde{g}^{i\ell}\widetilde{g}^{jk}\Big)\times\sqrt{-\widetilde{g}}\,\widetilde{g}^{k\ell}\Big|\Omega\Big\rangle_{\mathrm{tail}}(aa')^{\frac{D}{2}-2}\partial_j\partial_0' i\Delta(x;x')\Bigg\}$$
$$-\partial_i\partial_j'\Bigg\{\Big\langle\Omega\Big|\sqrt{-\widetilde{g}}\Big(\widetilde{g}^{im}\widetilde{g}^{k\ell}-\widetilde{g}^{i\ell}\widetilde{g}^{mk}\Big)\times\sqrt{-\widetilde{g}}\Big(\widetilde{g}^{jn}\widetilde{g}^{k\ell}-\widetilde{g}^{j\ell}\widetilde{g}^{kn}\Big)\Big|\Omega\Big\rangle_{\mathrm{tail}}$$
$$\times(aa')^{\frac{D}{2}-2}\partial_m\partial_n' i\Delta(x;x')\Bigg\}, \quad (42)$$

$$= -\kappa^2\vec{\nabla}\cdot\vec{\nabla}'\Bigg\{(aa')^{\frac{D}{2}-2} i\delta\Delta_A(x;x')\Bigg[2\Big(\frac{D^2-5D+5}{D-3}\Big)\vec{\nabla}\cdot\vec{\nabla}'+(D-2)(D-1)\partial_0\partial_0'\Bigg]$$
$$\times i\Delta(x;x')\Bigg\} + (D-2)^2\kappa^2\partial_0\partial_i'\Bigg\{(aa')^{\frac{D}{2}-2} i\delta\Delta_A(x;x')\partial_0\partial_i' i\Delta(x;x')\Bigg\}$$
$$+(D-2)^2\kappa^2\partial_i\partial_0'\Bigg\{(aa')^{\frac{D}{2}-2} i\delta\Delta_A(x;x')\partial_i\partial_0' i\Delta(x;x')\Bigg\}$$
$$-(D-4)(D-1)\kappa^2\partial_i\partial_j'\Bigg\{(aa')^{\frac{D}{2}-2} i\delta\Delta_A(x;x')\partial_i\partial_j' i\Delta(x;x')\Bigg\}, \quad (43)$$

where some of the terms from the first line of (43) derive from the operator expressions on the last line of (42) and spatial translation invariance has been exploited. It remains to extract the inner derivatives using relation (58) and solve for $G_{3t}(x;x')$,

$$G_{3t}(x;x') = \frac{\kappa^2 H^2 \partial\cdot\partial'}{32\pi^4}\left[\frac{\ln(\frac14 H^2\Delta x^2)+2}{\Delta x^2}\right] . \tag{44}$$

This result agrees with the $\ln(\frac14 H^2\Delta x^2)$ term reported in equation (132) of [16]. However, it has neither the ultraviolet divergence reported in equation (131) of that paper, nor the associated factor of $\ln(\mu^2\Delta x^2)$ reported in equation (132). These terms come from the non-tail part of the graviton propagator.

3.3 Tail Renormalization

The right hand diagram of Fig. 1 stands for renormalization counterterms. Their contributions to the two structure functions was given in equation (24). We must bear in mind the fact that the coefficients $\overline{C}$ and ΔC are not those appropriate to the full vacuum polarization [16] but rather just the parts needed to cancel the divergences in our tail results (30) and (39) for $F(x;x')$ and (33) and (44) for $G(x;x)$.

Based on expressions (30) and (39) the best choice for the $\overline{C}$ counterterm is,

$$\overline{C} = \frac{\kappa^2 H^{D-4}}{(4\pi)^{\frac{D}{2}}}\left\{\frac{D(D-5)\Gamma(D-1)\pi\cot(\frac{\pi D}{2})}{16\Gamma(\frac{D}{2})} + \frac{(D-1)\Gamma(\frac{D}{2}+1)}{4(D-3)(D-4)} + 1\right\} . \tag{45}$$

After combining with the primitive results (30) and (39) and taking the unregulated limit we obtain,

$$F_{\rm tail}(x;x') = \frac{\kappa^2 H^2}{(2\pi)^4}\left\{2\pi^2\ln(a) i\delta^4(x-x') + \frac14 \partial^2\left[\frac{\ln(\frac14 H^2\Delta x^2)}{\Delta x^2}\right]\right.$$
$$\left. + \partial_0^2\left[\frac{\ln(\frac14 H^2\Delta x^2)+2}{\Delta x^2}\right]\right\} . \tag{46}$$

Expression (46) agrees exactly with the intrinsically de Sitter part of the full $F(x;x')$ structure function (25), including even the parts on the second line which play no role in either the secular enhancement of dynamical photons [19] or the logarithmic running of the Coulomb potential [17].

Based on expressions (33) and (44) the best choice for the nocovariant ΔC counterterm is,

$$\Delta C = \frac{\kappa^2 H^{D-4}}{(4\pi)^{\frac{D}{2}}} \Biggl\{ -\frac{(D^2-4D+1)\Gamma(D-1)\pi \cot(\frac{\pi D}{2})}{4(D-3)\Gamma(\frac{D}{2})} + 1 \Biggr\} \, . \tag{47}$$

The unregulated limit of the renormalized tail contribution to $G(x;x')$ is,

$$G_{\rm tail}(x;x') = \frac{\kappa^2 H^2}{(2\pi)^4} \Biggl\{ -4\pi^2 \ln(a) i\delta^4(x-x') - \frac{1}{2}\partial^2 \Biggl[\frac{\ln(\frac{1}{4}H^2\Delta x^2)}{\Delta x^2} \Biggr] \Biggr\}. \tag{48}$$

Expression (48) does not agree with (26) because the primitive 3-point tail contribution (44) lacks both the divergence and the associated μ-dependent logarithm of the full 3-point result [16].

4 Discussion

Our aim has been to see how much of the intrinsically de Sitter part (25-26) of the vacuum polarization arises from replacing the full graviton propagator (10) with just its tail part (19). Our result is that the tail reproduces all of (25) but not all of (26). This means that the graviton tail is responsible for the the secular enhancement of dynamical photons (3), but not for all of the logarithmic running of the Coulomb potential (2). The remaining parts of (26) come from using the most singular part of the graviton propagator in the 3-point contribution. Although these terms have no factor of H^2, they do contain $\frac{1}{aa'} = H^2 \eta\eta'$. When the inner derivatives are passed through this term they can act on the $\eta\eta'$ and leave the required factor of H^2.

Our result means that the tail term is *not* responsible for all the interesting secular effects mediated by the one loop vacuum polarization. This may not be the setback it would seem for the crucial task of extending Satrobinsky's stochastic technique [20, 23] to quantum gravity. The large logarithms of interest derive from three sources:

1. Explicit factors of $\ln(aa')$ and $\ln(H^2\Delta x^2)$ in the tail part of the graviton propagator (19);

2. Factors of $(aa')^{\frac{D}{2}-2}/(D-4)$ and $(\Delta x)^{D-4}/(D-4)$ which arise either in primitive ultraviolet divergences or in the counterterms which remove them; and

3. The integration of interaction vertices which one must do in higher loop diagrams.

The one loop tail contributions (46) and (48) that we have computed come from the first two sources. The reason (48) does not give all the interesting parts (26) of the $G(x;x')$ structure function is that we have missed some ultraviolet divergences from the most singular part of the propagator. *These sorts of terms are easy to recover using renormalization group techniques.* The "hard" contributions — the ones for which one loop divergences do not predict higher loop results — are those from the other two sources. So perhaps the key to dealing with the large logarithms is to combine Starobinsky's technique with the renormalization group.

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5 Appendix A: $i\Delta_{\Sigma I}(x;x')$ Expansions

The infinite series expansions for the scalar propagators (14) are:

$$
\begin{aligned}
i\Delta_{\Sigma A}(x;x') &= \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \sum_{n=1}^{\infty} \Bigl(\frac{aa'H^2\Delta x^2}{4}\Bigr)^n \\
&\qquad \times \Biggl\{ \frac{\Gamma(n+D-1)}{n\,\Gamma(n+\frac{D}{2})} - \frac{\Gamma(n+\frac{D}{2}+1)}{(n-\frac{D}{2}+2)(n+1)!} \Bigl(\frac{4}{aa'H^2\Delta x^2}\Bigr)^{\frac{D}{2}-2} \Biggr\}, \qquad (49) \\
i\Delta_{\Sigma B}(x;x') &= \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \sum_{n=1}^{\infty} \Bigl(\frac{aa'H^2\Delta x^2}{4}\Bigr)^n
\end{aligned}
$$

$$\times\Big\{\frac{\Gamma(n+D-2)}{\Gamma(n+\frac{D}{2})}-\frac{\Gamma(n+\frac{D}{2})}{(n+1)!}\Big(\frac{4}{aa'H^2\Delta x^2}\Big)^{\frac{D}{2}-2}\Big\}, \quad (50)$$

$$i\Delta_{\Sigma C}(x;x') = \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\sum_{n=1}^{\infty}\Big(\frac{aa'H^2\Delta x^2}{4}\Big)^n$$

$$\times\Big\{\frac{(n+1)\Gamma(n+D-3)}{\Gamma(n+\frac{D}{2})}-\frac{(n-\frac{D}{2}+3)\Gamma(n+\frac{D}{2}-1)}{(n+1)!}\Big(\frac{4}{aa'H^2\Delta x^2}\Big)^{\frac{D}{2}-2}\Big\}. \quad (51)$$

6 Appendix B: Extracting Derivatives

Evaluating the 3-point contributions requires that we wish pass derivatives of the photon propagator to the left of $(aa')^{\frac{D}{2}-2}i\delta\Delta_A(x;x')$ in expressions of the form,

$$(aa')^{\frac{D}{2}-2}i\delta\Delta_A(x;x')\partial_\mu\partial'_\nu i\Delta(x;x') \; . \quad (52)$$

The propagator $i\Delta(x;x')$ goes like $1/\Delta x^{D-2}$. From equation (16) we see that $(aa')^{\frac{D}{2}-2}i\delta\Delta_A(x;x')$ contains three distinct sorts of coordinate dependence. The result passing derivatives through each of these terms is,

$$\frac{1}{\Delta x^{D-4}}\,\partial_\mu\partial'_\nu\frac{1}{\Delta x^{D-2}} = \frac{[D\partial_\mu\partial'_\nu-\eta_{\mu\nu}\partial\cdot\partial']}{4(D-3)}\frac{1}{\Delta x^{2D-6}}\;, \quad (53)$$

$$(aa')^{\frac{D}{2}-2}\,\partial_\mu\partial'_\nu\frac{1}{\Delta x^{D-2}} = \Big[\partial_\mu-\Big(\frac{D}{2}-2\Big)Ha\delta^0{}_\mu\Big] \times\Big[\partial'_\nu-\Big(\frac{D}{2}-2\Big)Ha'\delta^0{}_\nu\Big]\Big[\frac{(aa')^{\frac{D}{2}-2}}{\Delta x^{D-2}}\Big]\;, \quad (54)$$

$$\ln(aa')\,\partial_\mu\partial'_\nu\frac{1}{\Delta x^{D-2}} = \partial_\mu\partial'_\nu\Big[\frac{\ln(aa')}{\Delta x^{D-2}}\Big]-\partial_\mu\Big[\frac{Ha'\delta^0{}_\nu}{\Delta x^{D-2}}\Big]-\partial'_\nu\Big[\frac{Ha\delta^0{}_\mu}{\Delta x^{D-2}}\Big]\;. \quad (55)$$

The first terms on the right hand side of relations (53-55) give the derivatives acting on the product $(aa')^{\frac{D}{2}-2}i\delta\Delta_A(x;x')i\Delta(x;x')$. That product is integrable for $D=4$ so we can take its unregulated limit. The secondary terms of relations (54-55) cancel in $D=4$ dimensions, so it only remains to consider the second term on the right of relation (53),

$$\partial\cdot\partial'\frac{1}{\Delta x^{2D-6}} = \partial\cdot\partial'\Big[\frac{1}{\Delta x^{2D-6}}-\frac{\mu^{D-4}}{\Delta x^{D-2}}\Big]-\frac{4\pi^{\frac{D}{2}}\mu^{D-4}i\delta^D(x-x')}{\Gamma(\frac{D}{2}-1)}\;, \quad (56)$$

$$= -\Big(\frac{D-4}{2}\Big)\partial\cdot\partial'\Big[\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\Big]+O\Big((D-4)^2\Big)-\frac{4\pi^{\frac{D}{2}}\mu^{D-4}i\delta^D(x-x')}{\Gamma(\frac{D}{2}-1)}\;. \quad (57)$$

Setting $\mu = \frac{1}{2}H$ and putting everything together gives,

$$(aa')^{\frac{D}{2}-2} i\delta\Delta_A(x;x')\partial_\mu\partial'_\nu i\Delta(x;x') = -\frac{H^2\partial_\mu\partial'_\nu}{32\pi^4}\left[\frac{\ln(\frac{1}{4}H^2\Delta x^2)+2}{\Delta x^2}\right]$$
$$+\frac{H^2\eta_{\mu\nu}\partial\cdot\partial'}{128\pi^4}\left[\frac{\ln(\frac{1}{4}H^2\Delta x^2)}{\Delta x^2}\right]+\frac{H^{D-2}\eta_{\mu\nu}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(\frac{D}{2}+1)\, i\delta^D(x-x')}{2(D-3)(D-4)}+O(D-4). \quad (58)$$

References

[1] E. Lifshitz, J. Phys. (USSR) **10**, 116 (1946) [Gen. Rel. Grav. **49**, no. 2, 18 (2017)]. doi:10.1007/s10714-016-2165-8

[2] L. P. Grishchuk, Sov. Phys. JETP **40**, 409 (1975) [Zh. Eksp. Teor. Fiz. **67**, 825 (1974)].

[3] L. H. Ford and L. Parker, Phys. Rev. D **16**, 1601 (1977). doi:10.1103/PhysRevD.16.1601

[4] A. A. Starobinsky, JETP Lett. **30**, 682 (1979) [Pisma Zh. Eksp. Teor. Fiz. **30**, 719 (1979)].

[5] B. S. DeWitt and R. W. Brehme, Annals Phys. **9**, 220 (1960). doi:10.1016/0003-4916(60)90030-0

[6] Y. Z. Chu and G. D. Starkman, Phys. Rev. D **84**, 124020 (2011) doi:10.1103/PhysRevD.84.124020 [arXiv:1108.1825 [astro-ph.CO]].

[7] T. Tanaka, Y. Mino, M. Sasaki and M. Shibata, Phys. Rev. D **54**, 3762 (1996) doi:10.1103/PhysRevD.54.3762 [gr-qc/9602038].

[8] Y. Mino, M. Sasaki and T. Tanaka, Phys. Rev. D **55**, 3457 (1997) doi:10.1103/PhysRevD.55.3457 [gr-qc/9606018].

[9] T. C. Quinn and R. M. Wald, Phys. Rev. D **56**, 3381 (1997) doi:10.1103/PhysRevD.56.3381 [gr-qc/9610053].

[10] C. L. Wang and R. P. Woodard, Phys. Rev. D **92**, 084008 (2015) doi:10.1103/PhysRevD.92.084008 [arXiv:1508.01564 [gr-qc]].

[11] M. B. Fröb and E. Verdaguer, JCAP **1603**, no. 03, 015 (2016) doi:10.1088/1475-7516/2016/03/015 [arXiv:1601.03561 [hep-th]].

[12] A. F. Radkowski, Annals Phys. **56**, 319 (1970).

[13] D. M. Capper, M. J. Duff and L. Halpern, Phys. Rev. D **10**, 461 (1974). doi:10.1103/PhysRevD.10.461

[14] N. C. Tsamis and R. P. Woodard, Commun. Math. Phys. **162**, 217 (1994). doi:10.1007/BF02102015

[15] R. P. Woodard, gr-qc/0408002.

[16] K. E. Leonard and R. P. Woodard, Class. Quant. Grav. **31**, 015010 (2014) doi:10.1088/0264-9381/31/1/015010 [arXiv:1304.7265 [gr-qc]].

[17] D. Glavan, S. P. Miao, T. Prokopec and R. P. Woodard, Class. Quant. Grav. **31**, 175002 (2014) doi:10.1088/0264-9381/31/17/175002 [arXiv:1308.3453 [gr-qc]].

[18] K. E. Leonard and R. P. Woodard, Phys. Rev. D **85**, 104048 (2012) doi:10.1103/PhysRevD.85.104048 [arXiv:1202.5800 [gr-qc]].

[19] C. L. Wang and R. P. Woodard, Phys. Rev. D **91**, no. 12, 124054 (2015) doi:10.1103/PhysRevD.91.124054 [arXiv:1408.1448 [gr-qc]].

[20] A. A. Starobinsky, Lect. Notes Phys. **246**, 107 (1986). doi:10.1007/3-540-16452-9_6

[21] R. P. Woodard, Nucl. Phys. Proc. Suppl. **148**, 108 (2005) doi:10.1016/j.nuclphysbps.2005.04.056 [astro-ph/0502556].

[22] N. C. Tsamis and R. P. Woodard, Nucl. Phys. B **724**, 295 (2005) doi:10.1016/j.nuclphysb.2005.06.031 [gr-qc/0505115].

[23] A. A. Starobinsky and J. Yokoyama, Phys. Rev. D **50**, 6357 (1994) doi:10.1103/PhysRevD.50.6357 [astro-ph/9407016].

[24] S. P. Miao and R. P. Woodard, Phys. Rev. D **74**, 044019 (2006) doi:10.1103/PhysRevD.74.044019 [gr-qc/0602110].

[25] T. Prokopec, N. C. Tsamis and R. P. Woodard, Annals Phys. **323**, 1324 (2008) doi:10.1016/j.aop.2007.08.008 [arXiv:0707.0847 [gr-qc]].

[26] S. P. Miao and R. P. Woodard, Class. Quant. Grav. **25**, 145009 (2008) doi:10.1088/0264-9381/25/14/145009 [arXiv:0803.2377 [gr-qc]].

[27] H. Kitamoto and Y. Kitazawa, Phys. Rev. D **83**, 104043 (2011) doi:10.1103/PhysRevD.83.104043 [arXiv:1012.5930 [hep-th]].

[28] H. Kitamoto and Y. Kitazawa, Phys. Rev. D **85**, 044062 (2012) doi:10.1103/PhysRevD.85.044062 [arXiv:1109.4892 [hep-th]].

[29] S. P. Miao and R. P. Woodard, Class. Quant. Grav. **23**, 1721 (2006) doi:10.1088/0264-9381/23/5/016 [gr-qc/0511140].

[30] S. P. Miao and R. P. Woodard, Phys. Rev. D **74**, 024021 (2006) doi:10.1103/PhysRevD.74.024021 [gr-qc/0603135].

[31] D. Glavan, S. P. Miao, T. Prokopec and R. P. Woodard, Class. Quant. Grav. **32**, no. 19, 195014 (2015) doi:10.1088/0264-9381/32/19/195014 [arXiv:1504.00894 [gr-qc]].

[32] T. Prokopec, O. Tornkvist and R. P. Woodard, Annals Phys. **303**, 251 (2003) doi:10.1016/S0003-4916(03)00004-6 [gr-qc/0205130].

[33] K. E. Leonard, T. Prokopec and R. P. Woodard, Phys. Rev. D **87**, no. 4, 044030 (2013) doi:10.1103/PhysRevD.87.044030 [arXiv:1210.6968 [gr-qc]].

[34] K. E. Leonard, T. Prokopec and R. P. Woodard, J. Math. Phys. **54**, 032301 (2013) doi:10.1063/1.4793987 [arXiv:1211.1342 [gr-qc]].