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Black hole scalar charge from a topological horizon integral in Einstein-dilaton-Gauss-Bonnet gravity

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In theories of gravity that include a scalar field, a compact object's scalar charge is a crucial quantity since it controls dipole radiation, which can be strongly constrained by pulsar timing and gravitational wave observations. However in most such theories, computing the scalar charge requires simultaneously solving the coupled, nonlinear metric and scalar field equations of motion. In this article we prove that in linearly-coupled Einstein-dilaton-Gauss-Bonnet gravity, a black hole's scalar charge is completely determined by the horizon surface gravity times the Euler characteristic of the bifurcation surface, without solving any equations of motion. Within this theory, black holes announce their horizon topology and surface gravity to the rest of the universe through the dilaton field. In our proof, a 4-dimensional topological density descends to a 2-dimensional topological density on the bifurcation surface of a Killing horizon. We also comment on how our proof can be generalized to other topological densities on general G-bundles, and to theories where the dilaton is non-linearly coupled to the Euler density.

1. INTRODUCTION

Despite the theoretical beauty and continued consistency with observations [1] of general relativity (GR), there are strong motivations for studying theories of gravity beyond GR. These range from attempts at quantum theories of gravity [2, 3], to trying to explain some phenomenon or fix some problem (inflation [4, 5], dark matter [6, 7], dark energy [7, 8], black hole information [9, 10]) by changing the gravity theory, or exploring the theory space to better understand gravity theories.

Almost all beyond-GR theories include additional degrees of freedom and a large number of proposed beyond-GR theories include one or more massless or very light scalar fields [11]. With a long-ranged scalar field in the theory, compact objects (e.g. black holes or neutron stars) may acquire a *scalar charge*: the spherically-symmetric, $1/r$ component of the scalar field sourced by a body. Scalar charges are crucial in the dynamics of a compact-object binary system, as they control the presence or absence of scalar dipole radiation. The presence of dipole radiation would dominate over the otherwise-leading quadrupolar emission of gravitational waves, a so-called "pre-Newtonian" correction. Such an effect can be strongly constrained by both pulsar timing and the direct detection of gravitational waves [12].

In most theories with scalar fields, computing a black hole's scalar charge requires solving the coupled set of metric and scalar field equations in the nonlinear gravity regime. On occasion, this can be accomplished analytically with symmetry reduction [13, 14], and/or simplifying assumptions such as a perturbative treatment away from GR [15]. However, the general case requires numerics for fully nonlinear partial differential equations [16].

A dramatic simplification occurs in linearly-coupled Einstein-dilaton-Gauss-Bonnet gravity (EdGB). Already in [17, 18], the authors had found through explicit calculations, perturbative in α, a coupling parameter, that the dilaton charge q on the Kerr background is (in our conventions)

$$q = \alpha \frac{a^2 - M^2 + M\sqrt{M^2 - a^2}}{2Ma^2} + \mathcal{O}(\alpha^2)\,. \tag{1.1}$$

However it was not noticed that this particular combination of mass and spin is equal to the Kerr surface gravity (Eq. 12.5.4 of [19]), so that in fact $q = \alpha\kappa_{\text{Kerr}} + \mathcal{O}(\alpha^2)$.

We prove (Theorem 1) that in this theory the dilaton scalar charge is given by the rather simple expression

$$q = \tfrac{1}{2}\alpha\kappa \text{Euler}(B)\,, \tag{1.2}$$

where κ is the surface gravity of a Killing horizon, and Euler(B) is the Euler characteristic of the bifurcation surface. Our proof is valid to all orders in α, and moreover does not require explicitly solving any field equations. In fact the metric does not need to satisfy any equations of motion; the proof is valid on any asymptotically flat, stationary-axisymmetric black hole spacetime. Thus in linearly-coupled EdGB, the horizon topology and surface gravity of black holes are known at spatial infinity by looking at the asymptotic falloff of the dilaton.

We present the proof of our main result in § 2, which relies on a certain divergence identity (Eq. 2.8) for a massless scalar field linearly coupled to a topological form and the Killing symmetry of the black hole spacetime. Combining these ingredients allows the 4-dimensional Euler (or Gauss-Bonnet) topological density to descend to the 2-dimensional Euler characteristic on the bifurcation

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surface. In § 3, we discuss the implications of this proof, and how it can be generalised to theories where the dilaton is nonlinearly-coupled to the Euler density. We also show how to extend our proof to higher dimensions and other invariants, such as Chern characters of complex G-bundles. We demonstrate this with an example of an axion coupled to electromagnetism, wherein the axion charge measures the electromagnetic potential of the horizon times the magnetic monopole charge of the black hole.

NOTATION AND CONVENTIONS

We follow the factor and sign conventions in [19]. Tensor fields on spacetime will be denoted by abstract indices $\mu, \nu, \lambda, \ldots$ from the lowercase Greek alphabet, and we use lowercase letters from the beginning of the Latin alphabet $a, b, c, \ldots$ to denote tensors on the orthonormal frame bundle, summarized in Appendix A. When using an index-free notation for differential forms and vector fields we denote differential forms by a bold-face symbol. When translating differential forms to and from an index notation, we use the symbol $\equiv$ to denote such a translation, for example, for a p-form $\boldsymbol{A}$ we have $(\boldsymbol{A})_{\mu_1\ldots\mu_p} \equiv A_{\mu_1\ldots\mu_p}$. Our conventions for the *volume form* $\boldsymbol{\varepsilon}_4$, the *Hodge dual* $*$ and the *interior product* are as follows

$$\boldsymbol{\varepsilon}_4 \equiv \varepsilon_{\mu_1\ldots\mu_4} \tag{1.3a}$$

$$\varepsilon^{\mu_1\ldots\mu_k\mu_{k+1}\ldots\mu_4}\varepsilon_{\mu_1\ldots\mu_k\nu_{k+1}\ldots\nu_4} = -(4-k)!k!\delta^{[\mu_{k+1}}_{\nu_{k+1}}\cdots\delta^{\mu_4]}_{\nu_4} \tag{1.3b}$$

$$(*\boldsymbol{A})_{\mu_1\ldots\mu_{4-p}} \equiv \tfrac{1}{p!}\varepsilon^{\nu_1\ldots\nu_p}{}_{\mu_1\ldots\mu_{4-p}}A_{\nu_1\ldots\nu_p} \tag{1.3c}$$

$$(X\cdot\boldsymbol{A})_{\mu_1\ldots\mu_{p-1}} \equiv X^\nu A_{\nu\mu_1\ldots\mu_{p-1}} \tag{1.3d}$$

where $\boldsymbol{A}$ is a p-form and X^μ is some vector field.

2. LAGRANGIAN AND THE DILATON CHARGE

We consider a theory with gravity on a 4-dimensional spacetime M with a Lorentzian metric $g_{\mu\nu}$ and a scalar dilaton field ϑ. The dynamics of the theory is given by the Lagrangian 4-form

$$\boldsymbol{L} = \boldsymbol{L}_{gravity} + \boldsymbol{L}_\vartheta \tag{2.1}$$

where $\boldsymbol{L}_{gravity}$ is some gravitational Lagrangian which is independent of the dilaton ϑ.

There are several theories in the literature which are referred to as *Einstein-dilaton-Gauss-Bonnet*, commonly with an exponential coupling [20, 21] between a dilaton and the Euler density (defined below). For our main result we will consider a linear coupling (which admits a shift symmetry $\vartheta \to \vartheta + \text{const.}$), and comment on the extension to more general couplings in § 3. We take the dilaton-Gauss-Bonnet Lagrangian $\boldsymbol{L}_\vartheta$ to be

$$\begin{aligned}\boldsymbol{L}_\vartheta &= \tfrac{1}{2}(*d\vartheta)\wedge d\vartheta + \tfrac{\alpha}{8}\vartheta\boldsymbol{\mathcal{E}}\\ &= \boldsymbol{\varepsilon}_4\left(-\tfrac{1}{2}\nabla_\mu\vartheta\nabla^\mu\vartheta + \tfrac{\alpha}{8}\vartheta\mathcal{E}\right)\end{aligned} \tag{2.2}$$

where the 4-form $\boldsymbol{\mathcal{E}}$ corresponds to the 4-dimensional *Euler density* as

$$\begin{aligned}&\boldsymbol{\mathcal{E}} = \boldsymbol{\varepsilon}_4\ \mathcal{E}\\ \text{with}\quad &\mathcal{E} = -({}^*R^*)^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho}\\ &\quad = R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - 4R_{\mu\nu}R^{\mu\nu} + R^2\,,\end{aligned} \tag{2.3}$$

and the double dual of the Riemann tensor is defined as

$$({}^*R^*)^{\mu\nu\lambda\rho} := \tfrac{1}{4}\varepsilon^{\mu\nu\sigma\tau}R_{\sigma\tau\gamma\delta}\varepsilon^{\gamma\delta\lambda\rho}\,. \tag{2.4}$$

While our final result can be presented in terms of tensor fields on spacetime, it will be, instead, convenient to use orthonormal tetrads $\boldsymbol{e}^a$ and a connection $\boldsymbol{\omega}^a{}_b$. It is more natural in the following analysis to treat $\boldsymbol{e}^a$ and $\boldsymbol{\omega}^a{}_b$ as *globally* well-defined fields on a *principal bundle*, following the treatment in [22] (see also [23])—we summarise the essential points in Appendix A. Readers unfamiliar with the bundle formalism can skip to Eq. 2.12 where we present the tensorial form of the charge used in our main result Theorem 1.

In terms of the connection $\boldsymbol{\omega}^a{}_b$, we can write $\boldsymbol{\mathcal{E}}$ as an exact form [22, 24, 25]

$$\begin{aligned}&\boldsymbol{\mathcal{E}} = \epsilon_{abcd}\boldsymbol{R}^{ab}\wedge\boldsymbol{R}^{cd} = d\boldsymbol{\Upsilon}\\ &\text{with } \boldsymbol{\Upsilon} = \epsilon_{abcd}\boldsymbol{\omega}^{ab}\wedge\left(\boldsymbol{R}^{cd} - \tfrac{1}{3}\boldsymbol{\omega}^c{}_e\wedge\boldsymbol{\omega}^{ed}\right)\,,\end{aligned} \tag{2.5}$$

where $\boldsymbol{R}^a{}_b$ is the curvature 2-form of the connection (see Eq. A.6). We emphasise that the 3-form $\boldsymbol{\Upsilon}$ *cannot* be represented as a covariant tensor on spacetime—even if one writes $\boldsymbol{\Upsilon}$ in some local coordinate system it necessarily involves undifferentiated Christoffel symbols which are not covariant tensors. In the language of principal bundles, while $\boldsymbol{\Upsilon}$ is globally well-defined, it is not a horizontal form on the bundle.

Varying the Lagrangian with the dilaton ϑ gives the dilaton equation of motion

$$0 = \boldsymbol{E}_\vartheta = d*d\vartheta + \tfrac{\alpha}{8}\boldsymbol{\mathcal{E}} = \boldsymbol{\varepsilon}_4\left(\Box\vartheta + \tfrac{\alpha}{8}\mathcal{E}\right) \tag{2.6}$$

where $\Box := \nabla_\mu\nabla^\mu$ is the wave operator. Note that we *do not* impose the gravitational equations of motion obtained by varying the metric $g_{\mu\nu}$, that is the metric can be considered as a “background field”.

Using Eq. 2.5 the dilaton equation of motion can be written as an exact form, and one could attempt to integrate this over some region of spacetime bounded by two Cauchy surfaces to get the scalar charge. However, since $\boldsymbol{\Upsilon}$ is not covariant (as explained above), the result would depend on the choice of coordinate system, or equivalently on the choice of orthonormal tetrads.

To avoid using non-covariant quantities, we proceed instead as follows. Using Eq. A.9b we have, for any vector field X^μ

$$\begin{aligned}\epsilon_{abcd}\boldsymbol{R}^{ab}\wedge \pounds_X\boldsymbol{\omega}^{cd} &= \epsilon_{abcd}\boldsymbol{R}^{ab}\wedge D(X\cdot\boldsymbol{\omega}^{cd}) + \tfrac{1}{2}X\cdot\boldsymbol{\mathcal{E}} \\ &= d\left[\epsilon_{abcd}\boldsymbol{R}^{ab}(X\cdot\boldsymbol{\omega}^{cd})\right] + \tfrac{1}{2}X\cdot\boldsymbol{\mathcal{E}}\end{aligned} \tag{2.7}$$

Using this in Eq. 2.6, we get the divergence identity

$$\begin{aligned}\boldsymbol{\mathcal{J}}_X &= d\boldsymbol{\mathcal{Q}}_X + X\cdot\boldsymbol{E}_\vartheta \\ &= \pounds_X(*d\vartheta) + \tfrac{\alpha}{4}\epsilon_{abcd}\boldsymbol{R}^{ab}\wedge\pounds_X\boldsymbol{\omega}^{cd}\end{aligned} \tag{2.8a}$$

$$\text{where } \boldsymbol{\mathcal{Q}}_X = X\cdot(*d\vartheta) + \tfrac{\alpha}{4}\epsilon_{abcd}\boldsymbol{R}^{ab}(X\cdot\boldsymbol{\omega}^{cd}) \tag{2.8b}$$

The current $\boldsymbol{\mathcal{J}}_X$ is conserved for any vector field X^μ, i.e., $d\boldsymbol{\mathcal{J}}_X = 0$, on solutions to the dilaton equation of motion. Further, $\boldsymbol{\mathcal{J}}_X = 0$ whenever X^μ is a symmetry i.e. $\pounds_X g_{\mu\nu} = \pounds_X\vartheta = 0$. $\boldsymbol{\mathcal{J}}_X$ vanishes on symmetries because it is related to the Lie derivative of the Noether current for the shift symmetry $\vartheta\to\vartheta+\text{const}$. In this case, the expression for the charge can be simplified using Eq. A.10 to get

$$\boldsymbol{\mathcal{Q}}_X = X\cdot(*d\vartheta) - \tfrac{\alpha}{4}\epsilon_{abcd}\boldsymbol{R}^{ab}(e^c_\mu e^d_\nu\nabla^\mu X^\nu)\,. \tag{2.9}$$

We also provide a tensorial expression as follows. Define $\mathcal{Q}_X^{\mu\nu}$ from $\boldsymbol{\mathcal{Q}}_X$ by

$$(\boldsymbol{\mathcal{Q}}_X)_{\mu\nu} \equiv \mathcal{Q}_X^{\lambda\rho}\varepsilon_{\lambda\rho\mu\nu} \tag{2.10}$$

so that the current vector $\mathcal{J}_X^\mu$ is given by

$$\begin{aligned}(\boldsymbol{\mathcal{J}}_X)_{\mu\nu\lambda} &\equiv \mathcal{J}_X^\rho\varepsilon_{\rho\mu\nu\lambda} \\ \text{with}\qquad \mathcal{J}_X^\mu &= \nabla_\nu\mathcal{Q}_X^{\mu\nu} + X^\mu(\Box\vartheta + \tfrac{\alpha}{8}\mathcal{E})\,.\end{aligned} \tag{2.11}$$

For a symmetry X^μ we can compute (using Eq. 2.9)

$$\mathcal{Q}_X^{\mu\nu} = -2X^{[\mu}\nabla^{\nu]}\vartheta + \tfrac{\alpha}{2}({}^*R^*)^{\mu\nu\lambda\rho}\nabla_\lambda X_\rho\,. \tag{2.12}$$

It can be checked that $\mathcal{J}_X^\mu = 0$, using the identities $\nabla_\mu({}^*R^*)^{\mu\nu\lambda\rho} = 0$ (which follows from the Bianchi identity Eq. A.8), $\nabla_\mu\nabla_\nu X_\lambda = R_{\lambda\nu\mu}{}^\rho X_\rho$ for a Killing field X^μ (see Eq. C.3.6 [19]), and $C^{\mu\lambda\rho\sigma}C_{\nu\lambda\rho\sigma} = \frac{1}{4}\delta^\mu_\nu C^{\tau\lambda\rho\sigma}C_{\tau\lambda\rho\sigma}$ for the *Weyl tensor* $C_{\mu\nu\lambda\rho}$ in 4-dimensions [26]. The result of Theorem 1 can be obtained by evaluating the integral $0 = \int_\Sigma u_\mu\mathcal{J}_K^\mu$ where, K^μ is the horizon Killing field and u^μ is the future-pointing unit time-like normal to a Cauchy surface Σ, as described below.

* * *

We now consider an asymptotically flat, stationary-axisymmetric black hole spacetime $(M, g_{\mu\nu})$ shown in Fig. 1, with a stationary-axisymmetric dilaton field ϑ satisfying the equation of motion Eq. 2.6 (we take all fields to be smooth (C^∞) throughout M). We assume the spacetime has a bifurcate Killing horizon $\mathscr{H} := \mathscr{H}^+\cup\mathscr{H}^-$, with a bifurcation surface $B := \mathscr{H}^+\cap\mathscr{H}^-$. We assume that B is compact but do not assume other restrictions

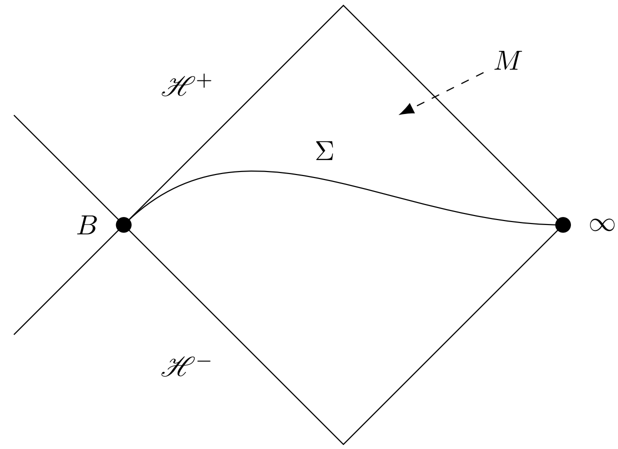

FIG. 1. Schematic diagram for black hole spacetime. Note this should not be considered as a Carter-Penrose diagram, in particular we make no assumptions about the existence of null infinity.

on its topology.[1] Let the Killing field generating $\mathscr{H}$ be $K^\mu = t^\mu + \Omega_{\mathscr{H}}\phi^\mu$ where t^μ denotes the time translation Killing field and ϕ^μ denotes the axial Killing field associated with the horizon rotation parameter $\Omega_{\mathscr{H}}$. Let Σ denote a Cauchy surface for the black hole exterior. We assume that Σ has one asymptotically flat end (with asymptotic conditions given by Eq. 2.14 below), and a boundary at B.

The asymptotic flatness conditions on our spacetime are as follows. There exist asymptotically Minkowskian coordinates $x^\mu = (t, x, y, z)$ such that the global Killing fields asymptote to the Minkowski ones at the rates

$$t^\mu = (\partial_t)^\mu + \mathcal{O}(1/r)\,,\quad \phi^\mu = (\partial_\phi)^\mu\big(1+\mathcal{O}(1/r)\big)\,, \tag{2.13}$$

the metric and dilaton asymptote at the rates

$$g_{\mu\nu} = \mathring{g}_{\mu\nu} + \mathcal{O}(1/r)\,,\quad \vartheta = \vartheta_\infty(\theta) + \frac{q(\theta)}{r} + \mathcal{O}(1/r^2) \tag{2.14}$$

where $\mathring{g}_{\mu\nu}dx^\mu dx^\nu \equiv -dt^2 + dx^2 + dy^2 + dz^2$ is the Minkowskian flat metric in these coordinates, and (r,θ,ϕ) are defined in terms of (x,y,z) in the standard way. In addition, all n^{th} derivatives of the above quantities (with respect to these coordinates) are required to fall off faster by an additional factor of $1/r^n$.[2]

For such spacetimes, we define the *scalar charge* to be the spherically-symmetric part of the asymptotic $1/r$

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[1] We need not assume that B is connected but, for notational convenience, we will assume that this is the case.

[2] Since we do not impose the gravitational equations of motion we do not need to ensure that such asymptotically flat spacetimes exist as solutions.

falloff (see also [13, 27])

$$q := -\frac{1}{4\pi}\int_\infty \boldsymbol{\varepsilon}_2 \partial_r \vartheta = \frac{1}{4\pi}\int_\infty \boldsymbol{\varepsilon}_2 \frac{q(\theta)}{r^2}\,. \tag{2.15}$$

Here, and henceforth, $\int_\infty$ means that the integral is evaluated over an asymptotic 2-sphere S_r of radius r and then one takes the limit $r \to \infty$, and $\boldsymbol{\varepsilon}_2$ is the induced area element on S_r.[3]

Now, we use Eq. 2.8 to prove our main result.

Theorem 1. *On any asymptotically flat, stationary-axisymmetric black hole spacetime (as defined above) the scalar charge Eq. 2.15 is given by*

$$q = \tfrac{1}{2}\alpha\kappa \mathrm{Euler}(B)\,, \tag{2.16}$$

where κ is the surface gravity of the black hole and Euler(B) *is the* Euler characteristic *of the bifurcation surface B.*

Proof. For $X^\mu = K^\mu$, the horizon Killing field, integrate $0 = \boldsymbol{\mathcal{J}}_K = d\boldsymbol{\mathcal{Q}}_K$ on the Cauchy surface Σ to get

$$\int_\infty \boldsymbol{\mathcal{Q}}_K = \int_B \boldsymbol{\mathcal{Q}}_K \tag{2.17}$$

where we have assumed that the induced orientations in both integrals are outward-pointing i.e. the space-like normals in Σ point towards ∞. We now evaluate each side of the above expression using Eq. 2.9.

To compute the contribution to the charge at infinity, let $\mathring{\boldsymbol{e}}^a = (dt, dx, dy, dz)$ be an asymptotic tetrad adapted to the asymptotically Minkowskian coordinates. From Eq. 2.14, we have near infinity

$$\boldsymbol{e}^a = \mathring{\boldsymbol{e}}^a + \mathcal{O}(1/r)\,, \quad \boldsymbol{\omega}^{ab} = \mathcal{O}(1/r^2)\,, \quad \boldsymbol{R}^{ab} = \mathcal{O}(1/r^3)\,. \tag{2.18}$$

Then, using Eq. 2.15, we find

$$\begin{aligned}\int_\infty \boldsymbol{\mathcal{Q}}_K &= \int_\infty K\cdot(*d\vartheta) = -\int_\infty \boldsymbol{\varepsilon}_2 \partial_r\vartheta \\ &= 4\pi q\,,\end{aligned} \tag{2.19}$$

where the curvature terms do not contribute due to the falloffs in Eq. 2.18, and the contribution from the ϕ^μ-part of K^μ vanishes since ϕ^μ is tangent to the spheres at infinity.

On B, we have $K^\mu|_B = 0$ and $\nabla_\mu K_\nu|_B = \kappa\tilde{\varepsilon}_{\mu\nu}$ where $\tilde{\varepsilon}_{\mu\nu}$ is the *binormal* to the bifurcation surface (see § 12.5 [19]). It can be shown that κ is a constant over *any* bifurcate Killing horizon $\mathscr{H}$ [28]. Using this in Eq. 2.9, and noting that $(\boldsymbol{\varepsilon}_2)_{\mu\nu} = -\frac{1}{2}\varepsilon^{\lambda\rho}{}_{\mu\nu}\tilde{\varepsilon}_{\lambda\rho}$ is the intrinsic area element to B, we get

$$\begin{aligned}\int_B \boldsymbol{\mathcal{Q}}_K &= \tfrac{\alpha}{2}\kappa\int_B \epsilon_{ab}\boldsymbol{R}^{ab} = \tfrac{\alpha}{2}\kappa\int_B \boldsymbol{\varepsilon}_2 R_2 \\ &= 2\pi\alpha\kappa\mathrm{Euler}(B)\,,\end{aligned} \tag{2.20}$$

where ϵ_{ab} is the tetrad component of $\boldsymbol{\varepsilon}_2$, R_2 is the intrinsic Ricci scalar of B, and the last line uses the 2-dimensional Gauss-Bonnet theorem [29].

Combining Eqs. 2.17, 2.19 and 2.20, we have our result Eq. 2.16. □

We note here that the contribution to the scalar charge from B (Eq. 2.20) can be written as

$$\int_B \boldsymbol{\mathcal{Q}}_K = -\tfrac{\alpha}{8}\kappa\int_B \boldsymbol{\varepsilon}_2 \frac{\delta\mathcal{E}}{\delta R_{\mu\nu\lambda\rho}}\tilde{\varepsilon}_{\mu\nu}\tilde{\varepsilon}_{\lambda\rho} \tag{2.21}$$

in analogy with the *Wald entropy formula* [30, 31]. This relation arises due to the second term in Eq. 2.8b.

3. DISCUSSION AND EXTENSION

A concise interpretation of this result is that black holes communicate their horizon topology and surface gravity to spatial infinity, by encoding this information in the asymptotic falloff of the dilaton.

When the bifurcation surface B is a topological 2-sphere, as is the case for a Kerr spacetime (or any continuous deformation of Kerr), we have $q = \alpha\kappa$, consistent with Eq. 1.1. However our result is valid to *all orders* in α, not just the decoupling limit, and at no point have we imposed the metric equations of motion. Our proof also generalises to stationary stars as long as the matter fields do not couple to ϑ (the matter Lagrangian is independent of ϑ). In this case there is no interior boundary and we get $q = 0$ (see [27] for an earlier approach based on the generalised-Gauss-Bonnet-Chern theorem; the main advantage of our proof is that it is based on local and covariant quantities).

Our proof can also be adapted to any theory (in any number of spacetime dimensions) where the dilaton field is linearly coupled to a topological density $\mathcal{T}$ which depends on the curvature with suitable modifications of Eq. 2.8 with terms of the form $\delta\mathcal{T}/\delta R_{\mu\nu\lambda\rho}$. We can also consider theories where a scalar field is linearly-coupled to a topological density of any G-bundle where G is some, possibly non-abelian, group. In this case, the charge contribution takes a form similar to Eq. 2.21 with the Riemann tensor replaced by the curvature in the G-bundle (see [22]). As an example we consider briefly a massless axion field φ coupled with strength g to electromagnetism through the second *Chern character* [29], via the Lagrangian

$$\begin{aligned}\boldsymbol{L}_\varphi &= \frac{1}{2}(*d\varphi)\wedge d\varphi + \frac{g}{2}\varphi\ \boldsymbol{F}\wedge\boldsymbol{F} \\ &= \frac{1}{2}(*d\varphi)\wedge d\varphi + \frac{g}{2}\varphi\ d(\boldsymbol{F}\wedge\boldsymbol{A})\,,\end{aligned} \tag{3.1}$$

[3] The induced area element $\boldsymbol{\varepsilon}_2$ grows as r^2 and Eq. 2.15 converges in the limit as $r \to \infty$.

where $\boldsymbol{A}$ is the electromagnetic vector potential, i.e. a $U(1)$-connection, and $\boldsymbol{F} := d\boldsymbol{A}$ is the field strength. For the analogue of Eq. 2.8, we then have

$$\boldsymbol{\mathcal{J}}^{(\varphi)}_X = \pounds_X(*d\varphi) + g\boldsymbol{F}\wedge\pounds_X\boldsymbol{A} \tag{3.2a}$$

$$\boldsymbol{\mathcal{Q}}^{(\varphi)}_X = X\cdot(*d\varphi) + g\boldsymbol{F}(X\cdot\boldsymbol{A})\,. \tag{3.2b}$$

Following through the proof of Theorem 1 under the assumption that $\boldsymbol{A} = \mathcal{O}(1/r)$ at spatial infinity, the axion scalar charge for a black hole is given by

$$q^{(\varphi)} = gV_{\mathscr{H}}Q_m \qquad \text{where} \qquad Q_m := \tfrac{1}{4\pi}\int_B \boldsymbol{F}\,, \tag{3.3}$$

and $V_{\mathscr{H}} := K\cdot\boldsymbol{A}|_B$ is the horizon potential, which is constant on $\mathscr{H}$ (see [32] and Theorem 1 [22]). Here Q_m is the black hole's magnetic charge, proportional to the first Chern number of the $U(1)$-bundle over B [29].

A number of recent investigations [33–35] focused on a non-linear coupling between the dilaton and Euler density, replacing $\vartheta\boldsymbol{\mathcal{E}} \to f(\vartheta)\boldsymbol{\mathcal{E}}$ in the Lagrangian Eq. 2.2 (with a nonlinear function $f(\vartheta)$, the theory no longer has the shift symmetry $\vartheta \to \vartheta + \text{const.}$). These authors pointed out that when $f'(\vartheta)$ vanishes at some value ϑ_0, such a theory admits standard ("no hair") GR solutions with a constant dilaton field $\vartheta = \vartheta_0$. However, if $f''(\vartheta_0) > 0$, these solutions can be unstable, and revert to a stable branch of black hole solutions with dilaton hair.

Analyzing this coupling, we again have Eq. 2.8a with

$$\begin{aligned}
\boldsymbol{E}_\vartheta &= d*d\vartheta + \tfrac{\alpha}{8}f'(\vartheta)\boldsymbol{\mathcal{E}}\\
\boldsymbol{\mathcal{Q}}_X &= X\cdot d\vartheta + \tfrac{\alpha}{4}f'(\vartheta)\epsilon_{abcd}\boldsymbol{R}^{ab}(X\cdot\boldsymbol{\omega}^{cd})\\
\boldsymbol{\mathcal{J}}_X &= \pounds_X(*d\vartheta) + \tfrac{\alpha}{4}f'(\vartheta)\epsilon_{abcd}\boldsymbol{R}^{ab}\wedge\pounds_X\boldsymbol{\omega}^{cd}\\
&\quad + \tfrac{\alpha}{4}f''(\vartheta)\epsilon_{abcd}\boldsymbol{R}^{ab}\wedge d\vartheta\ (X\cdot\boldsymbol{\omega}^{cd})\,.
\end{aligned} \tag{3.4}$$

Note that in the nonlinear case, $\boldsymbol{\mathcal{J}}_X$ is still conserved, but $\boldsymbol{\mathcal{J}}_X \neq 0$ even for a symmetry X^μ of the solution, because there is no more symmetry under the shift $\vartheta \to \vartheta + \text{const}$. Therefore, a bulk integral term will remain in the computation of the charge. A repetition of our proof gives the dilaton charge

$$\begin{aligned}
q &= \tfrac{\alpha}{8\pi}\kappa\int_B \boldsymbol{\varepsilon}_2 f'(\vartheta)R_2\\
&\quad - \tfrac{\alpha}{8\pi}\int_\Sigma \boldsymbol{\varepsilon}_3 f''(\vartheta)u_\mu({}^*R^*)^{\mu\nu\lambda\rho}\nabla_\nu\vartheta\nabla_\lambda K_\rho\,,
\end{aligned} \tag{3.5}$$

where $\boldsymbol{\varepsilon}_3$ is the induced volume element and u^μ is the unit timelike normal to the Cauchy surface Σ.

In Eq. 3.5, we can easily see the difference between the linearly coupled case and the non-linear case. Because of the lack of shift symmetry in the nonlinear case, the bulk term remains (i.e. $f''(\vartheta) \neq 0$), and thus the dilaton charge depends on the metric and dilaton solutions throughout the entire spacetime. We no longer get a relation between quantities evaluated purely on the boundaries. However, if the dilaton field has a small variation throughout spacetime, it may be possible to expand the theory [27] around some typical value $\vartheta_1 \neq \vartheta_0$, where $f'(\vartheta_1) \neq 0$. Then if we expand the coupling function to linear order around ϑ_1, we recover the shift-symmetric, linearly-coupled theory.

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Appendix A: Short primer on tetrads and spin connection

In this appendix we give a short introduction to tetrads and connections. We will use the language of principal bundles, for the details of which we refer the reader to the classic treatment[4] of [39–41] (see also the appendix of [22]).

On spacetime, the *oriented orthonormal tetrads* $(\boldsymbol{e}^a)_\mu \equiv e^a_\mu$ are defined by

$$g_{\mu\nu} = \eta_{ab}e^a_\mu e^b_\nu\,, \quad \varepsilon_{\mu\nu\lambda\rho} = \epsilon_{abcd}e^a_\mu e^b_\nu e^c_\lambda e^d_\rho\,, \tag{A.1}$$

where $\eta_{ab} = \text{diag}(-1,1,1,1)$ and ϵ_{abcd} are the metric and orientation in $\mathbb{R}^4$ with $\epsilon_{0123} = 1$. The "inverse" tetrads e^μ_a satisfy

$$e^\mu_a e^a_\nu = \delta^\mu_\nu\,, \quad e^\mu_b e^a_\mu = \delta^a_b\,. \tag{A.2}$$

The torsion-free *spin connection* $\boldsymbol{\omega}^a{}_b$ is given by

$$(\boldsymbol{\omega}^a{}_b)_\mu = e^a_\nu\nabla_\mu e^\nu_b\,. \tag{A.3}$$

Given a metric $g_{\mu\nu}$, the tetrads and the spin connection are only determined up to a local Lorentz transformation $\Lambda^a{}_b(x)$ which depends on the the point x in spacetime

$$\boldsymbol{e}^a \mapsto \Lambda^a{}_b\boldsymbol{e}^b\,, \quad \boldsymbol{\omega}^a{}_b \mapsto \Lambda^a{}_c\boldsymbol{\omega}^c{}_d(\Lambda^{-1})^d{}_b + \Lambda^a{}_c d(\Lambda^{-1})^c{}_b\,. \tag{A.4}$$

We note that the connection transforms non-covariantly. Due to this "internal gauge freedom" it is more natural to treat these as fields on a principal bundle with structure group given by the Lorentz group.

[4] Note that these references may use different conventions when converting differential forms to an index notation.

Differential forms on spacetime which are covariant under Lorentz transformations are represented by *horizontal* differential forms on the bundle. The connection is represented by a Lie algebra-valued 1-form (which by definition is not horizontal). The connection is then determined uniquely by the torsion-free condition

$$0 = D\boldsymbol{e}^a = d\boldsymbol{e}^a + \boldsymbol{\omega}^a{}_b \wedge \boldsymbol{e}^b\,, \tag{A.5}$$

where D is the *covariant exterior derivative* defined by $\boldsymbol{\omega}^a{}_b$. The curvature 2-form is defined by the horizontal form

$$\boldsymbol{R}^a{}_b := D\boldsymbol{\omega}^a{}_b = d\boldsymbol{\omega}^a{}_b + \boldsymbol{\omega}^a{}_c \wedge \boldsymbol{\omega}^c{}_b\,, \tag{A.6}$$

Since the curvature is horizontal it represents a covariant form on spacetime related to the Riemann tensor as

$$(\boldsymbol{R}^a{}_b)_{\mu\nu} \equiv R^\lambda{}_{\rho\mu\nu} e^a_\lambda e^\rho_b\,, \tag{A.7}$$

and the Bianchi identity reads

$$D\boldsymbol{R}^{ab} = 0 \implies \nabla_{[\sigma} R^{\lambda\rho}{}_{\mu\nu]} = 0\,. \tag{A.8}$$

The Lie derivative of the tetrads and connection with respect to vector fields on the bundle is

$$\pounds_X \boldsymbol{e}^a = D(X\cdot \boldsymbol{e}^a) - (X\cdot\boldsymbol{\omega}^a{}_b)\boldsymbol{e}^b \tag{A.9a}$$
$$\pounds_X \boldsymbol{\omega}^a{}_b = X\cdot \boldsymbol{R}^a{}_b + D(X\cdot\boldsymbol{\omega}^a{}_b) \tag{A.9b}$$

From the spacetime point of view, this encodes the fact that Lie derivatives of the tetrads and connection are only defined up to a local Lorentz transformation (encoded in the vertical part of the bundle vector field).

It can be shown (see Lemma A.2 [22]) that a bundle vector field which preserves the tetrads, $\pounds_X \boldsymbol{e}^a = 0$, projects to a Killing field X^μ of the metric on spacetime, and further satisfies

$$X\cdot\boldsymbol{\omega}^{ab} = -e^a_\mu e^b_\nu \nabla^\mu X^\nu\,, \tag{A.10}$$

where the left-hand-side is computed as a function on the bundle.

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