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Circular polarization of the cosmic microwave background (CMB) arises in the standard cosmological model from Faraday conversion of the linear polarization generated at the surface of last scatter by various sources of birefringence along the line of sight. If the sources of birefringence are generated at linear order in primordial density perturbations the principal axes of the index-of-refraction tensor are determined by gradients of the primordial density field. Since linear polarization at the surface of last scatter is generated at linear order in density perturbations, the circular polarization thus arises at second order in primordial perturbations. Here, we re-visit the calculation of the circular polarization using the total-angular-momentum formalism, which allows for some simplifications in the calculation of the angular power spectrum of the circular polarization—especially for the dominant photon-photon scattering contribution—and also provides some new intuition.

I. INTRODUCTION

The polarization of the cosmic microwave background (CMB) is the subject of considerable study, as its detailed characterization can constrain cosmological parameters and the mass distribution at intermediate and low redshifts, and perhaps lead to insights about the early Universe [1, 2]. The polarization that is being sought is generally restricted, though, to the linear polarization that is generated at linear order in the amplitude of primordial perturbations.

Circular polarization is generated by Faraday conversion of the primordial linear polarization as CMB photon propagate through a birefringent medium along the line of sight [3]. There are several ways that birefringence can arise at linear order in the primordial density perturbation; for example, spin polarization of neutral hydrogen atoms during the cosmic dark ages [4] and light-light scattering induced by quantum-electrodynamic effects [4–8]—the latter is expected to be the dominant effect. Thus, circular polarization arises at second order in primordial perturbations. This circular polarization has been derived in considerable detail in recent work, especially that of Ref. [4], which provided the principal inspiration for this work.

Here we revisit the calculation of fluctuations in circular polarization using the total-angular-momentum formalism [9, 10]. This formalism was developed to facilitate the predictions of cosmological models for observables on the celestial sphere. The calculation of the circular polarization provides a nice example of the power of the formalism. While no new physics is presented in this paper, there may be calculational simplifications and physical insights presented by the approach. Generalization to exotic early-Universe sources of circular polarization (e.g., from birefringence associated with primordial magnetic fields [11], neutrinos [12] or primordial gravitational waves) may also be simplified with this approach. The utility will be less apparent for late-Universe effects like those involving the first stars [13, 14].

Current upper limits to the circular-polarization power spectrum are $l(l+1)C_l^{VV} \lesssim (200\,\mu\mathrm{K})^2$ over the multipole range $33 < l < 307$ [15] and $\lesssim (1000\,\mu\mathrm{K})^2$ at larger angular scales [16]. These are orders of magnitude above the expected primordial signal. Still, renewed attention on the circular polarization is warranted given the prospects with the Cosmology Large Angular Scale Surveyor (CLASS) [17] for improvement in the sensitivity, by several orders of magnitude, to CMB circular polarization and thus for the prospects of hitherto unanticipated signals from new physics.

Below, we begin with a brief review of the total-angular-momentum (TAM) formalism, focussing specifically on the ingredients relevant for the circular-polarization calculation. We then show how the circular-polarization power spectrum is obtained in the TAM formalism, using relevant physics results from prior work. We first discuss the power spectrum from photon-photon scattering, which provides the dominant effect. We then discuss the power spectrum from a subdominant contribution due to spin polarization in hydrogen atoms, as this illustrates other aspects of the TAM approach.

II. TOTAL ANGULAR MOMENTUM WAVES

The plane waves $e^{i\vec{k}\cdot\vec{x}}$ constitute a simple and familiar complete orthonormal basis for scalar functions in Euclidean 3-space, the geometry of comoving fixed-time hypersurfaces a flat FRW Universe. They are easily generalized to vector fields with the adoption of three polarization vectors $\epsilon_i(\vec{k})$ for each Fourier mode and likewise to symmetric trace-free tensor fields with symmetric trace-free polarization tensors $\epsilon_{ij}(\vec{k})$. The formalism becomes unwieldy for cosmological observations, though, when projecting these plane waves onto the spherical surface of the sky.

The total-angular-momentum (TAM) formalism [9, 10] provides an alternative set of basis functions, for scalar, vector, and tensor fields in Euclidean 3-space, that are easily amemable to the calculation of observables on the celestial sphere. As we will see, TAM waves also provides geometric insights in some cases. We begin with a scalar function $\phi(\vec{x})$ which can be expanded as

$$\phi(\vec{x}) = \sum_{klm} \phi^k_{lm} \Psi^k_{lm}(\vec{x}), \tag{1}$$

in terms of scalar TAM waves $\Psi^k_{lm}(\vec{x})$. Here k is a wavenumber, $\sum_k$ is a shorthand for $\int k^2\, dk/(2\pi)^3$, and the scalar TAM wave is $\Psi^k_{lm} = 4\pi i^l j_l(k|\vec{x}|) Y_{lm}(\hat{x})$ in terms of spherical Bessel functions and spherical harmonics.[1] The orthonormality properties of these functions imply that the scalar TAM-wave expansion coefficients are

$$\phi^k_{lm} = \int d^3x\, \phi(\vec{x}) \left[\Psi^k_{lm}(\vec{x})\right]^*. \tag{2}$$

If $\phi(\vec{x})$ is a realization of a statistically isotropic and homogeneous random field, then

$$\left\langle \phi^k_{lm} \left(\phi^{k'}_{l'm'}\right)^* \right\rangle = \delta_{kk'} \delta_{ll'} \delta_{mm'} P_\phi(k), \tag{3}$$

where $P_\phi(k)$ is the power spectrum for ϕ, and $\delta_{kk'}$ is shorthand for $(2\pi)^3 k^{-2} \delta_D(k-k')$, with $\delta_D(k-k')$ a Dirac delta function.

Now consider a symmetric ($T_{ij} = T_{ji}$) trace-free ($T^i_{\ i} = 0$) tensor $T_{ij}(\vec{x})$, which involves five degrees of freedom most generally.[2] This can be expanded in terms of five sets of tensor TAM waves $\Psi^{\alpha,k}_{(lm)ij}(\vec{x})$, for α =L,VE,VB,TE,TB. Here, L refers to the longitudinal mode, and VE/VB and TE/TB to vector and tensor modes, respectively. Of relevance here is that the longitudinal mode has no variation in the plane orthogonal to the direction of the gradient of T_{ij}, while the other four modes do. Given that all the physical effects we will deal with in this paper trace back to primordial scalar perturbations, which can have no variation in the plane orthogonal to the direction of the gradient, we will need in this paper only the longitudinal TAM wave. This is obtained from the scalar TAM wave from [Eq. (81) in Ref. [9]],

$$-\nabla^2 \Psi^{\mathrm{L},k}_{(lm)ij}(\vec{x}) = \sqrt{\frac{3}{2}} \left(\nabla_i \nabla_j - \frac{1}{3} g_{ij} \nabla^2\right) \Psi_{lm}(\vec{x}), \tag{4}$$

where ∇ is a derivative with respect to $\vec{x}$ and g_{ij} a metric in Euclidean 3-space.

As we will see shortly, the circular polarization and index-of-refraction tensors are projections of a three-dimensional tensor onto the two-dimensional celestial sphere. That is, the polarization tensor in a direction $\hat{n}$ will be $P_{ab}(\hat{n}) \propto \mathcal{P}^{\ i}_a(\hat{n}) \mathcal{P}^{\ j}_b(\hat{n}) T_{ij}(\hat{n}\chi)$, where $\mathcal{P}^{ab}(\hat{n}) = g^{ab} - \hat{n}^a \hat{n}^b$ projects onto the plane orthogonal to $\hat{n}$. Since we deal here only with mechanisms that involve primordial density perturbations, we will deal only with longitudinal tensors. The final key TAM-wave result we need then comes from Eq. (94) in Ref. [9], which provides the projection of the longitudinal-tensor TAM wave onto the celestial sphere. Of the three terms in Eq. (94) in Ref. [9], we use the one that lives in the plane of the sky (the other two have components in the radial direction). The result is,

$$\mathcal{P}^{\ i}_a(\hat{n}) \mathcal{P}^{\ j}_b(\hat{n}) \Psi^{\mathrm{L},k}_{(lm)ij}(\vec{x}) = -\frac{4\pi i^l}{N_l} \sqrt{\frac{3}{2}} \frac{j_l(k\chi)}{(k\chi)^2} Y^{\mathrm{E}}_{(lm)ab}(\hat{n}), \tag{5}$$

where $\chi = |\vec{x}|$, $Y^{\mathrm{E}}_{(lm)ab}(\hat{n})$ is the E-mode tensor spherical harmonic [18, 19], normalized as in Ref. [1] to correspond to the conventions of Refs. [20, 21], and $N_l \equiv \sqrt{2(l-2)!/(l+2)!}$. Eq. (5) is the central TAM-wave result we will use. Eq. (5) also shows that the absence of B modes from density perturbations [19, 20] is related to the fact that the projection of a longitudinal-tensor field onto the sky has no B mode.

III. CALCULATION

The circular polarization induced in a photon observed from direction $\hat{n}$ with linear polarization described by Stokes parameter Q and U is

$$V(\hat{n}) = \phi_Q(\hat{n}) U(\hat{n}) - \phi_U(\hat{n}) Q(\hat{n}), \tag{6}$$

where

$$\phi_{Q,U}(\hat{n}) = \frac{2}{c} \int dr\, \omega(r) n_{Q,U}(\hat{n}r), \tag{7}$$

are phase shifts for the Q and U polarizations. Here, ω is the photon angular frequency, $n_{Q,U}(r)$ are the indexes of refraction for the Q and U polarizations at a distance r (related to the comoving distance χ by $dr = a d\chi$, where a is the scale factor. If we take $\hat{n}$ in the $\hat{z}$ direction, the index-of-refraction tensors are $n_Q = (1/2)(n_{xx} - n_{yy})$ and $n_U = n_{xy}$, where $n_{ij} \propto \chi_{ij} \propto \partial_i \mu_j$ are components of the Cartesian index-of-refraction tensor and χ_{ij} is the magnetic-susceptibility tensor.

The Stokes parameters are a spin-2 field under rotations about the $\hat{n}$ axis. Equivalently they can be represented as the components of a symmetric trace-free tensor,

$$P_{ab}(\hat{n}) = \frac{1}{\sqrt{2}} \begin{pmatrix} Q(\hat{n}) & U(\hat{n}) \\ U(\hat{n}) & -Q(\hat{n}) \end{pmatrix}. \tag{8}$$

Likewise, the phase shifts are components of a tensor,

$$\Phi_{ab}(\hat{n}) = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_Q(\hat{n}) & \phi_U(\hat{n}) \\ \phi_U(\hat{n}) & -\phi_Q(\hat{n}) \end{pmatrix}. \tag{9}$$

[1] Our definition of the harmonics differ by a factor of $4\pi i^l$ from that in Ref. [9], a choice that makes for more compact expression.

[2] We use indexes $ij\ldots$ for three-dimensional tensors and following Ref. [18] use indexes $ab\ldots$ for their projections onto the two-dimensional celestial sphere.

From these two tensors we can construct two scalars: a tensor analog $P^{ab}\Phi_{ab}$ of a dot product, and a tensor analog of a cross product, which we identify as the circular polarization,

$$V(\hat{n}) = \epsilon_{ac} P^{ab}(\hat{n}) \Phi_b{}^c(\hat{n}), \tag{10}$$

where ϵ_{ac} is the antisymmetric tensor on the celestial sphere (see, e.g., Ref. [18]).

If we consider arbitrary photon trajectories, the propagation of the different linear-polarization states will be described by a three-dimensional cartesian tensor n_{ij}, which we can take to be traceless $n^i_i = 0$, since the relevant observables in this paper will depend only on phase differences. If birefringence is generated by the primordial density field, the index-of-refraction tensor $n_{ij}(\vec{x})$ must be related to the density field by

$$n_{ij}(\vec{x}) \propto \left(\nabla_i \nabla_j - \frac{1}{3} g_{ij} \nabla^2 \right) \delta(\vec{x}). \tag{11}$$

This is the only STF tensor field that can be constructed from $\delta(\vec{x})$ that has no variation in the two directions orthogonal to $\vec{\nabla}\delta(\vec{x})$.

Now we return to the calculation of the circular-polarization power spectrum. This involves a linear-polarization field provided in terms of $Q(\hat{n})$ and $U(\hat{n})$ as a function of position $\hat{n}$ on the spherical sky. These are components of a spin-2 field and can thus be replaced by the geometrical invariants $E(\hat{n})$ and $B(\hat{n})$. This linear-polarization field is presumed to arise at linear order in primordial perturbations, and the polarization generated at position $\chi\hat{n}$ at a comoving distance χ in the direction $\hat{n}$ is related to the components of the quadrupole moment, orthogonal to $\hat{n}$, of the radiation field at that point. That is, the contribution to the polarization tensor at position $\chi\hat{n}$ is $P_{ab}(\hat{n}) \propto (\mathcal{P}\,\mathcal{P}\,n)_{ab}(\hat{n})$, as discussed in Section II. In words, the observed polarization field is related to the components of a longitudinal-tensor field constructed from the density field.

The same arguments apply also to Φ_{ab} and $\phi_{Q,U}$. To be precise (as it will be required in Section III B), the relation between the phase-shift tensor, on the celestial sphere, in terms of the projection $(\mathcal{P}\mathcal{P}n)_{ab}(\hat{n})$ is

$$\Phi_{ab}(\hat{n}) = \frac{\omega_0}{\sqrt{2}} \frac{2}{c} \int d\chi \, \Big[(\mathcal{P}\mathcal{P}n)_{ab}(\hat{n}\chi) - \frac{1}{2} (\mathcal{P}\mathcal{P}n)^c_c(\hat{n}\chi) g_{ab} \Big], \tag{12}$$

where ω_0 is the angular frequency today, and here g_{ab} is a metric on the 2-sphere. The trace in the second term will be irrelevant when we project in the next paragraph onto the traceless tensor spherical harmonics.

As discussed in Section II, the linear-polarization pattern generated by primordial density perturbations can be expanded,

$$P_{ab}(\hat{n}) = \sum_{lm} P_{lm} Y^{\rm E}_{(lm)ab}(\hat{n}), \tag{13}$$

in terms of the E-mode tensor spherical harmonics. The same arguments apply, however, also to the phase shifts—only ϕ_E is non-zero, and $\phi_B = 0$—and likewise to $n_{E,B}(r\hat{n})$. We thus conclude that we can expand the phase-shift tensor as

$$\Phi_{ab}(\hat{n}) = \sum_{lm} \Phi_{lm} Y^{\rm E}_{(lm)ab}(\hat{n}). \tag{14}$$

The spherical-harmonic coefficients of the circular polarization, given by Eq. (10), are then

$$V_{lm} = \sum_{l_1 m_1} \sum_{l_2 m_2} P_{l_1 m_1} \Phi_{l_2 m_2} \times \int d\hat{n} \, \epsilon^{ab} Y^{\rm E}_{(l_1 m_1)ac}(\hat{n}) Y^{\rm E}_{(l_2 m_2)b}{}^{c}(\hat{n}) Y^*_{lm}(\hat{n}). \tag{15}$$

The factor $\epsilon^{ab} Y^{\rm E}_{(l_1 m_1)ac}(\hat{n})$ in the integrand is a tensor with indices b and c that are then contracted with another tensor that is symmetric under $b \leftrightarrow c$. We can therefore replace $\epsilon^{ab} Y^{\rm E}_{(l_1 m_1)ac}(\hat{n})$ with its symmetrized version, which is $Y^{\rm B}_{(l_1 m_1)}{}^b{}_c(\hat{n})$. The integral over the product of the three spherical harmonics is thus, with the help of Eq. (21) in Ref. [22], evaluated to be

$$G^{lm}_{l_1 m_1 l_2 m_2} = -\xi^{lm}_{l_1, -m_1, l_2 m_2} H^l_{l_1 l_2}, \tag{16}$$

with $\xi^{lm}_{l_1 m_1 l_2 m_2}$ and $H^l_{l_1 l_2}$ as defined in terms of Wigner-3j symbols in Ref. [22]. Note that $G^{lm}_{l_1 m_1 l_2 m_2}$ is nonzero only for $l + l_1 + l_2 =$odd, and also that $G^{lm}_{l_1 m_1 l_2 m_2} = -G^{lm}_{l_2 m_2 l_1 m_1}$. We then find

$$V_{lm} = \sum_{l_1 l_2 m_1 m_2} P_{l_1 m_1} \Phi_{l_2 m_2} G^{lm}_{l_1 m_1 l_2 m_2}. \tag{17}$$

We now calculate the angular power spectrum C_l^{VV} for the circular polarization. Using the antisymmetry of $G^{lm}_{l_1 m_1 l_2 m_2}$ under $l_1 m_1 \leftrightarrow l_2 m_2$, the circular-polarization power spectrum is then,

$$C_l^{VV} = \sum_{l_1 m_1 l_2 m_2} \left(C^{PP}_{l_1} C^{\Phi\Phi}_{l_2} - C^{P\Phi}_{l_1} C^{P\Phi}_{l_2} \right) \left| G^{lm}_{l_1 m_1 l_2 m_2} \right|^2. \tag{18}$$

The minus sign arises because the circular polarization is a cross-product between the two fields; if there is a cross-correlation between P and Φ, it reduces the contribution to C_l^{VV} from what it would be if they were uncorrelated. The sum over m_1, m_2 can be can be done using results from Ref. [22], from which follows,

$$C_l^{VV} = \sum_{l_1 l_2} \frac{(2l_1+1)(2l_2+1)}{4\pi} \left[C^{PP}_{l_1} C^{\Phi\Phi}_{l_2} - C^{P\Phi}_{l_1} C^{P\Phi}_{l_2} \right] \left(H^l_{l_1 l_2} \right)^2, \tag{19}$$

where C_l^{PP} and $C_l^{\Phi\Phi}$ are the polarization and Φ power spectra, respectively. Here, $C_l^{P\Phi}$ is the power spectrum for the cross-correlation between P_{ab} and Φ_{ab}; i.e.,

$$\langle P_{lm} \Phi^*_{l'm'} \rangle = C_l^{P\Phi} \delta_{ll'} \delta_{mm'}. \tag{20}$$

The flat-sky limit of this expression can be obtained in the limit $l, l_1, l_2 \gg 1$ following the discussion in Section IV in Ref. [22],

$$C_l^{VV} = \int \frac{d^2 l_1}{(2\pi)^2} \sin^2 2\varphi_{\vec{l}_1, \vec{l}-\vec{l}_1} \left(C_{l_1}^{PP} C_{|\vec{l}-\vec{l}_1|}^{\Phi\Phi} - C_{l_1}^{P\Phi} C_{\vec{l}-\vec{l}_1|}^{P\Phi} \right), \tag{21}$$

and the rms circular polarization is,

$$\langle V^2 \rangle = \int \frac{d^2 l}{(2\pi)^2} C_l^{VV} = \frac{1}{2} \left(\langle P^2 \rangle \langle \Phi^2 \rangle - \langle P\Phi \rangle^2 \right). \tag{22}$$

To summarize, the principal result of this Section is the representation of C_l^{VV} in terms of the power spectra for P and Φ and their cross-correlation.

A. Photon-photon scattering

All that remains to do is to calculate the power spectra, but the hard atomic-physics/QED work has been done for us already [4]. The advantage we will obtain here is from the calculation of the angular fluctuations. We treat first the dominant contribution, from photon-photon scattering, to the birefringence.

The primordial curvature perturbation $\zeta(\vec{x})$ is expanded as in Eq. (1) in terms of curvature TAM coefficients ζ_{lm}^k. Any observable on the celestial sphere of quantum numbers lm then receives contributions only from ζ_{lm}^k of the same lm. The TAM-wave amplitudes are related to the more familiar Fourier amplitudes $\zeta(\vec{k})$ through $\zeta_{lm}^k = \int d\hat{k} \zeta(\vec{k})$, and the transfer function depends only on the magnitude $k = |\vec{k}|$, not its orientation $\hat{k}$. Therefore, the transfer function that determines the contribution of each TAM wave ζ_{lm}^k to the observable is the same for the TAM wave as for the Fourier mode of the same k. We can thus write, using Eq. (5), for $X = \{P, \Phi\}$,

$$X_{lm} = 4\pi N_l \int d\chi \sum_k \zeta_{lm}^k T_X(k, \eta_0 - \chi) \frac{j_l(k\chi)}{(k\chi)^2}. \tag{23}$$

The integral here is along the line of sight, or equivalently, in conformal time, and $T_X(k, \eta_0 - \chi)$ is a transfer function that determines the contribution from conformal time $\eta_0 - \chi$ (where η_0 is the conformal time today) to the polarization or phase Φ from a TAM wave (or Fourier mode), of wavenumber k, of the primordial curvature perturbation ζ.

The transfer function for the polarization is provided by numerical Boltzmann codes and inferred, e.g., in the line-of-sight formalism from Eq. (17) in Ref. [21]. If we take the last-scattering surface to be thin, then

$$T_P(k, \eta_0 - \chi) = \frac{3}{4} g(\chi) \Pi(k, \eta_0 - \chi), \tag{24}$$

where χ is a comoving distance along the line of sight, the second argument of Π is the conformal time at that comoving distance. Also, $g(\chi)$ is the visibility function which for our purposes can be approximated by Dirac delta function $\delta_D(\chi - \chi_{\rm ls})$. Here, $\Pi(k, \eta)$ is the polarization source function, defined as in Ref. [21] (this is $2\mathcal{P}^{(0)}$ in the `CLASS` (Cosmic Linear Anisotropy Solving System) code); cf. Ref. [23]). As the discussion around Eqs. (5.19)–(5.23) in Ref. [4] indicates, the transfer function for the rotation is

$$T_\Phi(k, r) = \frac{3\bar{A}}{4} (1+z)^4 \Pi(k, r_0 - r), \tag{25}$$

where

$$\bar{A} = 1.76 \times 10^{-38} \left(\frac{\nu_0}{100\,\mathrm{GHz}} \right) \mathrm{m}^{-1}. \tag{26}$$

The polarization and phase-shift power spectra (for $X_1 X_2 = \{PP, \Phi\Phi, P\Phi\}$) are then

$$C_l^{X_1 X_2} = (4\pi N_l)^2 \frac{9}{16} \int \frac{k^2\, dk}{(2\pi)^3} P_\zeta(k) F_l^{X_1}(k) F_l^{X_2}(k), \tag{27}$$

where the polarization window function is

$$F_l^P(k) = \Pi(k, \eta_{\rm ls}) \frac{j_l(k\chi_{\rm ls})}{(k\chi_{\rm ls})^2}, \tag{28}$$

and the phase-shift window function is

$$F_l^\Phi(k) = \bar{A} \int d\chi\ (1+z)^4 \Pi(k, \eta_0 - \chi) \frac{j_l(k\chi)}{(k\chi)^2}. \tag{29}$$

The results for C_l^{PP}, $C_l^{\Phi\Phi}$, and $C_l^{P\Phi}$ are then plugged into Eq. (21) to obtain C_l^{VV}. The numerical results should agree with those shown in Fig. 7 in Ref. [4].

We close by estimating the range of multipoles l over which the cancellation from cross-correlation of P and Φ will be significant. For large l, the spherical Bessel functions are approximated by $j_l(kx) = 0$ for $kx < l$ and $j_l(kx) \sim (1/kx)\sin(kx - l\pi/2)$ for $kx > l$. There is thus a strong contribution to the cross-correlation for wavenumbers $k(\chi - \chi_{\rm ls}) \lesssim 1$, and the contribution to C_l is dominated by wavenumbers $k \sim l/\chi_{\rm ls}$. Since the phase-shift integrand is weighted by $(1+z)^4$, about half the signal is contributed for redshifts $900 \lesssim z \lesssim 1100$ which corresponds to a comoving distance ~ 50 Mpc. Using $r_{\rm ls} \simeq 14{,}000$ Mpc, we infer that the cross-correlation will suppress the circular-polarization signal that would otherwise arise, in the absence of the cross-correlation, for multipoles $l \lesssim 300$.

B. Spin polarization of neutral hydrogen

We now consider the calculation of the phase-shift power spectrum for the birefringence due to spin polarizations of hydrogen atoms. While this provides a subdominant effect, the calculation will illustrate some additional aspects of the TAM formalism.

Ref. [4] derives, using a spherical-tensor basis, a relation between the $n_{xx}-n_{yy}$ components of the index-of-refraction tensor (for propagation of a photon in the $\hat{z}$ direction) from a single Fourier mode of the density field with wavevector $\vec{k}$. This relation is

$$(n_{xx}-n_{yy})_{\vec{k}} = -\frac{cpW}{\omega_0}\delta(\vec{k})\left[Y_{22}(\hat{k})+Y_{2,-2}(\hat{k})\right], \quad (30)$$

where p and W are as defined in their paper. Using

$$Y_{22}(\hat{k})+Y_{2,-2}(\hat{k}) = 2\sqrt{\frac{15}{32\pi}}(k_x^2-k_y^2), \quad (31)$$

we infer that in configuration space, the symmetric traceless index-of-refraction tensor is related to the matter perturbation $\delta(\vec{x})$ through,

$$\nabla^2 n_{ij}(\vec{x}) = \sqrt{\frac{15}{8\pi}}\frac{cpW}{\omega_0}\left(\nabla_i\nabla_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\delta(\vec{x}), \quad (32)$$

with p and W defined as in Ref. [4]. We expand

$$\delta(\vec{x}) = \sum_{klm}\delta^k_{lm}\Psi^k_{lm}(\vec{x}), \quad (33)$$

in terms of scalar TAM waves and

$$n_{ij}(\vec{x}) = \sum_{klm} n^k_{lm}\Psi^{\mathrm{L},k}_{(lm)ij}(\vec{x}), \quad (34)$$

in terms of longitudinal-tensor TAM waves. Then, from Eq. (4) we infer,

$$n^k_{lm} = -\sqrt{\frac{5}{4\pi}}\frac{pW}{\omega_0}\delta^k_{lm}. \quad (35)$$

The projection of $n_{ij}(\vec{\chi})$, at a (comoving) position $\vec{\chi} = \hat{n}\chi$, onto the plane of the sky is, using Eq. (5),

$$(\mathcal{P}\,\mathcal{P}\,n)_{ab}(\hat{n}) = \sum_{klm} n^k_{lm}\frac{-4\pi i^l}{N_l}\sqrt{\frac{3}{2}}\frac{j_l(k\chi)}{(k\chi)^2}Y^{\mathrm{E}}_{(lm)ab}(\hat{n}). \quad (36)$$

We then infer from Eq. (35) and Eq. (36) that the contribution of any given klm TAM wave to the spherical-harmonic coefficient Φ_{lm} is, using Eq. (12),

$$\Phi^k_{lm} = 2\sqrt{15\pi}i^l\frac{p}{N_l}\int d\chi\, W(\chi)\frac{j_l(k\chi)}{(k\chi)^2}\delta^k_{lm}. \quad (37)$$

The angular power spectrum is then obtained by summing over all k for this given lm; i.e.,

$$C_l^{\Phi\Phi} = \frac{60\pi p^2}{N_l^2}\int\frac{k^2\,dk}{(2\pi)^3}P(k,\eta_0) \times\left[\int d\chi\, W(\chi)D(\eta_0-\chi)\frac{j_l(k\chi)}{(k\chi)^2}\right]^2, \quad (38)$$

where we have written the conformal-time η dependence of the matter power spectrum as $P(k,\eta) = P(k,\eta_0)\left[D(\eta)\right]^2$ in terms of the linear-theory growth factor $D(\eta)$. For the broad line-of-sight distribution here, we can use the Limber approximation [24, 25] [see, e.g., Eqs. (2.20) and (2.21) in Ref. [26]] together with $N_l \simeq 2l^{-4}$ for $l \gg 1$. We then note that the Limber approximation sets $k\chi = l$ to obtain

$$C_l^{\Phi\Phi} = \frac{15}{8\pi}p^2\int\frac{c\,dz}{\left[\chi(z)\right]^2 H(z)}\left[W\left(\chi\left(z\right)\right)\right]^2 P\left(\frac{l}{\chi(z)},\eta-\chi\right). \quad (39)$$

This then agrees with Eq. (3.22) in Ref. [4], given that their $C_l^{\phi Q}$ is half of $C_l^{\Phi\Phi}$.

IV. CONCLUSION

Here we have used the total-angular-momentum formalism to re-derive results from prior work on the power spectrum of circular polarization that arises, at second order in cosmological perturbations, in the standard cosmological model. This alternative derivation may be useful to provide a different perpective on the result and possibly some additional intuition; and it also (once the TAM formalism has been digested) simplifies some of the calculation.

The TAM formalism used here applies only for a flat Universe, where the spatial hypersurfaces are Euclidean 3-space. The TAM formalism has not yet—but can in principle—be extended to closed or open geometries. That said, Eqs. (13)–(22) follow from the spin-2 nature of the polarization and index-of-refraction tensors on the celestial sphere and so apply even if the Universe is open or closed.

In some sense, the true power of the TAM formalism is lost on problems that involve only scalar perturbations—it is far more powerful for calculations that involve vector and tensor perturbations. Thus, for example, it may prove of particular value in models for circular polarization that involve magnetic fields and/or primordial tensor perturbations. Still, the relative simplicity of the problem considered here may be valuable as a simple application of the TAM formalism.

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