

This is the accepted manuscript made available via CHORUS. The article has been published as:

Dimensional reduction in manifoldlike causal sets

J. Abajian and S. Carlip

Phys. Rev. D **97**, 066007 — Published 9 March 2018

DOI: [10.1103/PhysRevD.97.066007](https://doi.org/10.1103/PhysRevD.97.066007)

September 2017
revised February 2018

Dimensional reduction in manifold-like causal sets

J. ABAJIAN*
Department of Physics
Trinity University
San Antonio, TX 78212
USA

S. CARLIP†
Department of Physics
University of California
Davis, CA 95616
USA

Abstract

We investigate the behavior of small subsets of causal sets that approximate Minkowski space in three, four, and five dimensions, and show that their effective dimension decreases smoothly at small distances. The details of the short distance behavior depend on a choice of dimensional estimator, but for a reasonable version of the Myrheim-Meyer dimension, the minimum dimension is $d \approx 2$, reproducing results that have been seen in other approaches to quantum gravity.

*email: jabajian@trinity.edu

†email: carlip@physics.ucdavis.edu

1. Introduction

Despite their enormous differences, many competing approaches to quantum gravity share a common feature, a prediction that the effective dimension of spacetime decreases near the Planck scale, typically to $d \approx 2$ [1]. This phenomenon was first noted in high temperature string theory [2], but has subsequently been seen in causal dynamical triangulations [3], asymptotic safety [4,5], loop quantum gravity [6], the short distance Wheeler-DeWitt equation [7], minimum length scenarios [8], and a variety of other approaches; see [9] for a recent review.

Based in part on the properties of very small causal sets, it was suggested in [10] that causal set theory might exhibit similar phenomenon of dimensional reduction. Added evidence has come from the short-distance behavior of the causal set d’Alembertian [11] and perhaps from the appearance of “asymptotic silence” at short distances [12]. But while there are some hints from earlier investigations of small subsets of manifold-like causal sets [13], the question has not yet been systematically investigated.*

In this paper, we fill this gap. Starting with a large sample of Minkowski-like causal sets in dimension $d = 3, 4$, and 5 , we evaluate the dimensions of successively smaller causal diamonds. As a dimensional estimator, we use the Myrheim-Meyer dimension d_M [15,16], essentially a “box-counting” dimension adopted to Lorentzian signature. There is an ambiguity in the definition of d_M , depending on how one treats disconnected points. We find that the dimension of subsets decreases smoothly but rapidly at small volume, either to $d_M \approx 2$ for one definition or to $d_M \approx 0$ for another. The former behavior mimics that seen in other approaches to quantum gravity.

2. Causal sets and Myrheim-Meyer dimension

A causal set [17] is the natural Lorentzian version of a discrete spacetime, a discrete set of events with prescribed causal relations. Such a set can be described by a partial order \prec , where $x \prec y$ means “ x is to the past of y ,” obeying the conditions

1. transitivity: $x \prec y$ and $y \prec z \Rightarrow x \prec z$;
2. acyclicity: $x \prec y$ and $y \prec x \Rightarrow x = y$;
3. local finiteness: given any two elements x and y , the number of elements z lying between them (i.e., $x \prec z \prec y$) is finite.

The causal relations \prec may be viewed as determining “most” of the metric. In general, the causal structure of a globally hyperbolic manifold determines the metric up to a conformal factor [18]; for a causal set, the missing conformal factor is simply the number of points in a region.

Most causal sets are not at all manifold-like, and it is an open question whether one can find a dynamical principle that limits sets to those that look like nice spacetimes. The converse process, however—finding a causal set that approximates a given manifold M —is straightforward. Starting with a finite-volume region of a globally hyperbolic manifold M with metric g , we select

*Amusingly, related results have been seen in an entirely different context by Clough and Evans [14], who use causal set theory to analyze citation networks.

a “sprinkling” of points by a Poisson process such that the probability of finding m points in any region of volume V is

$$P_V(m) = \frac{(\rho V)^m}{m!} e^{-\rho V} \quad (2.1)$$

for a discreteness scale ρ^{-1} . We assign to these points the causal relations determined by the metric g , and then “forget” the original manifold, keeping only a set of points and their relations. At scales larger than ρ^{-1} , the resulting causal set is expected to be a good approximation of M . In particular, if M is Minkowski space, such a causal set preserves statistical Lorentz invariance [19], a highly nontrivial result.

In this paper, we will limit ourselves to causal sets obtained from such sprinklings in Minkowski space. This is implicitly a dynamical claim: we are assuming that whatever dynamics underlies causal set theory, it will pick out manifold-like sets. On large scales, the quantity we use to measure the dimension requires corrections to account for curvature [20], but as long as the curvature scale is much larger than the Planck scale, our small-distance results should hold for any manifold-like causal set.

As in other approaches to quantum gravity, it is not immediately obvious how to define the dimension of a causal set [9]. Causal sets are inherently Lorentzian, and we should presumably look for a “dimensional estimator” that takes this into account. The standard choice is the Myrheim-Meyer dimension, which is based on counting causally related points.

Start with a causal set obtained from a Poisson sprinkling on d -dimensional Minkowski space. Select an Alexandrov interval, or “causal diamond,” \mathcal{A} —that is, a set consisting of the intersection of the future of a point p and the past of another point q . Define $\langle C_1 \rangle$ to be the average number of points in \mathcal{A} , and $\langle C_2 \rangle$ to be the average number of causal relations, that is, pairs x, y such that $x \prec y$. The quantities $\langle C_1 \rangle$ and $\langle C_2 \rangle$ depend on the volume and the discreteness scale, but a suitable ratio depends only on the dimension:

$$\frac{\langle C_2 \rangle}{\langle C_1 \rangle^2} = \frac{\Gamma(d+1)\Gamma(\frac{d}{2})}{4\Gamma(\frac{3d}{2})} \quad (2.2)$$

The right-hand side of (2.2) is a monotonic function, and can be inverted. For an arbitrary causal set, the Myrheim-Meyer dimension d_M is then defined to be the value d for which (2.2) holds.

As noted earlier, there is an ambiguity in this definition when a causal set contains an isolated point, a point with no causal relations with any others. A single point should presumably have dimension $d = 0$, but the left-hand side of (2.2) is zero for such a point, which would correspond to $d \rightarrow \infty$ on the right-hand side. There seem to be two natural ways to treat such isolated points: we can ascribe a dimension of zero to them, or we can simply neglect them, on the grounds that a completely causally disconnected point is not really part of spacetime. For large causal sets, the choice makes no appreciable difference to the Myrheim-Meyer dimension, but as we shall see, for small enough sets it matters.

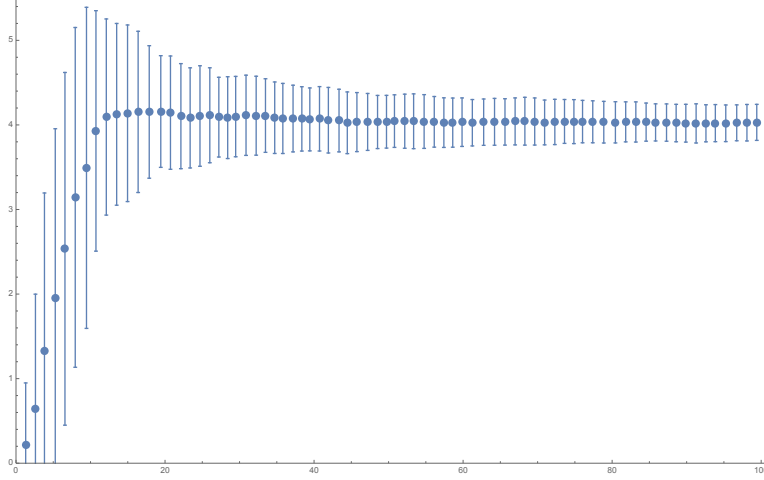


Figure 1: Myrheim-Meyer dimension for subsets of a relatively large causal set

3. Approach

To generate the causal sets used in our analysis, we created a Mathematica notebook [21] that allowed us to select random points uniformly from a causal diamond in Minkowski space, initially in four dimensions. We calculated the Myrheim-Meyer dimension and verified that it agreed with the dimension of the background manifold in the limit of dense sprinklings. As is evident from figure 1, this limit was already reached with sprinklings of about 20 points, so we used this as a typical size.

To investigate the dependence of dimension on volume, one must choose a way to select causal sets of “small” volume. Since the volume of a causal set is determined by the number of points in the set, it is tempting to simply average over all subsets containing a given number of points. This can be misleading, though: while such sets are “small” if viewed outside the context of the background spacetime, most of them do not come from a small region of the background spacetime, but include points spread across a large region of the background manifold. In particular, two points with a lightlike separation can be “adjacent” in a causal set even if they are widely separated in spacetime.

As an alternative, for each of our sprinklings we considered successively smaller sub-diamonds in the background spacetime. The points in each sub-diamond constitute a new causal set, whose volume and Myrheim-Meyer dimension we computed. We repeated the process for 10,000 sprinklings, and then averaged the dimension at each volume. We initially applied this analysis to four-dimensional Minkowski space, but subsequently repeated it for $d = 3$ and $d = 5$.

As noted above, the Myrheim-Meyer dimension is not well-defined for single points or causal sets with no edges. While this concern is unimportant for large causal sets, it must be confronted for the very small sets we are interested in. We explored two reasonable possibilities: taking the volume of an isolated point to be zero (they are, after all, single points) or dropping edgeless causal sets from our counting (they are causally disconnected from the rest of spacetime).

4. Results

In each of the background dimensions we studied, we found that dimensional reduction does indeed occur as the volume decreases. As shown in figures 2–4, the process appears to be smooth, but has a rather abrupt onset. The transition to lower dimension starts at a volume of approximately $V = 8$ points in three dimensions, $V = 16$ points in four dimensions, and $V = 22$ points in five dimensions.[†]

At volumes above the transition, the Myrheim-Meyer dimension remains stable and equal to the dimension of the background Minkowski space. Below the transition, the decrease is quite rapid. For each of the background dimensions we considered, the minimum Myrheim-Meyer dimension falls to $d_M \approx 0$ if edgeless causal sets are taken to have dimension zero, and $d_M \approx 2$ if they are omitted. We can understand the latter result by noting that the smallest causal set with an edge—two points with one relation—has a Myrheim-Meyer dimension of two.

Figures 2–4 show 1σ error bars. We believe these are not a result of poor statistics, but are rather a consequence of our definition of volume. A causal diamond of a given volume in a background Minkowski space can contain many different causal sets, which will not all have identical Myrheim-Meyer dimensions. This leads to a genuine statistical fluctuation in dimension, especially at small volumes.

The end point $d_M \approx 2$ is reminiscent of the behavior seen in other investigations of quantum gravity [9]. More precisely, when edgeless causal sets are discarded, we find a minimum dimension of $d_M = 2.08 \pm .26$ in three background dimensions, $d_M = 2.13 \pm .39$ in four background dimensions, and $d_M = 2.19 \pm .40$ in five background dimensions. It would be interesting to understand the fluctuations better, especially since a few other approaches to quantum gravity suggest a minimum dimension of $3/2$ [22, 23] or $5/2$ [24, 25].

The alternative endpoint $d_M \approx 0$, coming from a slightly different treatment of edgeless sets, may also be of interest. Although most approaches to quantum gravity that exhibit dimensional reduction have a short-distance dimension of $d \approx 2$, there are exceptions. In particular, certain theories with minimum lengths [26], higher derivatives [27], or a particular kind of nonlocality [28] have spectral dimensions that drop to below two—in some cases to zero—at very short distances. One conclusion, at least, is clear from the present work: the notion of “dimension” does not have a unique definition at very small scales, and near the Planck scale even seemingly minor changes in the definition can make a dramatic difference.

We would also like to understand what determines the scale at which dimensional reduction sets in. For three and four background dimensions, the transition seems to occur at a characteristic length of about twice the sprinkling length—that is, $V \sim 2^d$ points—but this pattern appears to break down for background dimension five. We also plan to investigate the behavior of another standard dimensional estimator, midpoint scaling dimension [13].

Ideally, we would like to do more. The results we have presented here have the awkward feature of relying on the background Minkowski space to define the small regions whose dimension we measure. This was necessary to avoid picking out causal subsets that were “small” in the sense

[†]Fractional volumes appear in the graphs because at a given background volume in Minkowski space, causal sets with varying numbers of points may be present.

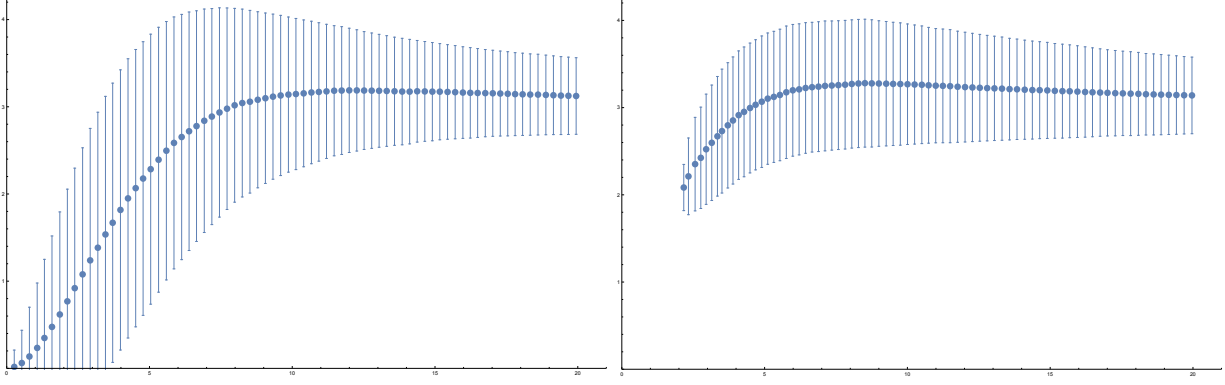


Figure 2: Myrheim-Meyer dimension in a three-dimensional background, with edgeless sets counted as dimension zero (left) or omitted (right)

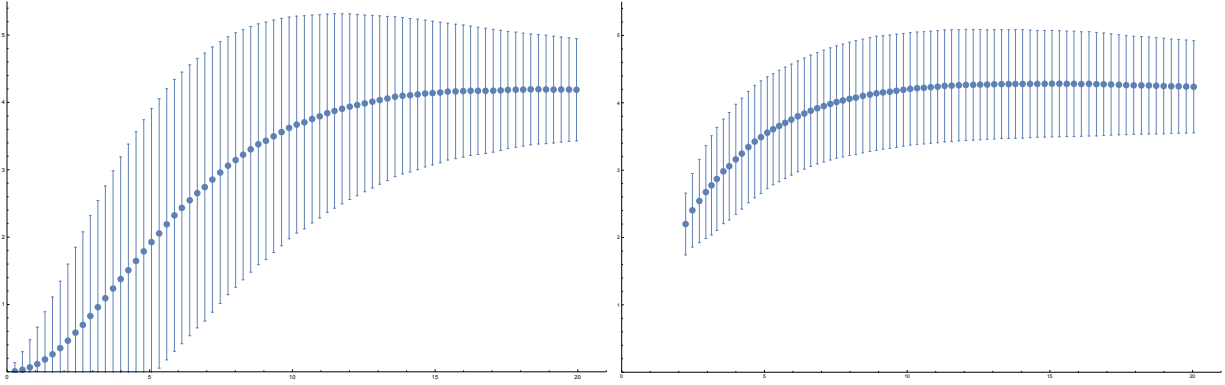


Figure 3: Myrheim-Meyer dimension in a four-dimensional background, with edgeless sets counted as dimension zero (left) or omitted (right)

of having few points, but “large” in the sense of occupying a highly extended region. Recently, some progress has been made in defining “local” regions entirely in the context of causal sets, without reference to any background [29]. It might be possible to use this work to investigate dimensional reduction more intrinsically.

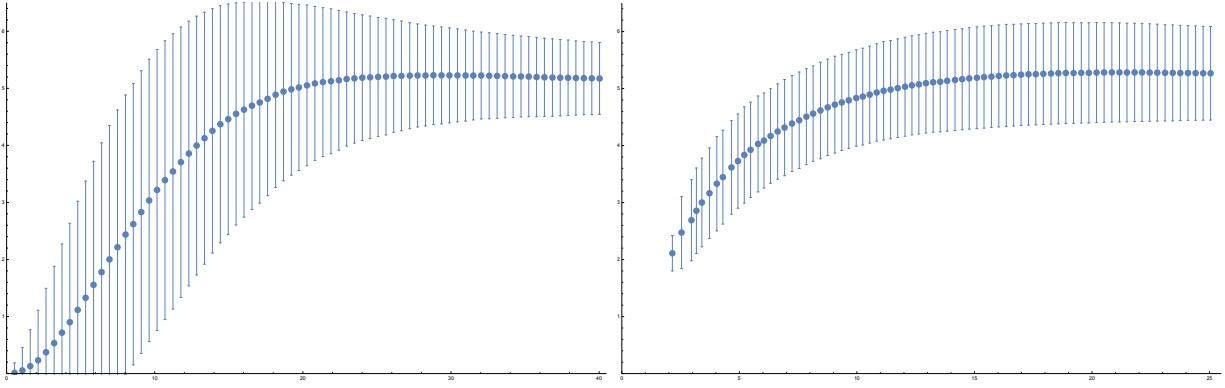


Figure 4: Myrheim-Meyer dimension in a five-dimensional background, with edgeless sets counted as dimension zero (left) or omitted (right)

Acknowledgments

S. C. was supported in part by Department of Energy grant DE-FG02-91ER40674. J. A. was supported in part by National Science Foundation grant PHY-1560482.

References

- [1] S. Carlip, Int. J. Mod. Phys. D25 (2016) 1643003, arXiv:1605.05694.
- [2] J. J. Atick and E. Witten, Nucl. Phys. B310 (1988) 291.
- [3] J. Ambjørn, J. Jurkiewicz, and R. Loll, Phys. Rev. Lett. 95 (2005) 171301, arXiv:hep-th/0505113.
- [4] O. Lauscher and M. Reuter, Phys. Rev. D65 (2002) 025013, arXiv:hep-th/0108040.
- [5] R. Percacci and D. Perini, Class. Quant. Grav. 21 (2004) 5035, arXiv:hep-th/0401071.
- [6] L. Modesto, Class. Quant. Grav. 26 (2009) 242002, arXiv:0812.2214.
- [7] S. Carlip, in *Proc. of the 25th Max Born Symposium: The Planck Scale*, AIP Conf. Proc. 1196 (2009) 72, arXiv:1009.1136.
- [8] T. Padmanabhan, S. Chakraborty, and D. Kothawala, Gen. Rel. Grav. 48 (2016) 55, arXiv:1507.05669.
- [9] S. Carlip, Class. Quant. Grav. 34 (2017) 193001arXiv:1705.05417.
- [10] S. Carlip, Class. Quant. Grav. 32 (2015) 232001, arXiv:1506.08775.

- [11] A. Belenchia, D. M. T. Benincasa, A. Marciano, and L. Modesto, Phys. Rev. D93 (2016) 044017, arXiv:1507.00330.
- [12] A. Eichhorn, S. Mizera, and S. Surya, arXiv:1703.08454.
- [13] D. D. Reid, Phys. Rev. D67 (2003) 024034, arXiv:gr-qc/0207103.
- [14] J. R. Clough and T. S. Evans, Physica A 448 (2016) 235, arXiv:1408.1274.
- [15] J. Myrheim, 1978 CERN preprint TH-2538.
- [16] D. A. Meyer, Ph.D. thesis, MIT (1989), <http://hdl.handle.net/1721.1/14328>.
- [17] L. Bombelli, J. Lee, D. Meyer, and R. Sorkin, Phys. Rev. Lett. 59 (1987) 521.
- [18] D. Malament, J. Math. Phys. 18 (1977) 1399.
- [19] L. Bombelli, J. Henson, and R. D. Sorkin, Mod. Phys. Lett. A24 (2009) 2579, arXiv:gr-qc/0605006.
- [20] M. Roy, D. Sinha, and S. Surya, Phys. Rev D87 (2013) 044046 arXiv:1212.0631.
- [21] Wolfram Research, Inc., Mathematica, Version 11.1, Champaign, IL (2017).
- [22] D. Coumbe and J. Jurkiewicz, JHEP 1503 (2015) 151, arXiv:1411.7712.
- [23] J. Laiho, S. Bassler, D. Coumbe, D. Du, and J. T. Neelakanta, arXiv:1604.02745.
- [24] V. Husain, S. S. Seahra, and E. J. Webster, Phys. Rev. D88 (2013) 024014, arXiv:1305.2814.
- [25] M. Ronco, Adv. High Energy Phys. 2016 (2016) 9897051, arXiv:1605.05979.
- [26] L. Modesto and P. Nicolini, Phys. Rev. D81 (2010) 104040, arXiv:0912.0220.
- [27] G. Calcagni, L. Modesto, and G. Nardelli, Int. J. Mod. Phys. D25 (2016) 1650058, arXiv:1408.0199.
- [28] L. Modesto, Phys. Rev. D86 (2012) 044005, arXiv:1107.2403.
- [29] L. Glaser and S. Surya, Phys. Rev. D 88 (2013) 124026, arXiv:1309.3403.