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Phys. Rev. D **97**, 024009 — Published 9 January 2018

DOI: 10.1103/PhysRevD.97.024009

Black hole perturbation under $2+2$ decomposition in the action

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(Dated: December 8, 2017)

Black hole perturbation theory is useful for studying the stability of black holes and calculating ringdown gravitational waves after the collision of two black holes. Most previous calculations were carried out at the level of the field equations instead of the action. In this work, we compute the Einstein-Hilbert action to quadratic order in linear metric perturbations about a spherically symmetric vacuum background in Regge-Wheeler gauge. Using a $2+2$ splitting of spacetime, we expand the metric perturbations into a sum over scalar, vector, and tensor spherical harmonics, and dimensionally reduce the action to two dimensions by integrating over the two sphere. We find that the axial perturbation degree of freedom is described by a two dimensional massive vector action, and that the polar perturbation degree of freedom is described by a two dimensional dilaton massive gravity action. Varying the dimensionally reduced actions, we rederive covariant and gauge-invariant master equations for the axial and polar degrees of freedom. Thus, the two dimensional massive vector and massive gravity actions we derive by dimensionally reducing the perturbed Einstein-Hilbert action describe the dynamics of a well studied physical system: the metric perturbations of a static black hole. The $2+2$ formalism we present can be generalized to $m+n$ dimensional spacetime splittings, which may be useful in more generic situations, such as expanding metric perturbations in higher dimensional gravity. We provide a self-contained presentation of $m+n$ formalism for vacuum spacetime splittings.

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I. INTRODUCTION

The theory of metric perturbations of static black hole spacetimes is an old and well studied subject. The field began with the work of Regge and Wheeler [1], who were the first to study linear metric perturbations of the Schwarzschild background. In particular, Regge and Wheeler derived a closed form expression, the Regge-Wheeler equation, for linear axial perturbations. The field was further developed by several workers, most notably Vishveshwara [2] and Zerilli [3], the latter of whom derived a closed form expression, the Zerilli equation, for linear polar perturbations. Black hole perturbation theory was first presented in a gauge invariant manner by Moncrief [4], who also wrote down a Hamiltonian for axial and polar perturbations. Gerlach and Sengupta [5, 6] later formulated a covariant and gauge invariant formalism to describe the metric and matter perturbations of a generic spherically symmetric spacetime. A thorough exposition of the state of the field up until the mid 1980's can be found in Chandrasekhar's monograph on the subject [7], while a more modern, covariant, and gauge invariant formulation of the theory of static black hole perturbations including source terms is presented by Martel and Poisson in [8].

While much progress has been made in understanding and reformulating the equations of motion of metric perturbations of static black holes, less work has been done on understanding the structure of the perturbed Einstein-Hilbert action in this background (see, however [9–16]). The purpose of this paper is to further develop this aspect of metric perturbation theory. There are several reasons why deriving the action for black hole perturbations may be useful, two of which we briefly describe below.

Firstly, this formalism may be useful in constructing effective field theories of black hole ringdown. In the context of FLRW cosmologies, a $1+3$ decomposition is natural as the background can be naturally split into a three dimensional maximally symmetric spacelike hypersurface and a time direction. The maximally symmetric subspace of a Schwarzschild black hole is the two sphere. Adapting an effective field theory approach with a $2+2$ formalism may be more useful for this background, where a foliation by maximally symmetric subspaces would be by two dimensional spheres (see, for example [17] for a related discussion).

Secondly, deriving the action for black hole perturbations may also be useful in understanding the quantum physics of black holes. Two dimensional gravity has been used to study Hawking radiation and the quantum mechanics of black holes [18–20]. The actions in Eqs. (49) and (71) could be useful in this context; for example in the construction of a path integral formulation of Hawking radiation for the metric perturbations of a black hole.

In this article, we derive the perturbed Einstein-Hilbert action for spherically symmetric backgrounds. From this action, we derive the equations of motion for the Schwarzschild black hole. We derive the action using a $2+2$ spacetime splitting, which allows us to decouple the linear scalar, vector, and tensor (SVT) perturbations in the action. While this is not the first work that derives black hole perturbations from the action [9–16], to our knowledge the application of the $2+2$ formalism directly to the perturbed Einstein-Hilbert action is novel, and brings to light several interesting new conceptual points about the nature of black hole perturbations. For example, we find that the polar perturbations of a Schwarzschild black hole are described by a (1+1)-dimensional dilaton massive gravity model, which naturally emerges by dimensionally reducing the perturbed Einstein-Hilbert action in a spherically symmetric background. Additionally, the axial perturbations of a black hole are described by a (1+1)-dimensional massive vector field action. While we derive these actions in the Regge-Wheeler gauge, our results may trivially be reexpressed in a gauge invariant fashion (see, for example, section IV B of [8]).

The organization of this paper is as follows. In Sec. II, we briefly review the $m+n$ formalism for vacuum spacetime splittings as applied to the Einstein-Hilbert action; more details are given in Appendices A and B. In Sec. III, we set our notation and review metric perturbation theory in a spherically symmetric spacetime. In Secs. IV B and IV C we derive the action for axial and polar perturbations respectively for a spherically symmetric background, which we derive in the Regge-Wheeler gauge [1]. From the axial and polar equations of motion, we rederive covariant and gauge invariant expressions for the axial and polar degrees of freedom, respectively. We discuss our results and conclude in Sec. V. We review the mathematics of the geometry of surfaces of arbitrary codimension in Appendix A, compute the Einstein-Hilbert action in ADM-like variables adapted to higher codimension spacetime splittings in Appendix B, and provide a summary of the properties of scalar, vector, and tensor spherical harmonics in Appendix C.

Our sign conventions for the metric and Riemann tensor follow that of Misner, Thorne, and Wheeler [21]: for a Lorentzian manifold the metric signature is $-++\cdots$, and $R^{\alpha}{}_{\mu\beta\nu} = \partial_\beta \Gamma^{\alpha}_{\mu\nu} - \cdots$. We work in reduced Planck units: $8\pi G = c = \cdots = 1$.

II. EINSTEIN-HILBERT ACTION IN THE $m+n$ FORMALISM

In this section we briefly review the $m+n$ formalism for vacuum spacetime splittings. A more detailed description of this formalism is presented in Appendices A and B.

We begin with a d dimensional (semi-)Riemannian manifold M with metric $g_{\mu\nu}$ and connection ∇_μ. We assume that the topology of M is $\mathbb{R}^m \times \Sigma$, so that we may foliate M with a family of $n = d - m$ dimensional submanifolds $\{\Sigma_{\mathbf{t}}\}_{\mathbf{t}\in\mathbb{R}^m}$. Unless otherwise noted we will drop the index subscript $\mathbf{t}$ from $\Sigma_{\mathbf{t}}$. For every point $p \in \Sigma$, the tangent space of p naturally splits into a tangent and transverse space, $T_p(M) = T_p(\Sigma) \oplus T_p(^\perp\Sigma)$. We define the tangent projector on to the tangent space $h_\mu{}^\nu$ and the transverse projector $l_\mu{}^\nu = \delta_\mu{}^\nu - h_\mu{}^\nu$.

Let us define the notion of tangent and transverse in more detail. A tensor *component* is called tangent if its contraction with the transverse projector is zero; e.g. if $l_\mu{}^\nu P^{\mu\alpha} = 0$ then we say the μ component of $P^{\mu\alpha}$ is tangent. Likewise a component of a tensor is called transverse if its contraction with the tangent projector is zero. A tensor is called tangent (transverse) if all of its components are tangent (transverse). We define the tangent extrinsic curvature $K^\gamma{}_{\mu\nu}$ and the transverse extrinsic curvature $A^\gamma{}_{\mu\nu}$

$$K^\gamma{}_{\mu\nu} \equiv h_\mu{}^\alpha h_\nu{}^\beta \nabla_\beta l_\alpha{}^\gamma\,, \tag{1}$$

$$A^\gamma{}_{\mu\nu} \equiv l_\mu{}^\alpha l_\nu{}^\beta \nabla_\beta h_\alpha{}^\gamma\,. \tag{2}$$

We define a tangent derivative operator ${}^\parallel\nabla_\mu$ as the tangent projection of the action of ∇_μ on a tangent tensor; e.g. for some $v_\mu \in T_p^*(\Sigma)$ we would have ${}^\parallel\nabla_\mu v_\nu = h_\mu{}^\alpha h_\nu{}^\beta \nabla_\alpha v_\beta$. Likewise we define the transverse derivative operator ${}^\perp\nabla_\mu$ as the transverse projection the action of ∇_μ on a transverse tensor. As the transverse spaces will generally not integrate to form a set of submanifolds, the transverse derivative will generally not be torsion free. We define curvature tensors for the tangent and transverse tensors as follows. For any $v_\mu \in T_p^*(\Sigma)$, we define

$${}^\parallel R_{\alpha\beta\gamma}{}^\delta v_\delta \equiv 2\,{}^\parallel\nabla_{[\alpha}{}^\parallel\nabla_{\beta]} v_\gamma\,. \tag{3}$$

Similarly, for any $v_\mu \in T_p^*(^\perp\Sigma)$, we define

$${}^\perp R_{\alpha\beta\gamma}{}^\delta v_\delta \equiv 2\,{}^\perp\nabla_{[\alpha}{}^\perp\nabla_{\beta]} v_\gamma + F^\lambda{}_{\alpha\beta} l_\gamma{}^\delta \nabla_\lambda v_\delta\,, \tag{4}$$

where the transverse torsion tensor is defined by $F^\gamma{}_{\alpha\beta} \equiv 2A^\gamma{}_{[\alpha\beta]}$. With these definitions at hand, we can rewrite the Ricci scalar as follows

$$\begin{aligned} R &= \left(h^{\alpha\gamma}h^{\beta\delta} + l^{\alpha\gamma}l^{\beta\delta} + 2h^{\alpha\gamma}l^{\beta\delta}\right) R_{\alpha\beta\gamma\delta} \\ &= {}^\parallel R + {}^\perp R + K_\lambda K^\lambda - K_{\lambda\alpha\beta}K^{\lambda\beta\alpha} + A_\lambda A^\lambda - A_{\lambda\alpha\beta}A^{\lambda\beta\alpha} - 2\nabla_\lambda\left(K^\lambda + A^\lambda\right), \end{aligned} \tag{5}$$

where $K^\lambda \equiv K^\lambda{}_\mu{}^\mu$ and $A^\lambda \equiv A^\lambda{}_\mu{}^\mu$.

At this point we choose a basis adapted to the $m+n$ foliation. Our discussion here most closely follows that of [22]. The coordinates $\{x^\alpha\}$ of some chart of the spacetime manifold M are written as functions of two sets of variables, $\{u^a\}$ and $\{\theta^A\}$, so $x^\alpha \equiv x^\alpha(u^a, \theta^A)$. Our notation is as follows: Greek indices run from $0, ..., d-1$, lower case Latin indices run from $0, ..., m-1$ and upper case Latin indices from from $m, ..., d-1$. Einstein summation notation will apply to all different index types. Derivatives with respect to the variables $\{u^a\}$ will be denoted by $\partial_a \equiv \partial/\partial u^a$, while derivatives with respect to the variables $\{\theta^A\}$ will be denoted by $\partial_A \equiv \partial/\partial\theta^A$. We set the variables $\{\theta^A\}$ to be intrinsic to the leaf Σ. We define a basis of frame vectors $e^\alpha_A \equiv \partial_A x^\alpha$ which span $T_p(\Sigma)$. The first fundamental form of Σ is $\gamma_{AB} = g_{\alpha\beta}e^\alpha_A e^\beta_B$; the inverse of γ_{AB} is γ^{AB}, and the metric compatible induced covariant derivative is denoted by ${}^\parallel\nabla_A$. Upper case Latin indices are raised/lowered by γ^{AB} and γ_{AB}, respectively. The variables $\{u^a\}$, which may also be thought of as functions on the chart, are constant on each leaf. We define a congruence of vector fields $u^\gamma_c \equiv \partial_c x^\gamma$ upon which the frame vectors $\{e^\alpha_A\}$ are Lie transported. We next define a basis for $T_p^*(^\perp\Sigma)$, $n^a_\alpha \equiv \partial_\alpha u^a$. The components of the inner product matrix of the forms n^a_α is written as $\alpha^{ab} = g^{\alpha\beta}n^a_\alpha n^b_\beta$. The matrix inverse of α^{ab} is denoted by α_{ab}. Formally, we will raise/lower lower case Latin indices with α^{ab} and α_{ab}, respectively. We note that generally α_{ab} is generically *not* the first fundamental form of any submanifold as the transverse spaces generally do not integrate to form a submanifold. We decompose the differential dx^α into terms tangent and transverse to the leaf Σ,

$$dx^\alpha = n^\alpha_a du^a + e^\alpha_A\left(d\theta^A + \beta^A_a du^a\right), \tag{6}$$

where we have defined the shift vectors $\{\beta^\alpha_a\}$. We now write down the line element for this adapted basis

$$ds^2 = \alpha_{ab}du^a du^b + \gamma_{AB}\left(d\theta^A + \beta^A_a du^a\right)\left(d\theta^B + \beta^B_b du^b\right), \tag{7}$$

where we recall $g_{\alpha\beta}e^\alpha_A n^\beta_b = 0$. With this line element the metric determinant factorizes as follows: $\det g = \det\alpha \det\gamma$. We can now compute the curvatures $K^\gamma{}_{\alpha\beta}$, $A^\gamma{}_{\alpha\beta}$, ${}^\parallel R_{\alpha\beta\gamma\delta}$, and ${}^\perp R_{\alpha\beta\gamma\delta}$ in terms of the metric components α_{ab}, γ_{AB}, and β^A_a:

$$K_{\gamma\alpha\beta} = e^A_\alpha e^B_\beta n^c_\gamma \mathcal{K}_{cAB}, \tag{8}$$

$$A_{\gamma\alpha\beta} = e_{C\gamma} n^a_\alpha n^b_\beta \mathcal{A}^C_{ab}, \tag{9}$$

$$^{\parallel}R_{\alpha\beta\gamma\delta} = e^A_\alpha e^B_\beta e^C_\gamma e^D_\delta {}^{\parallel}\mathcal{R}_{ABCD}, \tag{10}$$

$$^{\perp}R_{\alpha\beta\gamma\delta} = n^a_\alpha n^b_\beta n^c_\gamma n^d_\delta \left({}^{\perp}\mathcal{R}_{abcd} + 2\alpha_{ai}\alpha_{bj}\alpha_{ck}\alpha_{dl}\gamma^{CD}\mathcal{A}^{[ij]}_C \mathcal{A}^{lk}_D \right), \tag{11}$$

where

$$\mathcal{K}_{cAB} = \frac{1}{2}\left(\partial_c \gamma_{AB} - {}^{\parallel}\nabla_A \beta_{cB} - {}^{\parallel}\nabla_B \beta_{cA} \right), \tag{12}$$

$$\mathcal{A}^{ab}_C = \frac{1}{2}\left(\partial_C \alpha^{ab} - \alpha^{ac}\alpha^{bd}\gamma_{CD}\mathcal{F}^D_{cd} \right), \tag{13}$$

$$\mathcal{F}^C_{ab} = \partial_a \beta^C_b - \partial_b \beta^C_a + \beta^D_b \partial_D \beta^C_a - \beta^D_a \partial_D \beta^C_b, \tag{14}$$

$$^{\parallel}\mathcal{R}^D{}_{CAB} = \partial_A \Gamma^D{}_{CB} - \partial_B \Gamma^D{}_{CA} + \Gamma^D{}_{IA}\Gamma^I{}_{CB} - \Gamma^D{}_{IB}\Gamma^I{}_{CA}, \tag{15}$$

$$\Gamma_{CAB} = \frac{1}{2}\left(\partial_A \gamma_{CB} + \partial_B \gamma_{CA} - \partial_C \gamma_{AB} \right), \tag{16}$$

$$^{\perp}\mathcal{R}^d{}_{cab} = n^\mu_a \partial_\mu \Omega^d{}_{cb} - n^\mu_b \partial_\mu \Omega^d{}_{ca} + \Omega^d{}_{ia}\Omega^i{}_{cb} - \Omega^d{}_{ib}\Omega^i{}_{ca}, \tag{17}$$

$$\Omega_{cab} = \frac{1}{2}\left(n^\mu_a \partial_\mu \alpha_{cb} + n^\mu_b \partial_\mu \alpha_{ca} - n^\mu_c \partial_\mu \alpha_{ab} \right). \tag{18}$$

The Einstein-Hilbert action in this formalism can be written as

$$\begin{aligned} S_{EH} = \int d^m u d^n \theta \sqrt{\alpha}\sqrt{\gamma} \Big(& {}^{\parallel}\mathcal{R} + \alpha^{cd}\gamma^{AB}\gamma^{CD}\left(\mathcal{K}_{cAB}\mathcal{K}_{dCD} - \mathcal{K}_{cAC}\mathcal{K}_{dBD} \right) \\ & + {}^{\perp}\mathcal{R} + \gamma^{CD}\alpha_{ab}\alpha_{cd}\left(\mathcal{A}^{ab}_C \mathcal{A}^{cd}_D - \mathcal{A}^{ac}_C \mathcal{A}^{bd}_D \right) - 2\nabla_\lambda \left(K^\lambda + A^\lambda \right) \Big). \end{aligned} \tag{19}$$

We direct the reader to Appendices A and B for a more detailed discussion of the $m + n$ formalism, including a discussion of the relation of this formalism to the ADM $1 + (d - 1)$ formalism, and for derivations of the main results stated in this section.

III. METRIC PERTURBATIONS FOR SPHERICALLY SYMMETRIC BACKGROUND

In this section, we consider perturbations around a spherically symmetric four dimensional background spacetime. In a spherically symmetric spacetime, the full spacetime manifold naturally factorizes into the form $M = M^2 \times S^2$, where both M^2 and S^2 are submanifolds of M. S^2 is the two-sphere and roughly speaking M^2 is the '(t,r) plane' (see, for example the discussion in section II of [8]). For factorizable spacetimes the metric naturally factorizes as well; i.e. we can choose a background metric such that the shift vectors $\{\beta^\alpha_a\}$ are all zero.

We write the background metric as

$$ds^2 = {}^{(0)}\alpha_{ab} du^a du^b + {}^{(0)}\gamma_{AB} d\theta^A d\theta^B. \tag{20}$$

We identify ${}^{(0)}\alpha_{ab}$ and ${}^{(0)}\gamma_{AB}$ as the metrics for M^2 and S^2, respectively. The metric ${}^{(0)}\gamma_{AB}$ is equal to $r^2\Omega_{AB}$, where Ω_{AB} is the round metric. For a factorizable spacetime and metric we may also interpret ${}^{(0)}\alpha_{ab}$ as the induced metric on M^2, and define a metric compatible covariant derivative ${}^{\perp}\nabla_a$, with Ω_{cab} as the connection coefficients. See Appendix B 3 for a discussion of the $m + n$ formalism and factorizable spacetimes.

We begin by describing the geometry of a linearly perturbed spherically symmetric background. We write

$$\alpha_{ab} = {}^{(0)}\alpha_{ab} + \delta\alpha_{ab}, \tag{21}$$

$$\beta^A_a = \delta\beta^A_a, \tag{22}$$

$$\gamma_{AB} = {}^{(0)}\gamma_{AB} + \delta\gamma_{AB}. \tag{23}$$

The perturbations $\delta\alpha_{ab}$, $\delta\beta^A_a$, and $\delta\gamma_{AB}$ can be split into pieces that transform as scalars, vectors, and tensors with respect to the $SO(3)$ spacetime isometry. This is accomplished by decomposing $\delta\alpha_{ab}$, $\delta\beta_{aA} = {}^{(0)}\gamma_{AB}\delta\beta^B_a$, and $\delta\gamma_{AB}$ into a sum over spherical harmonics as

$$\delta\alpha_{ab}(u^a, \theta^A) = \sum_{l,m} h^{lm}_{ab}(u^a) Y^{lm}(\theta^A), \tag{24}$$

TABLE I. Gauge transformations for spherically symmetric background given by Eq. (20).

	variable(s)	gauge transformation
scalar	h_{ab}^{lm}	$h_{ab}^{lm} \to h_{ab}^{lm} + {}^{\perp}\nabla_a(\xi_{\mathcal{S}}^{lm})_b + {}^{\perp}\nabla_b(\xi_{\mathcal{S}}^{lm})_a$
	k^{lm}	$k^{lm} \to k^{lm} + \frac{1}{2}\frac{1}{r^2}(\xi_{\mathcal{S}}^{lm})^a\partial_a r^2 - \frac{1}{2}l(l+1)\xi_{\mathcal{E}}^{lm}$
vector	j_a^{lm}	$j_a^{lm} \to j_a^{lm} + \frac{1}{r^2}(\xi_{\mathcal{S}}^{lm})_a + \partial_a\xi_{\mathcal{E}}^{lm}$
	h_a^{lm}	$h_a^{lm} \to h_a^{lm} + \partial_a\xi_{\mathcal{B}}^{lm}$
tensor	G^{lm}	$G^{lm} \to G^{lm} + 2\xi_{\mathcal{E}}^{lm}$
	h_2^{lm}	$h_2^{lm} \to h_2^{lm} + 2\xi_{\mathcal{B}}^{lm}$

$$\delta\beta_{Aa}(u^a,\theta^A) = r^2\sum_{l,m}\left\{j_a^{lm}(u^a)E_A^{lm}(\theta^A) + h_a^{lm}(u^a)B_A^{lm}(\theta^A)\right\}, \tag{25}$$

$$\begin{aligned}\gamma_{AB} + \delta\gamma_{AB}(u^a,\theta^A) = r^2\sum_{l,m}\Big\{&\exp\left[2k^{lm}(u^a)\right]\Omega_{AB}Y^{lm}(\theta^A)\\ &+ G^{lm}(u^a)E_{AB}^{lm}(\theta^A) + h_2^{lm}(u^a)B_{AB}^{lm}(\theta^A)\Big\},\end{aligned} \tag{26}$$

where Y^{lm}, $\left\{E_A^{lm}, B_A^{lm}\right\}$, and $\left\{E_{AB}^{lm}, B_{AB}^{lm}\right\}$ are scalar, vector, and tensor spherical harmonics, respectively. We collect the basic properties of these functions in Appendix C. Our notation for the spherical harmonic decomposed perturbations follows Poisson and Martel [8], with the exceptions of K^{lm}, which we set to be $K^{lm} \equiv e^{2k^{lm}} - 1$ (see their equation (4.3)), and the perturbations j_a^{lm} and h_a^{lm}, which we multiply by r^2 (see their equations (4.2) and (5.2)). We further note that unlike Martel and Poisson [8], we raise/lower in indices A with γ_{AB}, and *not* the round metric Ω_{AB}. This includes the indices of the vector and tensor spherical harmonics. With the decomposition in Eqs. (24)–(26), we have rewritten the ten metric perturbation degrees of freedom into a sum over SVT spherical harmonics. We see that there are four scalar, four vector, and two tensor spherical harmonic degrees of freedom. In reduced Planck units the variables $\left\{h_{ab}^{lm}, k^{lm}, G^{lm}, h_2^{lm}\right\}$ are dimensionless, while the variables $\left\{j_a^{lm}, h_a^{lm}\right\}$ have dimensions of inverse length.

For completeness, we next review the gauge transformations of the perturbed quantities. Our treatment and notation most closely follows that of Martel and Poisson [8]. A linear gauge transformation can be written as the Lie derivative of the background metric along some arbitrary infinitesimal vector ξ^μ:

$$\pounds_{\xi^\alpha}g_{\mu\nu} = \xi^\alpha\partial_\alpha g_{\mu\nu} + g_{\mu\alpha}\partial_\nu\xi^\alpha + g_{\nu\alpha}\partial_\mu\xi^\alpha = \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu. \tag{27}$$

Under these transformations, and with our line element in Eqs. (20), (21), (22), and (23), we see that our perturbations $\delta\alpha_{ab}$, $\delta\beta_a^A$, and $\delta\gamma_{AB}$ transform as

$$\begin{aligned}\delta\alpha_{ab} &\to \delta\alpha_{ab} + {}^{\perp}\nabla_a\xi_b + {}^{\perp}\nabla_b\xi_a,\\ \delta\beta_{aA} &\to \delta\beta_{aA} + {}^{(0)}\alpha_{ac}\partial_A\xi^c + r^2\Omega_{AC}\partial_a\xi^C,\\ \delta\gamma_{AB} &\to \delta\gamma_{AB} + \Omega_{AB}\xi^c\partial_c r^2 + {}^{\parallel}\nabla_A\xi_B + {}^{\parallel}\nabla_B\xi_A.\end{aligned} \tag{28}$$

We can split the four-vector ξ^μ into terms that transform as scalars and vectors with respect to the $SO(3)$ isometry:

$$\begin{aligned}\xi_a(x^a,\theta^A) &= \sum_{l,m}(\xi_{\mathcal{S}}^{lm})_a(u^a)\,Y^{lm}(\theta^A),\\ \xi_A(x^a,\theta^A) &= r^2\sum_{l,m}\left\{\xi_{\mathcal{E}}^{lm}(u^a)\,E_A^{lm}(\theta^A) + \xi_{\mathcal{B}}^{lm}(u^a)\,B_A^{lm}(\theta^A)\right\},\end{aligned} \tag{29}$$

where the label $\mathcal{S}$, stands for 'scalar part', $\mathcal{E}$ for 'electric (polar)' part, and $\mathcal{B}$ for 'magnetic (axial)' part of the black hole perturbations. We see that ξ_μ has two scalar and two vector degree of freedom, one of which is axial and the other which is polar. Note that we have chosen to normalize the scalars and vectors so that in reduced Planck units the quantities $\left\{(\xi_{\mathcal{S}}^{lm})_a\right\}$ have the dimension of length, while quantities $\left\{\xi_{\mathcal{E}}^{lm}, \xi_{\mathcal{B}}^{lm}\right\}$ are dimensionless. In Table I we list how the SVT components of $\delta\alpha_{ab}$, β_a^A, and $\delta\gamma_{AB}$ transform under the gauge transformation in Eq. (27). Unlike in cosmological perturbation theory [23], the tensor perturbations with respect to the (spherically symmetric) background are not gauge invariant. Using the relations listed in Table I, one can construct gauge-invariant perturbations [4, 8], which we list for completeness

$$\begin{aligned}\tilde{h}_a^{lm} \equiv& h_a^{lm} - \frac{1}{2}\partial_a h_2^{lm}, & (30)\\ \tilde{h}_{ab}^{lm} \equiv& h_{ab}^{lm} - {}^{\perp}\nabla_a \epsilon_b - {}^{\perp}\nabla_b \epsilon_a, & (31)\\ \tilde{k}^{lm} \equiv& k^{lm} - \frac{1}{2}\epsilon^a \partial_a r^2 + \frac{1}{4}l(l+1)G^{lm}, & (32)\end{aligned}$$

where ϵ_a is defined to be [8]

$$\epsilon_a = r^2 j_a^{lm} - \frac{1}{2}r^2 \partial_a G^{lm}. \tag{33}$$

We see that $\tilde{h}_a^{lm}$ is an axial, while $\tilde{h}_{ab}^{lm}$ and $\tilde{k}^{lm}$ are polar gauge invariant perturbation variables.

In this paper, we adopt the Regge-Wheeler gauge. Such a gauge fixes the scalar and vector components of the gauge vector ξ_μ^{lm} as follows:

$$\begin{aligned}\xi_{\mathcal{B}}^{lm} =& -\frac{1}{2}h_2^{lm}, & (34)\\ \xi_{\mathcal{E}}^{lm} =& -\frac{1}{2}G^{lm}, & (35)\\ \left(\xi_{\mathcal{S}}^{lm}\right)_a =& -r^2\partial_a \xi_{\mathcal{E}}^{lm} - r^2 j_a^{lm}. & (36)\end{aligned}$$

While Regge and Wheeler worked with Schwarzschild coordinates [1], we see that their gauge choice does not depend on the detailed structure of the two-metric ${}^{(0)}\alpha_{ab}$, insofar that it has no functional dependence on the angular variables $\{\theta^A\}$ [8].

Importantly, as the gauge vector ξ_μ^{lm} is uniquely determined (e.g. with no integration constants) by the conditions in Eqs. (34)–(36), we can derive the correct perturbation and background equations of motion by imposing the gauge conditions first and then varying the expanded Einstein-Hilbert action [24]. The Regge-Wheeler gauge leaves us with the following six (two vector, four scalar) degrees of freedom: $\{h_a^{lm}, h_{ab}^{lm}, k^{lm}\}$.

Only one scalar and one vector degree of freedom, which correspond to the two polarizations of a gravitational wave, are dynamical degrees of freedom. The other three scalar degrees of freedom are either fixed by the equations of motion to be constants, or are absorbed into the definition of the Zerilli function Ψ_{even}^{lm}, which describes the dynamics of the polar perturbation [3, 4, 8]. For the remainder of this paper all of our calculations will be performed in the Regge-Wheeler gauge. From the gauge transformations listed in Table I, we see that we can rewrite our formulas in terms of the gauge invariant variables using the relations as follows: $h_a^{lm} \to \tilde{h}_a^{lm}$, $h_{ab}^{lm} \to \tilde{h}_{ab}^{lm}$, and $k^{lm} \to \tilde{k}^{lm}$, so that all the formulas we list can be cast into a gauge invariant form (see for example [8]).

IV. PERTURBED EINSTEIN-HILBERT ACTION IN REGGE-WHEELER GAUGE

In this section, we consider axial and polar perturbations of the Einstein-Hilbert action in the Regge-Wheeler gauge.

A. Background equations of motion

For completeness, we first derive the background equations of motion from unperturbed Einstein-Hilbert action in spherical symmetry. The unperturbed dimensionally reduced action is

$$S = \int d^2u\sqrt{\alpha}\left(\frac{r^2}{2}{}^{\perp}\mathcal{R} + (\partial_a r)^2 + 1\right). \tag{37}$$

Varying r and α^{ab}, we obtain the standard (see for example appendix B of [25]) equations of motion

$$\begin{aligned}0 =& r^{\perp}\mathcal{R} - 2^{\perp}\Box r, & (38)\\ 0 =& \left(2r^{\perp}\Box r + (\partial_c r)^2 - 1\right)\alpha_{ab} - 2r^{\perp}\nabla_a{}^{\perp}\nabla_b r. & (39)\end{aligned}$$

We note that we can split up Eq. (39) by computing its trace and trace free components. The trace gives us $r^{\perp}\Box r + (\partial r)^2 - 1 = 0$. We then use this in Eq. (39) to obtain ${}^{\perp}\nabla_a{}^{\perp}\nabla_b r = \frac{1}{2}\alpha_{ab}{}^{\perp}\Box r$ (see, for example Eq. (2.8) of [8] for a similar expression).

B. Axial perturbations

1. Axial action

Let us first consider axial perturbations. In the Regge-Wheeler gauge, the nonzero axial perturbations are completely described by the variable h_a^{lm}:

$$\delta\beta_{Aa} = r^2 \sum_{l,m} h_a^{lm} B_A^{lm}, \tag{40}$$

in other words we consider the line element

$$ds^2 = {}^{(0)}\alpha_{ab} du^a du^b + {}^{(0)}\gamma_{AB} \left(d\theta^A + \delta\beta_a^A du^a\right)\left(d\theta^B + \delta\beta_b^B du^b\right), \tag{41}$$

with $\delta\beta_a^A = \gamma^{AB}\delta\beta_{aB}$ given by Eq. (40). For the remainder of this subsection our notation will be ${}^{(0)}\alpha_{ab} \equiv \alpha_{ab}$. The Einstein-Hilbert action expanded to linear order in h_a^{lm} is zero in Regge-Wheeler gauge. So, we only need to consider the action expanded to quadratic order in h_a^{lm}. The terms of the Einstein-Hilbert action, Eq. (19), that are nonzero with line element Eq. (41) are

$$S_{odd}^{(2)} = \int d^4x \sqrt{\alpha}\sqrt{\gamma}\Big[\alpha^{cd}\gamma^{AB}\gamma^{CD}\left(\mathcal{K}_{cAB}\mathcal{K}_{dCD} - \mathcal{K}_{cAC}\mathcal{K}_{dBD}\right) - \frac{1}{4}\alpha^{ac}\alpha^{bd}\gamma_{CD}\mathcal{F}_{ab}^C\mathcal{F}_{cd}^D\Big]. \tag{42}$$

We will now rewrite Eq. (42) by integrating over the two sphere. Firstly, we record the components of $\mathcal{K}_{cAB}$ and $\mathcal{F}_{ac}^C$ subject to the perturbation Eq. (40)

$$\mathcal{K}_{cAB} = \frac{\partial_c r}{r}\gamma_{AB} - \frac{r^2}{2}\sum_{lm} h_c^{lm}\left(D_A B_B^{lm} + D_B B_A^{lm}\right), \tag{43}$$

$$\mathcal{F}_{ab}^C = \sum_{lm}(B^{lm})^C\left(\partial_a h_b^{lm} - \partial_b h_a^{lm}\right) + \mathcal{O}\left((h_a^{lm})^2\right), \tag{44}$$

where D_A is the covariant derivative on the two sphere (see Appendix C). Using the properties of the axial vector spherical harmonics recorded in Appendix C, and after several integrations by parts we obtain for the first two terms in Eq. (42) as

$$\begin{aligned}&\int d^2\Omega\sqrt{\gamma}\alpha^{cd}\gamma^{AB}\gamma^{CD}\left(\mathcal{K}_{cAB}\mathcal{K}_{dCD} - \mathcal{K}_{cAC}\mathcal{K}_{dBD}\right)\\ =&2\alpha^{ab}\partial_a r\partial_b r - \frac{r^2}{2}\sum_{l,m} l(l+1)\left[l(l+1)-2\right]\alpha^{ab}h_a^{lm}h_b^{lm}.\end{aligned} \tag{45}$$

We drop the order zero term $\alpha^{ab}\partial_a r\partial_b r$. We next dimensionally reduce the 'field strength' term (the one that depends on $(F_{ab}^D)^2$) and obtain

$$\int d^2\Omega\sqrt{\gamma}\gamma_{AB}\alpha^{ab}\alpha^{cd}F_{ac}^A F_{bd}^B = r^4\sum_{l,m} l(l+1)\alpha^{ab}\alpha^{cd}\mathcal{F}_{ac}^{lm}\mathcal{F}_{bd}^{lm}, \tag{46}$$

where we have defined

$$\mathcal{F}_{ab}^{lm} \equiv \partial_a h_b^{lm} - \partial_b h_a^{lm} = {}^\perp\nabla_a h_b^{lm} - {}^\perp\nabla_b h_a^{lm}. \tag{47}$$

We can remove the factor of r^4 from Eq. (46) (multiplied by $\sqrt{-\alpha}$ in Eq. (42)) by performing the following conformal transformation:

$$\hat{\alpha}_{ab} = \frac{1}{r^4}\alpha_{ab}. \tag{48}$$

Using Eqs. (45)–(48), we see that the dimensionally reduced Einstein-Hilbert action for axial perturbations about a spherically symmetric vacuum background is

$$S_{axial}^{(2)} = \sum_{l,m} l(l+1)\int d^2u\sqrt{-\hat{\alpha}^{lm}}\left[-\frac{1}{4}\hat{\alpha}^{ac}\hat{\alpha}^{bd}\mathcal{F}_{ab}^{lm}\mathcal{F}_{cd}^{lm} - \frac{1}{2}M_{lo}^2(r)\hat{\alpha}^{ab}h_a^{lm}h_b^{lm}\right], \tag{49}$$

where we have defined an effective mass $M_{lo}(r)$ to be

$$M_{lo}^2(r) = \left(l^2 + l - 2\right) r^2. \tag{50}$$

The action in Eq. (49) is the central result of this section. We again note that up until this point the only condition we have placed on the two metric α_{ab} is that it has no functional dependence on the angular variables $\{\theta^A\}$. We conclude that the action (Eq. (49)) describes the linear metric axial perturbations of the Einstein-Hilbert action in a spherically symmetric vacuum background.

We now derive the first order equations of motion by varying Eq. (49) with respect to h_a^{lm}:

$$0 = {}^{\perp}\hat{\Box} h_a^{lm} - \hat{\alpha}^{bc\perp}\hat{\nabla}_b{}^{\perp}\hat{\nabla}_a h_c^{lm} - M_{lo}^2(r) h_a^{lm}. \tag{51}$$

Here ${}^{\perp}\hat{\Box} \equiv \alpha^{ab}\nabla_a\nabla_b$ and ${}^{\perp}\hat{\nabla}_a$ are the derivative operators compatible with the background metric constructed from $\hat{\alpha}_{ab}$ instead of α_{ab}. Taking the divergence of Eq. (51), we obtain a constraint on the vector h_a^{lm} as

$$0 = {}^{\perp}\hat{\nabla}_a \left[M_{lo}^2(r)\hat{\alpha}^{ab} h_b^{lm}\right]. \tag{52}$$

Recall that we may relate the Regge-Wheeler variable h_a^{lm} to the gauge invariant variable under the simple substitution $h_a^{lm} \to \tilde{h}_a^{lm}$, so that to linear order in perturbation theory Eqs. (49), (51), and (52) under this relabeling become gauge invariant expressions.

2. *Master axial equation*

For completeness, we demonstrate that we can rewrite Eqs. (51), (52) as a single master equation (see, for example [5, 8, 26–28]). Firstly, we rewrite our equation of motion in the metric α_{ab}. Note that as $\sqrt{\hat{\alpha}}\hat{\alpha}^{ab} = \sqrt{\alpha}\alpha^{ab}$ and $M_{lo}^2(r) = (l+1)(l-2)r^2$ where $(l+1)(l-2)$ is a constant, we see that Eq. (52) is equivalent to

$$0 = {}^{\perp}\nabla_a \left(r^2 \alpha^{ab} h_b^{lm}\right). \tag{53}$$

We conclude that we can rewrite h_b^{lm} in terms of the master variable for the odd parity perturbation Ψ_{odd}^{lm} as

$$h_a^{lm} = \frac{1}{r^2}\epsilon_{ab}{}^{\perp}\nabla^b \left(r\Psi_{\text{odd}}^{lm}\right), \tag{54}$$

where ϵ_{ab} is the Levi-Civita tensor[1] for the Lorentzian metric α_{ab}. Next, we rewrite Eq (51) as

$$0 = {}^{\perp}\nabla^b \left[r^4 \left({}^{\perp}\nabla_b h_a^{lm} - {}^{\perp}\nabla_a h_b^{lm}\right)\right] - (l-1)(l+2)r^2 h_a^{lm}. \tag{55}$$

In a two dimensional manifold we have the identity

$$2\nabla^b\nabla_{[b}v_{a]} = \epsilon_{ab}\epsilon^{cd}\nabla^b\nabla_{[c}v_{d]}. \tag{56}$$

We use Eq. (56), along with Eq. (54) to rewrite Eq. (55) as

$$0 = \epsilon_{ab}{}^{\perp}\nabla^b \left\{r^4\epsilon^{cd\perp}\nabla_{[c}\left[\frac{1}{r^2}\epsilon_{d]p}{}^{\perp}\nabla^p\left(r\Psi_{\text{odd}}^{lm}\right)\right] - (l-1)(l+2)r\Psi_{\text{odd}}^{lm}\right\}. \tag{57}$$

We integrate this equation and choose the integration constant to be equal to zero. Expanding out our expression and using the background equations of motion we obtain

$$0 = \left[{}^{\perp}\Box - \frac{l(l+1)}{r^2} + \frac{3}{2}{}^{\perp}\mathcal{R}\right]\Psi_{\text{odd}}^{lm}. \tag{58}$$

We note that the master equation, Eq. (58) only holds in a vacuum spacetime, for which we have the Schwarzschild background. For the background we can write ${}^{\perp}\mathcal{R} = 4M/r^3$, and we recover the Regge-Wheeler equation [1] for axial perturbations.

We conclude that the variation of the dimensionally reduced action, Eq. (49), with respect to h_a^{lm} gives us the correct equations of motion for linear metric axial perturbations about a spherically symmetric vacuum spacetime. From these equations of motion we are able to derive a covariant and gauge-invariant master equation of motion for a scalar axial perturbation variable, as is done in, for example, [5, 8, 26–28].

---

[1] We note that the Levi-Cevita tensor ϵ^{ab} is related to the Levi-Cevita symbol $\tilde{\epsilon}^{ab}$ by $\epsilon^{ab} = \frac{-1}{\sqrt{-g}}\tilde{\epsilon}^{ab}$ for a Lorentzian spacetime, so that $\epsilon_{ab}\epsilon^{bc} = +\delta_a{}^c$.

C. Polar perturbations

1. Polar action

Next, let us look at polar perturbations. In the Regge-Wheeler gauge, there are four nonzero polar perturbations: $\{h^{lm}_{ab}, k^{lm}\}$. We begin by defining the following quantity

$$\Phi^2 = r^2 e^{2k}, \tag{59}$$

where (see Eq. (26))

$$e^{2k} \equiv \sum_{l,m} e^{2k^{lm}} Y^{lm}. \tag{60}$$

We next define

$$\phi^{lm} \equiv r \exp\left(k^{lm}\right), \tag{61}$$

so that

$$\Phi^2 = \sum_{l,m} \left(\phi^{lm}\right)^2 Y^{lm}. \tag{62}$$

Using Eq. (62), we can write the line element for a spherically symmetric spacetime with polar perturbations as

$$ds^2 = \left({}^{(0)}\alpha_{ab} + \delta\alpha_{ab}\right) du^a du^b + \Phi^2 d^2\Omega \equiv \alpha_{ab} du^a du^b + \Phi^2 d^2\Omega. \tag{63}$$

With the metric in Eq. (63) at hand, we now derive the dimensionally reduced Einstein-Hilbert action. First we will look at the terms which depend on $\mathcal{K}_{cAB}$ and $\mathcal{A}^{ab}_C$, which in the metric Eq. (63) evaluate to be

$$\mathcal{K}_{cAB} = \Omega_{AB} \sum_{l,m} Y^{lm} \phi^{lm} \partial_c \phi^{lm}, \tag{64}$$

$$\mathcal{A}^{ab}_C = \frac{1}{2} \sum_{l,m} (h^{lm})^{ab} \partial_C Y^{lm}. \tag{65}$$

Integrating over the two sphere we obtain

$$\int d^2u d^2\Omega \sqrt{-\alpha}\sqrt{\gamma} \left(\mathcal{K}_d \mathcal{K}^d - \mathcal{K}_{dAB}\mathcal{K}^{dAB}\right) = \sum_{l,m} 2 \int d^2u \sqrt{-\alpha^{lm}} \left(\partial_d \phi^{lm}\right)^2, \tag{66}$$

$$\int d^2u d^2\Omega \sqrt{-\alpha}\sqrt{\gamma} \left(\mathcal{A}_D \mathcal{A}^D - \mathcal{A}_{Dab}\mathcal{A}^{Dab}\right) = \sum_{l,m} \frac{l(l+1)}{4} \int d^2u \sqrt{-\alpha^{lm}} \left[\left(h^{lm}\right)^2 - \left(h^{lm}_{ab}\right)^2\right], \tag{67}$$

where $h^{lm} \equiv {}^{(0)}\alpha^{ab} h^{lm}_{ab}$. Note that the dimensionally reduced action for the $\mathcal{A}^D_{ab}$ terms in Eq. (67) is the Fierz-Pauli graviton mass [29]. We next compute ${}^{\parallel}\mathcal{R}$; firstly we compute

$$\Phi^{2\,\parallel}\mathcal{R} = 2 - 2\Omega^{AB} D_A D_B \ln\Phi, \tag{68}$$

where D_A is the covariant derivative for the round metric Ω_{AB} (see Appendix C). Expanding Φ in terms of spherical harmonics and to second order in the perturbations k^{lm}, $\Phi = re^k = r\sum_{lm} Y^{lm} e^{k^{lm}} = r\sum_{lm} Y^{lm}(1 + k^{lm} + \frac{1}{2}(k^{lm})^2)$, and integrating over the two sphere we obtain

$$\int d^2u d^2\Omega \sqrt{-\alpha}\sqrt{\gamma}\ {}^{\parallel}\mathcal{R} = \sum_{l,m} \int d^2u \sqrt{-\alpha^{lm}} \left[2 + 2l(l+1)k^{lm} + \mathcal{O}\left((k^{lm})^3\right)\right]. \tag{69}$$

As all the terms in h^{lm}_{ab} are scalars under the $SO(3)$ group action, we can straightforwardly dimensionally reduce ${}^{\perp}\mathcal{R}$:

$$\int d^2u d^2\Omega \sqrt{-\alpha}\sqrt{\gamma}\ {}^{\perp}\mathcal{R} = \sum_{l,m} \int d^2u \sqrt{-\alpha^{lm}} \left(\phi^{lm}\right)^2\ {}^{\perp}\mathcal{R}^{lm}. \tag{70}$$

In Eq. (70) we have not expanded out ${}^{\perp}\mathcal{R}$ into a background piece and pieces linear and quadratic in the perturbation h_{ab}^{lm}. Combining Eqs. (66)–(70), we obtain the dimensionally reduced action for linear polar perturbations of a spherically symmetric vacuum background in Regge-Wheeler gauge given by

$$S^{(2)}_{polar} = \sum_{l,m} \int d^2u \sqrt{-\alpha^{lm}} \left\{ \frac{(\phi^{lm})^2}{2} \, {}^{\perp}\mathcal{R}^{lm} + \left(\partial_d \phi^{lm}\right)^2 + 1 \right. \\ \left. - \frac{l(l+1)}{8} \left[(h_{ab}^{lm})^2 - (h^{lm})^2 \right] + l(l+1) k^{lm} \right\}. \tag{71}$$

Equation (71) is the action for a (1+1)-dimensional dilaton massive gravity model (see [30] for another example of such a model, but without a dilaton field). Note that by setting $h_{ab}^{lm} = 0$ and $k^{lm} = 0$, the action in (71) reduces to the standard dimensionally reduced gravity action for a spherically symmetric vacuum background, Eq. (37) (see for example Appendix B of [25]). For notational purposes, it is simpler to combine the linear and quadratic perturbations into the same action, and in Eq. (71) we have *not* expanded out ${}^{\perp}\mathcal{R}^{lm}$ or ϕ^{lm} into a background plus linear perturbation.

We next derive the equations of motion that describe the dynamics of polar metric perturbations about a spherically symmetric vacuum background. In the equations of motion one can disentangle the background and perturbation degrees of freedom more easily than in the action. Varying Eq. (71) by k^{lm}, we have

$$0 = (\phi^{lm})^2 \, {}^{\perp}\mathcal{R}^{lm} - 2\phi^{lm} {}^{\perp}\Box \phi^{lm} + \frac{1}{2} l(l+1) h^{lm}. \tag{72}$$

Here we have defined $h^{ab} \equiv {}^{(0)}\alpha^{ac} \, {}^{(0)}\alpha^{bd} h_{cd}$, and $h \equiv {}^{(0)}\alpha^{ab} h_{ab}$. The derivative operators ${}^{\perp}\nabla_a$ are treated as covariant derivative operators compatible with the metric $\alpha_{ab} = {}^{(0)}\alpha_{ab} + \delta\alpha_{ab}$. (see Appendix B 3 for a discussion of the $m+n$ formalism and factorizable spacetimes). Three more independent equations of motion are derived by varying Eq. (71) by $(\alpha^{lm})^{ab}$,

$$0 = \left[\frac{1}{2} (\partial \phi^{lm})^2 + \phi^{lm} {}^{\perp}\Box \phi^{lm} - \frac{1}{2} l(l+1) k^{lm} - \frac{1}{2} \right] \alpha_{ab} \\ - \phi^{lm} {}^{\perp}\nabla_a {}^{\perp}\nabla_b \phi^{lm} + \frac{l(l+1)}{4} \left(h_{ab}^{lm} - \alpha_{ab} h^{lm} \right). \tag{73}$$

We have not fully expanded out the metric, covariant derivatives, and $\phi \equiv r e^k$ in this expression. The right hand side of Eqs. (72) and (73) can be related to certain combinations of components of the full four dimensional Einstein tensor $G_{\mu\nu}$. Namely, Eq. (72) corresponds to $-(r\,Y^{lm})^{-1} \left(G_{\theta\theta} + G_{\phi\phi}/\sin^2\theta \right)$, while Eq. (73) corresponds to $r^2 (2Y^{lm})^{-1} G_{ab}$. We recall that the Regge-Wheeler variables h_{ab}^{lm} and k^{lm} can be related to the gauge invariant variables $\tilde{h}_{ab}^{lm}$ and $\tilde{k}^{lm}$ with the simple substitution $h_{ab}^{lm} \to \tilde{h}_{ab}^{lm}$ and $k^{lm} \to \tilde{k}^{lm}$, so that to linear order in perturbation theory Eqs. (71), (73), and (75) under this relabeling are gauge invariant expressions.

2. Master polar equation

For completeness we demonstrate that we can rewrite Eqs. (72), (73), and Eq. (75) as a single master equation (see, for example [8, 26–28]). We set

$$r^a \equiv {}^{\perp}\nabla^a r, \tag{74}$$

where the a index is raised/lowered with α^{ab}/α_{ab}, respectively.

We can take a divergence of Eq. (73), and use Eq. (72) to obtain the conditions

$$0 = 2 {}^{\perp}\nabla_a k^{lm} - {}^{\perp}\nabla^b h_{ab}^{lm} + {}^{\perp}\nabla_a h^{lm} - \frac{{}^{\perp}\nabla_a r}{r} h^{lm}. \tag{75}$$

For higher dimensional massive gravity in flat space (for example, in $1 + (d-1)$ dimensions), one can show that the addition of the Fierz-Pauli mass term to the Einstein-Hilbert action implies that the metric perturbation $\delta g_{\mu\nu}$ obeys a similar looking relation [31], namely $\nabla^\mu \delta g_{\mu\nu} - g^{\alpha\beta} \nabla_\nu \delta g_{\alpha\beta} = 0$.

We next expand out Eq. (72). Using the background equations of motion, Eqs (38) and (39), along with Eq. (75) and the fact that in two dimensions $R_{ab} = \frac{1}{2}\alpha_{ab} R$, we see that Eq. (72) reduces to

$$0 = \frac{1}{2} \left[-2 + l(1+1) \right] h^{lm}, \tag{76}$$

for $l > 1$, we conclude that the metric perturbation is traceless. Lower l values require special treatment (e.g. [8, 26]); we do not consider $l = 0, 1$ in this article.

We expand out Eq. (73) to first order in metric perturbations. Using $h = 0$ and the background equations of motion, this reduces to

$$
\begin{aligned}
0 = &\left[-rr^{c\perp}\nabla^d h^{lm}_{cd} - \frac{1}{2}r^c r^d h^{lm}_{cd} + r^{2\perp}\Box k^{lm} + 3rr^{c\perp}\nabla_c k - \frac{1}{2}(l-1)(l+2)k^{lm} \right] \alpha_{ab} \\
&+ \frac{1}{2}rr^c \left(2^\perp\nabla_{(a}h^{lm}_{b)c} - {}^\perp\nabla_c h^{lm}_{ab} \right) + \frac{1}{4}\left[{}^\perp\mathcal{R} + l(l+1) \right] h^{lm}_{ab} - r^{2\perp}\nabla_a{}^\perp\nabla_b k^{lm} - 2rr_{(a}{}^\perp\nabla_{b)}k^{lm}.
\end{aligned} \tag{77}
$$

From Eqs. (75), (76), and (77), we can construct the Zerilli-Moncrief function, which is a covariant and gauge-invariant scalar which describes the dynamics of the one independent polar degree of freedom. See, for example the discussions in [8, 28] [2]. The Zerilli-Moncrief function in our notation is

$$
\Psi^{lm}_{\text{even}} = \frac{2r}{l(l+1)} \left[2k^{lm} + \frac{2}{\Lambda}\left(r^a r^b h^{lm}_{ab} - 2rr^{a\perp}\nabla_a k^{lm} \right) \right], \tag{79}
$$

where we have defined [8] the function

$$
\Lambda = (l-1)(l+2) + \frac{3}{2}r^{2\perp}\mathcal{R}. \tag{80}
$$

The Zerilli-Moncrief function obeys the Zerilli equation,

$$
0 = \left({}^\perp\Box - V^{lm}_{\text{even}} \right) \Psi^{lm}_{\text{even}}, \tag{81}
$$

where

$$
V^{lm}_{\text{even}} = \frac{1}{\Lambda^2}\left[(l-1)^2(l+2)^2 \left(\frac{(l-1)(l+2)+2}{r^2} + \frac{3}{2}{}^\perp\mathcal{R} \right) + \frac{9}{4}r^2 \left({}^\perp\mathcal{R} \right)^2 \left((l-1)(l+2) + \frac{1}{2}r^{2\perp}\mathcal{R} \right) \right]. \tag{82}
$$

Note that ${}^\perp\mathcal{R} = 4M/r^3$ as the background is a Schwarzschild black hole spacetime; substituting this value in for ${}^\perp\mathcal{R}$ gives us a standard expression for the Zerilli potential. We refer the reader to [28] for details on how to derive the Zerilli-Moncrief function and Zerilli equation from Eqs. (75), (76), and (77).

We conclude that the variation of the dimensionally reduced action, Eq. (71), with respect to k and α^{ab} gives us the correct equations of motion for linear metric polar perturbations about a spherically symmetric vacuum spacetime, i.e. a Schwarzschild black hole. From these equations of motion we are able to derive a covariant and gauge-invariant master equation of motion for a scalar axial perturbation variable, as is done in [8, 26–28].

V. DISCUSSION AND CONCLUSION

In this work, we derived the action for linear perturbations about a spherically symmetric vacuum background in general relativity (Eqs. (49) and (71)) using a $2+2$ spacetime splitting. By dimensionally reducing the $2+2$ Einstein-Hilbert action to $(1+1)$ dimensions using the Regge-Wheeler gauge, we found that the axial perturbations are described by a massive vector field action (Eq. (49)), while the polar perturbations are described by a dilaton massive gravity action (Eq. (71)). Varying the actions Eqs. (49) and (71), we are able to rederive covariant and gauge invariant master equations for the axial and polar degree of freedom, respectively. While in this article we worked in a vacuum spacetime, with the addition of a cosmological constant or matter source our results could be extended to study other backgrounds, such as the Schwarzschild (anti)-de Sitter spacetime, or the Reissner-Nordström spacetime.

To our knowledge, Eq. (71) is a novel (1+1)-dimensional massive gravity action (for another example of a two dimensional dilaton massive gravity model, see for example [30]). The fact that we recover a massive gravity model from dimensionally reducing Einstein gravity may not come as a surprise: some four dimensional massive gravity models also arise from dimensionally reducing higher dimensional gravity theories [31, 36]. One interesting feature of this model is that it describes dynamics of linear gravitational waves about a Schwarzschild black hole. We note that

[2] Our Eq. (77) is equivalent to Eq. (4.13) in [8] once we take into account the identity

$$
\nabla^c\nabla_{(a}p_{b)c} - \frac{1}{2}\Box p_{ab} - \frac{1}{2}\alpha_{ab}\nabla^c\nabla^d p_{cd} = \frac{R}{2}p_{ab}, \tag{78}
$$

which holds for any traceless symmetric tensor p_{ab} in a two dimensional manifold [5, 28].

since Schwarzschild black holes are classically stable to linear perturbations, the massive gravity theory as described by Eq. (71) is also classically linearly stable in that background. Two dimensional (dilaton) gravity has been used to study Hawking radiation and the quantum mechanics of black holes for 'S-wave' scalar field perturbations (see, for example, [18–20]). The actions in Eqs. (49) and (71) could be useful in extending this program to investigating the quantum mechanics of gravitational wave perturbations about Schwarzschild black holes; for example in constructing the path integral formulation of Hawking radiation for metric perturbations of a Schwarzschild black hole.

The $m+n$ formalism is not limited to four dimensions and can be applied to a spacetime of arbitrary metric signature and arbitrary dimensionality. We caution that the $m+n$ formalism we present may be less useful in understanding the perturbations of spacetimes that cannot be foliated by subspaces that are maximally symmetric under the isometries of the full spacetime, i.e. spacetimes where one cannot write the background metric in the form of Eq. (20). In these backgrounds the background frame vectors n^α_a do not form an involution (e.g. $\beta^A_a \neq 0$), the quantity α_{ab} is not the induced metric of a submanifold, and calculating and varying quantities such as ${}^\perp\mathcal{R}_{abcd}$ become much more cumbersome. In particular, in the nonextremal Kerr spacetimes one cannot write the background metric in a form such that $\beta^\alpha_a = 0$ on the background. Because of this fact, other formalisms such as the Newman-Penrose formalism [38] may ultimately remain more useful for understanding the dynamics and perturbations of backgrounds such as the nonextremal Kerr spacetime.

ACKNOWLEDGMENTS

We thank Lasha Berezhiani, Emanuele Berti, Eric Poisson, Frans Pretorius, and Teruaki Suyama for fruitful and interesting discussions, reading through an earlier draft of this work, and for providing comments on that draft. Additionally, we thank an anonymous referee, whose useful comments greatly helped us in improving our presentation of the $m+n$ formalism. We thank another referee for helpful comments with regards to the master axial and polar equations of motion. KY acknowledges support from JSPS Postdoctoral Fellowships for Research Abroad, NSF grant PHY-1305682 and the Simons Foundation.

Appendix A: Geometry of arbitrary codimension foliations

In this section we most closely follow the treatment of this subject by [39]; we review and extend their calculations here to set our notation and to make this article more self-contained. Assume that we have a d dimensional manifold M that has the topology $\mathbb{R}^m \times \Sigma$. Furthermore, assume that M can be foliated by an $n = d - m$ dimensional family of spacelike submanifolds which we index with the label $\mathbf{t} \in \mathbb{R}^m$, $(\Sigma_\mathbf{t})_{\mathbf{t}\in\mathbb{R}^m}$. Greek indices will run from $0, ..., d-1$. For any point $p \in M$, the tangent space can split into $T_p = T_p(\Sigma_\mathbf{t}) \oplus T_p({}^\perp\Sigma_\mathbf{t})$, where ${}^\perp\Sigma_\mathbf{t}$ is called the transverse space to $\Sigma_\mathbf{t}$ and does not generally integrate to form a submanifold. From now on we will drop the subscript $\mathbf{t}$ from $\Sigma_\mathbf{t}$ and ${}^\perp\Sigma_\mathbf{t}$; the use of the symbols Σ and ${}^\perp\Sigma$ will refer to a specific leaf of the foliation unless otherwise noted. We define the tangent projection operator $h^\mu{}_\nu$ and the transverse projection operator $l^\mu{}_\nu = \delta^\mu{}_\nu - h^\mu{}_\nu$ which project vectors $v^\mu \in T_p(M)$ to $T_p(\Sigma)$ and $T_p({}^\perp\Sigma)$, respectively. A tensor *component* is called tangent if its contraction with the transverse projector is zero; e.g. if $l_\mu{}^\nu P^{\mu\alpha} = 0$ then we say the μ component of $P^{\mu\alpha}$ is tangent. Likewise a component of a tensor is called transverse if its contraction with the tangent projector is zero. A tensor is called tangent (transverse) if all of its components are tangent (transverse). For example, consider a tensor $P^{\rho_1\cdots\mu_r}_{\sigma_1\cdots\sigma_s}$ at a point $p \in \Sigma$. This tensor is tangent to the leaf at this point if

$$h^{\mu_1}{}_{\rho_1}\cdots h^{\mu_r}{}_{\rho_r} h^{\sigma_1}{}_{\nu_1}\cdots h^{\sigma_s}{}_{\nu_s} P^{\rho_1\cdots\rho_r}_{\sigma_1\cdots\sigma_s} = P^{\mu_1\cdots\mu_r}_{\nu_1\cdots\nu_s}. \tag{A1}$$

and is transverse to the leaf at this point if

$$l^{\mu_1}{}_{\rho_1}\cdots l^{\mu_r}{}_{\rho_r} l^{\sigma_1}{}_{\nu_1}\cdots l^{\sigma_s}{}_{\nu_s} P^{\rho_1\cdots\rho_r}_{\sigma_1\cdots\sigma_s} = P^{\mu_1\cdots\mu_r}_{\nu_1\cdots\nu_s}. \tag{A2}$$

1. Tangent/transverse derivatives and curvature tensors

We next define tangent derivatives and tangent extrinsic curvature. We introduce a metric $g_{\mu\nu}$ and metric compatible covariant derivative ∇_μ on M. For tangent tensors $P^{\mu_1\cdots\mu_r}_{\nu_1\cdots\nu_s} \in T_p(\Sigma)^{\otimes r} \otimes T^*_p(\Sigma)^{\otimes s}$, the tangent derivative operator ${}^\parallel\nabla_\mu$ is defined as the projection of the covariant derivative ∇_μ by $h^\mu{}_\nu$

$${}^\parallel\nabla_\alpha P^{\mu_1\cdots\mu_r}_{\nu_1\cdots\nu_s} \equiv h_\alpha{}^\beta h_{\rho_1}{}^{\mu_1}\cdots h_{\rho_r}{}^{\mu_r} h_{\nu_1}{}^{\sigma_1}\cdots h_{\nu_s}{}^{\sigma_s}\nabla_\beta P^{\rho_1\cdots\rho_r}_{\sigma_1\cdots\sigma_s}. \tag{A3}$$

The tangent extrinsic curvature $K^{\alpha}{}_{\mu\nu}$ can be defined as follows. Consider $v^{\mu} \in T_p(\Sigma)$, then

$$h_{\mu}{}^{\rho}\nabla_{\rho}v^{\nu} \equiv {}^{\parallel}\nabla_{\mu}v^{\nu} - K^{\nu}{}_{\lambda\mu}v^{\lambda}, \tag{A4}$$

in other words we have

$$K^{\nu}{}_{\lambda\mu} \equiv h_{\lambda}{}^{\sigma}h_{\mu}{}^{\rho}\nabla_{\rho}l_{\sigma}{}^{\nu}. \tag{A5}$$

The tangent extrinsic curvature is also known as the second fundamental form. Following similar terminology to that of Carter [40], we write $K^{\lambda} \equiv K^{\lambda}{}_{\alpha}{}^{\alpha}$, which we call the tangent curvature vector. As Σ is a submanifold, $K^{\lambda}{}_{\mu\nu}$ is symmetric under $\mu \leftrightarrow \nu$; Carter [40] refers to this property as the generalized Weingarten-Frobenius identity . From the definition in Eq. (A5) we see that

$$h_{\lambda}{}^{\sigma}K^{\lambda}{}_{\alpha\beta} = l_{\alpha}{}^{\mu}K^{\lambda}{}_{\mu\beta} = l_{\beta}{}^{\mu}K^{\lambda}{}_{\alpha\mu} = 0\,. \tag{A6}$$

The transverse derivative operator ${}^{\perp}\nabla_{\mu}$ and the transverse extrinsic curvature are defined in a similar manner to what is done for ${}^{\parallel}\nabla_{\mu}$. Consider a transverse tensor $P^{\mu_1\cdots\mu_r}_{\nu_1\cdots\nu_s} \in T_p({}^{\perp}\Sigma)^{\otimes r} \otimes T^*_p({}^{\perp}\Sigma)^{\otimes s}$, then

$${}^{\perp}\nabla_{\alpha}P^{\mu_1\cdots\mu_r}_{\nu_1\cdots\nu_s} \equiv l_{\alpha}{}^{\beta}l_{\rho_1}{}^{\mu_1}\cdots l_{\rho_r}{}^{\mu_r}l_{\nu_1}{}^{\sigma_1}\cdots l_{\nu_s}{}^{\sigma_s}\nabla_{\beta}P^{\rho_1\cdots\rho_r}_{\sigma_1\cdots\sigma_s}. \tag{A7}$$

The transverse extrinsic curvature $A^{\alpha}{}_{\mu\nu}$ is defined as follows. Consider $v^{\mu} \in T_p({}^{\perp}\Sigma)$, then

$$l_{\mu}{}^{\rho}\nabla_{\rho}v^{\nu} \equiv {}^{\perp}\nabla_{\mu}v^{\nu} - A^{\nu}{}_{\lambda\mu}v^{\lambda}, \tag{A8}$$

in other words we have

$$A^{\nu}{}_{\lambda\mu} \equiv l_{\lambda}{}^{\sigma}l_{\mu}{}^{\rho}\nabla_{\rho}h_{\sigma}{}^{\nu}. \tag{A9}$$

We write $A^{\lambda} \equiv A^{\lambda}{}_{\alpha}{}^{\alpha}$, which we call the transverse curvature vector. From the definition in Eq. (A9) we see that

$$l_{\lambda}{}^{\sigma}A^{\lambda}{}_{\alpha\beta} = h_{\alpha}{}^{\mu}A^{\lambda}{}_{\mu\beta} = h_{\beta}{}^{\mu}A^{\lambda}{}_{\alpha\mu} = 0\,. \tag{A10}$$

As the transverse space ${}^{\perp}\Sigma$ does not generally integrate to form a submanifold, the transverse extrinsic curvature $A^{\alpha}{}_{\mu\nu}$ is generally not symmetric in $\mu \leftrightarrow \nu$. This is reflected by the fact that the action of two transverse derivatives ${}^{\perp}\nabla_{\mu}$ on a scalar function f generally do not commute. We define the transverse torsion tensor $F^{\lambda}{}_{\alpha\beta}$, where

$$\begin{aligned} F^{\lambda}{}_{\alpha\beta}\nabla_{\lambda}f \equiv& -2{}^{\perp}\nabla_{[\alpha}{}^{\perp}\nabla_{\beta]}f \\ =& -2A^{\lambda}{}_{[\alpha\beta]}\nabla_{\lambda}f. \end{aligned} \tag{A11}$$

We see that the transverse torsion tensor is the antisymmetric component of the transverse extrinsic curvature $A^{\lambda}{}_{\mu\nu}$. The transverse torsion tensor $F^{\lambda}{}_{\alpha\beta}$ is also known as the twist connection.

We now define the curvature tensors for the derivative operators ${}^{\parallel}\nabla_{\mu}$ and ${}^{\perp}\nabla_{\mu}$. Consider a form $v_{\mu} \in T^*_p(\Sigma)$, we then define

$$\begin{aligned} {}^{\parallel}R_{\alpha\beta\gamma}{}^{\delta}v_{\delta} \equiv& 2{}^{\parallel}\nabla_{[\alpha}{}^{\parallel}\nabla_{\beta]}v_{\gamma} \\ =& 2h_{[\alpha}{}^{\mu}h_{\beta]}{}^{\nu}h_{\gamma}{}^{\lambda}\nabla_{\mu}\left(h_{\nu}{}^{\alpha}h_{\lambda}{}^{\rho}\nabla_{\alpha}v_{\rho}\right). \end{aligned} \tag{A12}$$

The curvature tensor for the operator ${}^{\perp}\nabla_{\mu}$ is defined similarly, except that we need to take into account that it generally will have nonzero torsion. Consider a form $v_{\mu} \in T^*_p({}^{\perp}\Sigma)$, we then define

$$\begin{aligned} {}^{\perp}R_{\alpha\beta\gamma}{}^{\delta}v_{\delta} \equiv& 2{}^{\perp}\nabla_{[\alpha}{}^{\perp}\nabla_{\beta]}v_{\gamma} + F^{\lambda}{}_{\alpha\beta}l_{\gamma}{}^{\delta}\nabla_{\lambda}v_{\delta} \\ =& 2l_{[\alpha}{}^{\mu}l_{\beta]}{}^{\nu}l_{\gamma}{}^{\lambda}\nabla_{\mu}\left(l_{\nu}{}^{\alpha}l_{\lambda}{}^{\rho}\nabla_{\alpha}v_{\rho}\right) - 2A^{\lambda}{}_{[\alpha\beta]}l_{\gamma}{}^{\delta}\nabla_{\lambda}v_{\delta}\,. \end{aligned} \tag{A13}$$

Note that the derivative acting on v_{δ} contracted with the torsion tensor is *not* ${}^{\perp}\nabla_{\mu}$ as $l_{\lambda}{}^{\rho}A^{\lambda}{}_{\alpha\beta} = 0$.

2. Projections of the Riemann tensor

With the definitions in Eqs. (A5), (A9), (A12), and (A13), we can rewrite the projections of the Riemann tensor by $h_\mu{}^\nu$ and $l_\mu{}^\nu$ entirely in terms of the tensors $K^\alpha{}_{\mu\nu}$, $A^\alpha{}_{\mu\nu}$, ${}^{\parallel}R_{\alpha\beta\gamma\delta}$, and ${}^{\perp}R_{\alpha\beta\gamma\delta}$. These are summarized below:

$$h_\alpha{}^\mu h_\beta{}^\nu h_\gamma{}^\lambda h_\delta{}^\rho R_{\mu\nu\lambda\rho} = {}^{\parallel}R_{\alpha\beta\gamma\delta} - K^\lambda{}_{\gamma\alpha}K_{\lambda\delta\beta} + K^\lambda{}_{\gamma\beta}K_{\lambda\delta\alpha}, \tag{A14}$$

$$l_\alpha{}^\mu l_\beta{}^\nu l_\gamma{}^\lambda l_\delta{}^\rho R_{\mu\nu\lambda\rho} = {}^{\perp}R_{\alpha\beta\gamma\delta} - A^\lambda{}_{\gamma\alpha}A_{\lambda\delta\beta} + A^\lambda{}_{\gamma\beta}A_{\lambda\delta\alpha}, \tag{A15}$$

$$\begin{aligned} l_\alpha{}^\mu h_\beta{}^\nu l_\gamma{}^\lambda h_\delta{}^\rho R_{\mu\nu\lambda\rho} = & - l_\alpha{}^\mu h_\beta{}^\nu l_\gamma{}^\lambda h_\delta{}^\rho \nabla_\mu K_{\lambda\rho\nu} - l_\alpha{}^\nu h_\beta{}^\mu l_\gamma{}^\lambda h_\delta{}^\rho \nabla_\mu A_{\rho\lambda\nu} \\ & - K_\alpha{}^\lambda{}_\beta K_{\gamma\delta\lambda} - A_\beta{}^\lambda{}_\alpha A_{\delta\gamma\lambda}, \end{aligned} \tag{A16}$$

$$h_\alpha{}^\mu h_\beta{}^\nu h_\gamma{}^\lambda l_\delta{}^\rho R_{\mu\nu\lambda\rho} = 2h_{[\alpha}{}^\mu h_{\beta]}{}^\nu h_\gamma{}^\lambda \nabla_\mu K_{\delta\lambda\nu}, \tag{A17}$$

$$l_\alpha{}^\mu l_\beta{}^\nu l_\gamma{}^\lambda h_\delta{}^\rho R_{\mu\nu\lambda\rho} = 2l_{[\alpha}{}^\mu l_{\beta]}{}^\nu l_\gamma{}^\lambda \nabla_\mu A_{\delta\lambda\nu} + 2A^\sigma{}_{[\alpha\beta]}K_{\gamma\delta\sigma}. \tag{A18}$$

Eq. (A14) is the $m+n$ generalization of the Gauss equation, Eq. (A16) is the $m+n$ generalization of the Ricci equation, and Eq. (A17) is the $m+n$ generalization of the Codazzi equation. Eqs. (A15) and (A18) are identically zero in codimension one spacetime splittings. We provide a derivation of Eqs. (A15) and (A16) below; the derivation of the other projections follow a similar procedure. Similar expressions projections of the Riemann tensor are presented in Appendix A of [39].

a. *Derivation of Eq.* (A15)

To show Eq. (A15), let us consider $v^\mu \in T_p({}^{\perp}\Sigma)$. We then have

$$\begin{aligned} l_\alpha{}^\mu l_\beta{}^\nu l_\gamma{}^\lambda R_{\mu\nu\lambda\sigma} v^\sigma = & 2l_{[\alpha}{}^\mu l_{\beta]}{}^\nu l_\gamma{}^\lambda \nabla_\mu \nabla_\nu v_\lambda \\ = & 2l_{[\alpha}{}^\mu l_{\beta]}{}^\nu l_\gamma{}^\lambda \nabla_\mu \left[\left(l_\nu{}^\rho + h_\nu{}^\rho\right)\left(l_\lambda{}^\sigma + h_\lambda{}^\sigma\right) \nabla_\rho v_\sigma \right] \\ = & 2A^\rho{}_{[\beta\alpha]} l_\gamma{}^\sigma \nabla_\rho v_\sigma + 2l_{[\alpha}{}^\mu l^\rho_{\beta]} l_\gamma{}^\lambda \nabla_\mu h_\lambda{}^\sigma \nabla_\rho v_\sigma + 2{}^{\perp}\nabla_{[\alpha}{}^{\perp}\nabla_{\beta]} v_\gamma \\ = & 2\,{}^{\perp}\nabla_{[\alpha}{}^{\perp}\nabla_{\beta]} v_\gamma + F^\rho{}_{\alpha\beta} l_\gamma{}^\sigma \nabla_\rho v_\sigma - 2l_{[\alpha}{}^\mu l_{\beta]}{}^\rho h_\lambda{}^\sigma \left(\nabla_\mu l_\gamma{}^\lambda\right) \nabla_\rho \left(l_\sigma{}^\kappa v_\kappa\right) \\ = & {}^{\perp}R_{\alpha\beta\gamma\delta} v^\delta + 2l_{[\alpha}{}^\mu l_{\beta]}{}^\rho l_{\sigma\delta} \left(\nabla_\mu l_\gamma{}^\lambda\right)\left(\nabla_\rho h_\lambda{}^\sigma\right) v^\delta. \end{aligned} \tag{A19}$$

Consider the last term:

$$\begin{aligned} \left(l_\alpha{}^\mu l_\beta{}^\rho - l_\beta{}^\mu l^\rho_\alpha\right) l_{\sigma\delta} \left(\nabla_\mu l_\gamma{}^\lambda\right)\left(\nabla_\rho h_\lambda{}^\sigma\right) v^\delta = & \left[\left(h_\kappa{}^\lambda l_\alpha{}^\mu \nabla_\mu l_\gamma{}^\kappa\right) A_{\lambda\delta\beta} - \left(h_\kappa{}^\lambda l_\beta{}^\mu \nabla_\mu l_\gamma{}^\kappa\right) A_{\lambda\delta\alpha} \right] v^\delta \\ = & \left(-A^\lambda{}_{\gamma\alpha}A_{\lambda\delta\beta} + A^\lambda{}_{\gamma\beta}A_{\lambda\delta\alpha}\right) v^\delta. \end{aligned} \tag{A20}$$

We conclude that Eq. (A15) holds,

$$l_\alpha{}^\mu l_\beta{}^\nu l_\gamma{}^\lambda l_\delta{}^\rho R_{\mu\nu\lambda\rho} = {}^{\perp}R_{\alpha\beta\gamma\delta} - A^\lambda{}_{\gamma\alpha}A_{\lambda\delta\beta} + A^\lambda{}_{\gamma\beta}A_{\lambda\delta\alpha}. \tag{A21}$$

b. *Derivation of Eq.* (A16)

To show Eq. (A16), let us consider $v^\mu \in T_p(\Sigma)$. We compute

$$\begin{aligned} l_\alpha{}^\mu h_\beta{}^\nu l_\gamma{}^\lambda R_{\mu\nu\lambda\delta} v^\delta = & 2l_\alpha{}^{[\mu} h_\beta{}^{\nu]} l_\gamma{}^\lambda \nabla_\mu \nabla_\nu v_\lambda \\ = & 2l_\alpha{}^{[\mu} h_\beta{}^{\nu]} l_\gamma{}^\lambda \Big((\nabla_\mu h_\lambda{}^\rho)(\nabla_\nu v_\rho) + (\nabla_\nu h_\lambda{}^\rho)(\nabla_\mu v_\rho) \\ & + h_\lambda{}^\rho \nabla_\mu \nabla_\nu v_\rho + v_\rho \nabla_\mu \nabla_\nu h_\lambda{}^\rho \Big) \\ = & 2l_\alpha{}^{[\mu} h_\beta{}^{\nu]} l_\gamma{}^\lambda \left(\nabla_\mu \nabla_\nu h_\lambda{}^\rho\right) v_\rho \\ = & 2l_\alpha{}^{[\mu} h_\beta{}^{\nu]} l_\gamma{}^\sigma \left[\nabla_\mu \left(l_\sigma{}^\lambda \nabla_\nu h_{\lambda\delta}\right) - \left(\nabla_\mu l_\sigma{}^\lambda\right)\left(\nabla_\nu h_{\lambda\delta}\right) \right] v^\delta. \end{aligned} \tag{A22}$$

We next split this calculation into two different parts. We first look at

$$2l_\alpha{}^{[\mu} h_\beta{}^{\nu]} l_\gamma{}^\sigma \left[\nabla_\mu \left(l_\sigma{}^\lambda \nabla_\nu h_{\lambda\delta}\right) \right] v^\delta = 2l_\alpha{}^{[\mu} h_\beta{}^{\nu]} l_\gamma{}^\sigma \left[\nabla_\mu \left(\left(h_\nu{}^\kappa + l_\nu{}^\kappa\right) l_\sigma{}^\lambda \nabla_\kappa h_{\lambda\delta} \right) \right] v^\delta$$

$$=2l_\alpha{}^{[\mu}h_\beta{}^{\nu]}l_\gamma{}^\sigma\left[\nabla_\mu\left(A_{\delta\sigma\nu}-K_{\sigma\delta\nu}\right)\right]v^\delta, \tag{A23}$$

where we have used $\delta_\nu{}^\kappa = l_\nu{}^\kappa + h_\nu{}^\kappa$. We further split this term into two more pieces

$$\begin{aligned}2l_\alpha{}^{[\mu}h_\beta{}^{\nu]}l_\gamma{}^\sigma\left(\nabla_\mu A_{\delta\sigma\nu}\right)v^\delta &= \left[-l_\alpha{}^\nu h_\beta{}^\mu l_\gamma{}^\sigma h_\delta{}^\lambda\nabla_\mu A_{\lambda\sigma\nu} + l_\alpha{}^\mu h_\beta{}^\nu\left(\nabla_\mu l_\nu{}^\xi\right)A_{\delta\gamma\xi}\right]v^\delta \\ &= \left(-l_\alpha{}^\nu h_\beta{}^\mu l_\gamma{}^\sigma h_\delta{}^\lambda\nabla_\mu A_{\lambda\sigma\nu} - A_{\delta\gamma\lambda}A_\beta{}^\lambda{}_\alpha\right)v^\delta.\end{aligned} \tag{A24}$$

Similarly we have

$$2l_\alpha{}^{[\mu}h_\beta{}^{\nu]}l_\gamma{}^\sigma\left(\nabla_\mu K_{\sigma\delta\nu}\right)v^\delta = \left(l_\alpha{}^\mu h_\beta{}^\nu l_\gamma{}^\sigma h_\delta{}^\lambda\nabla_\mu K_{\sigma\lambda\nu} + K_\alpha{}^\lambda{}_\beta K_{\gamma\delta\lambda}\right)v^\delta. \tag{A25}$$

Finally, we look at the last term on the right hand side of Eq. (A22),

$$\begin{aligned}2l_\alpha{}^{[\mu}h_\beta{}^{\nu]}l_\gamma{}^\sigma\left(\nabla_\mu l_\sigma{}^\lambda\right)\left(\nabla_\nu h_{\lambda\delta}\right)v^\delta =& 2l_\alpha{}^{[\mu}h_\beta{}^{\nu]}l_\gamma{}^\sigma\left(\nabla_\mu h_\sigma{}^\lambda\right)\left(\nabla_\nu l_{\lambda\delta}\right)v^\delta \\ =& 2l_\alpha{}^{[\mu}h_\beta{}^{\nu]}l_{\lambda\delta}\left(\nabla_\mu l_\gamma{}^\sigma\right)\left(\nabla_\nu h_\sigma{}^\lambda\right)v^\delta \\ =& 0,\end{aligned} \tag{A26}$$

which is zero as v^μ is a tangent vector. With this final relation we can recover Eq. (A16),

$$\begin{aligned}l_\alpha{}^\mu h_\beta{}^\nu l_\gamma{}^\lambda h_\delta{}^\rho R_{\mu\nu\lambda\rho} =& -l_\alpha{}^\mu h_\beta{}^\nu l_\gamma{}^\lambda h_\delta{}^\rho\nabla_\mu K_{\lambda\rho\nu} - l_\alpha{}^\nu h_\beta{}^\mu l_\gamma{}^\lambda h_\delta{}^\rho\nabla_\mu A_{\rho\lambda\nu} \\ &- K_\alpha{}^\lambda{}_\beta K_{\gamma\delta\lambda} - A_\beta{}^\lambda{}_\alpha A_{\delta\gamma\lambda}.\end{aligned} \tag{A27}$$

3. Projected Ricci tensor and projected Ricci scalar

Using Eqs. (A14), (A15), and (A16), we can rewrite the Ricci tensor in terms of ${}^{\parallel}R_{\alpha\beta\gamma\delta}$, ${}^{\perp}R_{\alpha\beta\gamma\delta}$, $A_{\gamma\alpha\beta}$, and $K_{\gamma\alpha\beta}$. Using the completeness relation $g_{\mu\nu} = l_{\mu\nu} + h_{\mu\nu}$, we have

$$\begin{aligned}h_\alpha{}^\mu h_\beta{}^\lambda R_{\mu\lambda} =& h_\alpha{}^\mu h_\beta{}^\lambda\left(h^{\nu\rho}+l^{\nu\rho}\right)R_{\mu\nu\lambda\rho} \\ =& -h_\alpha{}^\mu h_\beta{}^\lambda l^{\nu\rho}\nabla_\nu K_{\rho\lambda\mu} - h_\alpha{}^\nu h_\beta{}^\lambda l^{\mu\rho}\nabla_\nu A_{\lambda\rho\mu} \\ &- K^\lambda{}_{\beta\alpha}K_\lambda - A_\alpha{}^{\lambda\sigma}A_{\beta\sigma\lambda} + {}^{\parallel}R_{\alpha\beta},\end{aligned} \tag{A28}$$

$$\begin{aligned}l_\alpha{}^\mu l_\beta{}^\lambda R_{\mu\lambda} =& l_\alpha{}^\mu l_\beta{}^\lambda\left(l^{\nu\rho}+h^{\nu\rho}\right)R_{\mu\nu\lambda\rho} \\ =& -l_\alpha{}^\mu l_\beta{}^\lambda h^{\nu\rho}\nabla_\nu A_{\rho\lambda\mu} - l_\alpha{}^\nu l_\beta{}^\lambda h^{\mu\rho}\nabla_\nu K_{\lambda\rho\mu} \\ &- A^\lambda{}_{\beta\alpha}A_\lambda - K_\alpha{}^{\lambda\sigma}K_{\beta\sigma\lambda} + {}^{\perp}R_{\alpha\beta},\end{aligned} \tag{A29}$$

$$\begin{aligned}h_\alpha{}^\nu l_\beta{}^\rho R_{\nu\rho} =& h_\alpha{}^\nu l_\beta{}^\rho\left(h^{\mu\lambda}+l^{\mu\lambda}\right)R_{\mu\nu\lambda\rho} \\ =& 2l_{[\sigma}{}^\mu l_{\beta]}{}^\nu l^{\sigma\lambda}\nabla_\mu A_{\alpha\lambda\nu} + 2h_{[\sigma}{}^\mu h_{\alpha]}{}^\nu h^{\sigma\lambda}\nabla_\mu K_{\beta\lambda\nu} + 2A^\sigma{}_{[\mu\beta]}K^\mu{}_{\alpha\sigma}.\end{aligned} \tag{A30}$$

Eqs. (A28), (A29), and (A30) are the projected vacuum Einstein equations. In the context of a double null foliation in four dimensional spacetime (see section A 5), some authors have pointed out that Eq. (A30), with a suitable relabeling and interpretation of its variables resembles a Navier-Stokes equation [41–44] (see, e.g. [45] for a critique of this interpretation).

Calculating one further contraction gives us the projected Ricci scalar,

$$\begin{aligned}R =& \left(h^{\alpha\gamma}h^{\beta\delta}+l^{\alpha\gamma}l^{\beta\delta}+2h^{\alpha\gamma}l^{\beta\delta}\right)R_{\alpha\beta\gamma\delta} \\ =& {}^{\parallel}R + {}^{\perp}R + K_\lambda K^\lambda - K_{\lambda\alpha\beta}K^{\lambda\beta\alpha} + A_\lambda A^\lambda - A_{\lambda\alpha\beta}A^{\lambda\beta\alpha} - 2\nabla_\lambda\left(K^\lambda + A^\lambda\right).\end{aligned} \tag{A31}$$

One can similarly apply the Riemann projection formulas to rewrite scalar polynomials in the Riemann curvature, such as $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ in terms of the quantities ${}^{\parallel}R_{\alpha\beta\gamma\delta}$, ${}^{\perp}R_{\alpha\beta\gamma\delta}$, $A_{\gamma\alpha\beta}$, and $K_{\gamma\alpha\beta}$.

4. Codimension one foliations

Let us now consider a special case with codimension one foliations. For a codimension one surface, we can write $l^{\mu\nu} = \epsilon n^\mu n^\nu$. We choose n^μ to be normalized to $\epsilon \equiv \pm 1$ depending on whether n^μ is space- or time-like. The completeness relation for the projection operators then reads

$$g_{\mu\nu} = h_{\mu\nu} + \epsilon n_\mu n_\nu. \tag{A32}$$

In this case, we see that

$$\begin{aligned} K^{\lambda}{}_{\alpha\beta} =& \epsilon n^{\lambda} h_{\alpha}{}^{\mu} h_{\beta}{}^{\nu} \nabla_{\nu} n_{\mu} \\ =& \epsilon n^{\lambda} K_{\alpha\beta}, \end{aligned} \tag{A33}$$

$$A^{\lambda}{}_{\alpha\beta} = - a^{\lambda} n_{\alpha} n_{\beta}, \tag{A34}$$

$$^{\perp}R_{\alpha\beta\gamma\delta} = 0, \tag{A35}$$

where we have defined $a^{\mu} \equiv n^{\nu}\nabla_{\nu}n^{\mu}$, which is perpendicular to n_{μ} so that $a^{\mu}h_{\mu}{}^{\lambda} = a^{\lambda}$, and $K_{\mu\nu}$ is the standard second fundamental form for codimension one surfaces. We see that the torsion tensor $F^{\lambda}{}_{\alpha\beta} = 0$. The projected Ricci tensor components are

$$h_{\alpha}{}^{\mu} h_{\beta}{}^{\lambda} R_{\mu\lambda} = - \epsilon h_{\alpha}{}^{\mu} h_{\beta}{}^{\lambda} n^{\nu} \nabla_{\nu} K_{\lambda\mu} + \epsilon {}^{\parallel}\nabla_{\alpha} a_{\beta} - a_{\alpha} a_{\beta} - \epsilon K K_{\alpha\beta} + {}^{\parallel}R_{\alpha\beta}, \tag{A36}$$

$$n^{\mu} n^{\lambda} R_{\mu\lambda} = {}^{\parallel}\nabla_{\lambda} a^{\lambda} - \epsilon a^{\lambda} a_{\lambda} - h^{\mu\rho} n^{\nu} \nabla_{\nu} K_{\rho\mu} - K^{\sigma\lambda} K_{\sigma\lambda}, \tag{A37}$$

$$n_{\alpha}{}^{\nu} n^{\rho} R_{\nu\rho} = \epsilon \left({}^{\parallel}\nabla_{\lambda} K^{\lambda}{}_{\alpha} - {}^{\parallel}\nabla_{\alpha} K \right), \tag{A38}$$

from which one can derive the standard $1 + (d-1)$ projected Einstein equations. The projected Ricci scalar is

$$R = {}^{\parallel}R + \epsilon \left(K^2 - K_{\mu\nu} K^{\mu\nu} \right) - 2\epsilon \nabla_{\lambda} \left(n^{\lambda} K - a^{\lambda} \right). \tag{A39}$$

Here we have defined $K \equiv K_{\mu}{}^{\mu}$, and used the fact that $n^{\mu} a_{\mu} = 0$, so that $h_{\mu}{}^{\alpha} a_{\alpha} = a_{\mu}$.

5. Relation between double null and codimension two foliations

The $m+n$ formalism we have described is capable of describing the geometry of double null foliations. In a double null foliation, spacetime is foliated by a pair of lightlike surfaces, Σ^0 and Σ^1, which have the null generators $l^{(0)}_{\alpha}$ and $l^{(1)}_{\alpha}$, respectively [22]. The intersections of the foliations, $\{\Sigma^0\} \cap \{\Sigma^1\}$ form a spacelike foliation of codimension two, which we then identify as the foliation Σ. The transverse space $T_p({}^{\perp}\Sigma)$ for each point $p \in \Sigma$ is spanned by the two null generators $(l^{(0)})^{\alpha}$ and $(l^{(1)})^{\alpha}$. We can now define the transverse projection operator as

$$l_{\mu\nu} = l^{(0)}_{\mu} l^{(1)}_{\nu} + l^{(0)}_{\nu} l^{(1)}_{\mu}. \tag{A40}$$

The tangent projector can then be computed from the relation $h_{\mu\nu} = g_{\mu\nu} - l_{\mu\nu}$.

Appendix B: ADM-like variables for $m+n$ spacetime splitting

In this section, we set up a coordinate system adapted to the foliation $(\Sigma_{\mathbf{t}})_{\mathbf{t}\in\mathbb{R}^m}$. We then write down the tensors ${}^{\parallel}R_{\alpha\beta\gamma\delta}$, ${}^{\perp}R_{\alpha\beta\gamma\delta}$, $K^{\lambda}{}_{\alpha\beta}$, and $A^{\lambda}{}_{\alpha\beta}$ as functions of these coordinates. We closely follow the work of [22] in defining the basis vectors for $T_p(\Sigma)$ and $T_p({}^{\perp}\Sigma)$; see also [46, 47] for similar treatments of this subject.

We recall our notation: Greek indices run from $0, ..., d-1$, lower case Latin indices run from $0, ..., m-1$ and upper case Latin indices from from $m, ..., d-1$. Einstein summation notation will apply to all different index types.

1. Coordinate system and metric decomposition

We begin by setting up a coordinate system on our manifold M adapted to an $m+n$ spacetime foliation. The coordinates x^{α} of some chart of the spacetime manifold M are written as functions of two sets of variables, $\{u^a\}$ and $\{\theta^A\}$, $x^{\alpha} \equiv x^{\alpha}(u^a, \theta^A)$. Derivatives with respect to the variables u^a will be denoted by $\partial_a \equiv \partial/\partial u^a$, while derivatives with respect to the variables θ^A will be denoted by $\partial_A \equiv \partial/\partial\theta^A$. The set $\{\theta^A\}$ are the intrinsic coordinates on the leaf Σ. The $\{u^a\}$ are scalar fields, the level sets of which define a congruence of curves that intersect all the leafs Σ of the foliation. In other words, for the leaf $\Sigma_{\mathbf{t}\in\mathbb{R}^{\mathbf{m}}}$, we have

$$\mathbf{t} = (u^0, ..., u^{m-1}). \tag{B1}$$

We use this congruence to relate coordinates on each leaf to each other. For example, in the $1 + (d-1)$ formalism $\mathbf{t} = u^0 \equiv t$, the time function. Just as in the $1 + (d-1)$ formalism, we neither assume that the congruence of curves to be geodesics nor assume that they are orthogonal to the leafs Σ. The tangent vector for the congruence defined by u^c is denoted by

$$u_c^\gamma \equiv \partial_c x^\gamma . \tag{B2}$$

This is to be compared to the $1+(d-1)$ formalism, where the time tangent vector is often denoted by $t^\alpha \equiv \partial_t x^\alpha$.

We now define a coordinate basis on the leaf Σ as follows

$$e_A^\alpha \equiv \partial_A x^\alpha , \tag{B3}$$

from which we can construct the intrinsic metric on Σ

$$\gamma_{AB} \equiv g_{\alpha\beta} e_A^\alpha e_B^\beta . \tag{B4}$$

We will raise/lower capital Latin indices with γ_{AB} and γ^{AB} respectively, where γ^{AB} is the inverse of the induced metric γ_{AB}. The metric covariant derivative with respect to γ_{AB} will be denoted as ${}^{\parallel}\nabla_A$. At each point $p \in \Sigma$, we can define a basis for $T_p^*({}^{\perp}\Sigma)$ as follows:

$$n_\alpha^a \equiv \partial_\alpha u^a \, . \tag{B5}$$

The one-forms $\{n_\alpha^a\}$ need not be orthonormal with one another; we capture this lack of orthonormality with the following symmetric inner product matrix

$$\alpha^{ab} \equiv g^{\alpha\beta} n_\alpha^a n_\beta^b , \tag{B6}$$

which is symmetric in $a \leftrightarrow b$. As the $\{n_\alpha^a\}$ are form a basis for $T_p^*({}^{\perp}\Sigma)$, α^{ab} is invertible and we denote its matrix inverse by α_{ab}; $\alpha_{ac}\alpha^{cb} = \delta_a^b$, where δ_a^b is the Kronecker delta symbol. We emphasize that α^{ab} is *not* an induced metric on the transverse space ${}^{\perp}\Sigma$, as in general ${}^{\perp}\Sigma$ does not integrate to form a submanifold. We will formally raise/lower frame indices for the transverse spaces with the inner product matrices α^{ab} and α_{ab}, respectively. The spacetime scalar α_{ab} corresponds to a generalization of the lapse function α in the $1+(d-1)$ formalism. In particular, in the $1+(d-1)$ formalism we identify $\alpha_{00} = -\alpha^2$ and $\alpha^{00} = -\alpha^{-2}$. The unit normal forms to the leaves Σ are computed as follows,

$$n_{a\alpha} \equiv \alpha_{ab} \partial_\alpha u^b . \tag{B7}$$

We now introduce a generalization of the shift vector. With the above definitions in hand, we see that the vectors

$$\{u_a^\alpha - n_a^\alpha\}_{a=0,\ldots,m-1} \tag{B8}$$

are orthogonal to the one forms $\{n_\beta^b\}_{b=0,\ldots,m-1}$. From this we conclude that we can write the vector n_a^α as

$$n_a^\alpha \equiv u_a^\alpha - \beta_a^\alpha , \tag{B9}$$

where we have defined the shift vectors $\{\beta_a^\alpha\}$, which are orthogonal to the one forms $n_{a\alpha}$; i.e. $n_{a\alpha}\beta_b^\alpha = 0$. The shift vectors $\{\beta_a^\alpha\}_{a=0,\ldots,m-1}$ are a direct generalization of the shift vector β^α in the $1+(d-1)$ formalism. .

We next derive some useful relations for e_A^α and n_a^α. The relations Eq. (B3) and (B9) imply that in the coordinates (u^a, θ^A) we have

$$e_A^\alpha \stackrel{*}{=} \delta_A^\alpha , \tag{B10}$$
$$n_a^\alpha \stackrel{*}{=} \delta_a^\alpha - \beta_a^A \delta_A^\alpha , \tag{B11}$$

where the δ is the Kronecker delta symbol and $\stackrel{*}{=}$ means that this only holds in the specific coordinate choice $\{(u^a, \theta^A)\}$. We only use the symbol $\stackrel{*}{=}$ in this section; in the Sections I-V we work with the coordinate choices defined by Eqs. (B10) and (B11). In the $1+(d-1)$ formalism the equivalent coordinate choice would be $\{t, x^i\}$, where the $\{x^i\}_{i=1,2,3}$ are the three spatial directions. We see that in this basis the shift vectors have nonzero components only on their last n indices: $\beta_a^\alpha \stackrel{*}{=} (0, ..., 0, \beta_a^A)$. From Eqs. (B10) and (B11) we conclude that the frame vectors e_A^α are Lie transported along each of the congruences defined by the level sets of the functions u^c

$$\pounds_{u_c^\gamma} e_A^\alpha = 0. \tag{B12}$$

Since this expression is tensorial, it holds in any coordinate system. Other useful tensorial relations we can derive from the above expressions are

$$\pounds_{e_C^\gamma} n_\alpha^a = e_C^\gamma \nabla_\gamma n_\alpha^a + n_\gamma^a \nabla_\alpha e_C^\gamma = 0, \tag{B13}$$

$$\pounds_{e_C^\gamma} e_A^\alpha = e_C^\gamma \nabla_\gamma e_A^\alpha - e_A^\gamma \nabla_\gamma e_C^\alpha = 0, \tag{B14}$$

$$\pounds_{n_a^\alpha} n_b^\gamma = n_a^\alpha \nabla_\alpha n_b^\gamma - n_b^\beta \nabla_\beta n_a^\gamma = -\mathcal{F}_{ab}^\gamma, \tag{B15}$$

where we have defined the transverse torsion spacetime *vector* $\mathcal{F}_{ab}^\gamma$ to be

$$\mathcal{F}_{ab}^\gamma \equiv \partial_a \beta_b^\gamma - \partial_b \beta_a^\gamma + \beta_b^D \partial_D \beta_a^\gamma - \beta_a^D \partial_D \beta_b^\gamma. \tag{B16}$$

The vector $\mathcal{F}_{ab}^\gamma$ is orthogonal to the forms n_γ^c

$$n_\gamma^c \mathcal{F}_{ab}^\gamma = 0. \tag{B17}$$

As this expression is tensorial it holds in general coordinate system. In the adapted basis $\{(u^a, \theta^A)\}$ we may write $\mathcal{F}_{ab}^\gamma = e_C^\gamma \mathcal{F}_{ab}^C \overset{*}{=} \mathcal{F}_{ab}^C$ to reflect this fact.

We now see how the metric is $m+n$ decomposed. We begin by decomposing the differential dx^α into terms tangent and transverse to the leaf Σ [22]

$$dx^\alpha = n_a^\alpha du^a + e_A^\alpha \left(d\theta^A + \beta_a^A du^a \right). \tag{B18}$$

From which the spacetime line element can be written as

$$ds^2 = \alpha_{ab} du^a du^b + \gamma_{AB} \left(d\theta^A + \beta_a^A du^a \right) \left(d\theta^B + \beta_b^B du^b \right). \tag{B19}$$

We note that with the spacetime line element Eq. (B19) the metric determinant factorizes as follows

$$\det\left(g_{\mu\nu}\right) = \det\left(\alpha_{ab}\right) \det\left(\gamma_{AB}\right). \tag{B20}$$

We compare Eq. (B20) to the case in the $1+(d-1)$ formalism, where $\det(g) = \alpha^2 \det(\gamma_{ij})$. Furthermore, we have the following relations

$$h_{\alpha\beta} = \gamma_{AB} e_\alpha^A e_\beta^B, \tag{B21}$$

$$l_{\alpha\beta} = \alpha_{ab} n_\alpha^a n_\beta^b, \tag{B22}$$

so that the metric can be written as follows (see, for example [22, 48] for similar presentations of the metric tensor)

$$g_{\alpha\beta} = \alpha_{ab} n_\alpha^a n_\beta^b + \gamma_{AB} e_\alpha^A e_\beta^B. \tag{B23}$$

2. Rewriting curvature terms in ADM-like variables

In this section, we compute the components of $K_{\gamma\alpha\beta}$, ${}^{\parallel}R_{\alpha\beta\gamma\delta}$, $A_{\gamma\alpha\beta}$, and ${}^{\perp}R_{\alpha\beta\gamma\delta}$ in the adapted basis $\{(u^a, \theta^A)\}$, i.e. when the relations Eqs. (B10) and (B11) hold. The curvature terms $K_{\gamma\alpha\beta}$ and ${}^{\parallel}R_{\alpha\beta\gamma\delta}$ have direct analogues in the $1+(d-1)$ formalism, and can be computed as functions of the metric Eq. (B19) in a way analogous to what is done in the $1+(d-1)$ formalism. We have found a greater variety of functional forms for the curvature terms $A_{\gamma\alpha\beta}$ and ${}^{\perp}R_{\alpha\beta\gamma\delta}$ that have been presented in the literature. We recall that α_{ab} is generally not the induced metric for any submanifold, as the transverse space ${}^{\perp}\Sigma$ can only integrate to form a manifold in factorizable spacetimes (see Appendix B 3).

a. *Computing $K_{\gamma\alpha\beta}$*

We first compute $K_{\gamma\alpha\beta}$. We have

$$\begin{aligned} K_{\gamma\alpha\beta} =& h_\alpha{}^\mu h_\beta{}^\nu \nabla_\nu l_{\mu\gamma} \\ =& e_\alpha^A e_\beta^B n_\gamma^c \left(e_A^\mu e_B^\nu \nabla_\nu n_{c\mu} \right) \end{aligned}$$

$$=e^A_\alpha e^B_\beta n^c_\gamma \frac{1}{2}\left(e^\mu_A e^\nu_B \pounds_{n^\gamma_c} g_{\mu\nu}\right)$$
$$\equiv e^A_\alpha e^B_\beta n^c_\gamma \mathcal{K}_{cAB}. \tag{B24}$$

We now rewrite the Lie derivative of n^γ_c in terms of covariant derivatives acting on the shift vectors β^γ_c and the Lie derivative of u^γ_c. We compute

$$\begin{aligned}\pounds_{u^\gamma_c}\gamma_{AB} =&\pounds_{u^\gamma_c}\left(e^\alpha_A e^\beta_B g_{\alpha\beta}\right)\\ =&e^\alpha_A e^\beta_B \pounds_{u^\gamma_c} g_{\alpha\beta}\\ =&e^\alpha_A e^\beta_B\left(\nabla_\alpha u_{c\beta}+\nabla_\beta u_{c\alpha}\right)\\ =&e^\alpha_A e^\beta_B\left(\nabla_\alpha n_{c\beta}+\nabla_\beta n_{c\alpha}+\nabla_\alpha\beta_{c\beta}+\nabla_\beta\beta_{c\alpha}\right)\\ =&e^\alpha_A e^\beta_B \pounds_{n^\gamma_c} g_{\alpha\beta}+{}^{\parallel}\nabla_A\beta_{cB}+{}^{\parallel}\nabla_B\beta_{cA}.\end{aligned} \tag{B25}$$

From this we conclude that

$$\begin{aligned}K_{\gamma\alpha\beta} =&e^A_\alpha e^B_\beta n^c_\gamma \mathcal{K}_{cAB},\\ \mathcal{K}_{cAB} \equiv&\frac{1}{2}\left(\pounds_{u^\gamma_c}\gamma_{AB}-{}^{\parallel}\nabla_A\beta_{cB}-{}^{\parallel}\nabla_B\beta_{cA}\right).\end{aligned} \tag{B26}$$

Recall that lower case Latin letters act as labels, so that ${}^{\parallel}\nabla_A\beta_{cB}=\partial_A\beta_{cB}-\Gamma^C{}_{AB}\beta_{cC}$, where $\Gamma^C{}_{AB}$ is defined by Eq. (B27). Also note that as γ_{AB} is a spacetime scalar, in the coordinate adapted basis we have $\pounds_{u^\gamma_c}\gamma_{AB}=\partial_c\gamma_{AB}$. This is to be compared to the $1+(d-1)$ spacetime splitting formalism, where instead one has $\pounds_{t^\alpha}\gamma_{ij}\equiv\partial_t\gamma_{ij}$, and there is only one shift vector β^i.

b. *Computing* ${}^{\parallel}\mathcal{R}_{\alpha\beta\gamma\delta}$

Next, we compute ${}^{\parallel}\mathcal{R}_{\alpha\beta\gamma\delta}$. The connection coefficients for the induced covariant derivative on Σ is computed as follows:

$$\begin{aligned}\Gamma_{CAB} \equiv&e^\beta_B e_{C\alpha}\nabla_\beta e^\alpha_A\\ =&\frac{1}{2}\left(\partial_A\gamma_{BC}+\partial_B\gamma_{AC}-\partial_C\gamma_{AB}\right).\end{aligned} \tag{B27}$$

Note that $\Gamma^C{}_{AB}=\gamma^{CD}\Gamma_{DAB}$. We can now compute ${}^{\parallel}R_{\alpha\beta\gamma\delta}$ in terms of contractions and derivatives of the connection Γ_{CAB}.

$$\begin{aligned}{}^{\parallel}R_{\alpha\beta\gamma}{}^\delta e^D_\delta =&2h_{[\alpha}{}^\mu h_{\beta]}{}^\nu h_\gamma{}^\lambda\nabla_\mu\left(h_\nu{}^\eta h_\lambda{}^\rho\nabla_\eta e^D_\rho\right)\\ =&2e^A_{[\alpha}e^B_{\beta]}e^C_\gamma\left(e^\mu_A\nabla_\mu\Gamma_C{}^D{}_B+\Gamma_I{}^D{}_B\Gamma_C{}^I{}_A+\Gamma_C{}^D{}_I\Gamma_B{}^I{}_A\right)\\ \stackrel{*}{=}&2e^A_{[\alpha}e^B_{\beta]}e^C_\gamma\left(-\partial_A\Gamma^D{}_{CB}+\Gamma^D{}_{IB}\Gamma^I{}_{CA}\right).\end{aligned} \tag{B28}$$

To obtain the third line we used the property $\Gamma_B{}^I{}_A=-\Gamma^I{}_{BA}$. We also used the fact that γ_{CAB} is a spacetime scalar in M, and in our coordinate basis $e^\mu_A\stackrel{*}{=}\delta^\mu_A$ so that $e^\mu_A\nabla_\mu\Gamma_{CAB}\stackrel{*}{=}\partial_A\Gamma_{CAB}$. We conclude that

$${}^{\parallel}R_{\alpha\beta\gamma\delta}\stackrel{*}{=}e^A_\alpha e^B_\beta e^C_\gamma e^D_\delta\,{}^{\parallel}\mathcal{R}_{ABCD}, \tag{B29}$$

where

$${}^{\parallel}\mathcal{R}^D{}_{CAB}\equiv\partial_A\Gamma^D{}_{CB}-\partial_B\Gamma^D{}_{CA}+\Gamma^D{}_{IA}\Gamma^I{}_{CB}-\Gamma^D{}_{IB}\Gamma^I{}_{CA}. \tag{B30}$$

c. *Computing* $A_{\gamma\alpha\beta}$

Let us next compute $A_{\gamma\alpha\beta}$. We define the quantity $\mathcal{A}^C_{ab}\equiv n^\alpha_a n^\beta_b\nabla_\beta e^C_\alpha$, so that $A_{\gamma\alpha\beta}=e_{C\gamma}n^a_\alpha n^b_\beta\mathcal{A}^C_{ab}$. The antisymmetric part of $\mathcal{A}^C_{ab}$ (i.e. the transverse torsion) is

$$\mathcal{A}^C_{[ab]}=-\frac{1}{2}e^C_\gamma\left[n_b,n_a\right]^\gamma$$

$$= -\frac{1}{2} e^C_\gamma \mathcal{F}^\gamma_{ab}, \tag{B31}$$

where we obtained the second line using Eq. (B15). The symmetric part of $\mathcal{A}^{ab}_C$ is

$$\begin{aligned}\mathcal{A}^{(ab)}_C =& \frac{1}{2} n^a_\alpha n^b_\beta \pounds_{e^\gamma_C} g^{\alpha\beta} \\ =& \frac{1}{2} \pounds_{e^\gamma_C} \alpha^{ab}.\end{aligned} \tag{B32}$$

The second line holds as a result of Eq. (B13) and the definition of α^{ab}. Using Eqs. (B10) and (B11), we conclude that

$$\mathcal{A}^{ab}_C \overset{*}{=} \frac{1}{2} \left(\partial_C \alpha^{ab} - \alpha^{ac} \alpha^{bd} \gamma_{CD} \mathcal{F}^D_{cd} \right). \tag{B33}$$

d. Computing ${}^\perp\mathcal{R}_{\alpha\beta\gamma\delta}$

We now compute ${}^\perp\mathcal{R}_{\alpha\beta\gamma\delta}$. We define the quantity

$$\begin{aligned}\Omega_{cab} \equiv& n^\beta_b n_{c\alpha} \nabla_\beta n^\alpha_a \\ =& \frac{1}{2} \left(n^\mu_a \partial_\mu \alpha_{bc} + n^\mu_b \partial_\mu \alpha_{ac} - n^\mu_c \partial_\mu \alpha_{ab} \right).\end{aligned} \tag{B34}$$

To derive the second line of the above we used Eq. (B17). Note that in the coordinate adapted basis, Eq. (B11) we have $n^\mu_a \partial_\mu \Omega_{kij} \overset{*}{=} \partial_a \Omega_{kij} - \beta^A_a \partial_A \Omega_{kij}$. Similarly to Γ_{CAB}, whose first index can be raised with γ^{CD}, we can raise the first index of Ω_{cab} with α^{cd}, $\Omega^c{}_{ab} = \alpha^{cd} \Omega_{dab}$. We now look at

$${}^\perp R_{\alpha\beta\gamma}{}^\delta n^d_\delta = 2 l_{[\alpha}{}^\mu l_{\beta]}{}^\nu l_\gamma{}^\lambda \nabla_\mu \left(l_\nu{}^\eta l_\lambda{}^\rho \nabla_\eta n^d_\rho \right) - 2 A^\lambda{}_{[\alpha\beta]} l_\gamma{}^\delta \nabla_\lambda n^d_\delta. \tag{B35}$$

We first focus on the last term of this expression. Using Eq. (B13), we see that

$$-2 A^\lambda{}_{[\alpha\beta]} l_\gamma{}^\delta \nabla_\lambda n^d_\delta = 2 n_{a\alpha} n_{b\beta} n_{c\gamma} \gamma^{CD} \mathcal{A}^{[ab]}_C \mathcal{A}^{dc}_D. \tag{B36}$$

The first term of Eq. (B35) is

$$\begin{aligned}2 l_{[\alpha}{}^\mu l_{\beta]}{}^\nu l_\gamma{}^\lambda \nabla_\mu \left(l_\nu{}^\eta l_\lambda{}^\rho \nabla_\eta n^d_\rho \right) =& 2 n^a_{[\alpha} n^b_{\beta]} n^c_\gamma \left(n^\mu_a \nabla_\mu \Omega_c{}^d{}_b + \Omega_b{}^i{}_a \Omega_c{}^d{}_i + \Omega_c{}^j{}_a \Omega_j{}^d{}_b \right) \\ \overset{*}{=}& 2 n^a_{[\alpha} n^b_{\beta]} n^c_\gamma \left(n^\mu_b \partial_\mu \Omega^d{}_{ac} + \Omega^d{}_{ib} \Omega^i{}_{ac} \right).\end{aligned} \tag{B37}$$

To calculate the second line we have made use of the identities $\Omega_a{}^c{}_b = -\Omega^c{}_{ab}$ and $\Omega_{c[ab]} = 0$, which follow from Eq. (B34). We conclude that

$${}^\perp R_{\alpha\beta\gamma\delta} \overset{*}{=} n^a_\alpha n^b_\beta n^c_\gamma n^d_\delta \left({}^\perp\mathcal{R}_{abcd} + 2 \alpha_{ai} \alpha_{bj} \alpha_{ck} \alpha_{dl} \gamma^{CD} \mathcal{A}^{[ij]}_C \mathcal{A}^{lk}_D \right), \tag{B38}$$

where

$${}^\perp\mathcal{R}^d{}_{cab} \equiv n^\mu_a \partial_\mu \Omega^d{}_{cb} - n^\mu_b \partial_\mu \Omega^d{}_{ca} + \Omega^d{}_{ia} \Omega^i{}_{cb} - \Omega^d{}_{ib} \Omega^i{}_{ca}. \tag{B39}$$

e. Projected Einstein-Hilbert action

Having the above results at hand, we now rewrite the Einstein-Hilbert action in d dimensional spacetime

$$S = \frac{1}{2} \int d^d x \sqrt{-g} R, \tag{B40}$$

in an $m+n$ decomposition. Using Eqs. (B20), (B26), (B33), (B29), and (B38), we have

$$S \overset{*}{=} \int d^m u d^n \theta \sqrt{\alpha} \sqrt{\gamma} \Big({}^\parallel\mathcal{R} + \alpha^{cd} \gamma^{AB} \gamma^{CD} \left(\mathcal{K}_{cAB} \mathcal{K}_{dCD} - \mathcal{K}_{cAC} \mathcal{K}_{dBD} \right)$$

$$+ {}^{\perp}\mathcal{R} + \gamma^{CD}\alpha_{ab}\alpha_{cd}\left(\mathcal{A}^{ab}_{C}\mathcal{A}^{cd}_{D} - \mathcal{A}^{ac}_{C}\mathcal{A}^{bd}_{D}\right) - 2\nabla_{\lambda}\left(K^{\lambda} + A^{\lambda}\right)\Big). \tag{B41}$$

We can recover the complete Einstein equations by varying the Einstein-Hilbert action, Eq. (B41) with respect to $\{\alpha_{ab}\}_{a,b=0,...,m-1}$, $\{\beta^{\alpha}_{a}\}_{a=0,..,m-1}$, and γ_{AB}. This is to be compared to the $1+(d-1)$ formalism, where one varies the Einstein-Hilbert action with respect to α, β^i, and γ_{ij}, with α and β^i acting as constraint variables. Care must be taken when varying Eq. (B41) as in general α_{ab} cannot be treated as a metric so there is in general no well defined notion of a metric compatible connection for α_{ab}, and we have relations such as ${}^{\parallel}\nabla_A\alpha_{ab} \neq 0$. For a general spacetime with no symmetries, a potentially more straightforward approach to finding the Einstein equations in the $m+n$ formalism is to contract the projected Riemann tensor relations, Eqs. (A14), (A15), (A16), (A17), and (A18) to obtain the projected Ricci tensor relations.

3. $m+n$ splitting in a factorizable spacetime

In a factorizable spacetime the spacetime manifold can be written globally as $M = \Sigma^{(1)} \times \Sigma^{(2)}$, where both $\Sigma^{(i)}$ are submanifolds of M. In a factorizable spacetime, we see that we can think of either a family of submanifolds $\{\Sigma^{(2)}_{\mathbf{t}}\}$ foliating M, indexed by coordinates on $\Sigma^{(1)}$, or vice-versa. In the context of general relativity in four dimensions, an important class of a factorizable spacetimes are spherically symmetric spacetimes, which take the form $M = M^2 \times S^2$, where M^2 is a two dimensional Lorentzian manifold and S^2 is the two sphere. In factorizable spacetimes, we can choose an adapted basis to this foliation structure so that the shift vectors $\{\beta^{\alpha}_{a}\}_{a=0,...,m-1}$ all vanish, so that the metric can be written as

$$ds^2 = \alpha_{ab}du^a du^b + \gamma_{AB}d\theta^A d\theta^B. \tag{B42}$$

Unlike in the general $m+n$ decomposition, We can introduce a two metric compatible derivative for the submanifolds $\Sigma^{(1)}$ and $\Sigma^{(2)}$, which we denote by ${}^{\perp}\nabla_a$ and ${}^{\parallel}\nabla_A$, respectively. We see that Ω_{cab} takes on the role of the connection of the submanifold $(\Sigma^{(1)}, {}^{\perp}\nabla_a, \alpha_{ab})$. Writing down formulas for ${}^{\perp}\mathcal{R}^a{}_{bcd}$ and $\mathcal{A}_{Cab}$ become much simpler than in the general $m+n$ case as the shift vectors all vanish; in particular the directional derivatives along n^{α}_{a} become derivatives in the coordinate u^a; $n^{\alpha}_{a}\partial_{\alpha} \to \partial_a$.

Appendix C: Scalar, vector, and tensor spherical harmonics

In this section, we review the properties of the scalar, vector, and tensor spherical harmonics. We work on the two sphere S^2, with the round metric Ω_{AB} and metric compatible covariant derivative D_A: (S^2, Ω_{AB}, D_A).

We begin with the scalar spherical harmonics. Such harmonics satisfy the following eigenvalue equation:

$$\left\{\Omega^{AB}D_A D_B + l(l+1)\right\} Y^{lm} = 0. \tag{C1}$$

The scalar spherical harmonics form an orthogonal basis for functions in S_2. We choose the following normalization for Y^{lm}

$$\int d^2\Omega Y^{lm} Y^{l'm'} = \delta_{ll'}\delta_{mm'}. \tag{C2}$$

Next, we discuss vector spherical harmonics. The axial and polar spherical harmonics respectively are

$$E^{lm}_{A} = D_A Y^{lm}, \quad B^{lm}_{A} = \epsilon_A{}^B D_B Y^{lm}. \tag{C3}$$

Note that divergence of B^{lm}_{A} is zero, $D_A B^{A}_{lm} = 0$. The vector spherical harmonics satisfy the following eigenvalue equation:

$$\left\{\Omega^{AB}D_A D_B + [-1 + l(l+1)]\right\} V^{lm}_{C} = 0, \tag{C4}$$

where V^{lm}_{C} is either E^{lm}_{C} or B^{lm}_{C}. The vector spherical harmonics form an orthonormal basis for functions in S_2. The vector spherical harmonics are orthogonal to one another, and are normalized to obey

$$\int d^2\Omega\, \Omega^{AB} V^{lm}_{A} V^{l'm'}_{B} = l(l+1)\delta_{ll'}\delta_{mm'}. \tag{C5}$$

Finally, we introduce tensor spherical harmonics. We define such harmonics to be traceless; this choice follows, for example Poisson and Martel [8], but not Regge and Wheeler [1]. The traceless axial and polar tensor spherical harmonics respectively are

$$E^{lm}_{AB} = D_{(A}E^{lm}_{B)} + \frac{l(l+1)}{2}\Omega_{AB}Y^{lm}, \tag{C6}$$

$$B^{lm}_{AB} = D_{(A}B^{lm}_{B)}. \tag{C7}$$

The trace can be captured with $Y^{lm}\Omega_{AB}$, which behaves as a scalar under rotations. The tensor spherical harmonics satisfy the following eigenvalue equation:

$$\left\{\Omega^{AB}D_AD_B + [-2+l(l+1)]\right\}T^{lm}_{CD} = 0, \tag{C8}$$

where T^{lm}_{CD} is either E^{lm}_{CD} or B^{lm}_{CD}. The trace term $Y^{lm}\Omega_{AB}$ has the scalar spherical harmonic eigenvalue $l(l+1)$. Finally, the tensor spherical harmonics satisfy the following orthogonality relation

$$\int d^2\Omega\, \Omega^{AB}\Omega^{CD}T^{lm}_{AC}T^{l'm'}_{BD} = \frac{1}{2}l(l+1)\left[l(l+1)-2\right]\delta_{ll'}\delta_{mm'}. \tag{C9}$$

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[1] T. Regge and J. A. Wheeler, Phys. Rev. **108**, 1063 (1957).
[2] C. V. Vishveshwara, Phys. Rev. D **1**, 2870 (1970).
[3] F. J. Zerilli, Phys. Rev. Lett. **24**, 737 (1970).
[4] V. Moncrief, Annals of Physics **88**, 323 (1974).
[5] U. H. Gerlach and U. K. Sengupta, Phys. Rev. D **19**, 2268 (1979).
[6] U. H. Gerlach and U. K. Sengupta, Phys. Rev. D **22**, 1300 (1980).
[7] S. Chandrasekhar, *The mathematical theory of black holes*, Oxford classic texts in the physical sciences (Oxford Univ. Press, Oxford, 2002).
[8] K. Martel and E. Poisson, Phys. Rev. **D71**, 104003 (2005), arXiv:gr-qc/0502028 [gr-qc].
[9] T. Kobayashi, H. Motohashi, and T. Suyama, Phys. Rev. **D85**, 084025 (2012), arXiv:1202.4893 [gr-qc].
[10] T. Kobayashi, H. Motohashi, and T. Suyama, Phys. Rev. **D89**, 084042 (2014), arXiv:1402.6740 [gr-qc].
[11] A. De Felice, T. Suyama, and T. Tanaka, Phys. Rev. **D83**, 104035 (2011), arXiv:1102.1521 [gr-qc].
[12] H. Motohashi and T. Suyama, Phys. Rev. **D84**, 084041 (2011), arXiv:1107.3705 [gr-qc].
[13] H. Motohashi and T. Suyama, Phys. Rev. **D85**, 044054 (2012), arXiv:1110.6241 [gr-qc].
[14] H. Ogawa, T. Kobayashi, and T. Suyama, Phys. Rev. **D93**, 064078 (2016), arXiv:1510.07400 [gr-qc].
[15] K. Takahashi, T. Suyama, and T. Kobayashi, Phys. Rev. **D93**, 064068 (2016), arXiv:1511.06083 [gr-qc].
[16] K. Takahashi and T. Suyama, Phys. Rev. **D95**, 024034 (2017), arXiv:1610.00432 [gr-qc].
[17] R. Kase, L. A. Gergely, and S. Tsujikawa, Phys. Rev. **D90**, 124019 (2014), arXiv:1406.2402 [hep-th].
[18] W. G. Unruh, Phys. Rev. D **14**, 870 (1976).
[19] J. A. Harvey and A. Strominger, in *String Theory and Quantum Gravity '92*, edited by J. Harvey, R. Iengo, K. S. Narain, S. Randjbar-Daemi, and H. Verlinde (1993) p. 122, hep-th/9209055.
[20] E. C. Vagenas, Mod. Phys. Lett. **A17**, 609 (2002), arXiv:hep-th/0108147 [hep-th].
[21] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Macmillan, 1973).
[22] P. R. Brady, S. Droz, W. Israel, and S. M. Morsink, Class. Quant. Grav. **13**, 2211 (1996), arXiv:gr-qc/9510040 [gr-qc].
[23] J. M. Bardeen, Phys. Rev. D **22**, 1882 (1980).
[24] H. Motohashi, T. Suyama, and K. Takahashi, Phys. Rev. **D94**, 124021 (2016), arXiv:1608.00071 [gr-qc].
[25] V. Frolov and I. Novikov, *Black hole physics: basic concepts and new developments*, Vol. 96 (Springer Science & Business Media, 2012).
[26] O. Sarbach and M. Tiglio, Phys. Rev. **D64**, 084016 (2001), arXiv:gr-qc/0104061 [gr-qc].
[27] O. Sarbach, *On the generalization of the Regge-Wheeler equation for self-gravitating matter fields*, Ph.D. thesis, Universität Zürich (2000).
[28] K. Martel, *Particles and black holes: time-domain integration of the equations of the equations of black-hole perturbation theory*, Ph.D. thesis, The University of Guelph (2003).
[29] M. Fierz and W. Pauli, Proc. Roy. Soc. Lond. **A173**, 211 (1939).
[30] C. de Rham, G. Gabadadze, and A. J. Tolley, Phys. Rev. Lett. **106**, 231101 (2011), arXiv:1011.1232 [hep-th].
[31] K. Hinterbichler, Rev. Mod. Phys. **84**, 671 (2012), arXiv:1105.3735 [hep-th].
[32] R. H. Price, Phys. Rev. D **5**, 2419 (1972).
[33] R. M. Wald, Journal of Mathematical Physics **20**, 1056 (1979).
[34] M. Dafermos, G. Holzegel, and I. Rodnianski, (2016), arXiv:1601.06467 [gr-qc].
[35] D. G. Boulware and S. Deser, Phys. Rev. D **6**, 3368 (1972).

[36] C. de Rham, Living Rev. Rel. **17**, 7 (2014), arXiv:1401.4173 [hep-th].
[37] M. Dafermos and I. Rodnianski, *Evolution equations: Proceedings, Clay Mathematics Institute Summer School, Zurich, Switzerland, 23 Jun - 18 Jul, 2008*, Clay Math. Proc. **17**, 97 (2013), arXiv:0811.0354 [gr-qc].
[38] E. Newman and R. Penrose, Journal of Mathematical Physics **3**, 566 (1962).
[39] S. Ray, *Investigation of symmetries and conserved charges in general relativity*, Ph.D. thesis, University of Massachusetts-Amherst (2008).
[40] B. Carter, Journal of Geometry and Physics **8**, 53 (1992).
[41] E. Gourgoulhon, Phys. Rev. **D72**, 104007 (2005), arXiv:gr-qc/0508003 [gr-qc].
[42] T. Damour, in *Marcel Grossmann Meeting: General Relativity*, edited by R. Ruffini (1982).
[43] T. Damour, Phys. Rev. D **18**, 3598 (1978).
[44] T. Damour, *Quelues propriétés mécaniques, électromagnétiques, thermodynamiques et quantiques des trous noirs*, Ph.D. thesis, Université Pierre et Marie Curie, Paris 6 (1979).
[45] T. Padmanabhan, Phys. Rev. **D83**, 044048 (2011), arXiv:1012.0119 [gr-qc].
[46] E. Poisson, *A relativist's toolkit : the mathematics of black-hole mechanics* (Cambridge University Press, 2004).
[47] K. Hinterbichler, M. Trodden, and D. Wesley, Phys. Rev. D. **82**, 124018 (2010), arXiv:1008.1305 [hep-th].
[48] J. D. E. Grant and I. G. Moss, Phys. Rev. D **56**, 6284 (1997).