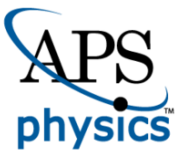

Large-scale structure perturbation theory without losing stream crossing

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Phys. Rev. D **97**, 023508 — Published 10 January 2018

DOI: 10.1103/PhysRevD.97.023508

Large-scale structure perturbation theory without losing stream crossing

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We suggest an approach to perturbative calculations of large-scale clustering in the Universe that includes from the start the stream crossing (multiple velocities for mass elements at a single position) that is lost in traditional calculations. Starting from a functional integral over displacement, the perturbative series expansion is in deviations from (truncated) Zel'dovich evolution, with terms that can be computed exactly even for stream-crossed displacements. We evaluate the one-loop formulas for displacement and density power spectra numerically in 1D, finding dramatic improvement in agreement with N-body simulations compared to the Zel'dovich power spectrum (which is exact in 1D up to stream crossing). Beyond 1D, our approach could represent an improvement over previous expansions even aside from the inclusion of stream crossing, but we have not investigated this numerically. In the process we show how to achieve effective-theory-like regulation of small-scale fluctuations without free parameters.

I. INTRODUCTION

Perturbation theory for large-scale gravitational evolution and clustering in the Universe [1] should be increasingly valuable as large-scale structure (LSS) surveys become increasingly large and precise [2, 3]. From the beginning [4], this kind of perturbation theory has had a nagging deficiency that there was no first-principles way to include "stream crossing" (often alternatively called "shell crossing"), i.e., multiple different velocities for mass at a single point in space. This is a purely technical problem – it is easy to write down the exact evolution equations for mass elements under effectively Newtonian gravity, and easy to see that stream crossing will happen, but we simply have not had any clean mathematical method to include this phenomenon and still produce perturbative analytic results for clustering statistics. In contrast, a wonderful thing about N-body simulations [5, 6] is they trivially include arbitrarily complex velocity structure at a point (pedantically, one can say that you never have more than one particle at a mathematical point, but we implicitly understand that the particles in simulations are really an approximation to some effectively continuous cloud). The key weakness appears in Lagrangian perturbation theory (LPT) [7–12] because density is approximated by the Taylor expansion of the determinant of the local deformation tensor, which is only correct before stream crossing. Eulerian perturbation theory (EPT) is derived by truncating the evolution equations for moments of the velocity distribution function after the first moment, i.e., velocity dispersion is set to zero [13–15].

One unavoidable criticism of various efforts to improve perturbation theory by effectively summing to higher orders [16–18] has been that we were summing a theory that was not exactly correct anyway, because of missing stream crossing [19]. [20] found only small effects from stream crossing in numerical simulations; however, this does not mean that missing stream crossing necessarily has only a small effect on a given perturbation theory calculation. One of our findings here will be that stream crossing self-regulates, in the following sense: If you ignore it in force calculations, e.g., in the Zel'dovich approximation, the extrapolated amount of stream crossing and its effect on the density power spectrum is large (the exact Zel'dovich power spectrum does include stream crossing for a given displacement field). This makes the Zel'dovich power spectrum inaccurate, while including the effect of stream crossing on forces, feeding back to suppress itself, greatly improves the results. Of course, it is also simply desirable to compute small effects to match increasingly high precision observations.

There have been many efforts to include the effects of stream crossing in calculations. [15, 21, 22] included equations for moments of the velocity distribution, but this requires a truncation of the moment hierarchy and convergence was never demonstrated. The adhesion approximation of [23–25] aimed to make crossing streams stop and stick instead of crossing. [26] derived evolution equations for explicitly smoothed fields in which a coarse-grained velocity dispersion appears, with contributions to it beyond EPT to be computed by simulations. [27–29] introduced general "effective field theory" counter-terms with free coefficients that allow fitting for the stream crossing effect in simulations or data.

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[30] computed the velocity dispersion tensor implied by LPT. While many of these approaches have had significant success, we would like to find a more direct calculation.

Many elements of this paper have appeared before. The basic math we exploit to compute forces including stream crossing has been hiding in plain sight in the calculation of the exact Zel'dovich power spectrum (apparently computed first, at least in correlation function form, in [31], which we could not find online – see [1]) which does include stream crossing but only for a given displacement field. [32] investigated the effect of stream crossing by comparing this Zel'dovich calculation to one where relative streams are stopped "by hand" when they would otherwise cross. Valageas and others have been using a similar functional integral formalism for many years [33–42]. The starting formalism of [43] is similar to ours except for discussing literal particles instead of a continuum limit, while [44] is even more similar, although without the functional integral formalism. [45] have a somewhat similar idea of integrating the force after stream crossing. It is generally understood that some form of damping of small-scale initial conditions is a good idea [46–54]. The decisive new feature in this paper is the specific straightforward perturbative expansion of the functional integral that we do, allowing concise calculation of statistical results. Stream crossing appears essentially effortlessly – in fact, if this was the first calculation a person ever saw, they probably would not realize stream crossing was a thing to worry about missing at all.

In the following sections we build up the calculation systematically, starting with nothing more than the basic equations for Newtonian gravitational evolution. Most of this is basically notation, that we think makes the calculations easier but isn't fundamentally connected to the inclusion of stream crossing. A reader who would like to try to understand the key math trick that we use to avoid losing stream crossing without learning any of the formalism may want to read Appendix A first, where we attempt to give a pedagogical taste of what is going on in the main calculation.

II. EVOLUTION EQUATIONS

We start with the exact equations for the displacement field $\psi(\mathbf{q})$ of mass elements labeled by their initial position (Lagrangian coordinate) $\mathbf{q}$, at physical position $\mathbf{x}(\mathbf{q}) = \mathbf{q} + \psi(\mathbf{q})$, and the velocity field $v \equiv \psi'$, where we use prime to indicate a derivative with respect to $\eta \equiv \ln a$, which we will use as our primary time variable (a is the expansion factor). We have, with dots for standard time derivative,

$$\ddot{\psi}(\mathbf{q}) + 2H\dot{\psi}(\mathbf{q}) = \mathbf{F}\left[\mathbf{x}\left(\mathbf{q}\right)\right] = -\partial_{\mathbf{x}} V\left[\mathbf{x}\left(\mathbf{q}\right)\right] = -\partial_{\mathbf{x}} V\left[\mathbf{q} + \psi\left(\mathbf{q}\right)\right] \tag{1}$$

with potential due to density fluctuations

$$V(\mathbf{x}) = \frac{3}{2}\Omega_m H^2 \partial_{\mathbf{x}}^{-2}\delta(\mathbf{x}) \tag{2}$$

(see [1] for a comprehensive introduction to LSS perturbation theory). Density at position $\mathbf{x}$ is

$$1 + \delta(\mathbf{x}) = \int d^d\mathbf{q}\ \delta^D\left[\mathbf{x} - \mathbf{q} - \psi\left(\mathbf{q}\right)\right] = \int d^d\mathbf{q} \int \frac{d^d\mathbf{k}}{(2\pi)^d} e^{-i\mathbf{k}\cdot\left[\mathbf{x}-\mathbf{q}-\psi(\mathbf{q})\right]}\ , \tag{3}$$

or in (Eulerian) Fourier space

$$\delta(\mathbf{k}) = \int d^d\mathbf{q}\ e^{i\mathbf{k}\cdot\mathbf{q}}\left[e^{i\mathbf{k}\cdot\psi(\mathbf{q})} - 1\right]\ . \tag{4}$$

The density at particle $\mathbf{q}$, i.e., at position $\mathbf{x}(\mathbf{q}) = \mathbf{q} + \psi\left(\mathbf{q}\right)$, is

$$1 + \delta\left[\mathbf{x}\left(\mathbf{q}\right)\right] = \int d^d\mathbf{q}' \int \frac{d^d\mathbf{k}}{(2\pi)^d} e^{-i\mathbf{k}\cdot\left[\mathbf{q}+\psi(\mathbf{q})-\mathbf{q}'-\psi(\mathbf{q}')\right]}\ . \tag{5}$$

So finally the force on mass element $\mathbf{q}$ at $\mathbf{x}(\mathbf{q})$ is

$$\mathbf{F}\left[\mathbf{x}\left(\mathbf{q}\right)\right] = -\frac{3}{2}\Omega_m H^2 \int d^d\mathbf{q}' \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{i\mathbf{k}}{k^2}\left[e^{-i\mathbf{k}\cdot\left[\mathbf{q}+\psi(\mathbf{q})-\mathbf{q}'-\psi(\mathbf{q}')\right]} - e^{-i\mathbf{k}\cdot\left[\mathbf{q}+\psi(\mathbf{q})-\mathbf{q}'\right]}\right]\ , \tag{6}$$

where we have included the second term, coming from the mean (background) part of the density, subtracted in the definition of $\delta \equiv \rho/\bar{\rho} - 1$, because it definitely does produce a force – the force that decelerates the Hubble flow – which must be subtracted out of the first term here. To help clarify: $\psi(\mathbf{q}')$ here represents the displacements of the mass

doing the forcing, while $\psi(\mathbf{q})$ is the displacement of the mass being forced, which could just as well be a negligible test mass, which is why it makes sense for it to appear in the homogeneous part. If $\psi(\mathbf{q}') = 0$ the forcing field is homogeneous so the peculiar acceleration is zero.

Note that when we write formulas with general dimensionality d here, we are not really implying a Universe with fundamentally d dimensions. We are following [55] in modeling a 3D background Universe with reduced dimensionality fluctuations, i.e., if $\psi(\mathbf{q})$ does not depend on a component of $\mathbf{q}$, it is easy to see that we can integrate this component out of all these equations, leaving the reduced d accounting for only directions that fluctuate. If it is not obvious that the qualification that the background Universe is still 3D matters, note that in a 1D background, with no cosmological constant, in the Newtonian limit, the force between particles is constant, i.e., does not diminish with expansion. This means that a truly 1D Universe will always turn around and collapse eventually, i.e., there is no concept of an Einstein-de Sitter (EdS) Universe with power law expansion.

Specializing to EdS for simplicity, and substituting in υ to produce a 1st order equation, we can write Eq. (1) as

$$\upsilon' + \frac{1}{2}\upsilon - \frac{3}{2}\psi = -\frac{3}{2}\partial_{\mathbf{x}}\partial_{\mathbf{x}}^{-2}\delta - \frac{3}{2}\psi \tag{7}$$

where note that we have subtracted $\frac{3}{2}\psi$ from both sides to make the left hand side correspond to the linearized Zel'dovich evolution, while the right hand side is the force beyond this. We can combine ψ and υ into a single vector

$$\phi \equiv \begin{pmatrix} \psi \\ \upsilon \end{pmatrix} \tag{8}$$

and, understanding $\phi = [\phi]_i^\alpha(\eta, \mathbf{q})$ as a vector in $\infty \times \infty^d \times d \times 2$ dimensional space, labeled by time, d spatial coordinates, their standard spatial vector direction, and ψ or υ, write the entire system compactly as

$$\mathbf{L}_0\phi + \mathbf{\Delta}_0(\phi) = \epsilon \ , \tag{9}$$

where $\mathbf{L}_0$ is a matrix acting in this $\infty \times \infty^d \times d \times 2$ dimensional space, with elements

$$[\mathbf{L}_0]_{i_2 i_1}(\mathbf{q}_2, \eta_2, \mathbf{q}_1, \eta_1) = \begin{pmatrix} \frac{\partial}{\partial\eta_2} & -1 \\ -\frac{3}{2} & \frac{\partial}{\partial\eta_2} + \frac{1}{2} \end{pmatrix} \delta^D(\eta_2 - \eta_1)\delta^K_{i_2 i_1}\delta^D(\mathbf{q}_2 - \mathbf{q}_1) \tag{10}$$

(where the explicit matrix is over (ψ, υ) and Kronecker-δ over spatial directions – note that in Fourier space with coordinate $\mathbf{p}$ this would have $(2\pi)^d\,\delta^D(\mathbf{p}_2 - \mathbf{p}_1)$ in place of $\delta^D(\mathbf{q}_2 - \mathbf{q}_1)$),

$$\mathbf{\Delta}_0(\phi) = \frac{3}{2}\begin{pmatrix} 0 \\ \partial_{\mathbf{x}}\partial_{\mathbf{x}}^{-2}\delta + \psi \end{pmatrix} \ , \tag{11}$$

and we have added a stochastic source $\epsilon = [\epsilon]_i^\alpha(\eta, \mathbf{q})$ with covariance matrix $\langle\epsilon\epsilon^t\rangle \equiv \mathbf{N}$ which can be used to set the standard initial conditions in the form of an early time impulse. We write subscript 0 because later the split between $\mathbf{L}_0$ and $\mathbf{\Delta}_0$ will be modified. ϵ in this paper will only represent standard differential equation initial conditions, so one could ask why bother allowing it to formally have arbitrary time dependence. We do this with an eventual Wilsonian renormalization group (RG) [56] picture in mind, where small scales are "integrated out", producing effectively stochastic differential equations, with $\epsilon(\eta)$ including noise that is not simply initial conditions (in this case we would have $\mathbf{N}_0$ and a renormalized $\mathbf{N}$, as in, e.g., [41, 42]). Even for this paper where it is not strictly needed, we think this formulation is slightly more elegant than introducing an explicit initial time, when, as we will see, that is never necessary.

To repeat for clarity: vectors like ϕ, $\mathbf{\Delta}$, and ϵ are generally understood to live in $\infty \times \infty^d \times d \times 2$ dimensional space, labeled by time, d spatial coordinates, their standard spatial vector direction, and ψ or υ. Matrices like $\mathbf{L}$ are matrices in this space, and $\mathbf{L}\phi$ means a matrix times vector product, which, if we want to, we can write out in explicit coordinates like this:

$$[\mathbf{L}\phi]_{i_2}^{\alpha_2}(\eta_2, \mathbf{q}_2) \equiv \int d\eta_1 \int d^d\mathbf{q}_1 [\mathbf{L}]_{i_2 i_1}^{\alpha_2\alpha_1}(\mathbf{q}_2, \eta_2, \mathbf{q}_1, \eta_1)[\phi]_{i_1}^{\alpha_1}(\eta_1, \mathbf{q}_1) \tag{12}$$

where i labels vector direction and α labels ψ or υ... we generally just write $\mathbf{L}\phi$ because it is less tedious. Similarly, something like $\mathbf{j}^t\phi$ is a dot product in this space, producing a scalar. Note that, when it matters (when using Fourier space coordinates), the t superscript should be understood to indicate complex conjugation as well as transpose, i.e., $\mathbf{j}^t\phi \equiv \int d^d\mathbf{q}\ \mathbf{j}^t(\mathbf{q})\phi(\mathbf{q}) \equiv \int \frac{d^d\mathbf{p}}{(2\pi)^d}\mathbf{j}^t(-\mathbf{p})\phi(\mathbf{p})$ (writing out only the coordinate part, and assuming real fields in $\mathbf{q}$

space). Basically, all of the equations can be understood as standard vector/matrix equations like you could implement numerically on a computer, if you mentally discretize time and space coordinates (integrals and derivatives are the limits of sums and finite differences as the grid spacing goes to zero).

Note that

$$[\mathbf{L}_0^{-1}]_{i_2 i_1}(\mathbf{q}_2,\eta_2,\mathbf{q}_1,\eta_1) = \frac{1}{5}\left[\begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix} e^{\eta_2-\eta_1} + \begin{pmatrix} 2 & -2 \\ -3 & 3 \end{pmatrix} e^{-\frac{3}{2}(\eta_2-\eta_1)}\right]\Theta(\eta_2-\eta_1)\delta^K_{i_2 i_1}\delta^D(\mathbf{q}_2-\mathbf{q}_1) \tag{13}$$

(if there is any doubt, multiply this from the left hand side with $\mathbf{L}_0$ to check). The Heaviside function $\Theta(\eta_2-\eta_1)$ enforces causality of propagation from η_1 to η_2.

III. FUNCTIONAL INTEGRAL

The statistical starting point for our system is that $\epsilon(\eta,\mathbf{q})$ is a Gaussian random field with mean zero and correlation $\mathbf{N}$. We can compute statistics of interest using a generating function

$$Z(\mathbf{j}) \equiv \int d\epsilon\; e^{-\frac{1}{2}\epsilon^t\mathbf{N}^{-1}\epsilon+\mathbf{j}^t\phi[\epsilon]}\;, \tag{14}$$

i.e., we can pull down powers of ϕ by taking derivatives with respect to $\mathbf{j}$ to give averages that we want, e.g.,

$$\langle\phi\rangle \equiv \frac{\int d\epsilon\;\phi[\epsilon]e^{-\frac{1}{2}\epsilon^t\mathbf{N}^{-1}\epsilon}}{\int d\epsilon\; e^{-\frac{1}{2}\epsilon^t\mathbf{N}^{-1}\epsilon}} = Z^{-1}(0)\left.\frac{\partial Z(\mathbf{j})}{\partial\mathbf{j}}\right|_{\mathbf{j}=0}\;. \tag{15}$$

More generally, Nth order connected correlation functions can be derived by taking N derivatives of $\ln Z(\mathbf{j})$. $Z(\mathbf{j})$ is simply a mathematical tool encoding all the information necessary to compute statistical averages like this – by manipulating it we can derive results for all possible averages at once, rather than computing them piecemeal. Note that $\phi[\epsilon]$ means $\phi(\eta,\mathbf{q})$ depending in principle on the full function $\epsilon(\eta,\mathbf{q})$, ϵ at all times and positions, although causality will of course limit ϕ in practice to depending on ϵ at earlier times.

Now we change integration variables to ϕ using Eq. (9), to give

$$Z(\mathbf{j}) \equiv \int d\phi\; e^{-S(\phi)+\mathbf{j}^t\phi} \tag{16}$$

with

$$S(\phi) = \frac{1}{2}\left[\mathbf{L}_0\phi+\mathbf{\Delta}_0(\phi)\right]^t\mathbf{N}^{-1}\left[\mathbf{L}_0\phi+\mathbf{\Delta}_0(\phi)\right]\;. \tag{17}$$

The Jacobian of the transformation is field-independent so we can drop it (see Appendix B). $\mathbf{N}$ is defined such that $\mathbf{L}_0^{-1}\mathbf{N}\mathbf{L}_0^{-t} = \mathbf{C}_0$ is the standard linear theory power spectrum with its standard time evolution.

It turns out to simplify calculations to introduce another field which can be integrated over to produce $S(\phi)$ (up to a field-independent normalization which is never relevant)

$$\int d\phi\; e^{-S(\phi)} \equiv \int d\phi\; d\chi\; e^{-S(\phi,\chi)} \tag{18}$$

where

$$S(\phi,\chi) = \frac{1}{2}\chi^t\mathbf{N}\chi + i\chi^t\mathbf{L}_0\phi + i\chi^t\mathbf{\Delta}_0(\phi)\;. \tag{19}$$

This just shuffles the non-linearity into a simpler single term. (Recall $\int d\mathbf{x}\exp\left(-\frac{1}{2}\mathbf{x}^t\mathbf{A}\mathbf{x}+\mathbf{b}^t\mathbf{x}\right) \propto \det\mathbf{A}^{-1/2}\exp\left(\frac{1}{2}\mathbf{b}^t\mathbf{A}^{-1}\mathbf{b}\right)$. We reached this point inspired by the idea of a Hubbard-Stratonovich transformation [57, 58]. It can also be inspired by the Martin-Siggia-Rose formalism [35, 59, 60], and derived as shown in Appendix B.) Note that, ignoring the $\mathbf{\Delta}_0$ term, evaluating the Gaussian integrals gives $\langle\phi\phi^t\rangle_g = \mathbf{C}_0$, $\langle\phi\chi^t\rangle_g = -i\mathbf{L}_0^{-1}$, and $\langle\chi\chi^t\rangle_g = 0$.

IV. PERTURBATION THEORY

The standard perturbative approach to this kind of functional integral is to split S into a quadratic part that leads to straightforward Gaussian integrals and a perturbation that we expand out of the exponential, i.e., with $S = S_g + S_p$, we will do $\exp(-S) = \exp(-S_g)(1 - S_p + S_p^2/2 + ...)$. In addition to the obvious move of putting the $\mathbf{\Delta}$ term in S_p, we know that small-scale displacements are not well-approximated by Zel'dovich at all times – generally they are damped. Therefore, it makes no sense to include them in the leading order Gaussian part of the calculation, where they can only cause trouble. We can self-consistently suppress them by moving them into the perturbation term S_p, i.e., we define

$$S_g \equiv \frac{1}{2}\chi^t \mathbf{N}\chi + i\chi^t[W^{-1}\mathbf{L}_0]\phi \equiv \frac{1}{2}\chi^t \mathbf{N}\chi + i\chi^t \mathbf{L}\phi \tag{20}$$

and

$$S_p \equiv i\chi^t \mathbf{\Delta}_0(\phi) + i\chi^t[(1 - W^{-1})\mathbf{L}_0]\phi \equiv i\chi^t \mathbf{\Delta}(\phi) \tag{21}$$

where $W(k)$ is some simple damping function like $W(k) = \exp(-c^2k^2/2)$. The key is that now the effective linear propagator $\mathbf{L}^{-1} \equiv \mathbf{L}_0^{-1}W(k)$ appropriately suppresses small-scale structure, while the term in S_p guarantees that this structure is not arbitrarily lost – its effects will enter as higher order corrections through $\mathbf{\Delta}(\phi) \equiv \mathbf{\Delta}_0(\phi) + (1 - W^{-1})\mathbf{L}_0\phi$. We will set c to minimize total higher order correction, i.e., an optimal level of suppression should be the one for which the leading order result is as close as possible to the final answer. It is not necessary to think too deeply about this on first reading – we are just treating small-scale propagation as a perturbation, not fundamentally different from how we routinely treat non-linear interactions as a perturbation. (We discuss below how we could be much more sophisticated than this simple Gaussian damping, including modifying $\mathbf{N}$, but for now we just want to be sure to capture the critical physical effect of generally suppressing high-k fluctuations.)

So we are set up to compute correlations using the generating function:

$$Z(\mathbf{j}, \mathbf{l}) = \int d\phi d\chi \; e^{-S_g(\phi,\chi) + \mathbf{j}^t\phi + \mathbf{l}^t\chi}(1 - S_p + S_p^2/2 + ...) \tag{22}$$

Note that Z is now function of $\mathbf{j}$ and $\mathbf{l}$ which allows us to pull down the ϕ or the χ term.

A. Leading order statistics

We start at lowest order, keeping only the Gaussian part

$$\begin{aligned} Z_0(\mathbf{j}, \mathbf{l}) &\equiv \int d\phi \; d\chi \; e^{-\frac{1}{2}\chi^t \mathbf{N}\chi - i\chi^t \mathbf{L}\phi + \mathbf{j}^t\phi + \mathbf{l}^t\chi} \propto \int d\phi \; e^{-\frac{1}{2}\left(\phi^t \mathbf{L}^t + i\mathbf{l}^t\right)\mathbf{N}^{-1}(\mathbf{L}\phi + i\mathbf{l}) + \mathbf{j}^t\phi} \\ &= \int d\phi \; e^{-\frac{1}{2}\phi^t \mathbf{C}^{-1}\phi - i\mathbf{l}^t\mathbf{N}^{-1}\mathbf{L}\phi + \frac{1}{2}\mathbf{l}^t\mathbf{N}^{-1}\mathbf{l} + \mathbf{j}^t\phi} \propto e^{\frac{1}{2}\left(\mathbf{j}^t - i\mathbf{l}^t\mathbf{N}^{-1}\mathbf{L}\right)\mathbf{C}\left(\mathbf{j} - i\mathbf{L}^t\mathbf{N}^{-1}\mathbf{l}\right) + \frac{1}{2}\mathbf{l}^t\mathbf{N}^{-1}\mathbf{l}} \\ &= e^{\frac{1}{2}\mathbf{j}^t\mathbf{C}\mathbf{j} - i\mathbf{j}^t\mathbf{L}^{-1}\mathbf{l}} \end{aligned} \tag{23}$$

where $\mathbf{C} \equiv \mathbf{L}^{-1}\mathbf{N}\mathbf{L}^{-t}$ is the linearly evolved, damped, displacement power spectrum.

Now suppose we want to compute statistics of the density field. By construction, we can pull a factor $\psi(\mathbf{q}, \eta)$ out of $Z(\mathbf{j})$ using the derivative operator $\partial_{\mathbf{j}_\psi(\mathbf{q},\eta)}$ and therefore we can pull out a factor of $\delta(\mathbf{k})$ using the operator $\int d^d\mathbf{q} \; e^{i\mathbf{k}\cdot\mathbf{q}}\left[e^{i\mathbf{k}\cdot\partial_{\mathbf{j}_\psi(\mathbf{q})}} - 1\right]$. Generally $\exp(\mathbf{x}\cdot\partial_\mathbf{j})f(\mathbf{j}) = f(\mathbf{j} + \mathbf{x})$, so our operator $e^{i\mathbf{k}\cdot\partial_{\mathbf{j}_\psi(\mathbf{q})}}$ adds $i\mathbf{k}\delta^D(\mathbf{q} - \mathbf{q}')$ to $\mathbf{j}_\psi(\mathbf{q}')$ within $Z(\mathbf{j})$ (time is also an index on all these vectors, but we suppress it because it isn't doing anything interesting).

Finally we can compute the density power spectrum:

$$\begin{aligned} \langle\delta(\mathbf{k}_1)\delta(\mathbf{k}_2)\rangle_0 &= \int d^d\mathbf{q}_1 \; e^{i\mathbf{k}_1\cdot\mathbf{q}_1}\left[e^{i\mathbf{k}_1\cdot\partial_{\mathbf{j}_\psi(\mathbf{q}_1)}} - 1\right]\int d^d\mathbf{q}_2 \; e^{i\mathbf{k}_2\cdot\mathbf{q}_2}\left[e^{i\mathbf{k}_2\cdot\partial_{\mathbf{j}_\psi(\mathbf{q}_2)}} - 1\right] e^{\frac{1}{2}\mathbf{j}^t\mathbf{C}\mathbf{j}}\Big|_{\mathbf{j}=0} \\ &= \int d^d\mathbf{q}_1 \; e^{i\mathbf{k}_1\cdot\mathbf{q}_1}\int d^d\mathbf{q}_2 \; e^{i\mathbf{k}_2\cdot\mathbf{q}_2} e^{-\frac{1}{2}[\mathbf{k}_1\cdot\mathbf{C}_\psi(0)\cdot\mathbf{k}_1 + 2\mathbf{k}_1\cdot\mathbf{C}_\psi(\mathbf{q}_1 - \mathbf{q}_2)\cdot\mathbf{k}_2 + \mathbf{k}_2\cdot\mathbf{C}_\psi(0)\cdot\mathbf{k}_2]} \; + k = 0 \text{ piece} \\ &= (2\pi)^d\delta^D(\mathbf{k}_1 + \mathbf{k}_2)\int d^d\mathbf{q} \; e^{i\mathbf{k}_1\cdot\mathbf{q}} e^{-\mathbf{k}_1\cdot[\mathbf{C}_\psi(0) - \mathbf{C}_\psi(\mathbf{q})]\cdot\mathbf{k}_1} \; + k = 0 \text{ piece} \\ &= (2\pi)^d\delta^D(\mathbf{k}_1 + \mathbf{k}_2)\int d^d\mathbf{q} \; \cos(\mathbf{k}_1\cdot\mathbf{q})\, e^{-\frac{1}{2}k_{1i}k_{1j}\sigma_{ij}^2(\mathbf{q})} \; + k = 0 \text{ piece} \end{aligned} \tag{24}$$

where $\sigma^2_{ij}(\mathbf{q}) \equiv \langle[\psi_i(\mathbf{q}') - \psi_i(\mathbf{q}'+\mathbf{q})][\psi_j(\mathbf{q}') - \psi_j(\mathbf{q}'+\mathbf{q})]\rangle_g$, i.e., the covariance of relative displacements for points with separation $\mathbf{q}$, with Gaussian weights. This is of course the standard exact Zel'dovich power spectrum [61], truncated by W. To be clear, W enters because σ^2_{ij} is a Gaussian expectation value, i.e., expectation value with weight given by S_g, and W is part of S_g.

B. First correction

Now we consider the first correction.

$$Z_1(\mathbf{j},\mathbf{l}) = -\int d\chi\ d\phi\ e^{-S_g(\chi,\phi)+\mathbf{j}^t\phi+\mathbf{l}^t\chi}\ S_p(\chi,\phi) = -i\int d\chi\ d\phi\ e^{-S_g(\chi,\phi)+\mathbf{j}^t\phi+\mathbf{l}^t\chi}\ \chi^t\mathbf{\Delta}(\phi)\ . \tag{25}$$

We can do this calculation by manipulating $Z_0(\mathbf{j},\mathbf{l})$ that we have already calculated.

1. Non-trivial piece

The most interesting piece is:

$$\begin{aligned} Z_1^{(a)}(\mathbf{j},\mathbf{l}) &= -\frac{3}{2}i\int d\chi d\phi e^{-S_g(\chi,\phi)+\mathbf{j}^t\phi+\mathbf{l}^t\chi}\chi^t_\upsilon\partial_{\mathbf{x}}\partial_{\mathbf{x}}^{-2}\delta \\ &= -\frac{3}{2}i\int d^d\mathbf{q}d\eta\int d^d\mathbf{q}'\frac{d^d\mathbf{k}}{(2\pi)^d}\frac{i\mathbf{k}}{k^2}\int d\chi d\phi e^{-S_g(\chi,\phi)+\mathbf{j}^t\phi+\mathbf{l}^t\chi}\chi_\upsilon(\mathbf{q},\eta)e^{-i\mathbf{k}\cdot\left[\mathbf{q}+\psi(\mathbf{q},\eta)-\mathbf{q}'-\psi(\mathbf{q}',\eta)\right]} - \text{mean part} \\ &= \frac{3}{2}\int d^d\mathbf{q}d\eta\int d^d\mathbf{q}'\frac{d^d\mathbf{k}}{(2\pi)^d}\frac{\mathbf{k}}{k^2}e^{-i\mathbf{k}\cdot(\mathbf{q}-\mathbf{q}')}\partial_{\mathbf{l}_\upsilon(\mathbf{q},\eta)}e^{-i\mathbf{k}\cdot\left[\partial_{\mathbf{j}_\psi(\mathbf{q},\eta)}-\partial_{\mathbf{j}_\psi(\mathbf{q}',\eta)}\right]}Z_0(\mathbf{j},\mathbf{l}) - \text{mean part} \\ &= -i\frac{3}{2}\int d^d\mathbf{q}d\eta\int d^d\mathbf{q}'\frac{d^d\mathbf{k}}{(2\pi)^d}\frac{\mathbf{k}}{k^2}e^{-i\mathbf{k}\cdot(\mathbf{q}-\mathbf{q}')}e^{-i\mathbf{k}\cdot\left[\partial_{\mathbf{j}_\psi(\mathbf{q},\eta)}-\partial_{\mathbf{j}_\psi(\mathbf{q}',\eta)}\right]}\mathbf{j}^t\mathbf{L}^{-1}_{*\upsilon(\mathbf{q},\eta)}Z_0(\mathbf{j},\mathbf{l}) - \text{mean part} \end{aligned} \tag{26}$$

$\mathbf{M}_{*\upsilon}$ means the right index of $\mathbf{M}$ is υ and left is summed over in a product with the adjacent object as usual. Note that χ was projected to its υ element when it was dotted with $\mathbf{\Delta}_0$ given by Eq. (11) (ultimately, because the force acts to change velocity).

Note that to compute correlations of observables we don't need χ so we can set $\mathbf{l}$ to zero after the derivative with respect to it. The Z_2 calculation will just involve inserting another set of these derivatives, and so on.

Now, the action of the derivative operator $e^{-i\mathbf{k}\cdot\left[\partial_{\mathbf{j}_\psi(\mathbf{q},\eta)}-\partial_{\mathbf{j}_\psi(\mathbf{q}',\eta)}\right]}$ on $f[\mathbf{j}]$ is to add $-i\mathbf{k}\left[\delta^D(\mathbf{q}-\hat{\mathbf{q}})-\delta^D(\mathbf{q}'-\hat{\mathbf{q}})\right]\delta^D(\eta-\hat{\eta})$ to $\mathbf{j}_\psi(\hat{\mathbf{q}},\hat{\eta})$. This leads to

$$\begin{aligned} Z_1^{(a)}(\mathbf{j}) = -i\frac{3}{2}\int d^d\mathbf{q}d\eta\int d^d\mathbf{q}'\frac{d^d\mathbf{k}}{(2\pi)^d}\frac{\mathbf{k}}{k^2}e^{-i\mathbf{k}\cdot(\mathbf{q}-\mathbf{q}')} \\ \mathbf{j}^t\mathbf{L}^{-1}_{*\upsilon(\mathbf{q},\eta)}e^{-i\mathbf{k}\cdot\left[\mathbf{C}_{\psi(\mathbf{q},\eta)*}-\mathbf{C}_{\psi(\mathbf{q}',\eta)*}\right]\mathbf{j}-\frac{1}{2}k_i\sigma^2_{ij}(\mathbf{q}-\mathbf{q}',\eta)k_j}Z_0(\mathbf{j}) \\ - \text{mean part} \end{aligned} \tag{27}$$

where note that $\mathbf{L}^{-1}_{\psi\upsilon}(\eta,\eta) = 0$ leads to significant simplification. The "mean part" is:

$$\begin{aligned} &-i\tfrac{3}{2}\int d^d\mathbf{q}d\eta\int d^d\mathbf{q}'\frac{d^d\mathbf{k}}{(2\pi)^d}\frac{\mathbf{k}}{k^2}e^{-i\mathbf{k}\cdot(\mathbf{q}-\mathbf{q}')}\mathbf{j}^t\mathbf{L}^{-1}_{*\upsilon(\mathbf{q},\eta)}e^{-i\mathbf{k}\cdot\mathbf{C}_{\psi(\mathbf{q},\eta)*}\mathbf{j}-\frac{1}{2}k_i\mathbf{C}^\psi_{ij}(0,\eta)k_j}Z_0(\mathbf{j}) \\ &\quad = -d^{-1}\frac{3}{2}\mathbf{j}^t\mathbf{L}^{-1}_{*\upsilon}\mathbf{C}_{\psi*}\mathbf{j}\ Z_0(\mathbf{j})\ . \end{aligned} \tag{28}$$

We will see that for $d = 1$ this precisely cancels the Zel'dovich force-compensating term below, so it will not enter the numerical calculations in this paper, although it will need to be included in higher dimensional calculations.

Note that we generally need the normalized generating function $Z(\mathbf{j})/Z(0)$. If we have perturbatively computed $Z(\mathbf{j}) = Z_0(\mathbf{j})+Z_1(\mathbf{j})+...$ we have perturbatively $Z(\mathbf{j})/Z(0) = Z_0(\mathbf{j})+Z_1(\mathbf{j})-Z_1(0)Z_0(\mathbf{j})$ where we have used $Z_0(0) = 1$. Of course, $Z_1^{(a)}(0) = 0$ here.

Let's first compute the displacement power spectrum, which will be the Fourier transform of the displacement correlation function:

$$\begin{aligned}\langle\psi_{i_1}(\mathbf{q}_1,\eta)\,\psi_{i_2}(\mathbf{q}_2,\eta)\rangle_1^{(a)} &= \partial_{j_{\psi_{i_1}}(\mathbf{q}_1,\eta)}\partial_{j_{\psi_{i_2}}(\mathbf{q}_2,\eta)}Z_1^{(a)}(\mathbf{j}) \\ &= -\frac{3}{2}\int d^d\mathbf{q}d\eta'\int d^d\mathbf{q}'\frac{d^d\mathbf{k}}{(2\pi)^d}\frac{k_jk_i}{k^2}e^{-i\mathbf{k}\cdot(\mathbf{q}-\mathbf{q}')}\left[\mathbf{L}^{-1}\right]^{i_1j}_{\psi v}(\mathbf{q}_1-\mathbf{q},\eta,\eta') \\ &\qquad\left[C_\psi^{ii_2}(\mathbf{q}-\mathbf{q}_2,\eta',\eta)-C_\psi^{ii_2}(\mathbf{q}'-\mathbf{q}_2,\eta',\eta)\right]e^{-\frac{1}{2}k_m\sigma^2_{mn}(\mathbf{q}-\mathbf{q}',\eta')k_n} \\ &\quad +1\leftrightarrow 2-\text{mean part} \\ &= -\frac{3}{2}\int d^d\mathbf{q}d\eta'\int d^d\mathbf{q}'\frac{d^d\mathbf{k}}{(2\pi)^d}\cos(\mathbf{k}\cdot\mathbf{q}')\left[\mathbf{L}^{-1}\right]^{i_1j}_{\psi v}(\mathbf{q},\eta,\eta')\frac{k_jk_i}{k^2} \\ &\qquad\left[C_\psi^{ii_2}(\mathbf{q}_{12}+\mathbf{q},\eta',\eta)-C_\psi^{ii_2}(\mathbf{q}_{12}+\mathbf{q}+\mathbf{q}',\eta',\eta)\right]e^{-\frac{1}{2}k_m\sigma^2_{mn}(\mathbf{q}',\eta')k_n} \\ &\quad +1\leftrightarrow 2-\text{mean part}\end{aligned} \tag{29}$$

where in the last step we have shifted some variable definitions around and defined $\mathbf{q}_{12}\equiv\mathbf{q}_1-\mathbf{q}_2$ (when we use one index label when there should be two we mean to duplicate the index value, e.g., $C_\psi\equiv C_{\psi\psi}$). Note that symmetry guarantees that the result, like $C_\psi^{ij}(\mathbf{q}_{12})$, takes the form $\delta^K_{ij}f_1(q_{12})+\hat{\mathbf{q}}^i_{12}\hat{\mathbf{q}}^j_{12}f_2(q_{12})$, e.g., if we align the coordinates along the direction of $\mathbf{q}_{12}$, the matrix is diagonal with value f_1+f_2 along the direction of $\mathbf{q}_{12}$ and f_1 in the transverse directions.

We do not write out the mean part because in the 1D calculations in this paper it is exactly canceled by the Zel'dovich-compensating piece below. Higher dimension calculations will need to include it.

We Fourier transform this with respect to $\mathbf{q}_{12}$ to compute the power spectrum:

$$\begin{aligned}&P_\psi^{(a)i_1i_2}(\mathbf{k}_1,\eta)= \\ &\quad -\frac{3}{2}\int d\eta'\left[\mathbf{L}^{-1}\right]^{i_1j}_{\psi v}(\mathbf{k}_1,\eta,\eta')P_\psi^{gii_2}(\mathbf{k}_1,\eta',\eta)\int d^d\mathbf{q}'\left[1-\cos(\mathbf{k}_1\cdot\mathbf{q}')\right]\int\frac{d^d\mathbf{k}}{(2\pi)^d}\frac{k_jk_i}{k^2}\cos(\mathbf{k}\cdot\mathbf{q}')\,e^{-\frac{1}{2}k_m\sigma^2_{mn}(\mathbf{q}',\eta')k_n} \\ &\quad +1\leftrightarrow 2-\text{mean part}\,,\end{aligned} \tag{30}$$

where $P^g(\mathbf{k})$ is the Fourier transform of $C(\mathbf{q}_{12})$.

For simplicity of obtaining numerical results in this paper, we will specialize to 1D where it is easy to do the $\mathbf{k}$ integral:

$$P_\psi^{(a)1D}(k,\eta)=-3\int d\eta'\left[\mathbf{L}^{-1}\right]_{\psi v}(k,\eta,\eta')P_\psi^g(k,\eta',\eta)\int dq'\left[1-\cos(kq')\right]\frac{e^{-\frac{1}{2}\frac{q'^2}{\sigma^2(q',\eta')}}}{\sqrt{2\pi\sigma^2(q',\eta')}}-\text{mean part}\,. \tag{31}$$

In the low-k limit this becomes

$$P_\psi^{(a)1D}(k\to 0,\eta)=-\frac{3}{2}k^2P_\psi^g(k,\eta)\int d\eta'\left[\mathbf{L}_0^{-1}\right]_{\psi v}(\eta,\eta')\,e^{\eta'-\eta}\int dq'q'^2\frac{e^{-\frac{1}{2}\frac{q'^2}{\sigma^2(q',\eta')}}}{\sqrt{2\pi\sigma^2(q',\eta')}}-\text{mean part} \tag{32}$$

i.e., for a given power spectrum the integrals over q' and η' give some damping scale. In addition to making calculations more straightforward, 1D is an especially clean test case for the introduction of stream crossing effects because the Zel'dovich approximation is exact in 1D up to stream crossing [55, 62] – any deviation at all from Zel'dovich must be a stream crossing effect.

We will do calculations for power law initial conditions with $\pi^{-1}k\,P(k)=(k/k_{\rm NL})^{n+1}$, with $n=$(2, 1, 0.5, 0), because [55] ran 1D N-body simulations for these slopes. For $n=0$, with no high-k suppression ($W(k)=1$) this boils down to $P_\psi^{(a)1D}(k\to 0,\eta=0)=-\left(1.85\frac{k}{k_{\rm NL}}\right)^2P_\psi^g(k,0)$, while for $n=1/2$ it is $P_\psi^{(a)1D}(k\to 0,\eta=0)=-\left(1.26\frac{k}{k_{\rm NL}}\right)^2P_\psi^g(k,0)$, i.e., intuitively reasonably, the damping scale roughly corresponds to the non-linear scale. For $n\geq 1$ $\sigma^2(q)$ diverges with no small-scale suppression so we will discuss those below, after computing the small-scale restoring term.

To be clear: there is no correction like this in LPT, where the displacement power simply follows linear theory, i.e., the Zel'dovich approximation. The effect here comes entirely from stream crossing. The equation makes intuitive sense, with $e^{-\frac{1}{2}\frac{q'^2}{\sigma^2(q',\eta')}}$ representing the probability that elements with Lagrangian separation q' have crossed at time η'.

2. Canceling the Zel'dovich force

We now compute the Zel'dovich force term that we need to subtract because it is included in the Gaussian part:

$$Z_1^{(b)}(\mathbf{j},\mathbf{l}) = -\frac{3}{2}i\int d\chi\ d\phi\ e^{-S_g(\chi,\phi)+\mathbf{j}^t\phi+\mathbf{l}^t\chi}\ \chi_v^t\psi = -\frac{3}{2}i\partial_{\mathbf{l}_v}^t\partial_{\mathbf{j}_\psi}Z_0(\mathbf{j},\mathbf{l}) = -\frac{3}{2}\mathbf{j}^t\mathbf{L}_{*v}^{-1}\mathbf{C}_{\psi *}\mathbf{j}\ Z_0(\mathbf{j}) \tag{33}$$

where at the end we have set $\mathbf{l}=0$. We see that for $d=1$ this exactly cancels the "mean part" of $Z_1^{(a)}$ (Eq. 28, which is subtracted from Z, so it appears with a positive sign). If it exactly cancels in Z, it will exactly cancel in all statistics.

For higher dimensions the cancellation against Eq. 28 is only partial, so the contribution of this term to statistics must be computed. It is interesting to compute the time dependence of, e.g., the $\psi(\mathbf{q}_1,\eta)$–$\psi(\mathbf{q}_2,\eta)$ correlation function contribution, $\propto\int d^d\mathbf{q}d\eta'\mathbf{L}_{\psi v}^{-1}(\mathbf{q}_1,\eta,\mathbf{q},\eta')\mathbf{C}_{\psi\psi}(\mathbf{q},\eta',\mathbf{q}_2,\eta)\propto e^{2\eta}\,(\eta-\eta_i-2/5)$ where we have included an initial time η_i in the term where it does not give zero in the $\eta_i\to-\infty$ (expansion factor $a_i\to 0$) limit. This is not a problem because it cancels a similar term coming out of the other part of Eq. (27), to produce a final result that is insensitive to the initial time. We just need to make sure to do numerical calculations in a way that preserves this cancellation.

3. Restoring the suppressed small-scale fluctuations

Finally we compute the term restoring the small-scale fluctuations that we suppressed in the Gaussian part:

$$\begin{aligned} Z_1^{(c)}(\mathbf{j},\mathbf{l}) = -i\int d\chi d\phi e^{-S_g(\chi,\phi)+\mathbf{j}^t\phi+\mathbf{l}^t\chi}\chi^t(1-W^{-1})\mathbf{L}_0\phi &= -i\partial_{\mathbf{l}}^t(1-W^{-1})\mathbf{L}_0\partial_{\mathbf{j}}Z_0(\mathbf{j},\mathbf{l}) \\ &= \left[\text{constant}+\mathbf{j}^t\left(1-W\left(k\right)\right)\mathbf{C}\mathbf{j}\right]Z_0(\mathbf{j})\ , \end{aligned} \tag{34}$$

where we again set $\mathbf{l}=0$ at the end because we don't need $\mathbf{l}$ to compute observable statistics. This is adding back suppressed power, at lowest order. The constant is an irrelevant normalization factor.

The displacement correlation contribution is simply

$$\left\langle\psi_{i_1}\left(\mathbf{q}_1,\eta\right)\psi_{i_2}\left(\mathbf{q}_2,\eta\right)\right\rangle_1^{(c)} = 2\ \mathbf{C}_{i_1i_2}^{\psi(1-W)}\left(\mathbf{q}_1-\mathbf{q}_2,\eta\right) \tag{35}$$

where $\mathbf{C}^{\psi(1-W)}$ is understood to mean $\mathbf{C}^{\psi\psi}$ computed with a factor $1-W(k)$ multiplying the power spectrum, i.e., the corresponding power spectrum contribution is

$$P_\psi^{(c)}(\mathbf{k},\eta) = 2\left[1-W\left(k\right)\right]P_\psi^g(\mathbf{k},\eta)\ . \tag{36}$$

For $W(k)=\exp\left(-k^2c^2/2\right)$, the low-$k$ limit is $P_\psi^{(c)}(k\to 0,\eta)=c^2k^2P_\psi^g(k,\eta)$. This is a natural counter-term for the coefficient computed by Eq. (32), i.e., we can set $(P_{1,\psi}^{(a)}+P_{1,\psi}^{(c)})(k\to 0,\eta=0)=0$ by choosing

$$c^2 = \frac{3}{2}\int d\eta'\left[\mathbf{L}_0^{-1}\right]_{\psi v}(0,\eta')\,e^{\eta'}\int dq'q'^2\frac{e^{-\frac{1}{2}\frac{q'^2}{\sigma^2(q',\eta')}}}{\sqrt{2\pi\sigma^2(q',\eta')}}\ . \tag{37}$$

This matching is self-regulating, i.e., it is an equation of the form $c^2=f(c)$, because $\sigma^2(q',\eta')$ depends on c. The formal divergence of the un-damped $\sigma^2(q)$ for $n\geq 1$ is not a problem – it should be seen as an artifact of the unphysical nature of pure Zel'dovich displacements. The solutions are $c=$(0.6, 0.62, 0.74, 1.35)$k_{\rm NL}^{-1}$ for $n=$(2, 1, 0.5, 0), respectively. (This matching is similar to but appears to differ somewhat from that in [63], in that they relied on the fact that calculation of the term to match converges for a linear theory ΛCDM power spectrum, rather than using the damping in their theory to cut it off.) Figures 1-4 show the final displacement power spectrum, relative to pure linear. A key thing to notice in these figures is the dependence of the power suppression on c^2. The larger value of c^2 results in *less* total suppression at 1-loop order, even though the direct leading order effect is more suppression. At small c^2, there is *too much* stream crossing at leading order, which leads to large 1-loop power suppression. The larger c^2 introduces an appropriate level of suppression at leading order, allowing 1-loop corrections to be small. Using an even larger value of c^2 than we plot here reverses the trend, giving leading order suppression too large so that the 1-loop correction must become large and positive to compensate.

This use of $W(k)$ with c^2 calibration is not the main point of the paper (in case it is not clear: it is not fundamentally related to the inclusion of stream crossing) so we defer most discussion to §V, but we address a couple questions here.

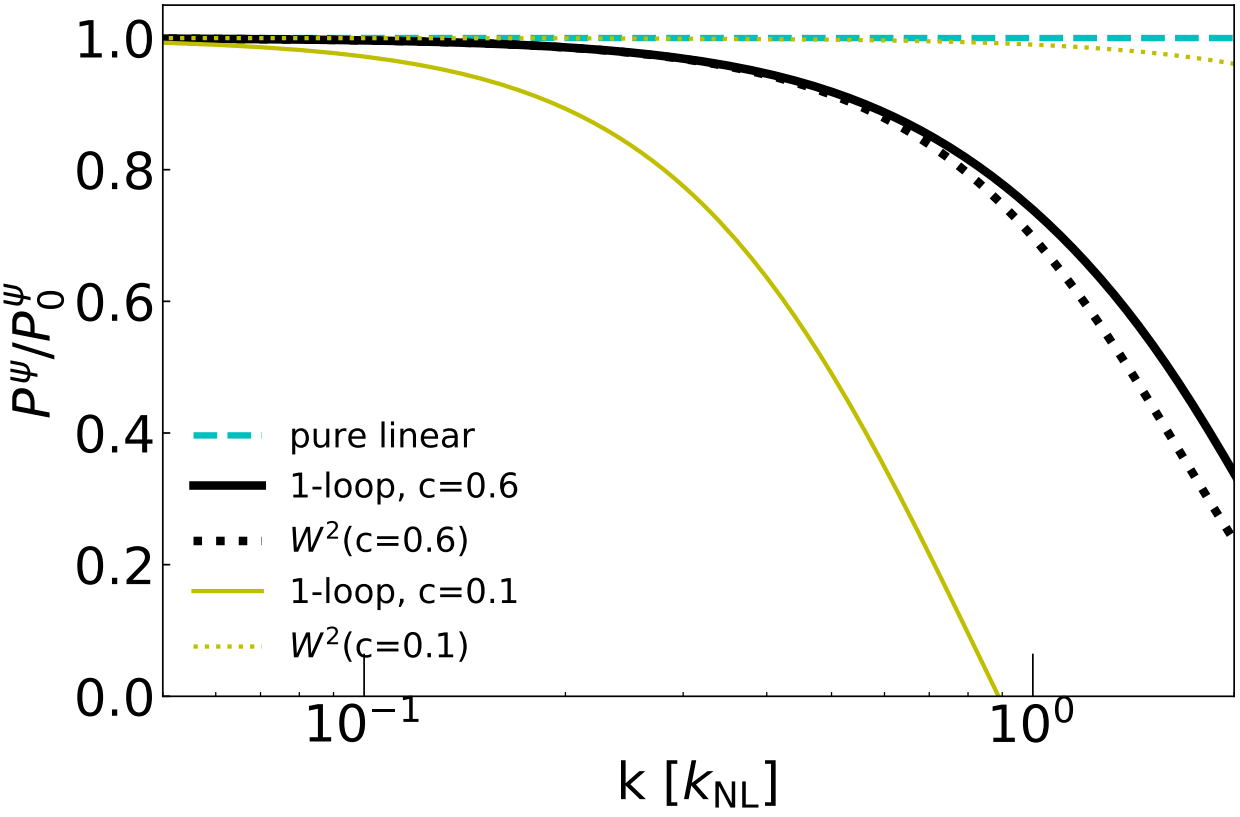

FIG. 1. One loop suppression of displacement power, in 1D, for $n = 2$. Dotted lines show the suppression of the linear power by $W(k) \equiv \exp\left(-k^2c^2/2\right)$, for the value of c that sets the one loop correction to zero in the $k \to 0$ limit (black) or a smaller value for contrast (yellow), i.e., these are effectively truncated Zel'dovich, while the denominator is bare Zel'dovich. Solid adds the 1-loop correction, i.e., dotted is the leading order result and the difference between solid and dotted is the correction. For $n = 2$ the correction in the $c \to 0$ limit is rapidly divergent. Note that the preferred c is fixed entirely by the principle that the correction goes to zero in the low-k limit – it is not a free parameter. In fact, we do not have simulations of the 1D displacement power, so the black curve is literally a (lowest order) prediction.

First note that, at least in this 1D calculation, c^2 is guaranteed to be positive, so we don't need to worry about $e^{-c^2k^2}$ diverging. As discussed below, we expect any serious 3D application to use an at least slightly more sophisticated replacement for the procedure used here, so there is no point in thinking about when the exact approach here might pathologically break down. Second, it would be very natural for c^2 to be time dependent, i.e., we would evaluate Eq. (37) at all η instead of $\eta = 0$. For an arbitrary power spectrum this would introduce significant complication as we'd need to solve the matching for a continuous function, with values at a given time depending on all values in the past, but for power law EdS models we would still only need to solve for one number because we know exactly what the time dependence must be. The only scale in the problem is the non-linear scale where $\Delta^2(k_{\rm NL}) = 1 \propto e^{2\eta}k^{n+d}$, so $k_{\rm NL}(\eta) \propto e^{-2\eta/(n+d)}$ (assuming $n + d > 0$). $c^2(\eta)$ must be $\propto k_{\rm NL}^{-2}(\eta) \propto e^{4\eta/(n+d)}$, with only the coefficient to be determined by matching. We expect that using this would make calculations at a given order more accurate by bringing the leading order closer to the truth, but it is important to understand that the calculation is mathematically consistent either way, because in either case the part that is taken away is restored as a perturbation. If we wanted to use a constant c^2 to make predictions in a model with arbitrary power spectrum, there would be nothing fundamentally wrong with simply computing a different value at each redshift we were interested in – in each case the fixed c^2 used in the calculation would represent a reasonable ∼time-averaged value for that redshift. There is certainly ambiguity here, but that is true of all perturbation theory, where the definition of orders is generally not unique – what we hope for is that the series we are computing converges to the same answer for any reasonable choice.

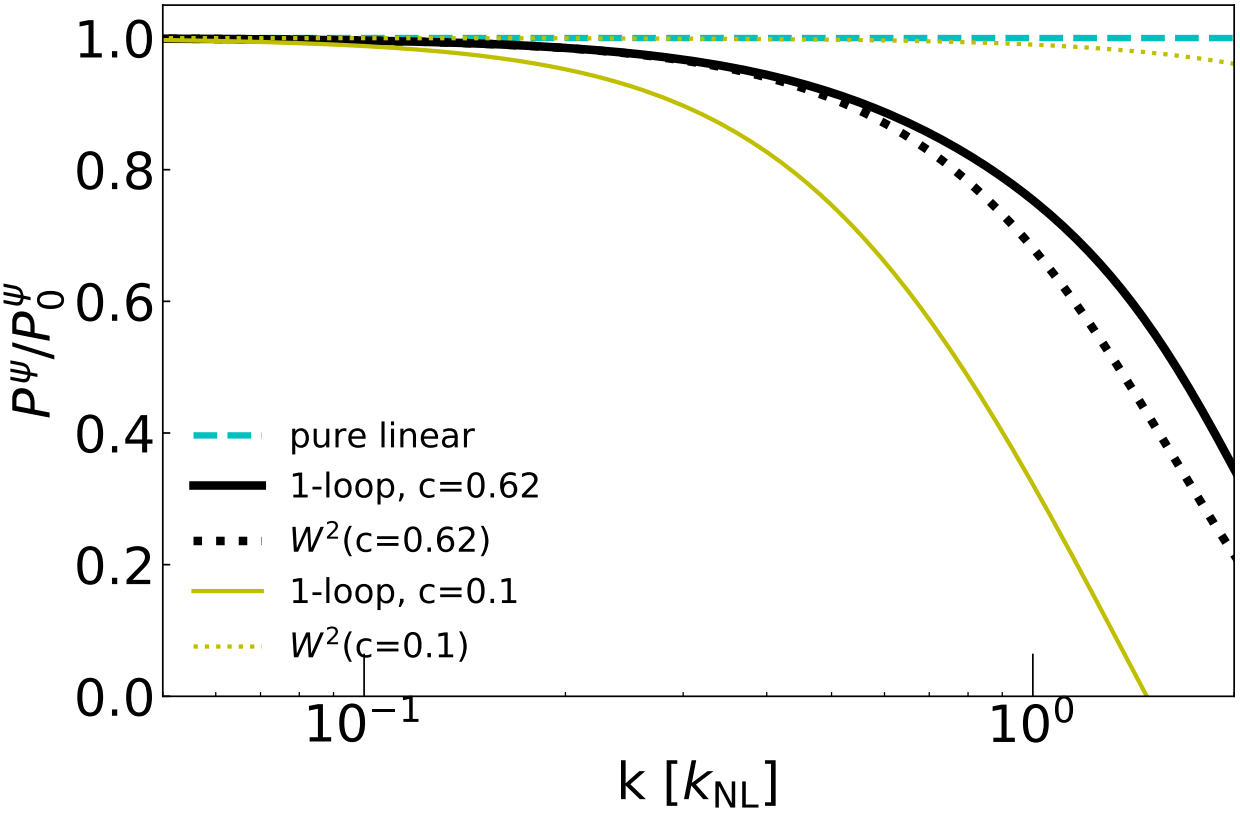

FIG. 2. One loop suppression of displacement power, in 1D, for $n = 1$, similar to Fig. 1.

4. *Density power spectrum*

The non-trivial density power spectrum contribution is

$$\langle\delta(\mathbf{k}_1,\eta)\delta(\mathbf{k}_2,\eta)\rangle_1^{(a)} = \int d^d\mathbf{q}_1 \int d^d\mathbf{q}_2\; e^{i\mathbf{k}_1\cdot\mathbf{q}_1+i\mathbf{k}_2\cdot\mathbf{q}_2} e^{i\mathbf{k}_1\cdot\partial_{\mathbf{j}_\psi(\mathbf{q}_1,\eta)}+i\mathbf{k}_2\cdot\partial_{\mathbf{j}_\psi(\mathbf{q}_2,\eta)}} Z_1^{(a)}(\mathbf{j})\mid_{\mathbf{j}=0} \tag{38}$$

$$= \frac{3}{2}(2\pi)^d\delta^D(\mathbf{k}_1+\mathbf{k}_2)\int d^d\mathbf{q}_{12}d^d\mathbf{q}d\eta'\int d^d\mathbf{q}'\frac{d^d\mathbf{k}}{(2\pi)^d}\frac{\mathbf{k}\cdot\mathbf{L}_{\psi v}^{-1}(\mathbf{q})\cdot\mathbf{k}_1}{k^2}e^{i\mathbf{k}_1\cdot\mathbf{q}_{12}}$$

$$\left[e^{i\mathbf{k}\cdot\mathbf{Q}(\mathbf{q},\mathbf{q}',\eta'\mathbf{q}_{12},\eta,\mathbf{k}_1)} - e^{-i\mathbf{k}\cdot\mathbf{Q}(\mathbf{q},\mathbf{q}',\eta'\mathbf{q}_{12},\eta,\mathbf{k}_1)}\right] e^{-\frac{1}{2}k_i\sigma^2_{ij}(\mathbf{q}',\eta')k_j} e^{-\frac{1}{2}k_{1i}\sigma^2_{ij}(\mathbf{q}_{12},\eta)k_{1j}} - \text{mean part}$$

where $\mathbf{Q}(\mathbf{q},\mathbf{q}',\eta'\mathbf{q}_{12},\eta,\mathbf{k}_1) \equiv \mathbf{q}' - i\boldsymbol{\Xi}(\mathbf{q},\mathbf{q}',\eta',\mathbf{q}_{12},\eta)\cdot\mathbf{k}_1$ and

$$\boldsymbol{\Xi}(\mathbf{q},\mathbf{q}',\eta',\mathbf{q}_{12},\eta) \equiv \mathbf{C}_\psi(\mathbf{q},\eta',\eta) - \mathbf{C}_\psi(\mathbf{q}+\mathbf{q}_{12},\eta',\eta) - \mathbf{C}_\psi(\mathbf{q}+\mathbf{q}',\eta',\eta) + \mathbf{C}_\psi(\mathbf{q}+\mathbf{q}'+\mathbf{q}_{12},\eta',\eta) \tag{39}$$

In 1D we can again integrate over k:

$$\langle\delta(k_1,\eta)\delta(k_2,\eta)\rangle_1^{(a),1D} = -(2\pi)\delta^D(k_1+k_2)\frac{3}{2}\mathrm{Im}\int dq_{12}dqd\eta' e^{ik_1q_{12}}\mathbf{L}_{\psi v}^{-1}(q,\eta,\eta') \tag{40}$$

$$k_1\int dq'\mathrm{erf}\left[\frac{q'-ik_1\Xi(q,q',\eta',q_{12},\eta)}{\sqrt{2}\sigma(q',\eta')}\right]e^{-\frac{1}{2}k_1^2\sigma^2(q_{12},\eta)} - \text{mean part}$$

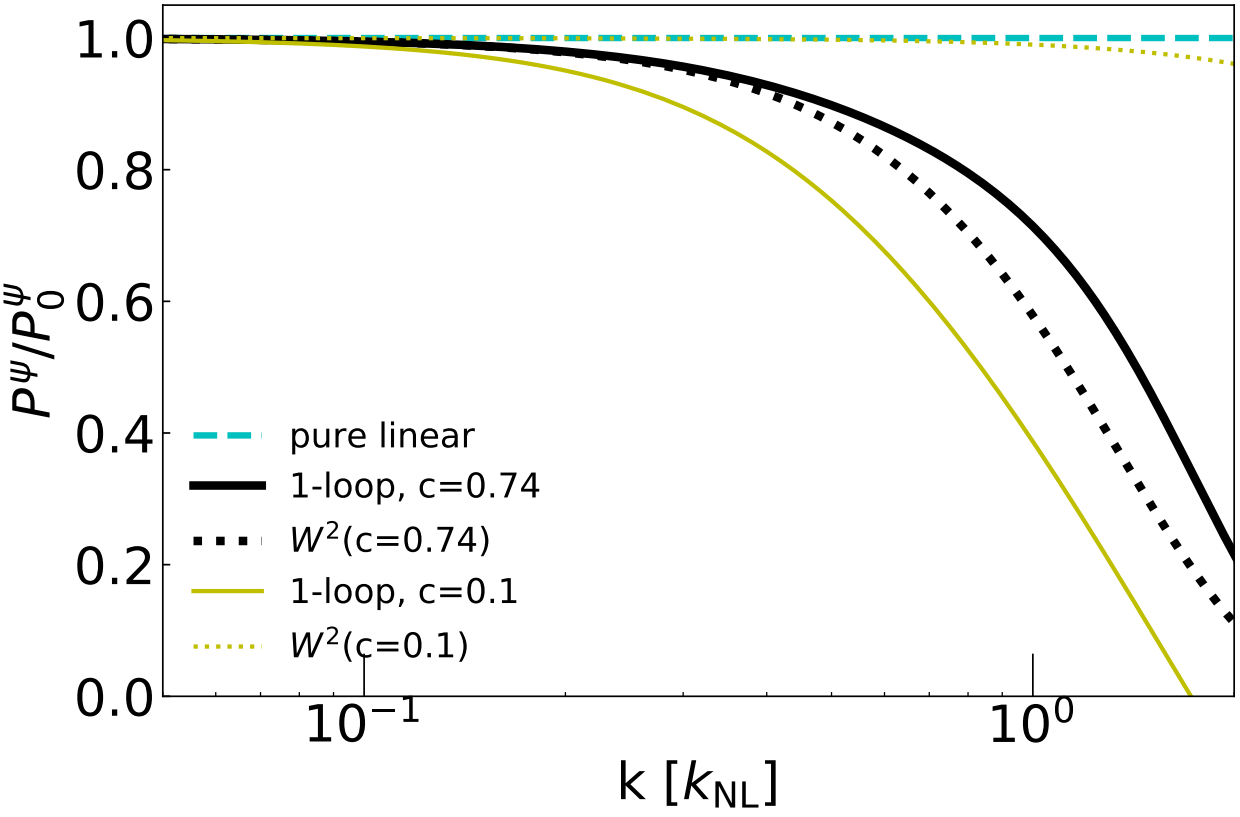

FIG. 3. One loop suppression of displacement power, in 1D, for $n = 0.5$, similar to Fig. 1.

The suppressed-power-restoring contribution to the density power spectrum is

$$\langle\delta(\mathbf{k}_1,\eta)\delta(\mathbf{k}_2,\eta)\rangle_1^{(c)} = \int d^d\mathbf{q}_1 \int d^d\mathbf{q}_2\; e^{i\mathbf{k}_1\cdot\mathbf{q}_1+i\mathbf{k}_2\cdot\mathbf{q}_2} e^{i\mathbf{k}_1\partial_{\mathbf{j}_\psi(\mathbf{q}_1,\eta)}+i\mathbf{k}_2\partial_{\mathbf{j}_\psi(\mathbf{q}_2,\eta)}} \left[Z_1^{(c)}(\mathbf{j}) - Z_1^{(c)}(0)Z_0(\mathbf{j})\right]|_{\mathbf{j}=0} \quad (41)$$

$$= \int d^d\mathbf{q}_1 \int d^d\mathbf{q}_2\; e^{i\mathbf{k}_1\cdot\mathbf{q}_1+i\mathbf{k}_2\cdot\mathbf{q}_2} e^{i\mathbf{k}_1\partial_{\mathbf{j}_\psi(\mathbf{q}_1,\eta)}+i\mathbf{k}_2\partial_{\mathbf{j}_\psi(\mathbf{q}_2,\eta)}} \mathbf{j}^t\,(1-W(k))\,\mathbf{C}\mathbf{j}\; Z_0(\mathbf{j})\,|_{\mathbf{j}=0}$$

$$= -(2\pi)\delta^D(k_1+k_2)\int d^d\mathbf{q}\;\cos(\mathbf{k}_1\cdot\mathbf{q})\;\; k_1^m\sigma^{2mn}_{1-W}(q,\eta)k_1^n\;\; e^{-\frac{1}{2}k_1^i\sigma^2_{ij}(q,\eta)k_1^j}$$

So our final density power spectrum predictions are shown in Figures 5-8. We see first that the numerical values of c computed by setting the one loop correction to the low-k displacement power to zero do a great job producing a perturbative expansion where the corrections are in fact small. This calibration is entirely internal to the theory. Even if we only had the Zel'dovich vs. corrected curves for density power here, it would be obvious that the calibrated values of c are a lot better than significantly different ones, just based on the principle that higher order terms in a perturbative expansion should be small. Compared to the N-body results of [55], we see that the predictions for $n = 1$ and $n = 1/2$ are excellent at low k, breaking down at the 10% level at slightly higher k than the free-parameter-fitted "EFT" curves shown in [55], and breaking down much more gracefully (in all cases gradually under-predicting the simulation power). The comparison for $n = 2$ is murky because [55] had trouble achieving convergence in the simulations – in any case, considering that the naive Zel'dovich prediction is divergent for $n = 2$, our theory seems like a very big step forward. Finally, $n = 0$ presents a somewhat greater challenge for the theory. While we successfully predict that the power will only start deviating from linear at higher k than Zel'dovich, we don't predict these deviations very well once they happen. It may be that this is because a Gaussian $W(k)$ is less appropriate for this spectrum with more low-k power. We see in Fig. 4 that our kernel matched in the asymptotic low-k limit is not doing a good job matching the predicted damping at higher k. This can be fixed by finding a kernel

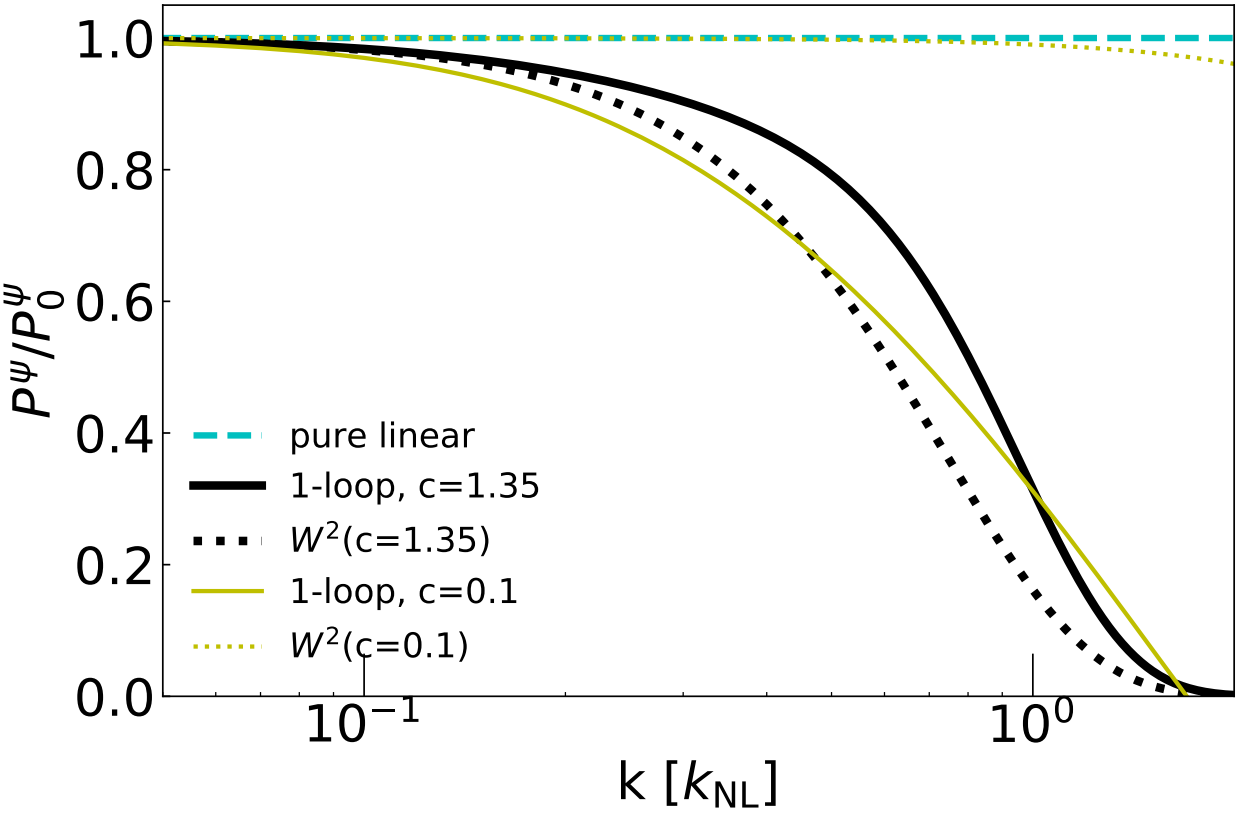

FIG. 4. One loop suppression of displacement power, in 1D, for $n = 0$.

that matches the calculated suppression better. There are many other possible refinements of the leading order theory that might improve the results, discussed in §V, but of course we may simply need another loop.

V. DISCUSSION

There are many obvious next steps for this formalism, e.g., computing displacement, density, etc. power spectra in 3D and comparing to numerical simulations [64–71], computing other statistics like the bispectrum [72–78], or power spectrum covariance matrix [79–82], etc.. Efforts to speed up PT numerical evaluation [83–87] could be reconsidered in this context. Here we discuss a few possibilities where we may have something useful to say about the path forward.

A. non-Gaussianity, Modified Gravity, neutrinos, baryons, bias, redshift space distortions, etc.

It should not be too hard to fit primordial non-Gaussianity [88–93] into this formalism. By definition the modification of the starting point statistical distribution for ϵ, Eq. (14), will add an extra non-Gaussian part to the perturbative piece S_p. Note that any kind of polynomial in $\phi \propto k^{-2}\delta$ can be constructed by repeated applications of the δ-generating operator.

Typical modifications of gravity [94, 95] should not present any new problem for this formalism. In standard perturbation theory, calculations are usually simplified by the good approximation for density statistics [96] that the growth at each order n is proportional to $D^n(t)$, where $D(t)$ is the linear growth factor. This means, as far as density statistics is considered, we have not needed to do numerical time integrals as part of PT (other than to compute $D(t)$). For momentum correlators this approximation starts to be less than percent accurate and corrections need to be taken

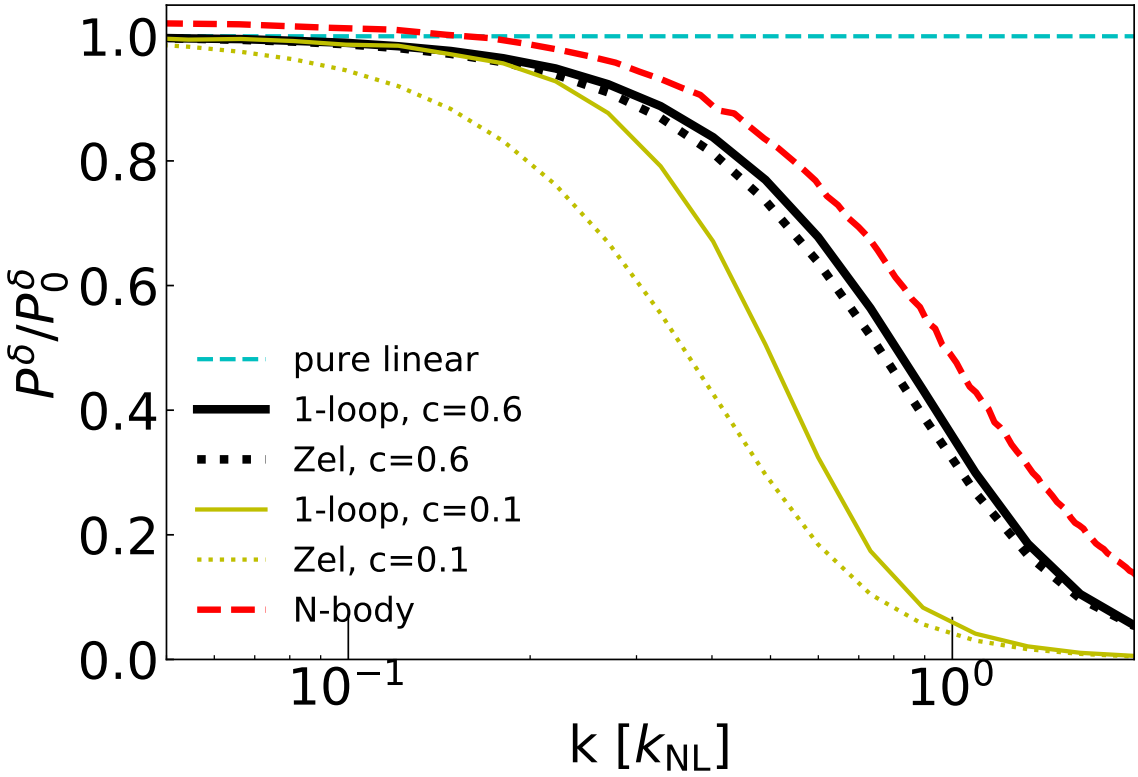

FIG. 5. One loop density power, in 1D, for $n = 2$, relative to the initial power law. Solid black shows the final one loop prediction, using the value of c fixed internally by the displacement power calculation as shown in Fig. 1. Black dotted shows the Zel'dovich power for the same c, i.e., the difference is the happily small one loop correction. Red dashed are the N-body results of [55], who say that their $n = 2$ simulations had not fully converged. E.g., the points at low k are almost surely a couple percent lower where they would agree well with our prediction (see the smaller-n figures that follow where they did not have this issue). For comparison, we show the same quantities for $c = 0.1$, where the one loop calculation makes a valiant effort to correct the massive failure of Zel'dovich, but it is clear that $c = 0.6$ is a much better starting point for a perturbative expansion.

into account [97]. In the calculations of this paper, however, we have abandoned this feature – time integration is an unavoidable part of the calculation including stream crossing. We also integrate over k-dependence of the propagator, which is not necessary in SPT. Having accepted these necessities, we can implement a modified gravity model that forces them on us with no additional complication. Non-linear effects can be included as perturbations.

Perturbation theory for massive neutrinos [98–102] may be an ideal application for this formalism. The only practical difference at late times between neutrinos and cold dark matter amounts to initial conditions – the initial conditions for neutrinos include large velocities that are uncorrelated down to arbitrarily small scales. It should be possible to represent the neutrinos with a displacement field just like we have discussed for CDM by imagining a snapshot at some early time and defining $\mathbf{N}$ to give appropriate random velocities in addition to the usual initial correlated perturbations. The first calculation to do will be the displacement power spectrum to determine the small-scale suppression kernel $W(k)$ for the neutrinos, which will be appropriately longer range. The key point is: because the formalism already fully includes stream crossing, and deals gracefully with naively very large dispersion in the initial conditions (e.g., infinite for $n = 2$ here), there should be no need for, e.g., special evolution equations for neutrinos. To be clear: we would now have separate ψ and υ for the neutrinos, driven by the combined density field, with $\mathbf{L}$ becoming a 4×4 matrix multiplying fields, etc.. To more faithfully represent all the relevant physics, this scenario may provide extra motivation for more detailed renormalization of $\mathbf{L}$ and renormalization of $\mathbf{N}$, as discussed below.

Unfortunately, what we have here is a theory for dark matter, which is not generally observable. Gravitational

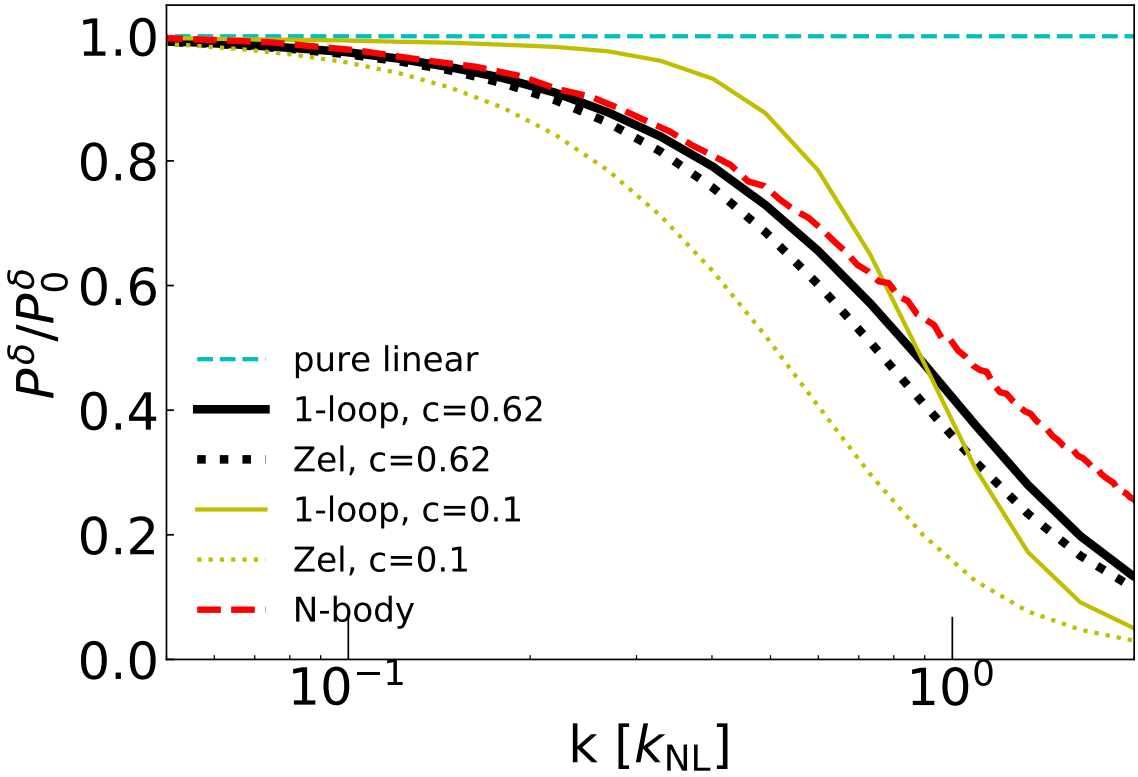

FIG. 6. One loop density power, in 1D, for $n = 1$, relative to the initial power law, similar to Fig. 5. Note that the values of c used in all figures were computed literally "blind" to the simulation results – it was clear that $W(k)$ was needed to keep blue spectra from diverging (and from common-sense understanding of the physics), and the exact numerical values used here were settled by the displacement power calculation before the density power was ever computed.

lensing [103–109] is typically cited as the most direct tracer, but even this is sensitive to baryon pressure effects on small-scales [110], and ultimately is an observation of galaxies (or some other source of photons), not directly mass, with corresponding biasing issues like intrinsic alignments [111–113]. Baryon displacement fields can easily be introduced to at least model the effect of the difference in initial conditions on large scales [114–120]. We could easily add explicit pressure terms for a given equation of state (this might be especially interesting for describing the Lyα forest [121–128], where little effort has been made to use higher order perturbation theory, even though its generally weak non-linearity would seem to make it a good candidate), but it is not clear that the perturbation theory as it stands will represent the physics fully correctly, because baryon streams should really never cross (they should shock instead), while the perturbation theory might effectively allow them to sometimes, at which point, e.g., the pressure force might be effectively pointing in the wrong direction. $W(k)$ the way we use it probably cannot entirely control this problem. It might be necessary to introduce something that more completely stops the possibility of crossing (e.g., [23–25, 32]), although we would want to be sure that this was a controlled modification of the perturbative expansion like our $W(k)$, not an arbitrary hack. Another possibility might be to use Eulerian-type fields for baryons while still using displacements for dark matter, as in Eulerian hydrodynamic simulations [129, 130]. While some kind of effective model with free or externally fixed parameters is inevitably needed for modeling temperatures any time they are affected by star formation or other complex physics, it would be interesting to see if any progress could be made including an explicit temperature field computed from a few simple principles to describe a relatively simple system like the IGM [131].

One of the most hopeful uses for perturbation theory is to organize our understanding of biasing models describing how galaxies and other observables trace dark matter [132–135]. The usual Eulerian bias prescriptions [132, 136–142]

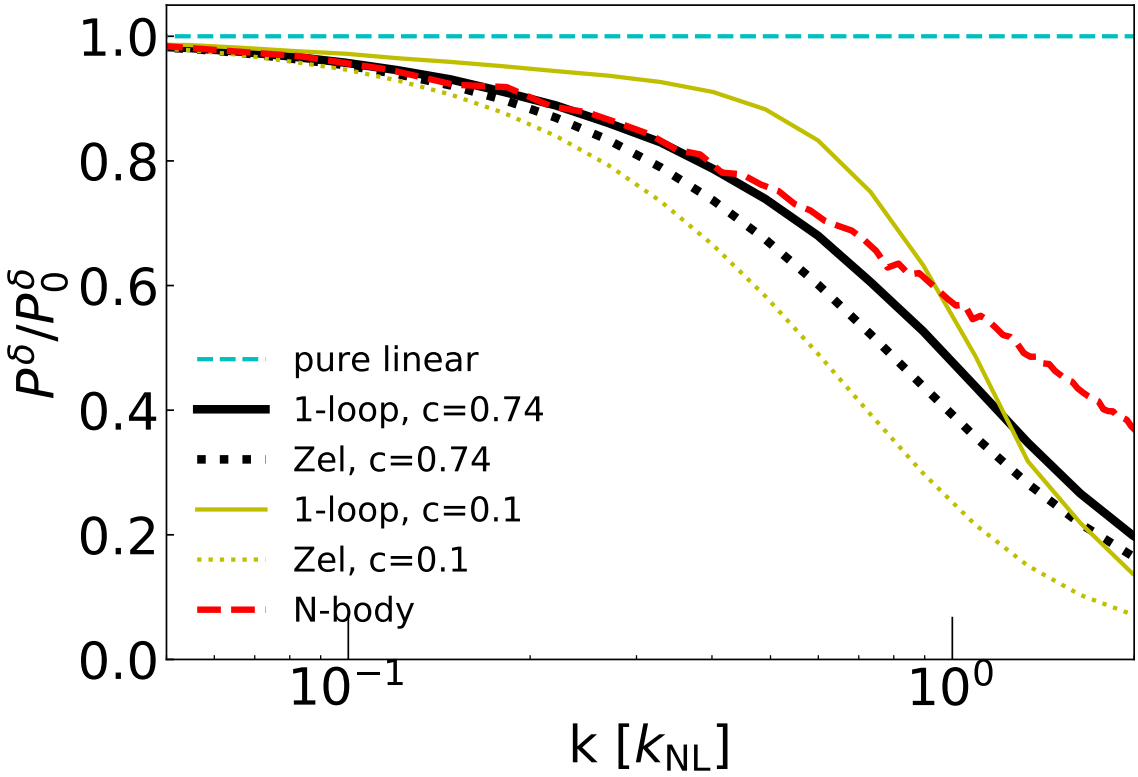

FIG. 7. One loop density power, in 1D, for $n = 0.5$, relative to the initial power law, similar to Fig. 5.

should be straightforward to implement in this formalism, with correlations involving powers of $\delta(\mathbf{x})$ and its derivatives computed by repeated applications of the δ-generating operator. For Lagrangian bias [143–145] we can always write some function of $\phi(\mathbf{q})$ as a weight next to the delta function in Eq. (3), and the tracers will stream-cross just like dark matter.

Redshift space distortions [143, 144, 146–158] can be included for dark matter with no additional approximation. We just add the appropriate apparent radial displacement $\propto v_\parallel$ in the exponential when constructing density fields. Redshift space distortions for tracers like galaxies are a little trickier because, while low-k modes of velocity should be the same as dark matter, this is not generally true on smaller scales, so some kind of generalized biasing model is needed to allow for the full range of possibilities.

B. Deriving effective theories

Suppose that instead of statistics we really want a theory for the Eulerian field $\delta(\mathbf{x})$. We can introduce it into Z using a delta function like this:

$$Z = \int d\delta_S\ d\phi\ d\chi\ \delta^D[\delta_S - \delta_S(\psi)]\ e^{-S(\phi,\chi)} = \int d\delta_S\ d\pi\ d\phi\ d\chi\ e^{i\pi^t[\delta_S - \delta_S(\psi)] - S(\phi,\chi)}\ , \tag{42}$$

where we have written δ_S to indicate that we probably want to construct a theory for a smoothed version of δ. We can now substitute Eq. (4) for $\delta(\psi)$ and if we can perform the integrals over ϕ and χ we will have a theory for δ_S, with π playing the role that χ does for ϕ. We can perform these integrals perturbatively by pulling the standard linear approximation out of $\delta(\psi)$ to write $\delta(\mathbf{k}) = i\mathbf{k}\cdot\psi(\mathbf{k}) + ...$ where the leading piece here will become part of the Gaussian integral over ψ and the rest will be included in the perturbative part of the integration. I.e., we can achieve

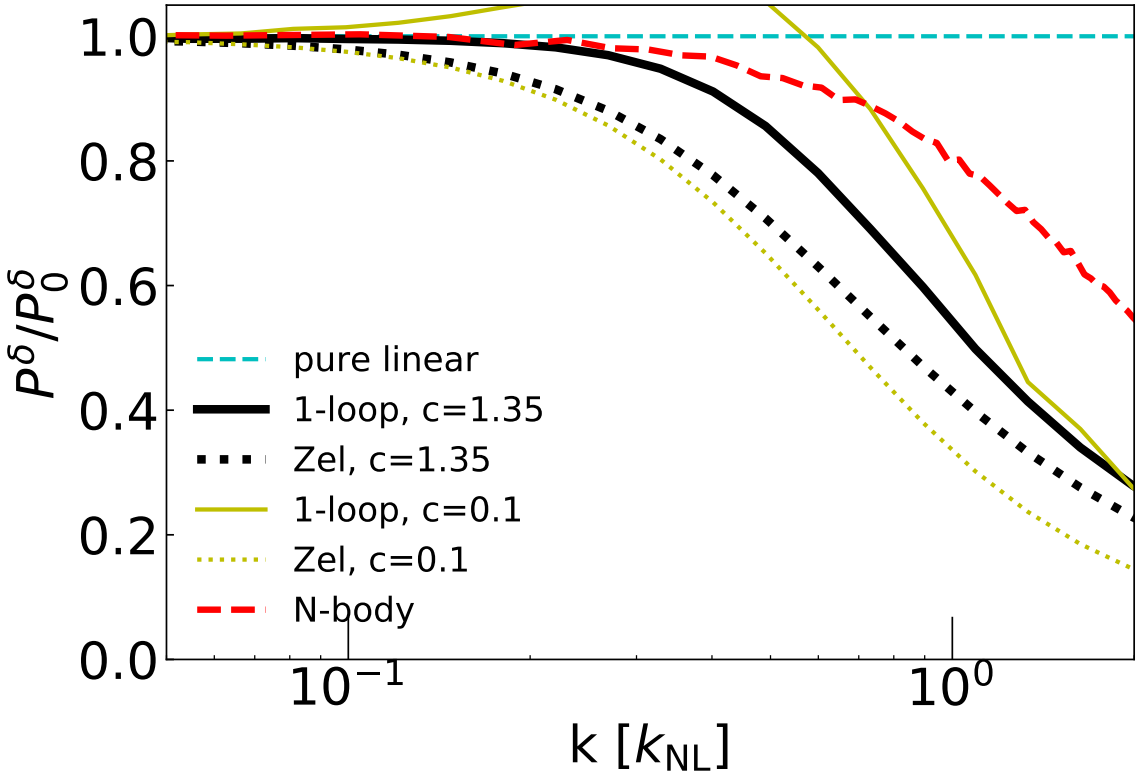

FIG. 8. One loop density power, in 1D, for $n = 0$, relative to the initial power law, similar to Fig. 5.

a generating function for δ statistics that can be made increasingly accurate by including higher order perturbative integral contributions. The formula will be for $\delta(t)$, and so imply evolution equations (and initial conditions) which could be used in different ways. If the calculation is done consistently for smoothed δ_S, the evolution equations will correctly represent dynamics entirely in terms of the smoothed fields. A similar calculation could be used to construct equations for other fields, e.g., moments of the velocity distribution function (momentum, energy, etc. [15]) can easily be written as integrals like the one for $\delta(\psi)$. One might ask "how likely is it that this could be an accurate theory, when it includes integration over small-scale ϕ?" That is not clear, but at least the size of perturbative terms will allow an internal estimate, and the $W(k)$-type suppression of small scales at leading order should aid convergence.

Another possibility, especially relevant if calculations in the full stream-crossing theory turn out to be slower than in traditional PT, would be to calibrate the free parameters of previous theories (e.g., [159]) by matching predictions to the stream crossing theory in the low-k limit. It should be possible to make the matching step fast because formulas simplify in the low-k limit.

C. More sophisticated renormalization

We intentionally tried to minimize the footprint of $W(k)$ in this paper, to avoid distracting from the main point about stream crossing, but clearly the issue it is addressing (of traditional linear theory being a terrible starting point for perturbation theory) is very generally important. Our approach here evolved from a more standard field theory starting point [160]. One of our original motivations was the observation that "EFT of LSS" proponents [27–29] did not seem to be taking their effective theory seriously enough, in that their prescription introduced terms intended to represent effective pressure/viscosity, but treated them as perturbations to the traditional fluid equations, rather than taking them seriously as part of the leading order linear evolution as one naturally would have done if given

a system with these terms already in the linearized equations. This was presumably for ease of computation, but means that the terms can only be used to cancel potentially divergent parts of the higher order corrections, rather than prevent the divergence from ever happening as they probably physically should do. We originally thought to improve their treatment by fully including $k^2\delta$-type terms in $\mathbf{L}$ where they would naturally damp the propagator [41, 42, 63]. We thought to follow a standard Wilsonian RG procedure [56, 160] of integrating out fields in shells of $\mathbf{k}$, absorbing the corrections into the coefficients of the effective theory. E.g., computing $\langle\chi\phi^t\rangle$ perturbatively will give you corrections to $\mathbf{L}^{-1}$, which can then be disentangled from corrections to $\mathbf{N}$ in $\langle\phi\phi^t\rangle$. Higher order interactions will also be generated. Because of the self-regulating nature of the propagator damping, i.e., that increasing a coefficient like c^2 generally decreases the calculated small-scale contribution to it, we expected to find c^2 values like we obtained here as some sort of fixed points of the RG evolution. At some point we realized that this full procedure might not be necessary – that we could capture the most relevant physics in the simpler way presented here. (We tell this historical story because it is helpful to understand the relation between our approach and standard field theory approaches.) A natural thought at this point would be "ok, what you have is a poor-person's version of the full Wilsonian treatment, so a next step must be to more fully implement that." But it is not clear that this is correct – it may be that, *for our problem*, standard field theory renormalization should be seen as the poor-person's version of what we do here. Renormalization in quantum field theory takes the form it does because of the infinities that are apparently unavoidable in the calculations, i.e., where we had an equation like $c^2 = f(c)$ to solve, they have $c^2 = f(c) + \infty$. Obviously solving this directly is no good, so they go on to observe that the derivative of c^2 with respect to some matching point (like our $k \to 0$) or similarly the contribution from integrating out a small shell of $\mathbf{k}$ in the Wilsonian picture, is finite. This allows them to calculate how c^2 changes with scale (where note that here we are using c^2 abstractly to represent any parameter, like a particle mass), but the initial condition for this running is still corrupted by the infinity and must be treated as a free parameter. The possible exception to this problem is if the parameter hits a fixed point of the RG evolution, so it acquires a value independent of the initial conditions, but this generally does not happen for all parameters. If we do not have any true deep-UV sensitivity in our system, i.e., do not have any divergence once enough correct physics is included for the system to self-regulate, we can hope to avoid free parameters, solving for parameters of the Gaussian model as we have done here for c^2 instead of only computing relative values between different scales. From the conventional point of view this is probably equivalent to saying all parameters flow to a fixed point. If this sounds like wishful thinking, note that we implicitly take for granted that it is possible to construct an effective theory of dark matter clustering that has no free parameters, while being finite and completely insensitive to small-scale degrees of freedom. We use this theory all the time: N-body simulations, where we generally take for granted that larger scale structure converges as the numerical resolution increases.

There is another reason to prefer the approach we use to more literally pursuing the Wilsonian picture of integrating out shells of modes: in those calculations, generally the integration results in a contribution that cannot be represented by a convenient, small number of terms in the Lagrangian (equivalent to terms in our S), so one is forced to "truncate the basis", i.e., ignore some of the calculation that does not project onto the terms you want to track. This is a potential source of error, not necessarily easy to control, which we avoid in our approach here where we do not throw anything away. Maybe the two approaches can be combined somehow – if nothing else, the exercise of integrating out small scale modes should help to identify and understand the useful modifications of the expansion.

Our modification of $\mathbf{L}^{-1}$ by a single overall suppression function $W(k)$ with semi-arbitrarily chosen Gaussian form was intended to absolutely minimally capture what are of course really more complicated effects. While the details should be filled in by higher order terms, this should be more efficient if the leading order model can be improved. The first extension one might make is to allow for separate $\psi\psi$, ψv, and vv kernels. While there is always a temptation to judge the value of this kind of detail based on whether it appears to improve agreement with simulations, this is really not necessary – these differences are an inevitable consequence of computing all the elements of $\langle\phi\phi^t\rangle$ and $\langle\phi\chi^t\rangle$, and can be calibrated entirely internally, i.e., if we compute something other than the $\langle\psi\psi^t\rangle$ that we computed here, using c^2 from $\langle\psi\psi^t\rangle$, we would find a less perfect match. By making $W(k)$ a matrix in (ψ, v), we can match all terms in $\langle\phi\phi^t\rangle$ at once. More generally, we should also match $\langle\phi\chi^t\rangle$, which is more directly identified with $\mathbf{L}^{-1}$, while $\langle\phi\phi^t\rangle$ is also sensitive to, and therefore can be used to make corrections to, the noise $\mathbf{N}$. The noise modifications should presumably be additive instead of multiplicative and go like k^4 at low k [13]. We could also attempt to match computed corrections to higher k by modifying the form of $W(k)$ (and similarly $\delta N(k)$ if using it). Similarly, these modifications are generally matrices in time, which can be calibrated in more detail by evaluating unequal time correlations. Also similarly, in higher dimensions $W(k)$ could be generalized to be asymmetric depending on the relation between its vector indices and $\mathbf{q}/\mathbf{k}$ (like the dependence of $C_{\psi_i\psi_j}(\mathbf{q})$ on whether the indices are along or transverse to $\mathbf{q}$). In other words: ultimately $\mathbf{L}^{-1}$ and $\mathbf{N}$ are general matrices in the full relevant space, and one could imagine working to make all loop corrections to them zero. Note, however, that we should not think that each new correction absorbed will make the final predictions more accurate by the full amount absorbed – once the corrections are small enough to take the form of a rapidly converging series, it should not matter much if they are absorbed at lowest order or computed perturbatively. To identify non-Gaussian terms, it will probably be useful to perform a

large/small scale split of the fields, perturbatively integrate out the small scale part, and see how to interpret the result as a modification of the large-scale equations (this would also tell us how to modify the Gaussian part, if we couldn't guess).

We tried to keep our $W(k)$ calibration simple by including the contribution to changing P_ψ from all scales, but, especially if we were isolating $\mathbf{L}^{-1}$ by computing $\langle\phi\chi^t\rangle$, we might encounter an issue known in more standard perturbation theory, that very large scale bulk flows can make significant (even divergent) contributions to matching calculations like we did here even though their effect on smaller, observation-scale fluctuations is not actually to damp them. In fact, for an asymptotically large-scale flow there should be no effect at all, by Galilean invariance. Even very large scale stream crossing will not necessarily have a simple damping effect on smaller scale fluctuations within the flow (it seems there will be a change in background density, plus noise from the uncorrelated fluctuations in the other stream, but not the direct damping you get from displacements streaming out of their own driving perturbations). If this is found to be an issue, it will probably be useful in the future to do something like splitting the fields into a small and large scale contribution, basically relative to the scale of observation, and only including the contribution from the small-scale part when modifying $\mathbf{L}^{-1}$. Because we are not permanently throwing anything away in these calculations, only shifting some of the naive Gaussian part of the functional integral into the perturbative part, the results should not be very sensitive to, e.g., where exactly you put the large/small split.

Finally, note that some attempts to implement Wilsonian RG [37, 161–163] ideas have been criticized for only integrating out small-scale fluctuations in the initial conditions, not in the evolving fields [164]. It is clear in the formalism presented here how one could integrate these fluctuations out of all fields at all times.

We thank Matt McQuinn for the N-body results from [55], and Anže Slosar and Zack Slepian for helpful comments on drafts.

Z.V. is supported in part by the U.S. Department of Energy contract to SLAC no. DE-AC02-76SF00515.

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Appendix A: Pedagogical explanation of the key math trick

The very simple math trick that makes this paper work may not be easy to identify within the formalism of our calculations, so we attempt to explain it more pedagogically here. Standard perturbation theory amounts to repeatedly performing the following type of integral:

$$\int dx\ e^{-\frac{1}{2}x^2+jx-\lambda x^3+\dots} = \int dx\ e^{-\frac{1}{2}x^2+jx}\left(1-\lambda x^3+\dots\right) \tag{A1}$$

i.e., it is based on the fact that one kind of integral we can do analytically is a Gaussian times a polynomial. If some more general function of x appears in the exponential here, one's first instinct is to Taylor expand it to obtain a polynomial. However, we observe that we can also do

$$\int dx\ e^{-\frac{1}{2}x^2+jx-\lambda e^{\alpha x}+\dots} = \int dx\ e^{-\frac{1}{2}x^2+jx}\left(1-\lambda e^{\alpha x}+\dots\right) = \int dx\ e^{-\frac{1}{2}x^2+jx}-\lambda e^{-\frac{1}{2}x^2+(j+\alpha)x}+\dots \tag{A2}$$

i.e., if the coefficients of the Gaussian include exponentials, we can simply absorb them back into the Gaussian and again do the integral analytically. In the main calculation in this paper the formula for density as exponential in displacement, Eq. (4), is analogous to $e^{\alpha x}$ here. We avoid losing stream crossing by *not* Taylor expanding $e^{i\mathbf{k}\cdot\psi}$ (which is tempting because it gives us a polynomial in ψ), but instead expanding this whole exponential factor out of the e^{-S} exponential and doing the calculation in the way of Eq. (A2). The "action of the derivative operator" that leads to Eq. (27) amounts to, underneath all the notation, an integration over force derived from density written in terms of exponentiated displacement. This is the key to the new result.

Appendix B: Jacobian of $\epsilon \to \phi$ variable change

Our evaluation of the Jacobian determinant factor in the change of variables from ϵ to ϕ follows similar calculations done in [33, 36].

$$\det\left(\frac{\partial\epsilon}{\partial\phi}\right) = \det\left(\mathbf{L}_0+\frac{\partial\mathbf{\Delta}_0(\phi)}{\partial\phi}\right) = \det\left(\mathbf{L}_0\right)\det\left(\mathbf{I}+\mathbf{L}_0^{-1}\frac{\partial\mathbf{\Delta}_0(\phi)}{\partial\phi}\right) \propto e^{\mathrm{Tr}\ln\left(\mathbf{I}+\mathbf{L}_0^{-1}\mathbf{M}\right)} \tag{B1}$$

where we have dropped the obviously field-independent $\det(\mathbf{L}_0)$, defined $\mathbf{M}\equiv\frac{\partial\mathbf{\Delta}_0}{\partial\phi}$, and used the identity $\det\mathbf{A} = e^{\mathrm{Tr}\ln\mathbf{A}}$. Note that $\mathbf{\Delta}_0$ has only an v component, but only depends on ψ, which means that in $\mathbf{M}$ only $\mathbf{M}_{v\psi}$ is non-zero. Taylor expanding in $\mathbf{L}_0^{-1}\mathbf{M}$ gives

$$\mathrm{Tr}\ln\left(\mathbf{I}+\mathbf{L}_0^{-1}\mathbf{M}\right) = -\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{Tr}\left[\left(-\mathbf{L}_0^{-1}\mathbf{M}\right)^n\right]\ . \tag{B2}$$

All of these terms take the form $\mathrm{Tr}\left[\mathbf{L}_{0\psi v}^{-1}\mathbf{M}_{v\psi}\mathbf{L}_{0\psi v}^{-1}\mathbf{M}_{v\psi}\dots\right]$. With $\mathbf{M}$ a delta function and $\mathbf{L}_0^{-1}$ a Heaviside function in the difference between their time indices, we see that the trace, which requires time to flow in a circle, i.e., $\int dt_1 dt_2 \dots dt_n A(t_1,t_2)\dots A(t_n,t_1)$, must give zero except possibly for the tricky limit where all times are the same so all Θ functions are evaluate at zero (with one integral over time remaining). For a general evolution equation, we would need a more subtle analysis of the equal time limit, but here it is simple because $\mathbf{L}_{0\psi v}^{-1}(\eta,\eta)=0$ independent of the Θ function, so every term is unambiguously zero.

1. Alternative derivation

We learned a different way to derive our basic functional integral starting point from [43]. Starting from the likelihood function for ϵ, Eq. (14), we can enforce the evolution Equation (9) using a delta function:

$$Z(\mathbf{j}) = \int d\phi\, d\epsilon\, \delta^D\left[\phi - \phi\left(\epsilon\right)\right]\, e^{-\frac{1}{2}\epsilon^t \mathbf{N}^{-1}\epsilon} \propto \int d\phi\, d\epsilon\, \delta^D\left[\epsilon - \epsilon\left(\phi\right)\right]\, e^{-\frac{1}{2}\epsilon^t \mathbf{N}^{-1}\epsilon} = \int d\phi\, d\epsilon\, d\chi\, e^{i\chi^t[\epsilon-\epsilon(\phi)]-\frac{1}{2}\epsilon^t \mathbf{N}^{-1}\epsilon} \quad \text{(B3)}$$

where we have used the same fact that the Jacobian of the ϵ-ϕ transformation is field independent to change arguments in the delta function. Note that this transformation is invertible, e.g., if we imagine discretizing the evolution equation, it is easy to see that knowing ϵ uniquely determines ϕ and vice versa (this is easier than inverting a final field to determine initial conditions – here we know the field at all times by definition). We can now insert Eq. (9) for $\epsilon(\phi)$ and perform the Gaussian integral over ϵ to obtain Eq. (18-19). χ plays a role enforcing the evolution equation instead of appearing through an apparently arbitrary Gaussian integral trick.

Appendix C: Low-k expansion of density power formula

We can consider a small k expansion of the erf part of Eq. (41). For small y, $\mathrm{erf}(x+iy) = \mathrm{erf}(x) + 2iy\exp\left(-x^2\right)/\sqrt{\pi} + ...$ The leading term gives zero in the q' integral, so we have:

$$\begin{aligned} &-\frac{3}{2}\mathrm{Im}\int dq_{12}dqd\eta' e^{ik_1 q_{12}}\mathbf{L}^{-1}_{\psi\upsilon}(q,\eta,\eta')k_1\int dq'\mathrm{erf}\left[\frac{q'-ik_1\Xi(q,q',\eta',q_{12},\eta)}{\sqrt{2}\sigma\left(q',\eta'\right)}\right]e^{-\frac{1}{2}k_1^2\sigma^2(q_{12},\eta)} \quad \text{(C1)}\\ &\simeq 3k_1^2\int dq_{12}dqd\eta'\cos\left(k_1 q_{12}\right)\mathbf{L}^{-1}_{\psi\upsilon}(q,\eta,\eta')\int dq'\Xi(q,q',\eta',q_{12},\eta)G\left[q',\sigma\left(q',\eta'\right)\right]e^{-\frac{1}{2}k_1^2\sigma^2(q_{12},\eta)}\\ &\simeq 3k_1^2\int dq_{12}\cos\left(k_1 q_{12}\right)e^{-\frac{1}{2}k_1^2\sigma^2(q_{12},\eta)}\int d\eta'[\mathbf{L}_0^{-1}]_{\psi\upsilon}(\eta,\eta')\int dq'\Xi_W(0,q',\eta',q_{12},\eta)G\left[q',\sigma\left(q',\eta'\right)\right] \end{aligned}$$

where in the last line we note that the q integral is a convolution between $W(q)$ in $\mathbf{L}^{-1}$ and Ξ and remove the q integral by defining Ξ_W to be Ξ with the power spectrum used to compute it multiplied by an extra factor of $W(k)$.

For the 1D cases we have tried, we find that, while the approximation is valid at low enough k, "low enough" is always too low to be useful.