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John L. Friedman[1], Lee Lindblom[2], Luciano Rezzolla[3,4], and Andrey I. Chugunov[5]

[1]*Leonard Parker Center for Gravitation, Cosmology and Astrophysics,*
University of Wisconsin-Milwaukee, P.O. Box 413, Milwaukee, Wisconsin 53201, USA[*]
[2]*Center for Astrophysics and Space Sciences, University of California at San Diego, La Jolla, CA 92093, USA*[†]
[3]*Institute for Theoretical Physics, Max-von-Laue-Straße 1, 60438 Frankfurt, Germany*[‡]
[4]*Frankfurt Institute for Advanced Studies, Ruth-Moufang-Str. 1, 60438 Frankfurt, Germany and*
[5]*Ioffe Institute, Polytekhnicheskaya 26, 194021 St.-Petersburg, Russia*[§]

At second order in perturbation theory, the unstable r-mode of a rotating star includes growing differential rotation whose form and growth rate are determined by gravitational-radiation reaction. With no magnetic field, the angular velocity of a fluid element grows exponentially until the mode reaches its nonlinear saturation amplitude and remains nonzero after saturation. With a background magnetic field, the differential rotation winds up and amplifies the field, and previous work where large mode amplitudes were considered [1], suggests that the amplification may damp out the instability. A background magnetic field, however, turns the saturated time-independent perturbations corresponding to adding differential rotation into perturbations whose characteristic frequencies are of order the Alfvén frequency. As found in previous studies, we argue that magnetic-field growth is sharply limited by the saturation amplitude of an unstable mode. In contrast to previous work, however, we show that if the amplitude is small, i.e., $\lesssim 10^{-4}$, then the limit on the magnetic-field growth is stringent enough to prevent the loss of energy to the magnetic field from damping or significantly altering an unstable r-mode in nascent neutron stars with normal interiors and in cold stars whose interiors are type II superconductors. We show this result first for a toy model, and we then obtain an analogous upper limit on magnetic field growth using a more realistic model of a rotating neutron star. Our analysis depends on the assumption that there are no marginally unstable perturbations, and this may not hold when differential rotation leads to a magnetorotational instability.

I. INTRODUCTION

Gravitational radiation drives an instability in the r-modes of rotating relativistic stars [2, 3] whose growth time [4] may be short enough to limit the angular velocity of old accreting neutron stars and may contribute to the spin-down of nascent neutron stars (see [4–9] for reviews and references). At second-order in perturbation theory, the unstable mode includes exponentially growing differential rotation [1, 10–14], whose form with no magnetic field was recently obtained by Friedman, Lindblom and Lockitch [14] (henceforth *Paper I*). Past work that considered r-modes saturated at large amplitudes in newly born and highly magnetized neutron stars has suggested that the resulting magnetic field windup could damp out or significantly alter the instability [1, 10, 11, 15–18]. The present paper, however, which considers smaller saturation amplitudes, finds restrictions on the growth of differential rotation that appear stringent enough to exclude significant damping of the instability by magnetic fields in old neutron stars spun up by accretion and in nascent, rapidly rotating stars. For the stable r-mode, with no radiation reaction, the secular drift is pure gauge

[19]: It can be removed by adding a second-order time-independent perturbation that adds differential rotation to the unperturbed equilibrium star.

The growth of an unstable mode is limited by non-linear saturation – that is, by loss of energy to other modes at a rate equal to the growth rate of the unstable mode. In their studies of magnetic field windup by an unstable r-mode in nascent neutron stars, Rezzolla *et al.* [1, 10, 11, 16] use a saturation amplitude $\alpha_{\rm sat}$ of order 10^{-1} or larger, as these were the typical values estimated to be relevant in newly born neutron stars [20]. Subsequent work in the context of second-order perturbation theory, however, finds an amplitude smaller than 10^{-4} [6, 21–24], and recent papers argue for still smaller limits based on observations of low-mass X-ray binaries and millisecond pulsars [9, 25]. Although a small saturation amplitude in itself sharply limits the effect of magnetic-field windup on the r-mode instability of young stars, Cuofano *et al.* [16, 17] find a substantial effect on r-mode evolution in old accreting neutron stars. They use the formalism developed by Rezzolla *et al.* [1, 10, 11]. They do not include nonlinear couplings, but the amplitude in their simulations remains below 10^{-4}. What these studies do not include is the back-reaction of magnetic field windup on the second-order perturbation associated with differential rotation, and that is the focus of the present work.

For a stationary star with no magnetic field and no viscosity, adding differential rotation is a time-independent perturbation: It simply changes a uniformly

[*]Electronic address: friedman@uwm.edu
[†]Electronic address: llindblom@ucsd.edu
[‡]Electronic address: rezzolla@itp.uni-frankfurt.de
[§]Electronic address: andr.astro@mail.ioffe.ru

rotating equilibrium to a neighboring equilibrium with a slightly different rotation law. Still in the absence of viscosity, but with a background magnetic field, however, a perturbation that adds differential rotation is a sum of axisymmetric modes with nonzero frequencies, modes restored by the magnetic Lorentz force – by the tension of stretched field lines. The periods of these modes are of order the Alfvén time t_A, which is essentially the time over which a perturbation in the magnetic field travels across a reference lengthscale in a plasma, which we take here to be the radius R of the star.

At second order in perturbation theory, differential rotation of an unstable star with negligible magnetic field is driven by a second-order radiation reaction force together with quadratic terms in the perturbed MHD-Euler equation (terms quadratic in the perturbed variables of the first-order r-mode). Before saturation, the effective driving force grows exponentially over a gravitational radiation-reaction timescale τ_{GR}, driving an exponentially growing differential rotation. After saturation, the driving force is constant, but the differential rotation maintains a power-law growth in time.

With a magnetic field large enough that $t_A \lesssim \tau_{GR}$ and a sufficiently small saturation amplitude, the picture is sharply altered. Now the driving force acts on a set of axisymmetric modes with frequencies of order $\omega_A = 2\pi/t_A$. Before saturation, the amplitudes of these modes again grow exponentially. But after saturation, each of the modes that comprise the differential rotation is effectively an oscillator acted on by a constant force: Its amplitude is the sum of its amplitude at saturation and a solution with harmonic time dependence. The combination of the small-saturation amplitude of the first-order r-mode and the fact that the growth of second-order differential rotation stops shortly after saturation, leads to a stringent constraint on differential rotation (on the secular drift of a fluid element) and hence on magnetic-field windup. We find that the increase in the magnetic field prior to saturation is smaller than the value needed to damp the unstable r-mode by a factor of order α; equivalently, the rate at which the magnetic field's energy drains energy from the r-mode is smaller by a factor of order α^2 than the rate at which the radiation-reaction force drives the unstable mode[1]. When $\alpha \sim \mathcal{O}(1)$, as assumed in the initial investigations of the instability [20] and in Refs. [1, 10, 11], this difference is small, but the situation changes considerably if $\alpha \sim 10^{-4}$, as in the present study.

The major results of this paper can be summarized

as follows. In Sec. II we qualitatively describe the fundamental physical processes that contribute to this problem: the timescales associated with the r-mode fluid oscillations, the timescales associated with magnetic field processes, and the timescale on which gravitational radiation drives an r-mode toward instability in neutron stars. We summarize in Sec. II previously published estimates of the magnetic field strength needed to suppress the growth of the gravitational radiation driven r-mode instability in neutron stars. The section ends with an outline of the argument that gives our main result.

In Sec. III, we introduce a modified version of a toy model due to Shapiro [26] that illustrates the main features we have just discussed. In Shapiro's model a cylinder of uniform-density fluid with an initial magnetic field and initial differential rotation has a time evolution given by the MHD-Euler system in the ideal-magnetohydrodynamics (MHD) limit (i.e., in a plasma with infinite conductivity). We add to the system a forcing term that mimics the second-order axisymmetric radiation-reaction force. Although the system is nonperturbative, the fluid displacement and magnetic field satisfy linear equations and can be written as a superposition of normal modes. We find an analytic solution for its evolution and use it to obtain a first estimate of the maximum angular displacement and magnetic field of the r-mode.

In Sec. IV, we develop the formalism governing the equilibrium and first- and second-order perturbations of a rotating star with a background magnetic field, in an ideal-magnetohydrodynamics (MHD) framework with radiation-reaction. We express perturbations in terms of a Lagrangian displacement and obtain the second-order MHD-Euler equation. In contrast to the toy model, the equation involves terms with first as well as second time derivatives, and we need a formalism developed by Dyson and Schutz [27], based on a conserved symplectic product [28], to express the amplitude of each mode in terms of the effective driving force.

In Sec. V, we obtain estimates of the maximum angular displacement of a fluid element and on the corresponding magnetic-field amplification for the second-order unstable r-mode itself. We assume that the perturbations are governed by a barotropic equation of state, that axisymmetric perturbations of the equilibrium star conserving angular momentum and baryon number are strictly stable, and that such axisymmetric perturbations can be written as a sum of discrete, non-degenerate modes. A brief discussion in Sec. VI summarizes our conclusions and considers implications of relaxed assumptions.

We relegate to Appendices details of the Lagrangian perturbation theory and of the formalism that obtains the amplitude of fluid modes in terms of a driving force.

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[1] In Ref. [19], Chugunov notes an analogous relation for the stable r-mode if one assumes that the arbitrarily chosen initial differential rotation is of order α^2. Here, for the unstable r-mode, the induced differential rotation is necessarily of order α^2, but one needs an additional constraint (Eq. (164) below) to keep the secular exponential growth of the magnetic field below its critical value.

II.   UNDERLYING MAGNITUDES

A.   A problem with four timescales

Four timescales are involved in this problem. In order of increasing size are: (1) the rotation period $2\pi/\Omega$ of the star; (2) the oscillation period $T_{\text{mode}} = 2\pi/\omega$ of an r-mode; and (3) the r-mode growth time τ_{mode}. Timescale (4), the Alfvén time t_A, may be larger or smaller than τ_{mode}, depending on the magnitude of the initial magnetic field and on whether the neutron-star's interior is superconducting.

The r-mode frequency ω is proportional to the star's angular velocity, having for slowly rotating Newtonian stars the form

$$\omega = -\frac{(\ell - 1)(\ell + 2)}{\ell + 1}\Omega, \qquad (1)$$

for a mode associated with the $\ell = m$ angular harmonic. The critical rotational frequency above which the $\ell = m = 2$ r-mode is unstable depends sensitively on temperature, but is likely to be above $f = \Omega/(2\pi) \simeq 500$ Hz, and the corresponding periods of rotation and oscillation are then of order 1-2 ms.

We define the Alfvén velocity v_A for a normal plasma by

$$v_A = \sqrt{\frac{B^2}{4\pi\rho}}. \qquad (2)$$

Using the radius R of the star as a characteristic wavelength gives the corresponding Alfvén angular frequency

$$\omega_A = 2\pi v_A/R = \frac{B}{R}\sqrt{\frac{\pi}{\rho}}, \qquad (3)$$

where ρ is an average rest-mass density [29–31]. In old and accreting neutron stars, such as those in X-ray binaries, the corresponding magnetic fields inferred from observation are in the range 10^8 - 10^9 G. The interior poloidal and toroidal fields may be higher, with the exterior poloidal field partly suppressed by the accreting material [29–31], and the relative size of the poloidal/toroidal magnetic-field components remains an open question [32]. Using the inferred values and typical sizes and densities for neutron stars, the typical Alfveń timescale for a normal plasma is

$$t_A \sim \frac{R}{v_A} = 7 \times 10^4 \, R_6 \, B_9^{-1} \, \sqrt{\rho_{14.6}} \,\, \text{s}, \qquad (4)$$

where B is an average magnetic field intensity, and the subscripts refer to Gaussian-cgs units, e.g., $R_6 := R/(10^6 \text{ cm})$. This timescale is considerably shorter if the neutron star interior is a type II superconductor, in which case the magnetic field is confined to flux tubes carrying fields of order $H_c \gtrsim 10^{15}$ G and the Alfvén time is of order

$$t_{A,SC} \sim R\sqrt{\frac{4\pi\rho}{BH_c}} \sim 70 \, R_6\sqrt{\frac{\rho_{14.6}}{B_9 \, H_{c,15}}} \,\, \text{s}. \qquad (5)$$

Nascent neutron stars have normal interiors and observed magnetic fields that range from 10^{12} to 10^{15} G.

Finally, the growth time τ_{mode} of the r-mode instability is set by a competition between gravitational radiation-reaction and local dissipation; the dominant contribution to local dissipation may be shear viscosity for a normal interior or at the core-crust interface, or mutual friction for a dominantly superfluid interior. In the absence of viscosity, the growth time of the instability is the gravitational radiation-reaction timescale, given for an equation of state with average polytropic index of order 0.5 by [4, 33]

$$\tau_{GR} \sim 2 \times 10^3 \, f_{500}^{-6} \frac{1.4 M_\odot}{M} R_6^{-4} \,\, \text{s}, \qquad (6)$$

where, adopting 500 Hz as a fiducial rotational frequency, we write $f_{500} := f/500$ Hz. Below a critical frequency, viscosity damps the instability. An accreting neutron star becomes unstable when accretion spins the star just beyond this critical frequency, with an initial near balance between viscosity and radiation reaction. After continued spin up, however, the radiation-reaction time can be short compared to the viscous damping time, and the mode will then grow with a timescale of order τ_{GR} until energy loss to other modes becomes important [6, 7]. From Eqs. (5) and (6), it follows that old neutron stars with a dominantly superconducting interiors have Alfvén times shorter than the growth time of the r-mode. In contrast, stars with a primarily normal interior have, by Eq. (4), Alfvén times comparable to or longer than the radiation-reaction time, if

$$B \lesssim 5 \times 10^{10} f_{500}^6 \left(\frac{M}{1.4 \, M_\odot}\right)^{3/2} R_6^{7/2}\text{G}. \qquad (7)$$

B.   Magnetic field needed to damp the r-mode instability

At first order in perturbation theory, the amplitude $\alpha(t)$ of the unstable r-mode grows exponentially

$$\alpha(t) = \alpha(0)e^{\beta t}, \qquad (8)$$

where $\beta = 1/\tau_{GR}$. At second order in perturbation theory, the unstable r-mode has axisymmetric differential rotation driven by a force comprising gravitational radiation-reaction and terms in the perturbed MHD-Euler equation that are quadratic in the first-order perturbation. The magnitude of the radiation-reaction force per unit mass is (see, e.g., Paper I)

$$|\boldsymbol{f}_{GR}| \sim \alpha^2(t)\,\beta\,\Omega\,R. \qquad (9)$$

Second-order contributions to viscous damping may reduce the magnitude of this effective driving force; because our goal is to set an upper limit on the second-order differential rotation, we do not include them.

The growth of magnetic-field energy can stop the growth of an unstable r-mode when the rate at which the differential rotation increases the energy of the *second-order* magnetic field, $\langle \delta B \rangle$, with $\langle \cdot \rangle$ indicating the axisymmetric part of a quantity, is equal to the rate of growth of energy of the *first-order r-mode*.

For a normal plasma, the growth rate of the magnetic-field energy density can be roughly estimated as

$$\frac{d\mathcal{E}_m}{dt} = 4\beta\mathcal{E}_m \sim \frac{1}{2\pi}\beta \langle \delta B \rangle^2 \,, \tag{10}$$

while the energy density of the linear r-mode grows at the rate

$$\frac{d\mathcal{E}_{\mathrm{mode}}}{dt} = 2\beta\,\mathcal{E}_{\mathrm{mode}} \sim \beta\,\rho\,[\alpha(t)\,\Omega\,R]^2 \,. \tag{11}$$

The critical value of the axisymmetric part of the perturbed magnetic field $\langle \delta B \rangle_{\mathrm{crit}}$ at which the two rates are equal is then

$$\langle \delta B \rangle_{\mathrm{crit}} \sim \alpha(t)\,\Omega\,R\,\sqrt{2\pi\rho} \sim 10^{13}\,\alpha_{-4}\,f_{500}\,R_6\,\sqrt{\rho_{14.6}}\,\mathrm{G}\,, \tag{12}$$

where we have taken as reference saturation amplitude $\alpha_{\mathrm{sat}} = 10^{-4}$. As noted in Sec. I, this is a conservative upper limit on the maximum value of α found in perturbative calculations [6, 21–24]), and it is much smaller than values $\alpha_{\mathrm{sat}} \sim 10^{-1}$ - 1 considered prior to the perturbative papers [1, 20].

Using the induction equation in the ideal MHD limit, it is not difficult to show that the secular drift of a fluid element associated with differential rotation in a normal core enhances an initial magnetic field B_0 by a factor of order

$$\delta B/B_0 \sim \xi^\phi, \tag{13}$$

with ξ^ϕ the angular displacement of the fluid element [1]. A value $\xi^\phi \gg 1$ is then needed to amplify an initial field of $B_0 \sim 10^8 - 10^{10}$ G to the critical value $\langle \delta B \rangle_{\mathrm{crit}} \sim 10^{13}$ G at which it can damp or significantly alter an unstable r-mode.

In Sec. V, we will show for an exponentially growing r-mode that ξ^ϕ has a bound of order $\alpha_{\mathrm{sat}}^2 \Omega/\omega_A$, which then leands to a bound on δB. The way it does so can be understood heuristically as follows. Using Eq. (13) and the expression (3) for the Alfvén frequency, we can write the perturbed magnetic field and the corresponding energy density as

$$\delta B \sim B_0\xi^\phi = \omega_A\sqrt{\frac{\rho}{\pi}}\,R\xi^\phi,$$

$$\frac{1}{8\pi}(\delta B)^2 \sim \frac{1}{8\pi^2}\rho\omega_A^2(R\xi^\phi)^2. \tag{14}$$

Then using Eq. (10) written in the form,

$$\frac{d\mathcal{E}_m}{dt} \sim \frac{1}{2\pi}\beta B_0^2(\xi^\phi)^2, \tag{15}$$

the bound on ξ^ϕ now gives

$$\frac{\langle \delta B_{\mathrm{sat}} \rangle}{\langle \delta B \rangle_{\mathrm{crit}}} \lesssim \alpha_{\mathrm{sat}}, \qquad \frac{d\mathcal{E}_m/dt}{d\mathcal{E}_{\mathrm{mode}}/dt}\bigg|_{\mathrm{sat}} \lesssim \alpha_{\mathrm{sat}}^2, \tag{16}$$

with numerical coefficients smaller than unity, where $\langle \delta B_{\mathrm{sat}} \rangle$ is the magnetic field generated by the fluid displacement ξ^ϕ when r-mode saturation occurs.

For a star with a superconducting interior, a given angular displacement ξ^ϕ produces a larger magnetic energy. However, because the Alfvén frequency is correspondingly higher and ξ^ϕ still has a bound of order $\alpha_{\mathrm{sat}}^2\Omega/\omega_A$, the bound on ξ^ϕ is more stringent. The net result is that two effects cancel, and the growth rate of magnetic energy again satisfies the bound (16).

We can define an average perturbed magnetic field, $\langle \delta B_{\mathrm{SC}} \rangle$, as a volume average for which $\delta\mathcal{E}_m = \langle \delta B_{\mathrm{SC}} \rangle^2/8\pi$. The critical magnetic field for which the growth rate of magnetic energy and of the linear r-mode are equal is then again given by Eq. (12).

III.   A TOY MODEL

We begin the discussion of differential rotation and magnetic field windup with a toy model that shows the main features of the evolution of the differential rotation and magnetic field that we claim for the nonlinear r-mode. In particular, in the model, a homogeneous incompressible rotating fluid with cylindrical symmetry has differential rotation driven by a force that mimics the radiation-reaction force driving the differential rotation of the unstable r-mode: It grows exponentially until a time t_{sat} corresponding to the saturation time of the r-mode and is then constant at its final value. This limit on the growth of the driving force leads to our main result, a stringent upper limit on the maximum angular displacement of a fluid element and a corresponding upper limit on magnetic field windup.

With a driving force per unit mass having maximum magnitude $\boldsymbol{f}_{\mathrm{max}}$, we will find an upper limit on the angular displacement ξ_{max}^ϕ of a fluid element of order

$$\xi_{\mathrm{max}}^\phi \sim \frac{|\boldsymbol{f}_{\mathrm{max}}|}{R\,\omega_A^2}, \tag{17}$$

where $\omega_A := 2\pi/t_A$ is the Alfvén angular frequency and R the radius of the model fluid. For a normal (i.e., not superconducting fluid), the corresponding maximum magnetic field is of order

$$B_{\mathrm{max}} \sim B_0\,\xi_{\mathrm{max}}^\phi\,. \tag{18}$$

The model is essentially that introduced by Shapiro [26], differing from it only by the addition of this driving force, and, as in Shapiro's model, the general solution to the MHD-Euler equation is analytic. The axisymmetric, homogeneous, incompressible model fluid has a purely azimuthal velocity field

$$\boldsymbol{v} = \Omega(t,\varpi)\boldsymbol{\phi}\,, \tag{19}$$

where $\boldsymbol{\phi}$ is the rotational symmetry vector

$$\boldsymbol{\phi} = \varpi\hat{\boldsymbol{\phi}} = x\hat{\boldsymbol{y}} - y\hat{\boldsymbol{x}}\,. \tag{20}$$

A magnetic field that is initially along the cylindrical radial vector field $\hat{\boldsymbol{\varpi}}$, is wound up by differential rotation driven by the exponentially growing forcing term. With no driving force, we will see that the dynamical equation governing the angular displacement of a fluid element is linear, and the fluid's displacement and angular velocity can be written as sums of normal modes with frequencies proportional to the Alfvén angular frequency (3).

With a driving force in the azimuthal direction, differential rotation continues to grow, and the radiation-reaction force continues to drive a growing magnetic field. Finally, when the driving force is time-independent (when the mode has reached saturation), the differential rotation becomes a sum of oscillatory modes, and the magnetic field oscillates about its final equilibrium value. For the nonlinear r-mode, the second-order radiation-reaction force includes a part that spins down the star. Because we are concerned here only with differential rotation, we will restrict consideration in the toy model to a driving force that preserves the total angular momentum of the fluid.

The toy model, like the stellar model, is governed by the MHD-Euler system in the ideal-MHD limit, comprising the source-free Maxwell equations and the Euler equation with a Lorentz force. For the incompressible fluid of the toy model, the source-free Maxwell equations are

$$\nabla \cdot \boldsymbol{B} = 0\,, \tag{21}$$

$$(\partial_t + \pounds_{\boldsymbol{v}})\boldsymbol{B} = \partial_t \boldsymbol{B} - \nabla \times (\boldsymbol{v} \times B) = 0\,, \tag{22}$$

and the Lorentz force per unit mass is

$$\boldsymbol{f}_m = \frac{1}{\rho}\boldsymbol{j} \times \boldsymbol{B} = \frac{1}{4\pi\rho}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}\,, \tag{23}$$

with $\boldsymbol{j}$ the electric current density, or, equivalently

$$f_m^i = \frac{1}{4\pi\rho}B_j(\nabla^j B^i - \nabla^i B^j)\,. \tag{24}$$

With no driving force, the MHD-Euler equation has the form

$$\boldsymbol{E} := \partial_t\boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \frac{\nabla p}{\rho} - \boldsymbol{f}_m = 0\,. \tag{25}$$

The differential rotation of the unstable r-mode is driven by the second-order axisymmetric radiation-reaction force. This is an azimuthal force, along $\boldsymbol{\phi}$, and we represent it in the toy model by a force $\boldsymbol{f}_{GR}$ per unit mass of the form

$$\boldsymbol{f}_{GR} = \alpha^2(t)f(\varpi)\hat{\boldsymbol{\phi}}\,, \tag{26}$$

where $f(\varpi)$ encodes the spatial dependence of the radiation-reaction force, while the mode amplitude can be modeled simply as given first by an exponential growth and then by constant after time t_{sat}

$$\alpha(t) = \begin{cases} \alpha(0)e^{\beta t}, & t \le t_{\text{sat}} \\ \alpha_{\text{sat}} \equiv \alpha(0)e^{\beta t_{\text{sat}}}, & t > t_{\text{sat}}. \end{cases} \tag{27}$$

The time evolution of the system is then determined by Eq. (22) and the driven MHD-Euler equation,

$$\partial_t\boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \frac{\nabla p}{\rho} - \boldsymbol{f}_m = \boldsymbol{f}_{GR}, \tag{28}$$

with

$$\nabla \cdot \boldsymbol{v} = 0\,, \tag{29}$$

because of the incompressibility assumption. Equation (29) is identically satisfied by a velocity field of the form (19), and the evolution equation for $\boldsymbol{B}$, Eq. (22), keeps $\boldsymbol{B}$ divergence free.

The ϖ and z components of Eq. (22) are $\partial_t B^\varpi = \partial_t B^z = 0$. The model has vanishing B^z, and Eq. (21) implies that B^ϖ has the temporally constant form

$$B^\varpi = \frac{R}{\varpi}B_0\,, \tag{30}$$

where R is the radius of the cylinder.

Only the ϕ component of the magnetic field is dynamical, and it is expressed in terms of ξ^ϕ by a first integral of the ϕ component of Eq. (22), namely

$$B^\phi = \frac{R}{\varpi}B_0\,\partial_\varpi\,\xi^\phi\,. \tag{31}$$

Hence, for a stationary system in which $\Omega(\varpi) = \partial_t\xi^\phi$ is constant in time, B^ϕ will simply grow linearly in time; this is the well-known magnetic-field "winding", producing a toroidal magnetic field out of a purely poloidal one [1, 15, 34].

As the instability develops and saturates, however, the evolution of the angular displacement ξ^ϕ is given by the ϕ component of (28)

$$\partial_t^2\xi^\phi - \omega_A^2\frac{R^4}{4\pi^2\varpi^3}\partial_\varpi(\varpi\partial_\varpi\xi^\phi) = \alpha^2(0)fe^{2\beta t}, \tag{32}$$

where ω_A is given by Eq. (3). Two remarks are worth making about Eq. (32). First, it has a simple mechanical equivalent in terms of a driven harmonic oscillator,

whose driving force first grows exponentially and then becomes time independent after $t_{\rm sat}$. Second, although it is derived from the MHD-Euler equation, it does not involve the pressure: The remaining ϖ component of the MHD-Euler equation determines p but is not needed for the evolution of ξ^ϕ, $\boldsymbol{B}$ or Ω.

We model crust pinning of the magnetic field by the boundary condition

$$B^\phi(\varpi = R) = 0\,, \tag{33}$$

and Eq. (31) then implies

$$\partial_\varpi \xi^\phi(\varpi = R) = 0\,. \tag{34}$$

Setting $\mathfrak{r} := \varpi^2/R^2$ allows us to write the homogeneous MHD-Euler equation in the form of a cylindrical wave equation $\partial_t^2 \xi^\phi - \pi^{-2}\omega_A^2 \mathfrak{r}^{-1}\partial_\mathfrak{r}(\mathfrak{r}\,\partial_\mathfrak{r}\xi^\phi) = 0$, whose solutions are proportional to Bessel functions of order 0,

$$\xi_n^\phi = J_0(k_n\varpi^2/R^2)e^{i\omega_n t}\,, \tag{35}$$

where k_n is the nth zero of J_0' and

$$\omega_n = \omega_A k_n/\pi\,. \tag{36}$$

Writing f and ξ^ϕ as sums of orthogonal eigenfunctions

$$f = \sum f_n J_0(k_n\varpi^2/R^2)\,, \tag{37a}$$

$$\xi^\phi = \sum c_n(t) J_0(k_n\varpi^2/R^2)\,, \tag{37b}$$

we obtain the exponentially growing solution to Eq. (32) prior to $t_{\rm sat}$,

$$\xi^\phi = \sum_n \frac{\alpha^2(t)}{4\beta^2 + \omega_n^2} f_n J_0(k_n\varpi^2/R^2). \tag{38}$$

On the other hand, when the driving force is time independent, representing an r-mode after nonlinear saturation is reached, the term $\alpha^2(0)fe^{2\beta t}$ in Eq. (32) is replaced by the the time-independent term $\alpha_{\rm sat}^2 f$. The solution for ξ^ϕ is now the sum of a time-independent term and harmonic functions of angular frequency ω_n With a constant force, the equilibrium value of ξ^ϕ is obtained by omitting $4\beta^2$ from the denominator of Eq. (38), and ξ^ϕ has the form

$$\begin{aligned} \xi^\phi &= \sum_n \frac{\alpha_{\rm sat}^2}{\omega_n^2} f_n J_0(k_n\varpi^2/R^2) \\ &+ \sum_n a_n J_0(k_n\varpi^2/R^2)\cos(\omega_n t + \eta_n)\,. \end{aligned} \tag{39}$$

The amplitude of the harmonic term depends on the transition from exponential growth to a time-independent driving force. A gradual approach to saturation reduces the amplitude, and we set an upper limit by adopting a driving force whose growth stops instantaneously, as given by Eqs. (26) and (27).

The oscillation amplitude is then

$$a_n = \alpha_{\rm sat}^2 \frac{2\beta}{\omega_n^2\sqrt{4\beta^2 + \omega_n^2}} f_n < \frac{\alpha_{\rm sat}^2}{\omega_n^2} f_n\,, \tag{40}$$

implying a maximum value of ξ^ϕ less than twice its equilibrium value. Here we have assumed that, prior to $t_{\rm sat}$, ξ^ϕ is dominated by the exponentially growing solution (38) associated with the unstable r-mode.

Equations (38) and (39) give us the toy-model's exact expressions for the angular displacement of a fluid element. We now consider its implications for the unstable r-mode, assuming that the behavior of the toy model's differential rotation is similar to that of the r-mode. The axisymmetric part of the r-mode's radiation-reaction force per unit mass is of order [cf., Eq. (11)]

$$\langle|\boldsymbol{f}_{GR}|\rangle \sim \alpha^2(t)\,\beta\,\Omega\,R\,. \tag{41}$$

For a normal interior, the Alfvén frequency (3) has magnitude

$$\omega_A = 0.9 \times 10^{-4}\,B_9\,R_6^{-1}\,\rho_{14.6}^{-1/2}\,. \tag{42}$$

With $f_n \sim \beta\,\Omega$ and hence $\alpha^2(t)\,R\,f_n$ of order $|\boldsymbol{f}_{GR}|$ and decreasing for large n, the sum in Eq. (38) is dominated by modes with $k_n \sim 1$ and $\omega_n \sim \omega_A$. Prior to saturation, we then have a bound that is independent of β,

$$\xi_{\rm sat}^\phi \lesssim \alpha_{\rm sat}^2 \frac{\beta\Omega}{4\beta^2 + \omega_A^2} < \alpha_{\rm sat}^2 \frac{\Omega}{4\omega_A}\,, \tag{43}$$

implied by the relation

$$\frac{\beta}{4\beta^2 + \omega_A^2} = \frac{1}{4\omega_A}\left[1 - \frac{(2\beta - \omega_A)^2}{4\beta^2 + \omega_A^2}\right] \le \frac{1}{4\omega_A}. \tag{44}$$

From Eq. (31), an angular displacement ξ^ϕ with characteristic wavelength of order R gives a magnetic field $B^{\hat\phi} \sim B_0\xi^\phi$, with a corresponding upper limit prior to saturation

$$\begin{aligned} B_{\rm sat}^{\hat\phi} &\sim \xi^\phi B_0 \\ &\lesssim \alpha_{\rm sat}^2 \frac{\Omega}{4\omega_A} B_0 = \frac{1}{4\sqrt{\pi}}\alpha_{\rm sat}^2 \Omega R \rho^{1/2}. \end{aligned}$$

or

$$B_{\rm sat}^{\hat\phi} \lesssim \frac{1}{4\sqrt{\pi}}\alpha_{\rm sat} B_{\rm crit}\,, \tag{45}$$

where we have used Eq. (3) for ω_A and Eq. (12) for the critical magnetic field needed to damp the r-mode. The corresponding inequality for the change in the magnetic energy density at quadratic order in ξ^ϕ is

$$\delta\mathcal{E}_{\rm sat} \lesssim \frac{1}{16\pi}\alpha_{\rm sat}^2 \delta\mathcal{E}_{\rm crit} \tag{46}$$

Then $\alpha_{\rm sat} \ll 1$ implies $B_{\rm sat}^{\hat\phi} \ll B_{\rm crit}$, or $\mathcal{E}_{\rm sat} \ll \mathcal{E}_{\rm crit}$. This is our main result.

After saturation, the linear r-mode is no longer growing. Energy gained from the first-order radiation-reaction is balanced by energy loss to daughter modes and to dissipation, and we now ask whether magnetic-field windup can play a significant role at this stage. In the post-saturation evolution of the angular displacement given by Eq. (39), ξ^ϕ reaches and oscillates about an equilibrium value that can be large if ω_A is small. That is, from Eq. (39), we have

$$\xi^\phi \lesssim \alpha_{\text{sat}}^2 \frac{\beta\Omega}{\omega_A^2}, \tag{47}$$

$$B^{\hat\phi} \lesssim \alpha_{\text{sat}}^2 \frac{\beta\Omega}{\omega_A^2} B_0. \tag{48}$$

Now, however, the growth rate of each mode is proportional to ω_n. Eq. (10) is then replaced by

$$\frac{d\mathcal{E}_m}{dt} \sim \omega_A \langle \delta B \rangle^2, \tag{49}$$

and the critical magnetic field for which the energy gained from radiation-reaction is comparable to the energy lost to magnetic-field windup field is given by

$$\begin{aligned}
\langle \delta B \rangle_{\text{crit}} &\sim \alpha_{\text{sat}} \Omega R \sqrt{\rho\beta/\omega_A} \\
&\geq 1.5 \times 10^{13} \alpha_{-4} \beta_{-3.3}^{1/2} f_{500} R_6^{3/2} \rho_{14.6}^{3/4} B_9^{-1/2} \ G,
\end{aligned} \tag{50}$$

where we have used $\omega_A \geq \omega_A(B_0) = (B_0/R)\sqrt{\pi/\rho}$. Eqs. (48) and (50) imply

$$\begin{aligned}
\frac{B^{\hat\phi}}{\langle \delta B \rangle_{\text{crit}}} &\lesssim \frac{\alpha_{\text{sat}}}{\pi^{3/4}} \beta^{1/2} R^{1/2} \rho^{1/4} B_0^{-1/2} \\
&= 1.3 \times 10^{-4} \alpha_{-4} \beta_{-3.3}^{1/2} R_6^{1/2} \rho_{14.6}^{1/4} B_9^{-1/2}.
\end{aligned} \tag{51}$$

To reach the critical magnetic field, one would need a normal interior with B_0 of order 20 G, more than 6 orders of magnitude smaller than the smallest estimated external magnetic field in an old neutron star (4.5×10^7 G, inferred from the period and spin-down of PSR J1938+2012 [35]).[2] Equation (51) implies that the post-saturation growth of an initial magnetic field of 10^8 or 10^9 G will continue to satisfy the saturation constraint (45).

The growth of a realistic initial magnetic field is then much too small to alter the r-mode. In particular, for neutron star whose interior is a normal plasma, the maximum angular displacement is of order

$$\xi_{\text{max}}^\phi \sim 2 \ \alpha_{-4}^2 \beta_{-3.3} f_{500} \rho_{14.6} R_6^2 B_9^{-2} \ \text{rad}, \tag{52}$$

and a corresponding maximum change in the magnetic field is

$$B_{\text{max}}^{\hat\phi} \sim \xi^\phi B_0 \lesssim 2 \times 10^9 \alpha_{-4}^2 \beta_{-3.3} f_{500} \rho_{14.6} R_6^2 B_9^{-1} \ \text{G}, \tag{53}$$

as implied by Eq. (48).

Again, two remarks are in order here. First, because Eq. (47) refers to a time after saturation has been achieved, the azimuthal displacement in (52) has a time-independent equilibrium value. Using again the mechanical equivalent discussed above, such a time-independent displacement corresponds to that of a harmonic oscillator subject to a constant and time-independent gravitational force. Second, in this toy model, because the poloidal component B^ϖ is constant, decoupled from the growth of the toroidal field, the frequencies ω_n of the modes are constant in time: They do not grow with the growth of the toroidal field. As a result, the quadratic dependence on the mode's amplitude in Eq. (53) can increase the magnetic field by six or more orders of magnitude if $\alpha_{\text{sat}} \simeq 0.1 - 1$, as was assumed in earlier work [1, 20].

The exact decoupling that keeps ω_n constant may be an artefact of the toy model: Sec. IV D displays the second-order MHD-Euler equations governing differential rotation generated by an unstable r-mode. In this more realistic model, we have checked that, for a generic background magnetic field, there is no analogous decoupling of poloidal and toroidal fields. Nevertheless, numerical evolutions of the MHD-Euler equations [36] show a poloidal field whose magnitude remains approximately constant while differential rotation winds up the magnetic field. We therefore do not assume that an increasing magnetic field produced by differential rotation results in increased frequency of modes associated with the field windup.

For cold neutron stars whose interior is a type II superconductor, we find in Sec. V that an essentially equivalent version of the constraint (45) holds both before *and* after nonlinear saturation. Before encountering the detailed calculation in Sec. V, we can understand the result heuristically as follows. The energy density of a stellar mode with displacement ξ and frequency ω is of order $\rho\omega^2\xi^2$. In particular, differential rotation drives modes whose energy density is of order $\delta\mathcal{E} \sim \rho\omega_A^2\xi^2$, where $\xi \sim R\xi^\phi$. The Alfvén frequency $\omega_{A,SC}$ of a superconducting interior is much larger than that of a normal plasma, and the rate of growth of magnetic energy is thus much larger for a given displacement ξ^ϕ. However, because the bound $\xi^\phi \lesssim \alpha_{\text{sat}}^2 \Omega/\omega_{A,SC}$ on ξ^ϕ is more stringent by the factor $\omega_A/\omega_{A,SC}$, the bound on $\delta\mathcal{E}_m$ remains the same:

$$\begin{aligned}
\delta\mathcal{E}_m &\lesssim \alpha_{\text{sat}}^4 \rho\Omega^2 R^2 \\
&\sim \alpha_{\text{sat}}^2 (\text{energy density of the linear } r\text{-mode}).
\end{aligned} \tag{54}$$

The constraint also holds after saturation because, as we noted in Sec. II A, $\omega_{A,SC} \gg \beta$, implying that the equilibrium displacement is within about a factor of two of the displacement at saturation. We conclude that,

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[2] Although interior fields below 100 G seem highly unlikely, field decay to that level has not, to our knowledge, been ruled out observationally.

for small saturation amplitudes ($\alpha_{\text{sat}} \lesssim 10^{-4}$), magnetic field windup from differential rotation is too small to produce magnetic fields that can damp or significantly alter the unstable r-mode.

IV. EQUILIBRIUM AND PERTURBATION EQUATIONS

We work in the approximation of Newtonian magnetohydrodynamics (MHD) with the star described by a perfect fluid with infinite conductivity. The version of the Euler equation that we use, Eq. (61), includes $\boldsymbol{f}_{GR}$, the post-Newtonian gravitational radiation-reaction force (per unit mass). This force plays a central role in the nonlinear evolution of the r-modes that is the primary focus of our paper. Because the old neutron stars we consider have spin-down times much longer than the gravitational radiation-reaction timescale of an r-mode (and may also be balanced by accretion), we neglect radiation reaction associated with the magnetic field.

We denote by $Q := \{\rho, \boldsymbol{v}, p, \Phi, \boldsymbol{B}, \mathbf{E}\}$ the collection of fields that determine the state of the fluid. Here ρ is the mass density, v^i the fluid velocity, p the pressure, Φ the gravitational potential, and $\mathbf{E}$ and $\boldsymbol{B}$ the electric and magnetic fields. For a barotropic equation of state, $p = p(\rho)$, the specific enthalpy h of the fluid is

$$h = \int_0^p \frac{dp}{\rho}\,, \qquad (55)$$

and we define a potential U by

$$U := h + \Phi\,, \qquad (56)$$

where Φ satisfies the Poisson equation

$$\nabla^2 \Phi = 4\pi\rho\,. \qquad (57)$$

The following equations govern the evolution of the fluid and its electromagnetic field. With g_{ij} the flat 3-metric and g its determinant, conservation of mass (the continuity equation) has the equivalent forms

$$(\partial_t + \pounds_{\boldsymbol{v}})(\rho\sqrt{g}) = 0 = \partial_t \rho + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v})\,, \qquad (58)$$

where we use the relation $\frac{1}{\sqrt{g}}\pounds_{\boldsymbol{v}}\sqrt{g} = \boldsymbol{\nabla}\cdot\boldsymbol{v}$. The vanishing of the electric field in a comoving frame is given by

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = 0\,, \qquad (59)$$

$$(\partial_t + \pounds_{\boldsymbol{v}})(\boldsymbol{B}\sqrt{g}) = 0 = \partial_t \boldsymbol{B} - \nabla \times (\boldsymbol{v} \times \boldsymbol{B})\,, \qquad (60)$$

and by expression (23) for the Lorentz force per unit mass. Recalling that $\boldsymbol{f}_{GR}$ is the radiation-reaction force per unit mass associated with gravitational radiation, we write the MHD-Euler equation in the form

$$\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{\nabla}U - \boldsymbol{f}_m = \boldsymbol{f}_{GR}\,. \qquad (61)$$

The radiation-reaction force per unit mass is given by [14, 37, 38]

$$\boldsymbol{f}_{GR} = \sum_{l\geq 2} \sum_{|m|\leq l} \frac{(-1)^{\ell+1}N_\ell}{32\pi} \Re\left\{ \frac{\boldsymbol{\nabla}(r^\ell Y^{\ell m})}{\sqrt{\ell}} \frac{d^{2\ell+1}I^{\ell m}}{dt^{2\ell+1}} \right.$$
$$\left. - \frac{2r^\ell \boldsymbol{Y}_B^{\ell m}}{\sqrt{\ell+1}} \frac{d^{2\ell+2}S^{\ell m}}{dt^{2\ell+2}} - \frac{2\boldsymbol{v} \times \boldsymbol{\nabla}(r^\ell Y^{\ell m})}{\sqrt{\ell}} \frac{d^{2\ell+1}S^{\ell m}}{dt^{2\ell+1}} \right\}, \qquad (62)$$

where $\Re(Z)$ denotes the real part of the complex quantity Z. The quantities $I^{\ell m}$ and $S^{\ell m}$ are the complex mass and current multiple moments of the fluid source (cf. Thorne [39] Eqs. 5.18a,b) defined by,

$$I^{\ell m} := \frac{N_\ell}{\sqrt{\ell}} \int \rho\, r^\ell Y^{*\ell m} d^3 x\,, \qquad (63)$$

$$S^{\ell m} := \frac{2N_\ell}{\sqrt{\ell+1}} \int \rho\, r^\ell\, \boldsymbol{v} \cdot \boldsymbol{Y}_B^{*\ell m} d^3 x\,, \qquad (64)$$

with N_ℓ the constant

$$N_\ell := \frac{16\pi}{(2\ell+1)!!}\sqrt{\frac{(\ell+2)(\ell+1)}{2(\ell-1)}}\,. \qquad (65)$$

The functions $Y^{\ell m}$ are the standard spherical harmonics, while the $\boldsymbol{Y}_B^{\ell m}$ are the magnetic-type vector harmonics defined by

$$\boldsymbol{Y}_B^{\ell m} := \frac{\boldsymbol{r} \times \boldsymbol{\nabla}Y^{lm}}{\sqrt{\ell(\ell+1)}}\,, \qquad (66)$$

with normalization $\int |Y^{\ell m}|^2 d\cos\theta\, d\phi = 1$ and $\int |\boldsymbol{Y}_B^{\ell m}|^2 d\cos\theta\, d\phi = 1$. In Cartesian coordinates, $\boldsymbol{r}$ is given by $\boldsymbol{r} = (x, y, z)$.

A. Equilibrium equations

We consider a uniformly rotating, axisymmetric equilibrium star with angular velocity Ω. Because the magnetic field is not in general aligned with the axis of symmetry, the equilibrium is stationary only in a rotating frame, satisfying

$$(\partial_t + \pounds_{\boldsymbol{v}})Q = 0\,, \qquad (67)$$

where

$$\boldsymbol{v} = \Omega\boldsymbol{\phi}\,, \qquad (68)$$

with $\boldsymbol{\phi}$ the generator of rotations about the z-axis. In Cartesian coordinates, $\boldsymbol{\phi} = (-y, x, 0)$, implying

$$\boldsymbol{\phi} \cdot \boldsymbol{\phi} = \varpi^2\,, \qquad (69)$$

where ϖ is the distance from the rotation axis.

We consider constant-mass sequences of stellar models, i.e., models whose exact mass perturbations, $\delta M = M(\alpha) - M(\alpha = 0)$ vanish identically for all values of α. The integrals of the nth-order density perturbations therefore vanish identically for these models:

$$0 = \frac{1}{n!} \left. \frac{d^n M(\alpha)}{d\alpha^n} \right|_{\alpha=0} = \int \delta^{(n)} \rho \sqrt{g} \, d^3x. \quad (70)$$

From Eq. (61) with $\boldsymbol{f}_{GR} = 0$, the Euler equation governing the equilibrium is

$$\nabla_i (U - \frac{1}{2}\varpi^2 \Omega^2) + \frac{1}{4\pi\rho} B^j (\nabla_i B_j - \nabla_j B_i) = 0, \quad (71)$$

where we have used the relation $(\partial_t + \pounds_{\boldsymbol{v}}) v_i = 0$.

B. Eulerian and Lagrangian perturbations

We denote by $Q(\alpha, t, x)$ a one-parameter family of stellar models. For each value of the parameter α, $Q(\alpha, t, x)$ satisfies the full nonlinear time-dependent Eqs. (57)–(61). The amplitude α is time independent and can be identified with the initial amplitude $\alpha(0)$ when we describe a growing mode by a time-dependent $\alpha(t)$.

The exact Eulerian perturbation δQ, defined as the difference between $Q(\alpha)$ and $Q(0)$, is defined everywhere on the intersection of the domains where $Q(\alpha)$ and $Q(0)$ are defined as

$$\delta Q(\alpha, t, x) := Q(\alpha, t, x) - Q(0, t, x) \quad (72a)$$
$$= \alpha \, \delta^{(1)} Q(t, x) + \alpha^2 \, \delta^{(2)} Q(t, x) + \mathcal{O}(\alpha^3), \quad (72b)$$

where the n-th order perturbation $\delta^{(n)} Q$ is

$$\delta^{(n)} Q(t, x) := \frac{1}{n!} \left. \frac{\partial^n Q(\alpha, t, x)}{\partial \alpha^n} \right|_{\alpha=0}. \quad (73)$$

Although the exact Eulerian perturbation has meaning only on the intersection of the support of the unperturbed and perturbed fluid, $\delta^{(n)} Q$ is well defined everywhere in the interior of the unperturbed star.

Exact Lagrangian perturbations can be defined by introducing a diffeomorphism χ_α that maps fluid elements in the equilibrium star $Q(0, t, x)$ to the corresponding elements in the solution $Q(\alpha, t, x)$. The exact Lagrangian change in a quantity Q is defined by,

$$\Delta Q(\alpha, t, x) := \chi_\alpha^* Q(\alpha, t, x) - Q(0, t, x) \quad (74)$$
$$= \alpha \Delta^{(1)} Q + \alpha^2 \Delta^{(2)} Q + \mathcal{O}(\alpha^3), \quad (75)$$

where χ_α^* is the pullback map (see Appendix A) and

$$\Delta^{(n)} Q(t, x) := \frac{1}{n!} \left. \frac{\partial^n \chi_\alpha^* Q(\alpha, t, x)}{\partial \alpha^n} \right|_{\alpha=0}. \quad (76)$$

We can write ΔQ in terms of a Lagrangian perturbation vector ξ^i in the manner

$$\Delta Q(\alpha, t, x) = \left(1 + \pounds_\xi + \frac{1}{2}\pounds_\xi^2\right) \left[Q(0, t, x) + \delta Q(\alpha, t, x)\right]$$
$$- Q(0, t, x) + \mathcal{O}(\alpha^3). \quad (77)$$

With

$$\xi^i = \alpha \xi^{(1)i} + \alpha^2 \xi^{(2)i} + \mathcal{O}(\alpha^3), \quad (78)$$

the first- and second-order Lagrangian perturbations are given by [see Appendix A 1, Eq. (A20)],

$$\Delta^{(1)} Q(t, x) = \left(\delta^{(1)} + \pounds_{\xi^{(1)}}\right) Q(0, t, x), \quad (79a)$$
$$\Delta^{(2)} Q(t, x) =$$
$$\left[\delta^{(2)} + \pounds_{\xi^{(2)}} + \pounds_{\xi^{(1)}} \delta^{(1)} + \frac{1}{2}\pounds_{\xi^{(1)}}^2\right] Q(0, t, x). \quad (79b)$$

The components of the vectors $\xi^{(1)i}$ and $\xi^{(2)i}$ are given in any coordinates by

$$\xi^{(1)i} = \left. \frac{\partial \chi_\alpha^i}{\partial \alpha} \right|_{\alpha=0}, \quad (80)$$
$$\xi^{(2)i} = \frac{1}{2} \left. \frac{\partial^2 \chi_\alpha^i}{\partial \alpha^2} \right|_{\alpha=0} - \frac{1}{2} \xi^{(1)j} \partial_j \xi^{(1)i}. \quad (81)$$

The commutator

$$\Delta(\partial_t + \pounds_{\boldsymbol{v}}) = (\partial_t + \pounds_{\boldsymbol{v}_0})\Delta$$

obtained as Eq. (A34a) of Appendix A, gives the perturbed mass-conservation equation (58) and induction equation (60) in the the forms

$$(\partial_t + \pounds_{\boldsymbol{v}_0})\Delta(\rho\sqrt{g}) = 0, \quad (82a)$$
$$(\partial_t + \pounds_{\boldsymbol{v}_0})\Delta(B^i\sqrt{g}) = 0, \quad (82b)$$

where v_0 is the unperturbed velocity field and Δ is the exact Lagrangian perturbation. These equations have first integrals

$$\Delta(\rho\sqrt{g}) = 0, \quad (83a)$$
$$\Delta(B^i\sqrt{g}) = 0, \quad (83b)$$

correct to all orders in α, implying

$$\Delta \frac{\boldsymbol{B}}{\rho} = 0. \quad (84)$$

The first- and second-order Lagrangian perturbations of g_{ij} and $\sqrt{g}$ are given by

$$\Delta^{(1)} g_{ij} = 2\nabla_{(i}\xi^{(1)}{}_{j)}, \quad (85a)$$
$$\Delta^{(2)} g_{ij} = 2\nabla_{(i}\xi^{(2)}{}_{j)} + \xi^{(1)k}\nabla_k\nabla_{(i}\xi^{(1)}{}_{j)}$$
$$+ \nabla_i\xi^{(1)k}\nabla_j\xi^{(1)}{}_k + \nabla_k\xi^{(1)}{}_{(i}\nabla_{j)}\xi^{(1)k}, \quad (85b)$$

$$\frac{1}{\sqrt{g}}\Delta^{(1)}\sqrt{g} = \boldsymbol{\nabla}\cdot\boldsymbol{\xi}^{(1)}\,, \tag{86a}$$

$$\frac{1}{\sqrt{g}}\Delta^{(2)}\sqrt{g} =$$
$$\boldsymbol{\nabla}\cdot\boldsymbol{\xi}^{(2)} + \frac{1}{2}(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}^{(1)})^2 + \frac{1}{2}\boldsymbol{\xi^{(1)}}\cdot\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{\xi}^{(1)}\,, \tag{86b}$$

and the corresponding perturbations of ρ and p are

$$\frac{\Delta^{(1)}\rho}{\rho} = -\boldsymbol{\nabla}\cdot\boldsymbol{\xi}^{(1)}, \tag{87a}$$

$$\frac{\Delta^{(2)}\rho}{\rho} = -\boldsymbol{\nabla}\cdot\boldsymbol{\xi}^{(2)}$$
$$+ \frac{1}{2}(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}^{(1)})^2 - \frac{1}{2}\boldsymbol{\xi^{(1)}}\cdot\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{\xi}^{(1)}\,, \tag{87b}$$

$$\frac{\Delta^{(1)}p}{\gamma p} = -\boldsymbol{\nabla}\cdot\boldsymbol{\xi}^{(1)}\,, \tag{88a}$$

$$\frac{\Delta^{(2)}p}{\gamma p} = -\boldsymbol{\nabla}\cdot\boldsymbol{\xi}^{(2)} + \frac{1}{2}\left(\gamma + \frac{\partial\log\gamma}{\partial\log\rho}\right)(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}^{(1)})^2$$
$$-\frac{1}{2}\boldsymbol{\xi^{(1)}}\cdot\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{\xi}^{(1)}\,, \tag{88b}$$

with $\gamma = d\log p/d\log\rho$ the adiabatic index.

The first- and second-order Lagrangian perturbations of the covariant and contravariant forms of the magnetic field are then

$$\Delta^{(1)}B^i = -B^i\nabla_j\xi^{(1)j}\,, \tag{89a}$$

$$\Delta^{(2)}B^i = -B^i\nabla_j\xi^{(2)j}$$
$$+ B^i\left[\frac{1}{2}(\nabla_j\xi^{(1)j})^2 - \frac{1}{2}\xi^{(1)k}\nabla_k\nabla_j\xi^{(1)j}\right]\,, \tag{89b}$$

and

$$\Delta^{(1)}B_i = B^j[2\nabla_{(i}\xi^{(1)}{}_{j)} - g_{ij}\nabla_k\xi^{(1)k}]\,, \tag{90a}$$

$$\Delta^{(2)}B_i = B^j[2\nabla_{(i}\xi^{(2)}{}_{j)} - g_{ij}\nabla_k\xi^{(2)k}]$$
$$+ B^j\left[\xi^{(1)k}\nabla_k\left(\nabla_{(i}\xi^{(1)}{}_{j)} - \frac{1}{2}g_{ij}\nabla_l\xi^{(1)l}\right)\right.$$
$$+ \nabla_i\xi^{(1)k}\nabla_j\xi^{(1)}{}_k + \nabla_k\xi^{(1)}{}_{(i}\nabla_{j)}\xi^{(1)k}$$
$$\left. - 2\nabla_{(i}\xi^{(1)}{}_{j)}\nabla_k\xi^{(1)k} + \frac{1}{2}g_{ij}\left(\nabla_k\xi^{(1)k}\right)^2\right]\,. \tag{90b}$$

Finally, the expressions for the Lagrangian changes in the contravariant and covariant velocity are (see Appendix A 2)

$$\Delta^{(1)}v^i = \partial_t\xi^{(1)i}\,, \tag{91a}$$

$$\Delta^{(2)}v^i = \partial_t\xi^{(2)i} + \tfrac{1}{2}\pounds_{\xi^{(1)}}\partial_t\xi^{(1)i}\,, \tag{91b}$$

implying

$$\Delta^{(1)}v_i = \partial_t\xi^{(1)}{}_i + 2\nabla_{(i}\xi^{(1)}{}_{j)}v^j\,, \tag{92a}$$

$$\Delta^{(2)}v_i = \partial_t\xi^{(2)}{}_i + 2\nabla_{(i}\xi^{(2)}{}_{j)}v^j + \partial_t\xi^{(1)j}\nabla_i\xi^{(1)}{}_j$$
$$+ \tfrac{1}{2}\partial_t(\xi^{(1)j}\nabla_j\xi^{(1)}{}_i) + (\xi^{(1)k}\nabla_k\nabla_{(i}\xi^{(1)}{}_{j)})$$
$$+ \nabla_k\xi^{(1)}{}_{(i}\nabla_{j)}\xi^{(1)k} + \nabla_i\xi^{(1)}{}_k\nabla_j\xi^{(1)k}v^j\,. \tag{92b}$$

C.   First-order perturbation equations

We now consider perturbations of the MHD-Euler system, at first order in the amplitude α. We use the formalism of Friedman and Schutz [28] and its extension to the MHD-Euler system by Glampedakis and Andersson [40]. To write the perturbed MHD-Euler equation (61),

$$\rho\Delta^{(1)}\mathcal{E}_i := \rho\Delta^{(1)}\left[(\partial_t + v^j\nabla_j)v_i + \frac{\nabla_i p}{\rho} + \nabla_i\Phi\right.$$
$$\left. + \frac{1}{4\pi\rho}B^j(\nabla_iB_j - \nabla_jB_i)\right] = \rho\delta^{(1)}f_{GR\,i}\,, \tag{93}$$

in terms of the Lagrangian displacement $\boldsymbol{\xi}^{(1)}$, we use the first-order part of Eq. (84),

$$\Delta^{(1)}\frac{B^i}{\rho} = 0\,, \tag{94}$$

and obtain for the term involving the perturbed Lorentz force the form

$$\rho\Delta^{(1)}\left[\frac{1}{4\pi\rho}B^j(\nabla_iB_j - \nabla_jB_i)\right]$$
$$= \frac{1}{4\pi}B^j(\nabla_i\Delta^{(1)}B_j - \nabla_j\Delta^{(1)}B_i)$$
$$= \frac{1}{2\pi}B^j\left[\nabla_i(B^k\nabla_{(k}\xi^{(1)}{}_{j)}) - \nabla_j(B^k\nabla_{(k}\xi^{(1)}{}_{i)})\right.$$
$$\left. - \nabla_{[i}(B_{j]}\nabla_k\xi^{(1)k})\right]\,, \tag{95}$$

where we have used Eq. (90a) and the fact that Lie and exterior derivatives commute.

The perturbed MHD-Euler equation (93) has the form

$$A_{ij}\partial_t^2\xi^{(1)j} + B_{ij}\partial_t\xi^{(1)j} + C_{ij}\xi^{(1)j} = \rho\delta^{(1)}f_{GR\,i}\,, \tag{96}$$

where

$$A_{ij} := \rho g_{ij}\,, \tag{97a}$$

$$B_{ij} := 2\rho g_{ij}v^k\nabla_k\,, \tag{97b}$$

$$C_{ij}\xi^j := \rho(v^j\nabla_j)^2\xi_i - \nabla_i(\gamma p\nabla_j\xi^j) + \nabla_i p\nabla_j\xi^j$$
$$- \nabla_j p\nabla_i\xi^j + \rho\xi^j\nabla_j\nabla_i\Phi + \rho\nabla_i\delta^{(1)}\Phi$$
$$+ \frac{1}{2\pi}B^j\left[\nabla_i(B^k\nabla_{(k}\xi_{j)}) - \nabla_j(B^k\nabla_{(k}\xi_{i)})\right.$$
$$\left. - \nabla_{[i}(B_{j]}\nabla_k\xi^{(1)k}) - \nabla_i\xi^{(1)k}\nabla_{[k}B_{j]}\right]\,. \tag{97c}$$

Here $\delta^{(1)}\Phi$ is the asymptotically vanishing solution to the perturbed Poisson equation

$$\nabla^2 \delta^{(1)}\Phi = 4\pi \delta^{(1)}\rho = -4\pi \boldsymbol{\nabla} \cdot (\rho \boldsymbol{\xi}^{(1)}) \,.$$

For vectors ξ^i and η^i that vanish at the boundary of the star, the operators A_{ij} and C_{ij} are self-adjoint in the sense

$$\int dV \eta^i C_{ij} \xi^j = \int dV \xi^i C_{ij} \eta^j \,, \tag{98}$$

and B_{ij} is anti-self-adjoint.

The exact perturbed gravitational radiation-reaction force $\delta \boldsymbol{f}_{GR}$ is given by [14]

$$\delta \boldsymbol{f}_{GR} = \sum_{l \geq 2} \sum_{|m| \leq l} \frac{(-1)^{\ell+1} N_\ell}{32\pi} \Re \left\{ \frac{\boldsymbol{\nabla}(r^\ell Y^{\ell m})}{\sqrt{\ell}} \frac{d^{2\ell+1}\delta I^{\ell m}}{dt^{2\ell+1}} \right.$$
$$- \frac{2r^\ell \boldsymbol{Y}_B^{\ell m}}{\sqrt{\ell+1}} \frac{d^{2\ell+2}\delta S^{\ell m}}{dt^{2\ell+2}} - \frac{2\Omega \boldsymbol{\phi} \times \boldsymbol{\nabla}(r^\ell Y^{\ell m})}{\sqrt{\ell}} \frac{d^{2\ell+1}\delta S^{\ell m}}{dt^{2\ell+1}}$$
$$\left. - \frac{2\delta \boldsymbol{v} \times \boldsymbol{\nabla}(r^\ell Y^{\ell m})}{\sqrt{\ell}} \frac{d^{2\ell+1}\delta S^{\ell m}}{dt^{2\ell+1}} \right\} \,, \tag{99}$$

where

$$\delta I^{\ell m} := \frac{N_\ell}{\sqrt{\ell}} \int \delta \rho \, r^\ell Y^{*\ell m} d^3 x \,, \tag{100a}$$

$$\delta S^{\ell m} := \frac{2 N_\ell}{\sqrt{\ell+1}} \int r^\ell \left[\rho \, \delta \boldsymbol{v} + \delta \rho \, (\Omega \boldsymbol{\phi} + \delta \boldsymbol{v}) \right] \cdot \boldsymbol{Y}_B^{*\ell m} d^3 x \,. \tag{100b}$$

D. Second-order axisymmetric perturbation

The second-order perturbation of the MHD-Euler equation (61) has the form

$$\Delta^{(2)} \mathcal{E}_i = (\partial_t + \pounds_{\boldsymbol{v}}) \Delta^{(2)} v_i + \nabla_i \Delta^{(2)} (U - \tfrac{1}{2}v^2)$$
$$+ \frac{1}{2\pi\rho} B^j (\nabla_{[i}\Delta^{(2)}B_{j]}) = \Delta^{(2)} f_{GR\,i} \,. \tag{101}$$

Here we have again used the commutation relation (A34a) together with the commutator (also derived in Appendix A)

$$\Delta d = d\Delta \,, \tag{102}$$

where d is the exterior derivative operator.

Equations (87-90) display the second-order perturbation of each variable as a sum of two parts. One part is linear in the second-order Lagrangian displacement $\xi^{(2)i}$, while the second part is quadratic in the first-order displacement $\xi^{(1)i}$. Each quantity is a sum of these two types of terms:

$$\Delta^{(2)} Q = \Delta_{\text{lin}}^{(2)} Q + \Delta_{\text{quad}}^{(2)} Q \,. \tag{103}$$

The linear part, $\Delta_{\text{lin}}^{(2)} Q$ is the linear perturbation of Q associated with the displacement $\xi^{(2)i}$: That is, $\Delta_{\text{lin}}^{(2)} Q$ is identical to $\Delta^{(1)} Q$ if one replaces $\xi^{(1)i}$ by $\xi^{(2)i}$. This is essentially the statement that, in the Taylor expansion of a function F of ξ^i,

$$F(\alpha \xi^{(1)} + \alpha^2 \xi^{(2)}) = F(0) + \frac{\partial F}{\partial \xi^j}\bigg|_{\xi=0} (\alpha \xi^{(1)j} + \alpha^2 \xi^{(2)j})$$
$$+ \frac{1}{2}\alpha^2 \frac{\partial^2 F}{\partial \xi^j \partial \xi^k}\bigg|_{\xi=0} \xi^{(1)j} \xi^{(1)k} + O(\alpha^3) \,,$$

$\alpha \xi^{(1)i}$ and $\alpha^2 \xi^{(2)i}$ have the same coefficient, namely the first derivative of F.

It follows that the second-order perturbation of the MHD-Euler equation is again the sum of a part linear in the second-order Lagrangian displacement $\xi^{(2)i}$ and a part quadratic in the first-order displacement $\xi^{(1)i}$; similarly, $\Delta_{\text{lin}}^{(2)} \mathcal{E}_i$ is the linear perturbation $\Delta^{(1)} \mathcal{E}_i$ of Eq. (96), with $\xi^{(1)i}$ replaced by $\xi^{(2)i}$. Including the second-order radiation-reaction term, the second-order equation (101) thus has the form

$$\rho \Delta^{(2)} \mathcal{E}_i = A_{ij} \partial_t^2 \xi^{(2)j} + B_{ij} \partial_t \xi^{(2)j} + C_{ij} \xi^{(2)j}$$
$$+ D_i(\xi^{(1)}, \xi^{(1)}) = \rho \Delta^{(2)} f_{GR\,i} \,, \tag{104}$$

where the operators A_{ij}, B_{ij}, and C_{ij} are given by Eqs. (97) and the quadratic operator D_i has the form

$$\rho^{-1} D_i(\xi^{(1)}, \xi^{(1)}) = (\partial_t + \pounds_{\boldsymbol{v}}) \Delta_{\text{quad}}^{(2)} v_i$$
$$+ \nabla_i \Delta_{\text{quad}}^{(2)} \left(h + \Phi - \frac{1}{2}v^2 \right) - \Delta_{\text{quad}}^{(2)} f_{m\,i} \,. \tag{105}$$

Here, with $\Delta_{\text{quad}}^{(2)} B_i$ and $\Delta_{\text{quad}}^{(2)} v_i$ displayed in Eqs. (90b) and (92b), we obtain

$$\Delta_{\text{quad}}^{(2)} h = \frac{1}{2}\frac{\gamma p}{\rho} \left[\left(\gamma - 1 + \frac{\partial \log \gamma}{\partial \log \rho} \right) (\boldsymbol{\nabla} \cdot \boldsymbol{\xi}^{(1)})^2 \right.$$
$$\left. - \boldsymbol{\xi}^{(1)} \cdot \boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \boldsymbol{\xi}^{(1)} \right] \,, \tag{106a}$$

$$\Delta_{\text{quad}}^{(2)} \Phi = \delta_{\text{quad}}^{(2)} \Phi + \boldsymbol{\xi}^{(1)} \cdot \boldsymbol{\nabla}\delta^{(1)}\Phi$$
$$+ \frac{1}{2}\boldsymbol{\xi}^{(1)} \cdot \boldsymbol{\nabla}(\boldsymbol{\xi}^{(1)} \cdot \boldsymbol{\nabla}\Phi) \,, \tag{106b}$$

$$\Delta_{\text{quad}}^{(2)} (\tfrac{1}{2}v^2) = \frac{1}{2}\left\{ \partial_t \xi^{(1)i} \partial_t \xi^{(1)}{}_i + v^i 2 \partial_t \xi^{(1)j} \nabla_i \xi^{(1)}{}_j \right.$$
$$+ v^i \partial_t (\xi^{(1)j} \nabla_j \xi^{(1)i}) + v^i v^j \left(\xi^{(1)k} \nabla_k \nabla_i \xi^{(1)}{}_j \right.$$
$$\left. \left. + \nabla_j \xi^{(1)k} \nabla_k \xi^{(1)}{}_i + \nabla_i \xi^{(1)k} \nabla_j \xi^{(1)}{}_k \right) \right\} \,, \tag{106c}$$

$$\Delta_{\text{quad}}^{(2)} f_{m\,i} = \frac{1}{2\pi\rho} B^j \nabla_{[j} \Delta_{\text{quad}}^{(2)} B_{i]} \,. \tag{106d}$$

In Eq. (106b) $\delta^{(1)}\Phi$ and $\delta_{\text{quad}}^{(2)}\Phi$ are the potentials asso-

ciated with $\delta^{(1)}\rho$ and with $\delta^{(2)}_{\text{quad}}\rho$:

$$\nabla^2\delta^{(1)}\Phi = 4\pi\delta^{(1)}\rho = -4\pi\nabla_i(\rho\xi^{(1)i}), \tag{107a}$$

$$\nabla^2\delta^{(2)}_{\text{quad}}\Phi = 4\pi\delta^{(2)}_{\text{quad}}\rho, \tag{107b}$$

$$\delta^{(2)}_{\text{quad}}\rho = \frac{1}{2}\rho\left[(\nabla_i\xi^{(1)i})^2 + \xi^{(1)i}\nabla_i\nabla_j\xi^{(1)j}\right]$$
$$+\xi^{(1)i}\nabla_i\rho\nabla_j\xi^{(1)j} + \frac{1}{2}\xi^{(1)i}\nabla_i(\xi^{(1)j}\nabla_j\rho), \tag{107c}$$

where the last expression is obtained from Eqs. (79b) and (87).

We now restrict consideration to an axisymmetric background star. Because the components of $\xi^{(1)i}$ have time dependence $\cos(m\phi + \omega t)e^{\beta t}$ and $\sin(m\phi + \omega t)e^{\beta t}$ (see [14] and Eq. (145)), the quadratic combination $D^i(\xi^{(1)}, \xi^{(1)})$ is a sum of terms of three kinds: terms with angular and temporal dependence $\cos[2(m\phi + \omega t)]e^{2\beta t}$, terms with dependence $\sin[2(m\phi + \omega t)]e^{2\beta t}$, and terms independent of ϕ, with time dependence $e^{2\beta t}$.

With the term $D_i(\xi^{(1)}, \xi^{(1)})$ moved to its right side, Eq. (104) has the form

$$A_{ij}\partial_t^2\xi^{(2)j} + B_{ij}\partial_t\xi^{(2)j} + C_{ij}\xi^{(2)j} = \Delta^{(2)}F_i, \tag{108}$$

where

$$\Delta^{(2)}F_i = \rho\Delta^{(2)}f_{GRi} - D_i(\xi^{(1)}, \xi^{(1)}). \tag{109}$$

Recalling that we use brackets $\langle\cdot\rangle$ to denote the axisymmetric part of a perturbation, we can write the axisymmetric part of the second-order MHD-Euler equation as

$$\langle\rho\Delta^{(2)}\mathcal{E}_i\rangle = A_{ij}\partial_t^2\langle\xi^{(2)j}\rangle + B_{ij}\partial_t\langle\xi^{(2)j}\rangle + C_{ij}\langle\xi^{(2)j}\rangle$$
$$= \langle\Delta^{(2)}F_i\rangle. \tag{110}$$

Axisymmetry of the background star implies axisymmetry of the operators A_{ij}, B_{ij}, and C_{ij}, allowing us to move the operators outside the brackets. Acting on axisymmetric perturbations, the operator B_{ij} has the form

$$B_{ij} = -2\rho\epsilon_{ijk}\Omega^k, \tag{111}$$

where Ω^i is the angular velocity vector. With the first-order perturbation $\xi^{(1)i}$ known, Eq. (110) is the equation for an axisymmetric linear perturbation of the star with a forcing term

$$\langle\Delta^{(2)}F_i\rangle := \rho\langle\Delta^{(2)}f_{GRi}\rangle - \langle D_i(\xi^{(1)}, \xi^{(1)})\rangle. \tag{112}$$

At second order in the perturbation, the star loses angular momentum to gravitational waves. We can decompose the second-order axisymmetric perturbation into two parts, one representing the spin down of the star, the other conserving total angular momentum. The first part, $\delta^{(2)}_{UR}Q$, is a perturbation that adds *uniform rotation* $\delta^{(2)}_{UR}\Omega < 0$ to the star and has total (negative) angular

momentum equal to the angular momentum lost in gravitational waves; the second part, $\delta^{(2)}_{DR}Q$, is the remaining, angular-momentum conserving part of the second-order axisymmetric perturbation that describes the addition of *differential rotation* with zero total angular momentum. We write the corresponding decomposition of the Lagrangian displacement in the form

$$\langle\xi^{(2)i}\rangle = \xi^{(2)i}_{UR} + \xi^{(2)i}_{DR}. \tag{113}$$

Finally, we can decompose the effective driving force $\langle\Delta^{(2)}F_i\rangle$ into an angular-momentum reducing part that drives the change in uniform rotation and an angular-momentum conserving part.

$$\langle\Delta^{(2)}F_i\rangle = \Delta^{(2)}_{UR}F_i + \Delta^{(2)}_{DR}F_i, \tag{114}$$

where

$$\Delta^{(2)}_{UR}F_i := (A_{ij}\partial_t^2 + B_{ij}\partial_t + C_{ij})\xi^{(2)b}_{UR}. \tag{115}$$

E. Symplectic product and the growth of driven modes.

We need an equation for the growth of the displacement $\xi^{(2)i}$ with a driving force and a background magnetic field. The simplicity of the toy model comes from fact that Eq. (32) governing the homogeneous solutions has the form

$$\partial_t^2\xi^\phi + C\xi^\phi = 0, \tag{116}$$

with the operator C self-adjoint. This allows one to write the solution to the inhomogeneous equation as a sum (38) of orthogonal eigenfunctions of the operator C; and in the exponentially growing solution, the coefficient of each eigenfunction of C is proportional to the inner product of f with the normalized eigenfunction. In contrast, the dynamical equation (108) governing the r-mode includes a first time derivative term with an operator $B^i{}_j$ that is anti-self-adjoint and does not commute with the self-adjoint operator $C^i{}_j$. If that first-time derivative were not present, solutions to the homogeneous equation could again be written as a superposition of eigenfunctions of $C^i{}_j$ and eigenfunctions ξ^i_n and ξ'^i_n with distinct eigenvalues would be orthogonal with respect to the inner product $\int dV\xi^*_i A\eta^i = \int dV\rho\xi^*_i \eta^i$. The presence of the first-time derivative term means that solutions to the homogeneous equation,

$$(A_{ij}\partial_t^2 + B_{ij}\partial_t + C_{ij})\xi^j = 0, \tag{117}$$

are not orthogonal in this sense. There is nevertheless a conserved *symplectic* product with respect to which modes of the homogeneous equation with distinct eigenvalues are orthogonal. We summarize the results here and relegate to Appendix B a detailed derivation based on Refs. [27, 28] and a summary by Schenk *et al.* [41].

Following Friedman and Schutz [28], we define the symplectic product of two complex solutions to the homogeneous equation $A\partial_t^2\xi + B\partial_t\xi + C\xi = 0$ by

$$W(\xi, \tilde{\xi}) := \langle \xi | \tilde{\pi} \rangle - \langle \pi | \tilde{\xi} \rangle, \qquad (118)$$

where π_i is the momentum conjugate to ξ^i,

$$\pi_i = \rho\partial_t\xi_i + \frac{1}{2}B_{ij}\xi^j, \qquad (119)$$

and $\langle\ |\ \rangle$ is the usual inner product

$$\langle \xi | \eta \rangle = \int dV \xi_i^* \ \eta^i. \qquad (120)$$

We use boldface angle brackets to distinguish the symbol for inner product from the ordinary typeface brackets in the expression $\langle Q \rangle$ for the axisymmetric part of Q.

We will restrict consideration to perturbations that conserve total angular momentum, mass and entropy; in particular, we use only the part $\langle \Delta_{DR}^{(2)} F_i \rangle$ of the driving force in the decomposition (114) because the addition of uniform rotation does not enhance the magnetic field. We also assume that the linear axisymmetric modes of the axisymmetric background star with magnetic field are stable, discrete and non-degenerate. Because the operators A, B and C are real, if ξ satisfies the homogeneous equation so does ξ^*. For a stable system with a complete set of discrete normal modes, the modes therefore come in pairs

$$\xi_n(t,x) = \tilde{\xi}_n(x)e^{i\omega_n t}, \qquad \xi_n^* = \tilde{\xi}_n^*(x)e^{-i\omega_n t}, \qquad (121)$$

and we will write frequencies as $\pm\omega_n$, with $\omega_n > 0$. Because we are assuming a stable Newtonian system, the frequencies are real. The fact that W is conserved implies that modes with different frequencies are symplectically orthogonal:

$$W(\xi_n, \xi_{n'}) = 0, \ \omega_n \neq \omega_{n'}, \qquad W(\xi_n^*, \xi_n) = 0. \qquad (122)$$

The proof is immediate: If $W(\xi_n, \xi_{n'})$ does not vanish, it has time-dependence $e^{i(\omega_{n'} - \omega_n)t}$, contradicting $dW/dt = 0$.

With our assumption that the spectrum has no continuous part, work by Dyson and Schutz [27], using symplectic orthogonality, shows that the modes are complete. For a driving term of the form

$$F_i(t,x) = \hat{F}_i(x)e^{2\beta t}, \qquad (123)$$

their work implies [see our Appendix B, Eq. (B25)] that the exponentially growing solution to the inhomogeneous equation,

$$(A_{ij}\partial_t^2 + B_{ij}\partial_t + C_{ij})\xi^j = F_i(t,x), \qquad (124)$$

is

$$\xi^i = \sum_n \Re \left[\frac{1}{i\kappa_n\omega_n(2\beta - i\omega_n)}\langle \hat{\xi}_n | F \rangle \hat{\xi}_n^i \right], \qquad (125)$$

where the modes $\hat{\xi}_n$ are normalized by

$$\langle \hat{\xi}_n | \rho \hat{\xi}_n \rangle = 1, \qquad (126)$$

and

$$\kappa_n := 1 - 2\frac{\Omega}{\omega_n} \ \Im \frac{\int dV \rho \tilde{\xi}_n^{\varpi *} \tilde{\xi}_n^{\phi}}{\int dV \rho |\tilde{\xi}_n|^2}. \qquad (127)$$

We have adopted the convention $\omega_n > 0$; taking the real part of the bracketed expression in Eq. (125) accounts for modes with frequency $-\omega_n$. After saturation, the driving force is constant, and the displacement oscillates about a constant equilibrium value given by Eq. (125) with $\beta = 0$ (Eq. (B26) of Appendix B),

$$\xi^i = \sum_n \Re \left[\frac{1}{\kappa_n\omega_n^2}\langle \hat{\xi}_n | F \rangle \hat{\xi}_n^i \right], \qquad (128)$$

where F is the value of the driving force at saturation.

Note that the canonical energy of the nth normalized mode is [28]

$$\begin{aligned} E_{cn} &= \frac{1}{2}W(\partial_t\hat{\xi}_n, \hat{\xi}_n) \\ &= -\frac{1}{2}i\omega_n W(\hat{\xi}_n, \hat{\xi}_n) = \omega_n^2\kappa_n\langle \hat{\xi}_n | \rho \hat{\xi}_n \rangle. \quad (129) \end{aligned}$$

If the unperturbed star is strictly stable against axisymmetric perturbations (having neither unstable nor zero-frequency axisymmetric perturbations that conserve angular momentum, baryon mass, and entropy), then $E_{cn} > 0$, implying $\kappa_n > 0$.

Finally, we break $\hat{\xi}_n^i$ into its real and imaginary parts,

$$\hat{\xi}_n^i = \hat{\xi}_{nR}^i + i\hat{\xi}_{nI}^i, \qquad (130)$$

to elucidate the dependence of different contributions to the sum on β, ω_n and Ω. A short calculation, beginning with the right side of Eq. (125) gives

$$\begin{aligned} \xi^i = \sum_n \frac{1}{\kappa_n(4\beta^2 + \omega_n^2)} &\left[\langle \hat{\xi}_{nR} | F \rangle \hat{\xi}_{nR}^i + \langle \hat{\xi}_{nI} | F \rangle \hat{\xi}_{nI}^i \right. \\ &\left. + \frac{2\beta}{\omega_n}\left(\langle \hat{\xi}_{nI} | F \rangle \hat{\xi}_{nR}^i - \langle \hat{\xi}_{nR} | F \rangle \hat{\xi}_{nI}^i \right) \right]. \quad (131) \end{aligned}$$

After saturation, Eq. (128) gives the equilibrium value

$$\xi^i = \sum_n \frac{1}{\kappa_n\omega_n^2}\left[\langle \hat{\xi}_{nR} | F \rangle \hat{\xi}_{nR}^i + \langle \hat{\xi}_{nI} | F \rangle \hat{\xi}_{nI}^i \right]. \qquad (132)$$

V. GROWTH OF DIFFERENTIAL ROTATION AND MAGNETIC FIELD WINDUP

To estimate the growth of differential rotation of an unstable r-mode, we use Eq. (131) to write the solution

$\langle \xi^{(2)\phi} \rangle$ to Eq. (110) at saturation in the form,

$$\sum_n \frac{1}{\kappa_n(4\beta^2 + \omega_n^2)} \times$$
$$\left[\langle \hat{\xi}_{nR} | \langle \Delta^{(2)} F \rangle \rangle \hat{\xi}_{nR}^\phi + \langle \hat{\xi}_{nI} | \langle \Delta^{(2)} F \rangle \rangle \hat{\xi}_{nI}^\phi \right.$$
$$\left. + \frac{2\beta}{\omega_n} \left(\langle \hat{\xi}_{nR} | \langle \Delta^{(2)} F \rangle \rangle \hat{\xi}_{nI}^\phi - \langle \hat{\xi}_{nI} | \langle \Delta^{(2)} F \rangle \rangle \hat{\xi}_{nR}^\phi \right) \right] ; \tag{133}$$

after saturation, we use Eq. (132) to write the equilibrium value of $\langle \xi^{(2)\phi} \rangle$ in the form

$$\sum_n \frac{1}{\kappa_n \omega_n^2} \left[\langle \hat{\xi}_{nR} | \langle \Delta^{(2)} F \rangle \rangle \hat{\xi}_{nR}^\phi + \langle \hat{\xi}_{nI} | \langle \Delta^{(2)} F \rangle \rangle \hat{\xi}_{nI}^\phi \right]. \tag{134}$$

We estimate the value of the inner product $\langle \hat{\xi}_n | \Delta^{(2)} F \rangle$ for modes ξ_n whose $B = 0$ limits are zero-frequency axisymmetric perturbations associated with differential rotation. Primary differences between Eq. (133) for the Lagrangian perturbation of the stellar model and Eq. (38) for the toy model are (1) the effective driving force $\Delta^{(2)} F$ includes the nonlinear terms D_i as well as the radiation-reaction force, and (2) the coefficient of the mode-expression for the Lagrangian displacement has the factor $1/\kappa_n$.

Although Eq. (133) involves a sum over all axisymmetric modes, modes with wavelengths much smaller than R should give negligible contributions, because the characteristic length of $\rho\langle \Delta^{(2)} F \rangle$ is of order R for the $\ell = m = 2$ r-mode. (For smooth vector fields f and g, the inner product $\langle g|f \rangle$ falls off exponentially as the wavelength of the Fourier components of g approach zero.) Of the axisymmetric modes with wavelengths of order R, the Alfvén modes have the lowest frequencies, with magnitudes for normal and superconducting interiors given by Eqs. (42) and (154). In particular, normal-fluid g-modes have frequencies of order the Brunt-Väisälä frequency of about 150 Hz (see, e.g.,[42, 43]), and a class of superfluid g-modes have higher frequency [44, 45]); inertial modes have frequencies of order Ω, and the frequencies of p- and f-modes are much higher. Because the coefficient of the mode sum is proportional to ω^{-2} for $\omega \gg \beta$, we assume that the estimate is dominated by modes with frequencies of order ω_A.

We will find that the inner products of these axisymmetric Alfvén-frequency modes with the two terms, $\rho\Delta^{(2)} f_{GR\,i}$ and $-D_i(\xi^{(1)}, \xi^{(1)})$ that comprise $\Delta^{(2)} F_i$ are of order

$$\langle \hat{\xi}_n | \rho | \Delta^{(2)} f_{GR} | \rangle \hat{\xi}_n^\phi \sim \beta \Omega e^{2\beta t}, \qquad \langle \hat{\xi}_n | D \rangle \hat{\xi}_n^\phi \sim \omega_A \Omega e^{2\beta t}; \tag{135}$$

As in the toy model, we set an upper limit on the maximum angular displacement by adopting a driving force whose growth stops instantaneously at $t = t_{\text{sat}}$. Setting $\beta = 0$ and $t = t_{\text{sat}}$ in Eq. (135) gives the equilibrium values reached after saturation.

The estimates (135) then imply (for $\beta, \omega_A \ll \Omega$) a maximum value of the angular displacement at saturation given by

$$\xi_{\text{sat}}^\phi = \alpha_{\text{sat}}^2 \xi^{(2)\phi} \sim \alpha_{\text{sat}}^2 \frac{\max(\omega_A, \beta)\,\Omega}{4\beta^2 + \omega_A^2} \tag{136}$$

and a maximum value after saturation

$$\xi_{\text{max}}^\phi \sim \alpha_{\text{sat}}^2 \frac{\max(\omega_A, \beta)\,\Omega}{\omega_A^2} . \tag{137}$$

As in the toy model, a larger post-saturation value of the displacement that arises when $\omega_A < \beta$ is mitigated by a larger critical magnetic field needed to alter the linear r-mode: That is after saturation, the critical magnetic field is given by Eq. (50) instead of Eq. (12).

We first outline the main ingredients that enter the estimates (136) and (137), then show how they are obtained. We assume the linear r-mode grows exponentially until a time t_{sat} and subsequently has constant amplitude.

- Prior to and at saturation, the radiation-reaction force per unit mass, $\Delta^{(2)} f_{GR\,i}$ is of order

$$|\Delta^{(2)} \boldsymbol{f}_{GR}| \sim \beta\,\Omega\,R\,e^{2\beta t_{\text{sat}}} . \tag{138}$$

This immediately gives the first estimate in Eq. (135).

- With no magnetic field the quadratic contribution $D_i(\xi^{(1)}{}_N, \xi^{(1)}{}_N)$ from the linear Newtonian r-mode ξ_N^i has no ϕ-component. With a generic magnetic field, $\langle D_{\hat\phi} \rangle$ is small compared to $\langle D_\varpi \rangle$ and $\langle D_z \rangle$:

$$\rho^{-1}\,|\langle D_\varpi \rangle| \sim \rho^{-1}\,|\langle D_z \rangle| \sim \Omega^2\,R\,e^{2\beta t_{\text{sat}}},$$
$$\rho^{-1}\left|\langle D_{\hat\phi} \rangle\right| \sim \max(\omega_A^2, \beta\,\Omega)\,R\,e^{2\beta t_{\text{sat}}} . \tag{139}$$

- For the first-order axisymmetric modes ξ_n^i associated with differential rotation, the part of ξ_n^i orthogonal to ϕ^i is small compared to $\xi_n^{\hat\phi}$:

$$|\xi_n^\varpi|, |\xi_n^z| \sim \frac{\omega_A}{\Omega}|\xi_n^{\hat\phi}| . \tag{140}$$

This comes from the fact that, with no magnetic field, a perturbation associated with adding differential rotation has the form $\Delta^{(1)} v^i = \partial_t \xi^{(1)\,i}$, along ϕ^i; Eq. (140) estimates the nonzero values of the components of $\boldsymbol{\xi}$ orthogonal to $\boldsymbol{\phi}$ for a magnetic field with $\omega_A \ll \Omega$.

- A consequence of the relations (140) is that the ratio of integrals that appears in the definition (127) of κ_n has an upper bound of order

$$\frac{\left| \int dV \rho \widetilde{\xi}_n^{\varpi *} \widetilde{\xi}_n^{\hat\phi} \right|}{\int dV \rho |\widetilde{\xi}_n|^2} \lesssim \frac{\omega_A}{\Omega} ,$$

and this in turn gives an upper bound of order unity on κ_n,

$$|\kappa_n| \lesssim 1. \tag{141}$$

Estimates (139) and (140) imply that the quantity $\langle \widehat{\xi}_n | D \rangle \langle \xi_n^{\hat{\phi}} \rangle$ has an upper bound of order $\omega_A \Omega$, giving the second estimate in Eq. (135). Finally, using the estimate (141) for κ_n, we obtain our main result, Eq. (136).

To obtain the estimates (138) and (139) for the two contributions to the effective driving force $\Delta^{(2)} F = |\Delta^{(2)} \boldsymbol{F}|$, we will use the slow-rotation forms of the radiation reaction force and the first-order Lagrangian displacement. Corrections are of order Ω/Ω_0, where $\Omega_0 = \sqrt{M/R^3}$. We use the slow rotation forms not because the corrections are negligible – for nascent stars with angular velocities near the Keplerian (mass-shedding) limit Ω_K, they could change the quantities we consider by factors of order unity – but because these corrections do not alter our order-of-magnitude estimates. We also neglect corrections to the linear r-mode and radiation-reaction force due to the background magnetic field; here the corrections *are* negligible for fields weaker than $10^{14} - 10^{15}$ G [19, 40, 46–52].

We consider first the second-order radiation-reaction force, $\langle \Delta^{(2)} f_{GR}^i \rangle$. Because the radiation-reaction force vanishes for the background star, Eq. (79b) gives as its second-order Lagrangian change

$$\Delta^{(2)} f_{GR}^i = \delta^{(2)} f_{GR}^i + \pounds_{\boldsymbol{\xi}^{(1)}} \delta^{(1)} f_{GR}^i. \tag{142}$$

For the $\ell = m$ angular harmonic, the axisymmetric part of $\delta^{(2)} f_{GR}^i$ is given by [see Paper I, Eq. (112)]

$$\langle \delta_R^{(2)} f_{GR}^i \rangle = -\frac{(\ell+1)^2}{4} \beta \Omega \left(\frac{\varpi}{R} \right)^{2\ell-2} e^{2\beta t} \phi^i, \tag{143}$$

at leading order in the star's angular velocity. The first-order radiation-reaction force $\delta^{(1)} f_{GR}^i$ appearing in Eq. (142) has the form

$$\delta^{(1)} f_{GR}^i = \beta \delta^{(1)} v^i + \delta_\perp^{(1)} f_{GR}^i, \tag{144}$$

where $\langle \boldsymbol{\xi}^{(1)}, \delta_\perp^{(1)} \boldsymbol{f}_{GR} \rangle = 0$ [see Paper I, Eq. (86)]. Because of this orthogonality, $\beta \delta^{(1)} v^i$, determines the growth rate of the linear mode $\xi^{(1)i}$.

At leading order in Ω, δv^i and ξ^i are orthogonal to $\widehat{\boldsymbol{r}}$, and their components along unit vectors $\hat{\theta}$ and $\hat{\phi}$ are

$$\delta^{(1)} v^{\hat{\theta}} = \widetilde{\delta^{(1)} v}^{\hat{\theta}} \cos(\ell\phi + \omega t) e^{\beta t}$$
$$= -\Omega R \left(\frac{r}{R} \right)^\ell \sin^{\ell-1}\theta \cos(\ell\phi + \omega t) e^{\beta t}, \tag{145a}$$
$$\delta^{(1)} v^{\hat{\phi}} = \widetilde{\delta^{(1)} v}^{\hat{\phi}} \sin(\ell\phi + \omega t) e^{\beta t}$$
$$= \Omega R \left(\frac{r}{R} \right)^\ell \sin^{\ell-1}\theta \cos\theta \sin(\ell\phi + \omega t) e^{\beta t}, \tag{145b}$$

$$\xi^{(1)\hat{\theta}} = \widetilde{\xi}^{\hat{\theta}} \sin(\ell\phi + \omega t) e^{\beta t}$$
$$= -\frac{\Omega}{\omega_r} R \left(\frac{r}{R} \right)^\ell \sin^{\ell-1}\theta \sin(\ell\phi + \omega t) e^{\beta t}, \tag{146a}$$
$$\xi^{(1)\hat{\phi}} = \widetilde{\xi}^{\hat{\phi}} \cos(\ell\phi + \omega t) e^{\beta t}$$
$$= -\frac{\Omega}{\omega_r} R \left(\frac{r}{R} \right)^\ell \sin^{\ell-2}\theta \cos\theta \cos(\ell\phi + \omega t) e^{\beta t}, \tag{146b}$$

where, to leading order in Ω, $\omega = -\frac{(\ell-1)(\ell+2)}{\ell+1} \Omega$ and $\omega_r = \frac{2}{\ell+1} \Omega$ is the frequency in a rotating frame. From Eqs. (145) and (146), the vectors $\xi^{(1)i}$ and $\delta^{(1)} v^i$ are of order

$$\xi^{(1)} \sim R e^{\beta t}, \qquad \delta^{(1)} v \sim \Omega R e^{\beta t}. \tag{147}$$

The divergence $\boldsymbol{\nabla} \cdot \boldsymbol{\xi}^{(1)}$ vanishes at lowest order in Ω, and is nonzero only at order Ω^2 [53], with

$$\boldsymbol{\nabla} \cdot \boldsymbol{\xi}^{(1)} \sim \frac{\Omega^2}{\Omega_0^2} e^{\beta t}, \tag{148}$$

where

$$\Omega_0 := \sqrt{\frac{GM}{R^3}} \sim \frac{v_s}{R}, \tag{149}$$

with v_s an average speed of sound in the star.

Prior to saturation, from Eqs. (143) and (144), $\delta^{(2)} f^i$ and $\delta^{(1)} f^i$ are of order $\beta \Omega R e^{2\beta t}$ and $\beta \Omega R e^{\beta t}$, respectively. Then Eq. (147) implies the term $\pounds_{\boldsymbol{\xi}^{(1)}} \delta^{(1)} \boldsymbol{f}_{GR}$ is of order

$$\left| \pounds_{\boldsymbol{\xi}^{(1)}} \delta^{(1)} \boldsymbol{f}_{GR} \right| \sim \beta \Omega R e^{2\beta t},$$

and we obtain the estimate (138), $|\Delta^{(2)} \boldsymbol{f}_{GR}| \sim \beta \Omega R e^{2\beta t}$.

We turn next to Eq. (139) for $\langle D_i(\xi^{(1)}, \xi^{(1)}) \rangle$, where $\xi^{(1)}$ is the Lagrangian displacement of the first-order unstable r-mode. To estimate $\langle D_i \rangle$, we use Eqs. (147) and (148), together with the estimate $\nabla Q \sim Q/R$. From Eq. (148), we have

$$\frac{\Delta^{(1)}\rho}{\rho} \sim \frac{\Delta^{(1)}p}{p} \sim \frac{\Omega^2}{\Omega_0^2} e^{\beta t}. \tag{150}$$

Equation (105) gives $\langle D_i(\xi^{(1)}, \xi^{(1)}) \rangle$ as a sum of three terms which we consider in order. The angle average removes both the ϕ-dependence and the harmonic dependence on t, leaving only the dependence $e^{2\beta t}$. We then have

$$|(\partial_t + \pounds_{\boldsymbol{v}}) \langle \Delta^{(2)} v_i \rangle_{\text{quad}}| = 2\beta |\langle \Delta^{(2)} v_i \rangle_{\text{quad}}| \sim \beta \Omega R e^{2\beta t}. \tag{151}$$

The ϕ component of the second term on the right of Eq. (105) vanishes by axisymmetry: $\partial_\phi \langle U - \frac{1}{2}v^2 \rangle = 0$; the components orthogonal to ϕ^i have magnitudes of order

$$|\nabla \langle \Delta^{(2)}_{\text{quad}} U \rangle| = |\nabla \langle \Delta^{(2)}_{\text{quad}}(h+\Phi) \rangle| \sim \Omega^2 \, R \, e^{2\beta t}, \tag{152a}$$

$$|\nabla \langle \Delta^{(2)}_{\text{quad}} \tfrac{1}{2} v^2 \rangle| \sim \Omega^2 \, R \, e^{2\beta t}. \tag{152b}$$

The last, magnetic term of Eq. (105) is of order

$$\left| \frac{1}{2\pi\rho} B^j (\nabla_{[i} \langle \Delta^{(2)}_{\text{quad}} B_{j]} \rangle) \right| \sim \omega_A^2 \, R \, e^{2\beta t}. \tag{153}$$

From Eq. (5), the Alfvén frequency for a type II superconductor has the form

$$\omega_{A,SC} = \frac{1}{R}\sqrt{\frac{\pi B_0 H_c}{\rho}} = 0.09 \, R_6^{-1} \sqrt{\frac{B_9 \, H_{c,15}}{\rho_{14.6}}} \; \text{s}^{-1}. \tag{154}$$

This and Eq. (3) for a normal fluid each imply $\omega_A < \Omega$ unless $B_0 > 10^{17}$ G. Then for both nascent neutron stars and old accreting neutron stars, rotating fast enough to be unstable to an r-mode, we have $\omega_A \ll \Omega$, and we recover Eq. (139),

$$\rho^{-1} \left| \langle D_\varpi(\xi^{(1)}, \xi^{(1)}) \rangle \right| \sim \rho^{-1} \left| \langle D_z(\xi^{(1)}, \xi^{(1)}) \rangle \right|$$
$$\sim \Omega^2 \, R \, e^{2\beta t}, \tag{155a}$$

$$\rho^{-1} \left| \langle D_{\hat\phi}(\xi^{(1)}, \xi^{(1)}) \rangle \right| \sim \max(\omega_A^2, \beta\Omega) \, R \, e^{2\beta t}. \tag{155b}$$

Finally, we justify the estimate (140). That is, we show that ξ^ϖ and ξ^z are of order $(\omega_A/\Omega)\xi^{\hat\phi}$ for an axisymmetric solution ξ^i to the perturbed MHD-Euler equation whose $B=0$ limit is a perturbation that describes a change in the rotation law – the addition of differential rotation to a uniformly rotating star. Like the vanishing of D_ϕ, the estimate is related to the form of the Euler equation for axisymmetric perturbations. Writing $\mathcal{E}_i$ for a general fluid with no magnetic field in the form

$$\mathcal{E}_i = (\partial_t + \pounds_{\boldsymbol{v}})v_i + \frac{\nabla_i p}{\rho} + \nabla_i(\Phi - \tfrac{1}{2}v^2), \tag{156}$$

we have

$$\mathcal{E}_\phi = (\partial_t + \pounds_{\boldsymbol{v}})v_\phi, \tag{157}$$

with $\mathcal{E}_\phi = 0$ expressing angular momentum conservation of each fluid ring. The commutator in Eq. (A34a) implies

$$\Delta \mathcal{E}_\phi = \partial_t \Delta v_\phi. \tag{158}$$

The fact that only the time-derivative term survives means, for a first-order axisymmetric perturbation described by a Lagrangian displacement $\xi^{(1)i}$

$$\phi^i \Delta^{(1)} \mathcal{E}_i = \phi^i \partial_t \Delta^{(1)} v_i = \phi^i (\partial_t^2 \xi^{(1)}{}_i + 2\epsilon_{ijk}\Omega^j \partial_t \xi^{(1)k}), \tag{159}$$

implying that the operator C_{ij} has no component along ϕ^i. When a background magnetic field is present, C_{ij} acquires a nonzero ϕ component given by the last line on the right of Eq. (97c), with magnitude

$$\rho^{-1} C_{\hat\phi j} \xi^{(1)j} \sim \frac{B^2 \xi^{(1)}}{\rho R^2} \sim \omega_A^2 \, R e^{\beta t}. \tag{160}$$

The corresponding magnitude of $\langle \xi^{(2)\varpi} \rangle$ can be seen from the $\hat\phi$ component of the 2nd-order Newtonian Euler equation:

$$\partial_t^2 \langle \xi^{(2)}{}_{\hat\phi} \rangle + 2\Omega \partial_t \langle \xi^{(2)\varpi} \rangle + \rho^{-1} C_{\hat\phi j} \langle \xi^{(2)j} \rangle$$
$$= -\rho^{-1} \langle D_{N\hat\phi}(\xi^{(1)}, \xi^{(1)}) \rangle. \tag{161}$$

The first-order axisymmetric modes satisfy

$$\partial_t^2 \xi_{n\hat\phi} + 2\Omega \partial_t \xi_n^\varpi + \rho^{-1} C_{\hat\phi j} \xi_n^j = 0. \tag{162}$$

We approximate the frequencies of the dominant modes by ω_A, writing $\partial_t \xi_n \sim \omega_A \xi_n$, $\partial_t^2 \xi_n \sim \omega_A^2 \xi_n$, and use Eq. (160) to write $\rho^{-1} C_{\hat\phi j} \xi_n^j \sim \omega_A^2 \xi_n$. We then have

$$\xi_n^\varpi \sim \left(\frac{\omega_A}{\Omega}\right) \xi_n^{\hat\phi}. \tag{163}$$

Finally, in the expression (127) for κ_n,

$$\kappa_n = 1 - 2\left(\frac{\Omega}{\omega_n}\right) \Im \frac{\int dV \rho \, \widetilde{\xi}_n^{\varpi *} \widetilde{\xi}_n^{\hat\phi}}{\int dV \rho \, |\widetilde{\xi}_n|^2},$$

the ratio of integrals is of order ω_A/Ω, giving a bound on κ_n of order unity. This completes our justification of the estimates (139), (140), and (141); and the argument following Eq. (141) then gives our main result, Eq. (136) for the angular displacement of a fluid element.

Normal interior

We turn now to the implications of this estimate. We first find bounds on magnetic field growth for a normal interior and then obtain equivalent bounds for an interior that is a type II superconductor. We obtain as follows a bound on the the maximum growth of δB similar to Eq. (53) of the toy model. In Eq. (136),

$$\frac{\max(\omega_A, \beta)}{4\beta^2 + \omega_A^2} = \max\left[\frac{\omega_A}{4\beta^2 + \omega_A^2}, \; \frac{\beta}{4\beta^2 + \omega_A^2}\right].$$

By inspection, $\dfrac{\omega_A}{4\beta^2 + \omega_A^2} < \dfrac{1}{\omega_A}$, and, using the inequality (44), we have

$$\frac{\max(\omega_A, \beta)}{4\beta^2 + \omega_A^2} < \frac{1}{\omega_A}.$$

Then the angular displacement and corresponding change in the magnetic field have upper limits

$$\langle \xi_{\text{sat}}^\phi \rangle \lesssim \alpha_{\text{sat}}^2 \frac{\Omega}{\omega_A} < \alpha_{\text{sat}}^2 \frac{\Omega R}{B_0}\sqrt{\frac{\rho}{\pi}}, \tag{164}$$

$$\langle \delta B_{\text{sat}}^{\hat{\phi}} \rangle \lesssim \alpha_{\text{sat}}^2 \Omega R \sqrt{\frac{\rho}{\pi}}, \qquad (165)$$

with the small numerical values

$$\langle \xi_{\text{sat}}^{\phi} \rangle \lesssim 0.4 \, \alpha_{-4}^2 f_{500} R_6 B_9^{-1} \rho_{14.6}^{1/2},$$

$$\langle \delta B_{\text{sat}}^{\hat{\phi}} \rangle \lesssim 4 \times 10^8 \, \alpha_{-4}^2 f_{500} R_6 \rho_{14.6}^{1/2} \text{ G}. \qquad (166)$$

Recalling Eq. (12) for the critical magnetic field and using Eq. (165), we obtain our main inequality,

$$\frac{\langle \delta B_{\text{sat}}^{\hat{\phi}} \rangle}{\langle \delta B \rangle_{\text{crit}}} \lesssim \alpha_{\text{sat}}, \qquad (167)$$

or, equivalently,

$$\frac{d\mathcal{E}_m/dt}{d\mathcal{E}_{\text{mode}}/dt} \lesssim \alpha_{\text{sat}}^2. \qquad (168)$$

When $\alpha_{\text{sat}} \sim \mathcal{O}(1)$, as assumed in the initial investigations of the instability [20] and in Refs. [1, 10, 11], then $\langle \delta B_{\text{sat}}^{\hat{\phi}} \rangle \sim \langle \delta B \rangle_{\text{crit}}$, and the magnetic field at saturation is similar to the critical field needed to damp or substantially alter the linear r-mode. However, for more realistic values of the saturation amplitude, and even for an unexpectedly large saturation amplitude, $\alpha_{\text{sat}} \sim 10^{-3}$, the change in the magnetic field at saturation is three orders of magnitude below the critical field.

After nonlinear saturation, the constraint on $\langle \delta B^{\hat{\phi}} \rangle$ corresponding to the limit (137) on the angular displacement is

$$\langle \delta B_{\text{max}}^{\hat{\phi}} \rangle \lesssim \alpha_{\text{sat}}^2 B_0 \begin{cases} \Omega/\omega_A, & \omega_A > \beta \\ \beta\Omega/\omega_A^2, & \omega_A < \beta. \end{cases} \qquad (169)$$

With the critical magnetic field now given by Eq. (50),

$$\langle \delta B \rangle_{\text{crit}} \sim \alpha_{\text{sat}} \Omega R \sqrt{\rho\beta/\omega_A} \geq \frac{\alpha_{\text{sat}}}{\pi^{1/4}} \beta^{1/2} \Omega R^{3/2} \rho^{3/4} B_0^{-1/2}, \qquad (170)$$

we have

$$\frac{\langle \delta B_{\text{max}}^{\hat{\phi}} \rangle}{\langle \delta B \rangle_{\text{crit}}} \lesssim \alpha_{\text{sat}} \sqrt{\frac{\omega_A}{\pi\beta}} \qquad \text{for} \quad \omega_A > \beta$$

$$\leq 2.4 \times 10^{-5} \alpha_{-4} \beta_{-3.3}^{-1/2} R_6^{-1/2} \rho_{14.6}^{-1/4} B_9^{1/2}, \qquad (171a)$$

$$\frac{\langle \delta B_{\text{max}}^{\hat{\phi}} \rangle}{\langle \delta B \rangle_{\text{crit}}} \lesssim \alpha_{\text{sat}} \sqrt{\frac{\beta}{\pi\omega_A}} \qquad \text{for} \quad \omega_A < \beta$$

$$\leq 1.3 \times 10^{-4} \alpha_{-4} \beta_{-3.3}^{1/2} R_6^{1/2} \rho_{14.6}^{1/4} B_9^{-1/2}. \qquad (171b)$$

The second case ($\omega_A < \beta$) is Eq. (51) of the toy model. For $\omega_A > \beta$, the present bound differs from that of the toy model because of the contribution to the effective driving force from the quadratic D term, but not by enough to alter our conclusion.

In particular, after saturation, the oscillation may allow $\xi_{DR}^{(2)\phi}$ to grow to about twice its equilibrium value, with a smaller value for a more gradual approach to saturation. Even with $\alpha_{\text{sat}} \sim 10^{-3}$, the initial magnetic field would need to be well below 100 G or above 10^{16} G before magnetic field windup could significantly alter the linear r-mode.

Superconducting interior

The r-mode instability has been studied most in the context of old neutron stars spun up by accretion. The interior of these stars is likely to be a type II superconductor, and we now turn to the corresponding limits on magnetic-field windup for such stars.

For a superconducting interior, the total energy of the magnetic field is given by

$$E_{m,SC} = \frac{1}{8\pi} \varphi_{\text{SC}} H_c \ell_f, \qquad (172)$$

where ℓ_f is the average length of a flux tube, and φ_{SC} is the total magnetic flux. Differential rotation stretches the flux tubes but leaves the flux in each tube and the number of tubes unchanged. Then φ_{SC} is constant, and the change in energy $E_{m,SC}$ is determined by the change in flux tube length ℓ_f. For a tube deformed by a small angular displacement $\langle \xi^{\phi} \rangle$, the change in length at quadratic order in ξ^{ϕ} is of order

$$\delta\ell_f \approx \ell_f \langle \xi^{\phi} \rangle^2. \qquad (173)$$

With $\ell_f \sim R$, the stretching rate at quadratic order is then

$$\frac{d\ell_f}{dt} \sim R\xi^{\phi} \frac{d\langle \xi^{\phi} \rangle}{dt} = 2\beta R(\xi^{\phi})^2, \qquad (174)$$

We define a field B_0 for which the total flux is

$$\varphi_{\text{SC}} = \pi R^2 B_0. \qquad (175)$$

The total magnetic energy is then

$$E_{m,SC} = \frac{1}{8} B_0 H_c \ell_f R^2, \qquad (176)$$

larger than its value for a normal plasma by a factor of order H_c/B_0, and the corresponding growth rate of magnetic energy density is

$$\frac{d\mathcal{E}_{m,SC}}{dt} \sim \frac{1}{30} \beta H_c B_0 (\xi^{\phi})^2, \qquad (177)$$

for a superconducting core of approximate radius R. A detailed calculation by Rezzolla *et al.* [1, 10, 11] for an initial dipole poloidal magnetic field B_0 gives the same relation with a somewhat smaller numerical coefficient,

$$\frac{d\mathcal{E}_{m,SC}}{dt} \sim \beta \frac{1}{60} B_0 H_c \langle \xi^{\phi} \rangle^2. \qquad (178)$$

We define an average perturbed magnetic field, $\langle \delta B_{\text{SC}} \rangle$, as a volume average for which $\langle \delta B_{\text{SC}} \rangle^2/8\pi :=$

$\delta\mathcal{E}_m$. The critical magnetic field for which the growth rate of magnetic energy and of the linear r-mode are equal is then again given by Eq. (12).

To obtain an approximate bound on $d\mathcal{E}_m/dt$ and $\langle\delta B_{\rm SC}\rangle$, we first write Eq. (178) in the form

$$\frac{d\mathcal{E}_{m,SC}}{dt} \sim \beta\frac{1}{60\pi}\rho\omega_{A,SC}^2(R\langle\xi^\phi\rangle)^2. \qquad (179)$$

The bound on $\langle\xi^\phi\rangle$ is given by Eq. (164) with ω_A replaced by $\omega_{A,SC}$,

$$\langle\xi_{\rm sat}^\phi\rangle \lesssim \alpha_{\rm sat}^2\frac{\Omega}{\omega_{A,SC}} < \alpha_{\rm sat}^2\Omega R\sqrt{\frac{\rho}{\pi B_0 H_c}}, \qquad (180)$$

with the small numerical value

$$\langle\xi_{\rm sat}^\phi\rangle \lesssim 6\times10^{-4}\alpha_{-4}^2 f_{500}\sqrt{\frac{\rho_{14.6}}{B_9 H_{c,15}}}. \qquad (181)$$

We then have

$$\frac{d\mathcal{E}_{m,SC}}{dt} \lesssim \frac{1}{60\pi}\alpha_{\rm sat}^4\beta\rho\Omega^2 R^2. \qquad (182)$$

Recognizing that the right side is proportional to the energy of the linear r-mode, as in Eq. (11), we obtain the inequalities

$$\frac{\langle\delta B_{\rm sat,SC}\rangle}{\langle\delta B_{SC}\rangle_{\rm crit}} \lesssim \frac{1}{\sqrt{60\pi}}\alpha_{\rm sat}\,, \qquad \frac{d\mathcal{E}_{m,SC}/dt}{d\mathcal{E}_{\rm mode}/dt} \lesssim \frac{1}{60\pi}\alpha_{\rm sat}^2\,. \qquad (183)$$

We are not entitled to claim bounds this stringent, however, because in deriving the bound $\langle\xi^\phi\rangle \lesssim \alpha_{\rm sat}^2\Omega/\omega_{A,SC}$, we used the rough approximation $\omega_n \sim \omega_{A,SC}$, while the coefficient $1/60\pi$ in the expression for $d\mathcal{E}_m/dt$ is consistent with the somewhat smaller frequency of long-wavelength Alfvén modes. What our estimates show are then the approximate bounds previewed in Sec. II,

$$\frac{\langle\delta B_{\rm SC}\rangle_{\rm sat}}{\langle\delta B_{\rm SC}\rangle_{\rm crit}} < \alpha_{\rm sat}\,, \qquad \frac{d\mathcal{E}_{m,SC}/dt}{d\mathcal{E}_{\rm mode}/dt} \lesssim \alpha_{\rm sat}^2. \qquad (184)$$

After saturation, because $\omega_{A,SC} \gg \beta$, the maximum displacement and magnetic field are within a factor of about 2 of their values at saturation.

Caveats: Continuous spectrum, zero-frequency modes, and MRI instability

The claim that magnetic field windup cannot damp or significantly alter the first-order r-mode comes with some caveats. The estimates of this section rely on two principal assumptions: That linear axisymmetric perturbations of the background star can be written in terms of a discrete non-degenerate spectrum, and that the background star has no unstable axisymmetric modes - or at least no unstable axisymmetric modes that wind up the magnetic field.

It may be that neither assumption is correct: There is no proof that discrete modes are complete for uniformly rotating stars, and, once differential rotation is established, the star is likely to encounter a magnetorotational instability (MRI). We briefly discuss the implications of relaxing the assumptions, beginning with a possible continuous part of the spectrum of linear modes.

Because the effective driving force $\Delta^{(2)}\boldsymbol{F}$ is a quadratic function of the linear r-mode, its value is unrelated to assumptions about the spectrum of linear axisymmetric perturbations. With a continuous spectrum, estimates (138) and (139) of its two parts are unchanged, and $\Delta^{(2)}\boldsymbol{F}$ retains its form, with magnitude

$$\Delta^{(2)}F_{\hat\phi} \sim \max(\omega_A^2, \beta\,\Omega)\,R\,e^{2\beta t}. \qquad (185)$$

Were we able to replace a sum over discrete modes by an integral over a continuous spectrum, we could regain our estimates for ξ^ϕ. We have no formal justification for this, because the time evolution of the system is described by an operator that is not self-adjoint. Simply discretizing the spatial operators, however, gives a system whose modes *are* discrete and for which the estimates hold. Because the estimates are independent of the discretization, they should hold in the continuum limit.

The assumption of a stable system is in question once differential rotation is established by a growing r-mode. That is, there appears to be an MRI instability when the magnetic field is smaller than about 10^{13} G [54, 55] and the drift angular velocity $\delta\Omega_{drift}$ satisfies

$$\frac{d(\delta\Omega_{drift}^2)}{d\varpi} < 0\,, \qquad (186)$$

in some region of the star. The instability is present only for perturbations that are not restored by negative buoyancy or by pressure. Buoyancy is governed by the Brunt-Väisälä frequency, which, for a neutron star, is of order 50-150 Hz (see, e.g., [42, 43]), much larger than $R\,d\Omega/d\varpi \sim \alpha(t)^2\Omega$. In the Balbus-Hawley analysis [55], this removes the instability for most modes, but leaves at least a set of unstable perturbations whose wavevector $\boldsymbol{k}$ is quasi-radial, along the Brunt-Väisälä vector $\boldsymbol{N}$, and there may also be modes with zero or near zero frequency. Because MRI-unstable perturbations cannot acquire more energy than is present in the small available differential rotation, we suspect that the presence of MRI-unstable or marginally unstable perturbations will not substantially alter our analysis. We should point out, however, that after saturation, the constant effective radiation-reaction force $\Delta^{(2)}F^a$ will drive the growth of any zero-frequency modes.

VI. CONCLUSIONS

Almost 20 years ago, r-modes oscillations in rotating neutron stars were shown to be unstable to the emission of gravitational waves [2, 3]. The impact of this finding on newly born neutron stars and in old neutron stars in

X-ray binaries was soon discussed in a long list of works starting with Ref. [20]. Among the many features of the nonlinear development of the stability, the development of differential rotation was pointed out early on, heuristically [15] and via perturbation theory [1], as was the amplification of strong magnetic fields and the possibility that this growth suppresses the instability [10, 11].

Building on a more realistic estimate of the saturation amplitude of the instability [6, 7] and on a more rigorous mathematical description of the development of differential rotation in unstable stars [14], we have here reconsidered the impact of differential rotation and magnetic field amplification on the growth of unstable r-modes. The instability may be present in nascent neutron stars and in old stars in X-ray binaries; in each case, nonlinear coupling to other modes limits the r-mode amplitude to a saturation amplitude $\alpha_{\rm sat} \lesssim 10^{-4}$. And in each case we find that the maximum enhancement of the average magnetic field is smaller by the factor $\alpha_{\rm sat}$ than the critical field needed to damp or significantly alter the r-mode. We have obtained this result following two different routes: First, using a simplified but exact toy model where the star is treated as an incompressible and homogeneous cylinder in the ideal-MHD limit; second,

using a formalism governing the equilibrium and first- and second-order perturbations of a rotating star with a background magnetic field and radiation-reaction.

In old neutron stars whose interior is a type II superconductor, we find that magnetic field growth stops soon after the mode reaches its saturation amplitude. In nascent neutron stars, before the interior has cooled below the superconducting transition temperature, continued magnetic field growth can follow nonlinear saturation. If the saturation amplitude is unexpectedly large, with $\alpha_{\rm sat} \sim 10^{-3}$, an initial small magnetic field of about 10^8 G could be amplified to 10^{11} G, before the remaining secular drift of a fluid element (that winds up the magnetic field) is restricted to less than a radian. Although still too small to damp the growth of the linear r-mode, this might be a contribution to magnetic field generation in nascent stars.

Although mathematically robust, our findings rest on the assumptions noted at the end of the last section. In particular, we assume that there are no marginally unstable perturbations, and this may not hold when differential rotation leads to a magnetorotational instability.

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Appendix A: Lagrangian perturbations

At first order in α the Lagrangian displacement vector $\boldsymbol{\xi} = \alpha \boldsymbol{\xi^{(1)}}$ can be viewed in two ways. $\boldsymbol{\xi}$ is a connecting vector from the position x of a fluid element in the unperturbed fluid to its position $\chi_\alpha(x)$ in the perturbed fluid; and $\boldsymbol{\xi^{(1)}}$ is the vector field tangent to the trajectories $\alpha \to \chi_\alpha(x)$ of the family of diffeomorphisms χ_α. At higher order the two viewpoints diverge and we have chosen the second approach, defining a Lagrangian displacement that depends only on the family of diffeomorphisms, not on the metric of flat space or on a choice of coordinates. The second-order formalism using the first approach is developed in Ref. [28].

1. First- and second-order Lagrangian perturbations

We derive here relations used in Sec. IV B to obtain first- and second-order Lagrangian perturbations, defined by Eq. (75).

Recall that the pullback map χ^* associated with a diffeomorphism χ is defined on scalars f by

$$\chi^* f(t, x) := f(t, \chi[t, x]) \,. \tag{A1}$$

On covariant and contravariant vectors w_i and w^i its action is given in any coordinate system by

$$\chi^* w_i(t, x) = \partial_i \chi^j \, w_j(t, \chi[t, x]) \,, \tag{A2a}$$

$$\chi^* w^i(t, x) = \partial_j \left(\chi^{-1} \right)^i w^j(t, \chi[t, x]) \,. \tag{A2b}$$

Acting on forms (antisymmetric covariant tensors) $\omega_{a \ldots b}$, it satisfies

$$[\chi^*, d]\omega = 0 \,, \tag{A3}$$

where d is the exterior derivative.

Given a family of diffeomorphisms $\chi_\alpha(x)$ of the unperturbed fluid to the perturbed fluid at a fixed time t, we can define a family of Lagrangian displacements $\xi(\alpha, x)$ in a way that is analogous to defining the velocity field $v^i(t, x)$ from the family of diffeomorphisms ψ_t that describe the fluid flow: In the fluid case the family of diffeomorphisms acts on both on the spatial coordinates x and the time coordinate t, while in our analogous case the parameter α plays the same role as the time coordinate in the fluid case. In the time-dependent fluid case ψ_τ maps a fluid element at x at a time t to its position $\psi_\tau(t, x)$ at time $t+\tau$. The velocity field $v^i(t, x)$ is tangent to the curve $c(\tau) = \psi_\tau(t, x)$.

$$v^i(t, x) = \left. \frac{d}{d\tau} c^i(\tau) \right|_{\tau=0} = \left. \frac{d}{d\tau} \psi_\tau^i(t, x) \right|_{\tau=0} \,. \tag{A4}$$

More concisely, the 4-dimensional diffeomorphism Ψ_τ,

$$\Psi_\tau(t,x) = (t+\tau, \psi_\tau(t,x)), \tag{A5}$$

moves the point (t,x) a parameter distance τ along an integral curve of the Newtonian 4-velocity

$$\boldsymbol{u}(t,x) = (1, v^i(t,x)). \tag{A6}$$

We now repeat the construction for the family of diffeomorphisms $\chi_\alpha(x)$. In this case, we include the parameter α as a coordinate and denote by (α, x) a point in the support of the perturbed fluid: The fluid element at $(0,x)$ in the unperturbed fluid is at the corresponding point $(\alpha, \chi_\alpha(x))$ in the perturbed fluid. As initially defined, χ_α maps a point x occupied by a fluid element in the unperturbed fluid to the location $\chi_\alpha(x)$ of that fluid element in the perturbed fluid. We extend χ_α to a family $\widetilde{\chi}_\alpha$ of diffeomorphisms that act on points in the perturbed fluid by writing

$$\widetilde{\chi}_\eta(\alpha, \chi_\alpha(x)) := \chi_{\eta+\alpha}(x). \tag{A7}$$

We define the vector field $\widetilde{\boldsymbol{\xi}}(\alpha, x)$ as the tangent to the curve $c(\eta) = \widetilde{\chi}_\eta(\alpha, x)$,

$$\widetilde{\xi}^i(\alpha, x) = \left.\frac{d}{d\eta} c^i(\eta)\right|_{\eta=0} = \left.\frac{d}{d\eta}\widetilde{\chi}^i_\eta(\alpha, x)\right|_{\eta=0}, \tag{A8}$$

to maintain a Lagrangian displacement $\boldsymbol{\xi}$ that is proportional to α at lowest order, we write

$$\boldsymbol{\xi} = \alpha\widetilde{\boldsymbol{\xi}}. \tag{A9}$$

Again our construction has a more concise form in terms of the 4-dimensional diffeomorphism X_η (the analog of Ψ_τ),

$$X_\eta(\alpha, x) = (\alpha+\eta, \widetilde{\chi}_\eta(\alpha, x)): \tag{A10}$$

X_η moves the point (α, x) a parameter distance η along an integral curve of the vector field

$$\boldsymbol{\Xi}(\alpha, x) = (1, \widetilde{\xi}^i(\alpha, x)). \tag{A11}$$

This is the statement that $\boldsymbol{\Xi}$ generates the family of diffeomorphisms X_α, and it leads to a simple expression, (A17) below, for the Lagrangian perturbation in the fluid variables Q at nth order in α. We begin by noting that the relation

$$\frac{d}{d\alpha}X_\alpha^* f(x) = \frac{d}{d\alpha}f(X_\alpha(x)) = \left.(\pounds_{\boldsymbol{\Xi}} f)\right|_{X_\alpha(x)}, \tag{A12}$$

for a scalar f, implies

$$\left.\frac{d^n}{d\alpha^n}f(X_\alpha(x))\right|_{\alpha=0} = \pounds_{\boldsymbol{\Xi}}^n f(x). \tag{A13}$$

The action of an analytic family of diffeomorphisms X_α on an analytic function is then given by a convergent Taylor series in α, namely

$$X_\alpha^* f = e^{\alpha\pounds_{\boldsymbol{\Xi}}} f. \tag{A14}$$

In our case, we have only a smooth family of diffeomorphisms acting on a smooth function, and the Taylor series at finite order in α gives the relation

$$X_\alpha^* f = \left[1 + \alpha\pounds_{\boldsymbol{\Xi}} + \cdots + \frac{1}{n!}(\alpha\pounds_{\boldsymbol{\Xi}})^n + o(\alpha^n)\right] f. \tag{A15}$$

It is straightforward to check that the same relation holds for the action of X_α^* on arbitrary smooth tensors.

From the definition (74) of the exact Lagrangian change in the fluid variables $Q(\alpha, x)$, we have

$$X_\alpha^* Q(0,x) = Q(\alpha, \chi_\alpha(x)), \tag{A16}$$

implying

$$\Delta Q = X_\alpha^* Q(0,x) - Q(0,x) = \sum_1^n \alpha^k \frac{1}{k!}\left.\pounds_{\boldsymbol{\Xi}}^k Q\right|_{\alpha=0} + o(\alpha^n). \tag{A17}$$

In particular, writing

$$\boldsymbol{\xi}^{(1)} = \left.\partial_\alpha\boldsymbol{\xi}\right|_{\alpha=0} = \left.\widetilde{\boldsymbol{\xi}}\right|_{\alpha=0}, \tag{A18a}$$

$$\boldsymbol{\xi}^{(2)} = \left.\frac{1}{2}\partial_\alpha^2\boldsymbol{\xi}\right|_{\alpha=0} = \left.\partial_\alpha\widetilde{\boldsymbol{\xi}}\right|_{\alpha=0}, \tag{A18b}$$

and $\boldsymbol{\Xi}_0 := \left.\boldsymbol{\Xi}\right|_{\alpha=0}$, we obtain

$$\Delta^{(1)}Q = \left.\pounds_{\boldsymbol{\Xi}} Q\right|_{\alpha=0}$$
$$= \left.(\partial_\alpha + \pounds_{\widetilde{\boldsymbol{\xi}}})Q\right|_{\alpha=0} = (\delta^{(1)} + \pounds_{\widetilde{\boldsymbol{\xi}}^{(1)}})Q, \tag{A19}$$

$$\Delta^{(2)}Q = \left.\left(\pounds_{\partial_\alpha\boldsymbol{\Xi}}Q + \frac{1}{2}\pounds_{\boldsymbol{\Xi}_0}^2 Q\right)\right|_{\alpha=0}$$
$$= \left.\left[\pounds_{\partial_\alpha\widetilde{\boldsymbol{\xi}}} + \frac{1}{2}(\partial_\alpha + \pounds_{\widetilde{\boldsymbol{\xi}}^{(1)}})^2 Q\right]\right|_{\alpha=0}$$
$$= \left.\left(\frac{1}{2}\partial_\alpha^2 + \pounds_{\partial_\alpha\widetilde{\boldsymbol{\xi}}} + \pounds_{\widetilde{\boldsymbol{\xi}}^{(1)}}\partial_\alpha + \frac{1}{2}\pounds_{\widetilde{\boldsymbol{\xi}}^{(1)}}^2\right)Q\right|_{\alpha=0}$$
$$= \left(\delta^{(2)} + \pounds_{\boldsymbol{\xi}^{(2)}} + \pounds_{\boldsymbol{\xi}^{(1)}}\delta^{(1)} + \frac{1}{2}\pounds_{\boldsymbol{\xi}^{(1)}}^2\right)Q. \tag{A20}$$

In these last two equations, we have used the definition (73) of $\delta^{(n)}Q$.

2. Perturbed fluid velocity

We will next find the expression for the Lagrangian change in the fluid velocity in terms of the Lagrangian displacement of the fluid, obtaining the form

$$\Delta v^i = \partial_t\xi^i + \tfrac{1}{2}\pounds_{\boldsymbol{\xi}}\partial_t\xi^i + \mathcal{O}(\alpha^3). \tag{A21}$$

Expanding this result in powers of α immediately gives

$$\Delta^{(1)}v^i = \partial_t\xi^{(1)i}, \tag{A22}$$

$$\Delta^{(2)}v^i = \partial_t\xi^{(2)i} + \tfrac{1}{2}\pounds_{\boldsymbol{\xi}^{(1)}}\partial_t\xi^{(1)i}. \tag{A23}$$

Equation (A21) can be derived by noting that the diffeomorphism χ maps trajectories in the unperturbed fluid to trajectories in the perturbed fluid. Denote by $\tau \mapsto c_0(t+\tau)$ the path of the fluid element in the unperturbed fluid that passes through the point $x = c_0(t)$ at time t. Then $\tau \mapsto \chi_\alpha(t+\tau, c_0(t+\tau))$ is the path of the fluid element in the perturbed flow, and it passes through $\chi_\alpha(t,x)$ at time t. The perturbed velocity is then given by

$$
\begin{aligned}
v^i_\alpha(t, \chi_\alpha(t,x)) &= \left. \frac{d}{d\tau} \chi^i_\alpha(t+\tau, c_0(t+\tau)) \right|_{\tau=0} \\
&= \partial_t \chi^i_\alpha + v^k_0 \partial_k \chi^i_\alpha\,. \quad (A24)
\end{aligned}
$$

The exact Lagrangian change in the fluid velocity is given by

$$
\begin{aligned}
\Delta v^i(t,x) &= \chi^*_\alpha v^i_\alpha(t, \chi_\alpha(t,x)) - v^i_0(t,x)\,, \quad (A25) \\
&= \partial_j (\chi^{-1}_\alpha)^i \big|_{(t,\chi_\alpha(t,x))} v^j_\alpha(t, \chi_\alpha(t,x)) - v^i_0(t,x)\,. \\
&\hspace{6cm} (A26)
\end{aligned}
$$

In all the remaining equations, each variable is evaluated at the point (t,x) unless the argument is explicitly shown. Note first that, by its definition (A7), $\widetilde{\chi}_\eta(\alpha, \chi_\alpha(x)) = \chi_{\eta+\alpha}(x)$. From Eq. (A8), we then have

$$
\begin{aligned}
\widetilde{\xi}^i(\alpha, \chi_\alpha(x)) &= \left. \frac{d}{d\eta} \chi^i_{\eta+\alpha}(x) \right|_{\eta=0} = \frac{d}{d\alpha} \chi^i_\alpha(x)\,, \\
\xi^{(1)i}(x) &= \left. \frac{d}{d\alpha} \chi^i_\alpha(x) \right|_{\alpha=0}\,. \quad (A27)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\left. \frac{d^2}{d\alpha^2} \chi^i_\alpha(x) \right|_{\alpha=0} &= \left. \frac{d}{d\alpha} \widetilde{\xi}^i(\alpha, \chi_\alpha(x)) \right|_{\alpha=0} \\
&= \left[\frac{d}{d\alpha} \widetilde{\xi}^i(\alpha, x) + \partial_j \widetilde{\xi}^i(0,x) \frac{d}{d\alpha} \chi^i_\alpha(x) \right]_{\alpha=0} \\
&= 2\xi^{(2)i} + \xi^{(1)j} \partial_j \xi^{(1)i}\,. \quad (A28)
\end{aligned}
$$

The expansion of the diffeomorphism χ_α,

$$
\chi^i_\alpha(x) = x^i + \alpha \partial_\alpha \chi^i_\alpha \big|_{\alpha=0} + \frac{1}{2}\alpha^2 \, \partial^2_\alpha \chi^i_\alpha \big|_{\alpha=0} + \mathcal{O}(\alpha^3)\,. \quad (A29)
$$

now gives

$$
\begin{aligned}
\chi^i_\alpha &= x^i + \xi^i + \tfrac{1}{2}\xi^j \partial_j \xi^i + \mathcal{O}(\alpha^3)\,, \quad (A30) \\
\chi^{-1\,i}_\alpha &= x^i - \xi^i + \tfrac{1}{2}\xi^j \partial_j \xi^i + \mathcal{O}(\alpha^3)\,. \quad (A31)
\end{aligned}
$$

Using these expressions, we obtain

$$
\partial_j (\chi^{-1}_\alpha)^i \big|_{(t,\chi_\alpha(t,x))} = \delta^i_{\ j} - \partial_j \xi^i - \xi^k \partial_k \partial_j \xi^i + \tfrac{1}{2}\partial_j \left(\xi^k \partial_k \xi^i \right)\,, \quad (A32)
$$

and

$$
\begin{aligned}
v^j_\alpha(t, \chi_\alpha(t,x)) &= \partial_t \chi^j_\alpha + v^k_0 \partial_k \chi^j_\alpha\,, \\
&= \partial_t \xi^j + \tfrac{1}{2}\partial_t \left(\xi^k \partial_k \xi^j \right) + v^j_0 + v^k_0 \partial_k \xi^j \\
&\quad + \tfrac{1}{2}v^\ell_0 \partial_\ell \left(\xi^k \partial_k \xi^j \right) + \mathcal{O}(\alpha^3)\,. \quad (A33)
\end{aligned}
$$

Substituting in Eq. (A26) the expressions from Eqs. (A32) and (A33) and keeping terms up to quadratic order in $\boldsymbol{\xi}$ yields the desired expression (A21) for $\Delta\boldsymbol{v}$.

3. Commutation relations

We now derive the commutation relations used in Sec. IV D, namely [3]

$$
\begin{aligned}
\Delta(\partial_t + \pounds_{\boldsymbol{v}}) &= (\partial_t + \pounds_{\boldsymbol{v}_0})\Delta\,, \quad &(A34a) \\
\Delta d &= d\Delta\,, \quad &(A34b)
\end{aligned}
$$

where the second relation is restricted to an action on forms.

We first show that Eq. (A34a) follows from a commutation relation between the diffeomorphism χ_α and the diffeomorphism generating the fluid flow. It is simplest to write the relation in terms of the corresponding 4-dimensional diffeomorphisms . Let $\mathcal{X}_\alpha$ be the spacetime diffeomorphism associated with χ_α,

$$
\mathcal{X}_\alpha(t,x) = (t, \chi_\alpha(t,x))\,, \quad (A35)
$$

and let

$$
t \mapsto C_\alpha(t) = (t, c_\alpha(t))\,, \quad (A36)
$$

be the trajectory of a fluid element in the perturbed fluid, with Newtonian 4-velocity $(1, \boldsymbol{v})$, where $v^i(t) = \dot{c}^i_\alpha(t)$. Then

$$
C_\alpha(t) = \mathcal{X}_\alpha \circ C_0(t)\,. \quad (A37)
$$

As in Eq. (A5), let $\Psi_{\tau,\alpha}$ be the spacetime diffeomorphism that maps a fluid element at time t in the perturbed fluid to its position at time $t+\tau$:

$$
\Psi_{\tau,\alpha} \circ C_\alpha(t) = C_\alpha(t+\tau)\,. \quad (A38)
$$

Then

$$
\Psi_{\tau,\alpha} \circ \mathcal{X}_\alpha \circ C_0(t) = C_\alpha(t+\tau) = \mathcal{X}_\alpha \circ \Psi_{\tau,0} \circ C_0(t)\,, \quad (A39)
$$

implying

$$
\Psi_{\tau,\alpha} \circ \mathcal{X}_\alpha = \mathcal{X}_\alpha \circ \Psi_{\tau,0}\,. \quad (A40)
$$

---

[3] At first order, Eq. (A34a) can be obtained by using the relation

$$
[\pounds_{\boldsymbol{\xi}}, \pounds_{\boldsymbol{v}}] = \pounds_{[\boldsymbol{\xi},\boldsymbol{v}]}\,,
$$

to write

$$
\left[\Delta^{(1)}, (\partial_t + \pounds_{\boldsymbol{v}}) \right] = -\pounds_{\partial_t \boldsymbol{v}} + \pounds_{[\boldsymbol{\xi}^{(1)},\boldsymbol{v}]} = \pounds_{-\partial_t \boldsymbol{v} + [\boldsymbol{\xi}^{(1)},\boldsymbol{v}]} = 0\,.
$$

This algebraic derivation can be extended to the more complicated second-order commutator, but it hides the simpler connection between the commutator (A34a) and the commutation relation of the diffeomorphisms, Eq. (A40).

The Lie derivative of a tensor T with respect to the 4-velocity $(1, \boldsymbol{v})$ is

$$(\partial_t + \pounds_{\boldsymbol{v}})T = \frac{d}{d\tau}\Psi^*_{\tau,\alpha}T\Big|_{\tau=0}, \qquad (A41)$$

where $\Psi^*_{\tau,\alpha}$ is the pullback map. By Eq. (A40) the corresponding pullbacks satisfy

$$\mathcal{X}^*_\alpha \Psi^*_{\tau,\alpha} = \Psi^*_{\tau,0}\mathcal{X}^*_\alpha. \qquad (A42)$$

Finally, taking the derivative of this relation with respect to τ at $\tau = 0$, we obtain Eq. (A34a) for tensors T that are functions of α and x:

$$\begin{aligned}(\partial_t + \pounds_{\boldsymbol{v}_0})\Delta Q &= \frac{d}{d\tau}\Psi^*_{\tau,0}(\mathcal{X}^*_\alpha Q_\alpha - Q_0)\Big|_{\tau=0}\\ &= \frac{d}{d\tau}(\mathcal{X}^*_\alpha \Psi^*_{\tau,\alpha}Q_\alpha - \Psi^*_{\tau,0}Q_0)\Big|_{\tau=0}\\ &= \Delta(\partial_t + \pounds_{\boldsymbol{v}})Q. \qquad (A43)\end{aligned}$$

The second commutation relation, Eq. (A34b), is immediate from the vanishing commutator of exterior derivative and pullback (acting on forms)

$$[d, \chi^*_\alpha] = 0. \qquad (A44)$$

Appendix B: Symplectic product and the growth of driven modes

We derive here Eq. (125) for the growth of a system satisfying an equation of the form

$$(A_{ij}\partial_t^2 + B_{ij}\partial_t + C_{ij})\xi^j = F_i(t, x). \qquad (B1)$$

This is essentially a summary of results due to Dyson and Schutz [27], included here because their work and the summary given by Schenk et $al.$ [41] are more elaborate, including in particular the Jordan chains that arise when there are degenerate modes. The treatment here is self-contained if one assumes that the discrete normal modes are a complete set for arbitrary initial data. Schutz and Dyson have a lengthy characterization of the spectrum that implies completeness of the discrete modes if one assumes only that the spectrum has no continuous part.

As noted Sec. IV E, orthogonality of nondegenerate modes follows from the fact that the symplectic product W of Eq. (118) is conserved. This is a property of any Hamiltonian system. Here, a quick computation, using only the self-adjointness properties of the operators, the homogeneous equation, and the definition (119) of π_i, gives a direct check that $d/dt\, W(\xi, \tilde{\xi}) = 0$.

For a nonrotating star, the quantity $iW(\xi_n, \xi_n)$ is real and is, for each mode with nonzero frequency, proportional to the usual norm $||\cdot||$, given by $||\xi||^2 = \langle\xi|A\xi\rangle = \int dV\rho|\xi|^2$. Because the constant of proportionality involves ω_n, and, even for spherical stars, $iW(\xi_n, \xi_n)$ has

no definite sign, we will use W itself to normalize ξ_n, writing

$$\begin{aligned}1 &= W(\xi_n, \xi_n)\\ &= \langle\xi_n|A\partial_t\xi_n + \frac{1}{2}B\xi_n\rangle - \langle A\partial_t\xi_n + \frac{1}{2}B\,\xi_n|\xi_n\rangle\\ &= \langle\xi_n|2i\omega_n A\xi_n + B\xi_n\rangle. \qquad (B2)\end{aligned}$$

We now assume that the modes are non-degenerate,

$$\omega_n \neq \omega_{n'}, \qquad \text{for } n \neq n', \qquad (B3)$$

implying the orthogonality relation (122),

$$W(\xi_n, \xi_{n'}) = 0, \quad \omega_n \neq \omega_{n'}, \qquad W(\xi_n^*, \xi_n) = 0,$$

and we assume that there are no zero frequency modes. We adopt the convention $\omega_n > 0$ and write a general solution to the homogeneous equation in the form

$$\begin{aligned}\xi &= \sum_n (C_{n+}\xi_n + C_{n-}\xi_n^*)\\ &= \sum_n \left(C_{n+}\tilde{\xi}_n e^{i\omega_n t} + C_{n-}\tilde{\xi}_n^* e^{-i\omega_n t}\right), \qquad (B4)\end{aligned}$$

where $\xi_n(t, x) = \tilde{\xi}_n(x)e^{i\omega_n t}$. The coefficients $C_{n\pm}$ are then given by

$$C_{n+} = W(\xi_n, \xi), \quad C_{n-} = W(\xi_n^*, \xi). \qquad (B5)$$

For a real solution, we have $C_{n-} = C_{n+}^*$.

The familiarity of an expansion in terms of orthonormal eigenfunctions belies a subtlety of the system: Completeness of the modes means completeness of the pairs of initial data

$$(\xi_{n\pm}, \partial_t\xi_{n\pm})|_{t=0} = (\tilde{\xi}_n, \pm i\omega_n\tilde{\xi}_n). \qquad (B6)$$

That is, arbitrary initial data $(\xi, \partial_t\xi)_{t=0}$ in the domain of the operators has a spectral decomposition of the form

$$\left.\begin{pmatrix}\xi\\ \partial_t\xi\end{pmatrix}\right|_{t=0} = C_{n+}\begin{pmatrix}\tilde{\xi}_n\\ i\omega_n\tilde{\xi}_n\end{pmatrix} + C_{n-}\begin{pmatrix}\tilde{\xi}_n^*\\ -i\omega_n\tilde{\xi}_n^*\end{pmatrix}. \qquad (B7)$$

The coefficients $C_{n\pm}$ in the expansion of ξ appear to determine the coefficients $\pm i\omega_n C_{n\pm}$ in the expansion of $\partial_t\xi$. How is this possible, when ξ and $\partial_t\xi$ are each arbitrary? The explanation is that the two sets of eigenfunctions $\{\tilde{\xi}_n\}$ and $\{\tilde{\xi}_n^*\}$ are not linearly independent; thus in Eq. (B7) the equation for ξ (or for $\partial_t\xi$) alone does not determine C_{n+} and C_{n-}. Each set $\{\tilde{\xi}_n\}$ and $\{\tilde{\xi}_n^*\}$ is separately a basis for the configuration space H of the system, and using both gives a basis $\{(\tilde{\xi}_n, i\omega_n\tilde{\xi}_n), (\tilde{\xi}_n^*, -i\omega_n\tilde{\xi}_n^*)\}$ for the set $H \times H$ of pairs $(\xi, \partial_t\xi)$.

This behavior – the fact that the set $\{\tilde{\xi}_n\}$ of vectors associated with $\{\omega_n\}$ and the set $\{\tilde{\xi}_n^*\}$ of vectors associated with $\{-\omega_n\}$ are each a basis for H is clear for the

homogeneous equation of a spherical star. Here a mode satisfies

$$-\omega_n^2 A\xi_n + C\xi_n = 0 \,. \tag{B8}$$

If the eigenvalue ω_n^2 is nondegenerate, then the normalized eigenvectors associated with ω_n and $-\omega_n$ differ only by a constant phase; they coincide as rays in a Hilbert space. In the more general case of a stable rotating star with a discrete spectrum, the fact that the sets $\{\xi_n\}$ and $\{\xi_n^*\}$ are each a basis for H is shown by Dyson and Schutz.

Consider now a solution $\xi(t)$ to the inhomogeneous equation (108). Completeness of the normal modes for data on each constant t hypersurface means that, at each time t, we can find coefficients $c_{n\pm}(t)$ that satisfy

$$\begin{pmatrix} \xi \\ \partial_t\xi \end{pmatrix} = \sum_n \left[c_{n+}(t)\begin{pmatrix} \widetilde{\xi}_n \\ i\omega_n\widetilde{\xi}_n \end{pmatrix} + c_{n-}(t)\begin{pmatrix} \widetilde{\xi}_n^* \\ -i\omega_n\widetilde{\xi}_n^* \end{pmatrix} \right]. \tag{B9}$$

By inserting the eigenfunction expansion into this equation and using the symplectic product W to project onto each mode $\widetilde{\xi}_n$, we will find for $c_{n\pm}(t)$ the dynamical equations

$$\dot{c}_{n+} - i\omega_n c_{n+} = \langle \widetilde{\xi}_n|F \rangle \,, \tag{B10a}$$

$$\dot{c}_{n-} + i\omega_n c_{n-} = \langle \widetilde{\xi}_n^*|F \rangle \,. \tag{B10b}$$

The derivation is as follows. From its definition (118), W can be regarded as acting on pairs $(\xi, \partial_t\xi)$ and $(\eta, \partial_t\eta)$ of data at a time t, with

$$W[(\xi,\partial_t\xi);(\eta,\partial_t\eta)] := W(\xi,\eta)$$
$$= \langle\xi|A\partial_t\eta + \tfrac{1}{2}B\eta\rangle - \langle A\partial_t\xi + \tfrac{1}{2}B\xi|\eta\rangle \,. \tag{B11}$$

For mode data $(\widetilde{\xi}_n, i\omega_n\widetilde{\xi}_n)$, the relations $A^\dagger = A, B^\dagger = -B$ give

$$W[(\widetilde{\xi}_n, i\omega_n\widetilde{\xi}_n);(\eta,\partial_t\eta)] = \langle\widetilde{\xi}_n|A\partial_t\eta + i\omega_n A\eta + B\eta\rangle \,, \tag{B12}$$

and Eq. (B9) then implies

$$c_{n+}(t) = W[(\widetilde{\xi}_n, i\omega_n\widetilde{\xi}_n);(\xi(t),\partial_t\xi(t))]$$
$$= \langle\widetilde{\xi}_n|A\partial_t\xi + i\omega_n A\xi + B\xi\rangle \,. \tag{B13}$$

Taking the time derivative of this equation and using Eq. (108) to replace $A\partial_t^2\xi$ by $-B\partial_t\xi - C\xi + F$, we obtain

$$\dot{c}_{n+}(t) = \langle -C\widetilde{\xi}_n|\xi\rangle + \langle\widetilde{\xi}_n|i\omega_n A\partial_t\xi_n + F\rangle \,. \tag{B14}$$

The homogeneous equation for the mode ξ_n implies

$$C\widetilde{\xi}_n = \omega_n^2 A\widetilde{\xi}_n - i\omega_n B\widetilde{\xi}_n \,, \tag{B15}$$

whence

$$\langle -C\widetilde{\xi}_n|\xi\rangle = \langle -\omega_n^2 A\widetilde{\xi}_n + i\omega_n B\widetilde{\xi}_n|\xi\rangle$$
$$= i\omega_n\langle\widetilde{\xi}_n|i\omega_n A\xi + B\xi\rangle \,. \tag{B16}$$

Finally, from Eqs. (B16) and (B14), we have

$$\dot{c}_{n+}(t) = i\omega_n\langle\widetilde{\xi}_n|A\partial_t\xi + i\omega_n A\xi + B\xi\rangle + \langle\widetilde{\xi}_n|F\rangle$$
$$= i\omega_n c_{n+}(t) + \langle\widetilde{\xi}_n|F\rangle \,, \tag{B17}$$

with Eq. (B13) used to obtain the last equality. The same steps with c_{n+}, ξ_n and ω_n replaced by c_{n-}, ξ_n^* and $-\omega_n$, respectively, yield the corresponding equation for $\dot{c}_{n-}(t)$. To summarize, the driven system is governed by the equations

$$\dot{c}_{n+} - i\omega_n c_{n+} = \langle\widetilde{\xi}_n|F\rangle \tag{B18a}$$

$$\dot{c}_{n-} + i\omega_n c_{n-} = \langle\widetilde{\xi}_n^*|F\rangle \,. \tag{B18b}$$

For an exponentially growing driving force $F_i(t,x) = \widetilde{F}_i(x)e^{2\beta t}$, the mode amplitudes of the particular solution ξ^i to Eq. (B1) with time dependence $e^{2\beta t}$ are given by

$$c_{n+}(t) = c_{n-}^*(t) = \frac{1}{2\beta - i\omega_n}\langle\widetilde{\xi}_n|\widetilde{F}\rangle e^{2\beta t} \,, \tag{B19}$$

and we have

$$\xi^i = \sum_n 2\Re\left[\frac{1}{2\beta - i\omega_n}\langle\widetilde{\xi}_n|F\rangle\widetilde{\xi}_n^i\right] \,. \tag{B20}$$

To estimate the magnitude of $\xi^{(2)}$ in Sec. V, it is helpful to rewrite this expression in terms of mode functions $\widehat{\xi}_n^i$ normalized by

$$\langle\widehat{\xi}_n|\rho\widehat{\xi}_n\rangle = 1 \,. \tag{B21}$$

We first find the symplectic norm of the mode functions $\widetilde{\xi}_n$. From Eqs. (97a), (111) and (B2), we have

$$\begin{aligned} 1 &= W(\widetilde{\xi}_n, \widetilde{\xi}_n) \\ &= \langle\widetilde{\xi}_n|2i\omega_n A\widetilde{\xi}_n + B\widetilde{\xi}_n\rangle \\ &= \langle\widetilde{\xi}_{n\,i}|2i\omega_n\rho\widetilde{\xi}_n^i - 2\rho\Omega\epsilon^i{}_j\widetilde{\xi}_n^j\rangle \\ &= 2i\left(\omega_n\int dV\rho|\widetilde{\xi}|^2 - 2\Omega\,\Im\int dV\rho\widetilde{\xi}_n^{\varpi*}\widetilde{\xi}_n^\phi\right) \\ &= 2i\omega_n\kappa_n\langle\widetilde{\xi}_n|\rho\widetilde{\xi}_n\rangle \,, \end{aligned} \tag{B22}$$

where

$$\kappa_n = 1 - 2\frac{\Omega}{\omega_n}\Im\frac{\int dV\rho\widetilde{\xi}_n^{\varpi*}\widetilde{\xi}_n^{\hat\phi}}{\int dV\rho|\widetilde{\xi}_n|^2} \,. \tag{B23}$$

The mode functions $\widetilde{\xi}_n$ are then given in terms of the $\widehat{\xi}_n$ of Eq. (B21) by

$$\widetilde{\xi}_n = \frac{1}{\sqrt{2i\omega_n\kappa_n}}\widehat{\xi}_n \,, \tag{B24}$$

and we obtain Eq. (125) for the exponentially growing solution prior to saturation,

$$\xi^{(2)i} = \sum_n \Re\left[\frac{1}{i\kappa_n\omega_n(2\beta - i\omega_n)}\langle\widehat{\xi}_n|F\rangle\widehat{\xi}_n^i\right] \,. \tag{B25}$$

After saturation, the displacement oscillates about an equilibrium position given by

$$\xi^{(2)\,i} = \sum_n \Re\left[\frac{1}{\kappa_n \omega_n^2}\langle\widehat{\xi}_n|F\rangle\widehat{\xi}_n^i\right]\,, \qquad \text{(B26)}$$

where F is the value of the forcing term at saturation.