Spectroscopy of exotic hadrons formed from dynamical diquarks

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Spectroscopy of Exotic Hadrons Formed from Dynamical Diquarks

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The dynamical diquark picture asserts that exotic hadrons can be formed from widely separated colored diquark or triquark components. We use the Born-Oppenheimer (BO) approximation to study the spectrum of states thus constructed, both in the basis of diquark spins and in the basis of heavy quark-antiquark spins. We develop a compact notation for naming these states, and use the results of lattice simulations for hybrid mesons to predict the lowest expected BO potentials for both tetraquarks and pentaquarks. We then compare to the set of exotic candidates with experimentally determined quantum numbers, and find that all of them can be accommodated. Once decay modes are also considered, one can develop selection rules of both exact (J^{PC} conservation) and approximate (within the context of the BO approximation) types and test their effectiveness. We find that the most appealing way to satisfy both sets of selection rules requires including additional low-lying BO potentials, a hypothesis that can be checked on the lattice.

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I. INTRODUCTION

In a period of less than 15 years, the number of observed heavy-quark exotic hadron candidates has grown from none to over 30 [1–3]. Even so, the nature of the substructure of these novel states remains hotly disputed. In addition to the possibility that some of the neutral exotics are heavy quark-antiquark ($Q\bar{Q}$) hybrid states [4], a variety of multiquark options have been advocated: hadronic molecules $(Q\bar{q}')(\bar{Q}q)$ of color-singlet hadrons, including kinematic enhancements due to the proximity of hadronic thresholds (reviewed in [5]); hadroquarkonium [6, 7], in which the $Q\bar{Q}$ pair forms a compact core surrounded by a larger light-quark $\bar{q}'q$ wave function; and diquark models (most notably, in Ref. [8]), in which the quark (and antiquark) pairs form close associations through the attractive color $\mathbf{3}\otimes\mathbf{3}\to\bar{\mathbf{3}}$ and $\bar{\mathbf{3}}\otimes\bar{\mathbf{3}}\to\mathbf{3}$ channels to form quasi-bound diquarks, $\delta\equiv(Qq)_{\bar{\mathbf{3}}}$ and $\bar{\delta}\equiv(\bar{Q}\bar{q}')_{\mathbf{3}}$, respectively.

The dynamical diquark *picture* [9] is a physical paradigm in which some of the light quarks $q, \bar{q}', \ldots$ created in the production process of a heavy quark-antiquark pair $Q\bar{Q}$ not only coalesce into diquarks through the color mechanism just described, but achieve a substantial spatial separation by virtue of recoils achieved through the large energies available in processes such as $b\to c$ decays or collider events. Originally posited as a natural mechanism for creating tetraquark states that remain strongly bound despite their large spatial extent (>1 fm) due to color confinement of the diquark-antidiquark pair, the picture can easily be extended to pentaquarks and beyond [10], by using the successive accretion of additional quarks through the color-triplet channel attraction. For example, pentaquarks can be interpreted as triquark-diquark $\bar{\theta}\delta\equiv[\bar{Q}(q_1q_2)_{\bar{\mathbf{3}}}]_{\mathbf{3}}(Qq_3)_{\bar{\mathbf{3}}}$ states. The substantial relative strength of the diquark color-triplet attraction compared to the quark-antiquark color-singlet attraction (which is a factor of $\frac{1}{2}$ at short distances) suggests that diquark formation should be a common feature of hadronic processes; for example, a simple treatment of a collection of quarks and antiquarks as a static ideal gas predicts diquark attraction to be the dominant interaction $O(10\%)$ of the time [11].

In order for this picture to be physically meaningful, the diquarks must be somewhat spatially compact and achieve some reasonable spatial separation, so that the state may exhibit some distinctive physical signature different from that of other structures, such as compact tetraquarks or hadronic molecules. "Reasonable" in this sense means that the bulk of the wave functions of the distinct diquarks do not significantly overlap. Alternatively, it is worth noting that diquarks have instead been considered as long-distance correlated quark pairs [12], not unlike electron Cooper pairs, but this scenario is not the one under scrutiny here.

The purpose of this paper is to initiate the development of a dynamical diquark *model*, thus turning the physical picture of rapidly separating colored constituents into a formalism from which quantifiable predictions for masses, selection rules for decay channels, and branching fractions can be made under various assignments of the relevant model parameters.

The first step in this direction is to describe how to express the quantization of the δ-$\bar{\delta}$ (or $\bar{\theta}$-δ) system in order to identify the appropriate combinations of quantum numbers leading to the spectrum of mass eigenstates. Already in the initial presentation of the dynamical diquark picture [9], such a configuration was described as a color flux tube connecting the δ-$\bar{\delta}$ pair, which eventually absorbs all of the available kinetic energy in the process until the pair comes relatively to rest. The principal calculation of Ref. [9] supposed that the $Z_c(4430)^-$ resonance appearing in $\Lambda_b\to(\pi^-\psi(2S))K^-$ results from a δ-$\bar{\delta}$ pair of known masses recoiling against the K^-,

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the potential between them assumed to be of the classic Coulomb-plus-linear Cornell type [13, 14]. The final separation of the δ-$\bar{\delta}$ pair was calculated to be 1.16 fm, comparable to (indeed, larger than) the expected spatial extent of the $\psi(2S)$ wave function but much larger than that of the J/ψ—providing a natural explanation why $Z_c(4430)^-$ decays far more to $\psi(2S)$ than to J/ψ [15].

That the system should not undergo significant hadronization prior to this moment is suggested by the Wentzel-Kramers-Brillouin (WKB) approximation, which favors transitions when the configuration lies near its classical turning point, since it gives an approximate wave function

$$\psi(x) \simeq \frac{C}{\sqrt{p(x)}} e^{\pm \frac{i}{\hbar} \int p(x) dx} , \qquad (1)$$

where $p(x) = \sqrt{2\mu[E - V(x)]}$ is the classical constituent relative momentum. Such a configuration—two color sources well separated and connected by strongly interacting field configurations that may carry nontrivial quantum numbers—is precisely the one used for heavy quarkonium hybrid studies, particularly those performed on the lattice, an approach begun decades ago [16]. In particular, the *Born-Oppenheimer* (BO) *approximation* [17], originally applied to atoms and molecules, proves instrumental for studying systems containing heavy, slow-moving and light, fast-moving degrees of freedom, which is the expectation for the δ-$\bar{\delta}$ state in its final moments prior to hadronization.

The dynamical diquark approach is rather different from those applied to hidden-color multiquark states in the past, in that the state is bound only by confinement until it can decay into hadrons, rather than existing as a (quasi-)static configuration. To provide context on related works with different perspectives, Sec. II briefly describes other directions for studying such states.

The BO approximation as applied to $Q\bar{Q}$ exotics is reviewed briefly in Sec. IV. We note immediately that its ground-state multiplet, conventionally denoted by the BO potential Σ_g^+, will be seen to coincide with the states one obtains from a δ-$\bar{\delta}$ Hamiltonian approach [8, 18], and we enumerate and develop a notation for these states even sooner, in Sec. III. However, in the BO approach one need not restrict to a single compact state with only contact interactions to obtain this result; the same lowest multiplet occurs even if the state has significant spatial extent. For example, in the case of $Q\bar{Q}$ hybrids, the Σ_g^+ multiplet simply represents conventional quarkonium states without excited glue (as well as states in which the glue content has all singlet quantum numbers). In that case, the mass gap between the lightest charmonium states (η_c, J/ψ) and the lightest true hybrids, as calculated on the lattice [19], is about 1.1–1.3 GeV. However, since the extended spatial structure of the Σ_g^+ states in the dynamical diquark picture is expected to be similar to that of the excited BO potential states, one may anticipate a smaller mass gap, say in the several hundred MeV range. They could be comparable to typical radial excitation energies in charmonium [$m_{\psi(2S)} - m_{J/\psi(1S)} \approx 600$ MeV], or even as small as typical orbital excitations [$m_{\chi_c(1P)} - m_{J/\psi(1S)} \approx 400$ MeV]. If this possibility is realized, then the dynamical diquark-generated phenomenological spectrum could be much richer than naively expected, due to the intermingling of orbitally, radially, and BO-excited exotic states.

By "light" quarks q, q' in this paper we mean only u or d, and by "heavy" quarks Q we mean only c or b. However, depending upon the circumstance, s quarks may be considered light (leading to a separate spectrum of, say, $c\bar{c}s\bar{s}$ exotics, for which the $X(3915)$ state was proposed as the $J^{PC} = 0^{++}$ ground state, and exotics with the observed decay channel $J/\psi\,\phi$ are natural candidates [20]) or heavy (for which one may seek interesting possible experimental signals of their production, such as the small forward and backward enhancements in the rate for $\gamma p \to \phi p$, which are consistent with a pentaquark-like structure [21]). For the heavy quarks, we have taken Q and $\bar{Q}$ to have the same flavor (b or c), but exotic B_c-like hadrons, as well as doubly heavy cc, bb, and bc states, can also be studied within the BO approach, although the treatment of the inversion quantum numbers P and C must be adjusted accordingly.

This paper is organized as follows. Section II presents a brief summary of alternate approaches studying multiquark configurations interacting through color flux tubes or potentials, or on the lattice. In Sec. III, we begin by enumerating the states in the ground-state BO potential [$\Sigma_g^+(1L)$, $L = 0, 1, 2, \ldots$] in both δ-$\bar{\delta}$ and heavy-quark spin bases. We then give a brief review of BO potentials as pertaining to $Q\bar{Q}q'\bar{q}$ states in Sec. IV, and in Sec. V identify the lowest expected potentials and catalogue the corresponding spectra of δ-$\bar{\delta}$ states in Table I. In Sec. VI we carry out this exercise for $\bar{\theta}$-δ pentaquark states, with the results listed in Table II. Section VII compares the list of exotics with known quantum numbers to the spectra predicted in the previous two sections, and then extends the analysis to discuss constraints from heavy-quark spin symmetry and decay selection rules, both exact and approximate. In Sec. VIII we summarize and indicate future directions of research.

II. ALTERNATE HIDDEN-COLOR MULTIQUARK APPROACHES

By "hidden color" here, we mean states containing subunits carrying nontrivial color charge, as opposed to only color-singlet (hadronic) subunits. These configurations were described many years ago in terms color flux-tube structures connected in different topologies, depending upon whether the shortest (energetically favored) tube configurations connect qq and $\bar{q}\bar{q}$ (diquark-like) pairs or $q\bar{q}$ (meson-like) pairs. The transition between the two was called a "flip-flop" [22–24]. Numerical simulations have indeed found that such diquark-type structures occur when the quarks are initially closer to each other than to antiquarks, even when the relative distances are not dramatically different [25–29]. If the coupling between flux-tube configurations is weak, however, Ref. [30] claims an absence of bound multiquark states.

Not every simulation uncovers a diquark-antidiquark-like structure, however. For example, Ref. [31] found no exotic $cc\bar{u}\bar{d}$ state, but note that this state contains cc rather than $c\bar{c}$, and the diquarks are $(cc)(\bar{u}\bar{d})$. On the other hand, Ref. [32] studies $\bar{c}c\bar{d}u$ but does not find evidence for an exotic Z_c^+ state, but the authors offer caveats that the basis of interpolating operators must be carefully considered to include all coupled meson-meson states, and that diquark-antidiquark operators having the same color structure after Fierz rearrangement as such states. Comments about the incompleteness of the operator basis are also voiced in Ref. [33].

Quark potential models with conventional two-body forces have also been found to support four-quark structures, particularly in the $Q\bar{Q}q\bar{q}$ case [34]. An extensive study of possible multiquark structures [35] finds the presence of hidden-color structures is quite feasible in the light-quark sector. Potentials based on confinement and instanton effects are used to study $qq\bar{q}\bar{q}$ bound states in Ref. [36], but no good observed candidates are found.

However, quark models based solely on string-type confining potentials [37] give more encouraging results for $QQ\bar{q}\bar{q}$, although the authors suggest that bound $Q\bar{Q}q\bar{q}$ bound states may require diquark substructure in order to occur [38]. $QQ\bar{q}\bar{q}$ bound states also appear in the more elaborate string-potential model calculations of Ref. [39].

From this discussion, it should be clear that the status of calculations of multiquark states is a delicate matter, depending upon the precise modeling of the state and its interactions. It should be equally clear that the dynamical diquark picture does not obviously appear to fit into any of these paradigms, especially as the system remains in a state of rapid change until the moment it decays. In order to model such a system appropriately, we therefore seek to describe it based not upon static structures, but upon symmetries: hence the introduction of the Born-Oppenheimer approximation. We start with the ground-state band where the symmetries are trivial, and then turn to the description of the excited states.

III. GROUND-STATE BAND

In order to identify the δ-$\bar{\delta}$ states of the Σ_g^+ BO potential, we employ notation as close as possible to that of Ref. [18]. Starting with the 4 quark spins and no orbital excitation, one may couple the angular momenta to obtain a state of total constituent spin S in several different orders, but the most convenient for our purpose are $(Q\bar{Q})+(q\bar{q})$ and $(qQ)+(\bar{q}\bar{Q})$. The first option uses eigenstates of heavy-quark spin, while the second uses eigenstates of diquark spin. Of course, all orders of coupling are connected by the relevant recoupling coefficients, which in this case are $9j$ symbols:

$$\begin{aligned} &\left\langle (s_q\, s_{\bar{q}})s_{q\bar{q}}, (s_Q\, s_{\bar{Q}})s_{Q\bar{Q}}, S \,\middle|\, (s_q\, s_Q)s_\delta, (s_{\bar{q}}\, s_{\bar{Q}})s_{\bar{\delta}}, S\right\rangle \\ &= \left([s_{q\bar{q}}][s_{Q\bar{Q}}][s_\delta][s_{\bar{\delta}}]\right)^{1/2} \begin{Bmatrix} s_q & s_{\bar{q}} & s_{q\bar{q}} \\ s_Q & s_{\bar{Q}} & s_{Q\bar{Q}} \\ s_\delta & s_{\bar{\delta}} & S \end{Bmatrix} , \end{aligned} \tag{2}$$

where $[s] \equiv 2s+1$ simply denotes the multiplicity of a spin-s state. Although we have here implicitly taken the light quarks $q, \bar{q}$ to form a charge-conjugate pair, it is simple to generalize to the case $q, \bar{q}'$, where $q, q' \in \{u, d\}$, which generates an $I=0$ and three $I=1$ states. The neutral $(I_3 = 0)$ isosinglet and isotriplet eigenstates carry definite C eigenvalues, and in both cases the C-parity of these states can be used to determine for all of the states the G-parity eigenvalues $C(-1)^I$.

Although at this stage we still consider only S-wave states, let us discuss the spatial-inversion parity eigenvalues P and C for arbitrary L. Using the usual reasoning applied to the corresponding eigenvalues for conventional $q\bar{q}$ mesons, P contains a factor $(-1)^L$ from the inversion properties of orbital wave functions and a (-1) from the intrinsic parity of each $q\bar{q}$ pair (both light and heavy). Charge conjugation of the $q\bar{q}$ pairs is equivalent to a combination of spatial inversion and the sign obtained from exchange of the q, $\bar{q}$ spins, $(-1)^{s_{q\bar{q}}+1}$. One therefore obtains the eigenvalues

$$P = (-1)^L \, , \quad C = (-1)^{L+s_{q\bar{q}}+s_{Q\bar{Q}}} \, . \tag{3}$$

In particular, all S-wave tetraquarks have $P=+$, and the $(q\bar{q}),(Q\bar{Q})$ basis is more convenient for identifying C (and G) eigenstates than the $\delta, \bar{\delta}$ basis. In the $(q\bar{q}),(Q\bar{Q})$ basis, and using the notation of Ref. [18], one obtains the following states:

$$\begin{aligned} J^{PC} = 0^{++}: \quad & X_0 \equiv \frac{1}{2}\left|0_{q\bar{q}}, 0_{Q\bar{Q}}\right\rangle_0 + \frac{\sqrt{3}}{2}\left|1_{q\bar{q}}, 1_{Q\bar{Q}}\right\rangle_0 \, , \\ & X_0' \equiv \frac{\sqrt{3}}{2}\left|0_{q\bar{q}}, 0_{Q\bar{Q}}\right\rangle_0 - \frac{1}{2}\left|1_{q\bar{q}}, 1_{Q\bar{Q}}\right\rangle_0 \, , \\ J^{PC} = 1^{++}: \quad & X_1 \equiv \left|1_{q\bar{q}}, 1_{Q\bar{Q}}\right\rangle_1 \, , \\ J^{PC} = 1^{+-}: \quad & Z \equiv \frac{1}{\sqrt{2}}\left(\left|1_{q\bar{q}}, 0_{Q\bar{Q}}\right\rangle_1 - \left|0_{q\bar{q}}, 1_{Q\bar{Q}}\right\rangle_1\right) \, , \\ & Z' \equiv \frac{1}{\sqrt{2}}\left(\left|1_{q\bar{q}}, 0_{Q\bar{Q}}\right\rangle_1 + \left|0_{q\bar{q}}, 1_{Q\bar{Q}}\right\rangle_1\right) \, , \\ J^{PC} = 2^{++}: \quad & X_2 \equiv \left|1_{q\bar{q}}, 1_{Q\bar{Q}}\right\rangle_2 \, , \end{aligned} \tag{4}$$

where outer subscripts indicate total component spin S.

Equivalently, using Eq. (2), the states in the $\delta, \bar{\delta}$ basis read:

$$\begin{aligned} J^{PC} = 0^{++}: \quad & X_0 = \left|0_\delta, 0_{\bar{\delta}}\right\rangle_0 \, , \quad X_0' = \left|1_\delta, 1_{\bar{\delta}}\right\rangle_0 \, , \\ J^{PC} = 1^{++}: \quad & X_1 = \frac{1}{\sqrt{2}}\left(\left|1_\delta, 0_{\bar{\delta}}\right\rangle_1 + \left|1_\delta, 0_{\bar{\delta}}\right\rangle_1\right) \, , \\ J^{PC} = 1^{+-}: \quad & Z = \frac{1}{\sqrt{2}}\left(\left|1_\delta, 0_{\bar{\delta}}\right\rangle_1 - \left|0_\delta, 1_{\bar{\delta}}\right\rangle_1\right) \, , \\ & Z' = \left|1_\delta, 1_{\bar{\delta}}\right\rangle_1 \, , \\ J^{PC} = 2^{++}: \quad & X_2 = \left|1_\delta, 1_{\bar{\delta}}\right\rangle_2 \, . \end{aligned} \tag{5}$$

Note that the pairs X_0, X_0' and Z, Z', carrying the same J^{PC}, can certainly mix. If one requires a basis of states with definite values of heavy-quark spin, then the most convenient combinations are:

$$\begin{aligned}
\tilde{X}_0 &\equiv \left|0_{q\bar{q}}, 0_{Q\bar{Q}}\right\rangle_0 = +\frac{1}{2}X_0 + \frac{\sqrt{3}}{2}X_0' \,, \\
\tilde{X}_0' &\equiv \left|1_{q\bar{q}}, 1_{Q\bar{Q}}\right\rangle_0 = +\frac{\sqrt{3}}{2}X_0 - \frac{1}{2}X_0' \,, \\
\tilde{Z} &\equiv \left|1_{q\bar{q}}, 0_{Q\bar{Q}}\right\rangle_1 = \frac{1}{\sqrt{2}}\left(Z' + Z\right) \,, \\
\tilde{Z}' &\equiv \left|0_{q\bar{q}}, 1_{Q\bar{Q}}\right\rangle_1 = \frac{1}{\sqrt{2}}\left(Z' - Z\right) \,. \qquad (6)
\end{aligned}$$

Whenever the symbols X_0, X_0' (Z, Z') appear below, it should be understood that $\tilde{X}_0$, $\tilde{X}_0'$ ($\tilde{Z}$, $\tilde{Z}'$) work equally well, while the forms with tildes are specified if states of definite $s_{Q\bar{Q}}$ eigenvalues are preferred.

Turning next to the $L > 0$ states in the ground-state band, one may use the usual rules of angular momentum addition to derive the spectrum based upon the states X_0, X_0', X_1, Z, Z', X_2 listed in Eq. (4) or (5), by appending a subscript letter for the L eigenvalue and a superscript number in parentheses for the total J eigenvalue:

$$\begin{aligned}
&\underline{J^{PC} = L^{(-1)^L,\,(-1)^L}} \\
&\quad X_{0\,L}^{(L)},\ X_{0\,L}'^{(L)}\,, \\
&\underline{J^{PC} = (L-1, L, L+1)^{(-1)^L,\,(-1)^L}} \\
&\quad X_{1\,L}^{(L-1)},\ X_{1\,L}^{(L)},\ X_{1\,L}^{(L+1)}, \\
&\underline{J^{PC} = (L-1, L, L+1)^{(-1)^L,\,(-1)^{L+1}}} \\
&\quad Z_L^{(L-1)},\ \ Z_L^{(L)},\ \ Z_L^{(L+1)}, \\
&\quad Z_L'^{(L-1)},\ Z_L'^{(L)},\ Z_L'^{(L+1)}, \\
&\underline{J^{PC} = (L-2, L-1, L, L+1, L+2)^{(-1)^L,\,(-1)^L}} \\
&\quad X_{2\,L}^{(L-2)},\ X_{2\,L}^{(L-1)},\ X_{2\,L}^{(L)},\ X_{2\,L}^{(L+1)},\ X_{2\,L}^{(L+2)}\,. \qquad (7)
\end{aligned}$$

The S-wave states of Eq. (4) or (5) are of course only those in Eq. (7) with the largest J value in each category: $X_0 = X_{0\,S}^{(0)}$, $X_0' = X_{0\,S}'^{(0)}$, $X_1 = X_{1\,S}^{(1)}$, $Z = Z_S^{(1)}$, $Z' = Z_S'^{(1)}$, $X_2 = X_{2\,S}^{(2)}$. For the P-wave states, one must also eliminate the first two states in the final category: Explicitly, one obtains the states $J^{PC} = 2 \times 1^{--}$ $[X_{0\,P}^{(1)}, X_{0\,P}'^{(1)}]$, $(0,1,2)^{--}$ $[X_{1\,P}^{(0),(1),(2)}]$, $2 \times (0,1,2)^{-+}$ $[Z_P^{(0),(1),(2)}, Z_P'^{(0),(1),(2)}]$, and $(1,2,3)^{--}$ $[X_{2\,P}^{(1),(2),(3)}]$. In this notation, the exhaustive list of 1^{--} states given in Ref. [18] obtained by allowing all values of L reads

$$\begin{aligned}
&Y_1 \equiv X_{0\,P}^{(1)}\,,\ Y_2 \equiv X_{1\,P}^{(1)}\,,\ Y_3 \equiv X_{0\,P}'^{(1)}\,, \\
&Y_4 \equiv X_{2\,P}^{(1)}\,,\ Y_5 \equiv X_{2\,F}^{(1)}\,. \qquad (8)
\end{aligned}$$

As a final illustration, the list of D-wave states reads $J^{PC} = 2 \times 2^{++}$ $[X_{0\,D}^{(2)}, X_{0\,D}'^{(2)}]$, $(1,2,3)^{++}$ $[X_{1\,D}^{(1),(2),(3)}]$, $2 \times (1,2,3)^{+-}$ $[Z_D^{(1),(2),(3)}, Z_D'^{(1),(2),(3)}]$, and $(0,1,2,3,4)^{++}$ $[X_{2\,D}^{(0),(1),(2),(3),(4)}]$.

Let us compare the number of conventional quarkonium states to the number of tetraquark states listed above, including isospin. For $L = 0, 1, 2, 3, \ldots$, one counts $2, 4, 4, 4, \ldots$ conventional states [the usual η, ψ (or Υ), h, χ combinations] and $24, 56, 64, 64, \ldots$ tetraquark states.[1] Again, this counting represents only the (radial) ground-state band (corresponding to a principal quantum number $n = 1$), but it does count all isospin states separately. Not counting $I_3 = \pm 1$ charge conjugates as distinct, the numbers reduce by 25%, to $18, 42, 48, 48, \ldots$. While to date, 28 bosonic charmoniumlike exotics have been observed (not counting charge conjugates), this number pales in comparison to that for potential future discoveries, should even a fraction of the predicted states actually exist. To emphasize this point, note that not even all of the $n = 1$ D-wave *conventional* quarkonium states have yet been seen.

A well-known problem for diquark models is their tendency to produce large numbers of unobserved states. This overabundance occurs because any Qq or $\bar{Q}\bar{q}$ pair is considered suitable for forming a diquark, regardless of its spin: The expectation in the light-quark sector for nature to prefer a "good" (spin-0) diquark over a "bad" (spin-1) diquark [46] is greatly reduced for heavy-quark systems (since these two types of quasiparticle differ in mass by a heavy-quark spin flip, which costs an energy proportional to $\Lambda_{\rm QCD}^2/m_Q$), so that both types of diquark are expected to be equally prevalent. Furthermore, since isospin symmetry of strong interactions implies that the replacement of $u \leftrightarrow d$ quarks makes little change to the heavy diquarks, then in the absence of significant isospin-dependent interactions between the diquarks, one naturally expects tetraquarks formed of such diquarks to appear in nearly degenerate $I = 0$ plus $I = 1$ quartets.

However, in the dynamical diquark picture as opposed to traditional Hamiltonian-based diquark models, significant isospin-dependent interactions may be quite natural due to the extended spatial size of the state. In the case of hadronic molecules, the long-distance color-singlet attraction is expected to be dominated by single-pion exchange since it is by far the lightest hadron, and in turn the pion is light and carries nontrivial isospin due to the Nambu-Goldstone (NG) theorem of chiral symmetry breaking. Interestingly, a version of the NG theorem exists even for colored particles (in the context of *color-flavor locking* [41]), so it is reasonable to expect interactions with both color and isospin dependence between the separated, colored diquarks. The analysis of Ref. [40] argued in the case of hadronic molecules that each of the two light-quark containing mesons contributes an isospin Pauli matrix $\boldsymbol{\tau}_{(k)}$ to the interaction, and $\boldsymbol{\tau}_{(1)} \cdot \boldsymbol{\tau}_{(2)} = -3, +1$ for $I = 0, 1$, meaning that the interaction is binding in one isospin channel and repul-

[1] This counting for the $L = 0$ and $L = 1$ states was carried out in Ref. [40].

sive in the other. Therefore, one expects *only one* of the $I=0$ or $I=1$ states for given angular momentum quantum numbers to be bound, which greatly reduces the expected number of tetraquark states, assuming a long-distance isospin-dependent interaction between the colored diquarks. Determining exactly which of the states are bound of course requires a detailed model.

IV. BORN-OPPENHEIMER POTENTIALS

The Born-Oppenheimer approximation amounts to a scale separation between heavy, slowly changing degrees of freedom (hence effectively acting as static sources) and light degrees of freedom (d.o.f.) that rapidly and adiabatically adjust to the configuration of the heavy ones. The full wave function then factors into a part due to the heavy sources, and a part described by *Born-Oppenheimer potentials* that carry only the quantum numbers of the light d.o.f. but parametrically depend upon the configuration of the heavy sources (hence the term "potentials"). In the original application to atoms and molecules, these d.o.f. are of course the nuclei (mass m_N) and electrons (mass m_e), respectively. The scale separation, expressed in powers of m_e/m_N, provides the necessary small parameter to recast the BO approximation into the modern language of effective field theories [42]. In heavy quarkonium, the $Q\bar{Q}$ pair provides the static sources, while the light d.o.f. are the gluon configuration (for hybrid mesons) or can also include light-quark d.o.f. (for multiquark mesons) [43]. The effective-field theory description arising from the BO approximation for the hybrid case (where the expansion parameter becomes $\Lambda_{\rm QCD}/m_Q$) was first considered in Ref. [44].

The configuration of the heavy d.o.f. is described both by the relative separations of the heavy components and by its symmetry. In the $Q\bar{Q}$ system with a relative separation r and a unit vector $\hat{\boldsymbol{r}}$ pointing from $\bar{Q}$ to Q, the BO potential depends only upon r, and the potentials are labeled by the irreducible representations of the group $D_{\infty h}$, which describes the symmetries of a cylinder with axis $\hat{\boldsymbol{r}}$. The conventional nomenclature [45] for these representations uses the quantum numbers $\Gamma \equiv \Lambda_\eta^\epsilon$, all of which refer to the $D_{\infty h}$ symmetry, as we now describe.

The basic angular momenta of the system are the total $\boldsymbol{J}_{\rm light}$ of the light d.o.f., the orbital angular momentum $\boldsymbol{L}_{Q\bar{Q}}$ of the heavy d.o.f., and spin $\boldsymbol{s}_{Q\bar{Q}}$ of the $Q\bar{Q}$ pair. Due to heavy-quark symmetry, $s_{Q\bar{Q}}$ is a good quantum number of the full state, but $\boldsymbol{J}_{\rm light}$ and $\boldsymbol{L}_{Q\bar{Q}}$ cannot be independently determined, although the Casimirs $J_{\rm light}$ and $L_{Q\bar{Q}}$ can be simultaneously specified. In this definition, the light-quark spin $s_{q\bar{q}}$ (in the case of multiquark hadrons) is incorporated into $\boldsymbol{J}_{\rm light}$. One then defines the total orbital angular momentum as

$$\boldsymbol{L} \equiv \boldsymbol{L}_{Q\bar{Q}} + \boldsymbol{J}_{\rm light}\,, \tag{9}$$

and finally, from coupling L and $s_{Q\bar{Q}}$, one obtains the total angular momentum quantum numbers J, J_z of the state. Since $\hat{\boldsymbol{r}} \cdot \boldsymbol{L}_{Q\bar{Q}} = 0$, the axial angular momentum $\hat{\boldsymbol{r}} \cdot \boldsymbol{J}_{\rm light} = \hat{\boldsymbol{r}} \cdot \boldsymbol{L}$ for the light d.o.f. provides a good quantum number for the system, its eigenvalues denoted by $\lambda = 0, \pm 1, \pm 2, \ldots$. Since the physical system is invariant under a reflection through any plane containing $\hat{\boldsymbol{r}}$ (under which $\lambda \to -\lambda$), its energy eigenvalues cannot depend upon the sign of λ, and from this fact one defines the first of the BO quantum numbers, $\Lambda \equiv |\lambda|$. Potentials with the eigenvalues $\Lambda = 0, 1, 2, \ldots$ are denoted by $\Sigma, \Pi, \Delta, \ldots$, in analogy to the labels $S, P, D, \ldots$ for the quantum numbers $L = 0, 1, 2, \ldots$. From Eq. (9) and $\hat{\boldsymbol{r}} \cdot \boldsymbol{L}_{Q\bar{Q}} = 0$, one immediately notes the constraint

$$L \geq |\hat{\boldsymbol{r}} \cdot \boldsymbol{L}| = |\hat{\boldsymbol{r}} \cdot \boldsymbol{J}_{\rm light}| = \Lambda\,. \tag{10}$$

The light d.o.f. also possess two reflection symmetries. The first is obtained by a reflection through the midpoint of the $Q\bar{Q}$ pair. Since this inversion exchanges the orientation of the light d.o.f. not just with respect to a coordinate origin but also with respect to Q and $\bar{Q}$, it is given not just by the parity operator $P_{\rm light}$, but in fact by the combination $(CP)_{\rm light}$. Its possible eigenvalues $\eta = +1, -1$, denoted by g, u, respectively, provide the second BO quantum number.

The system also possesses, as mentioned above, a symmetry under reflection $R_{\rm light}$ of the light d.o.f. through any plane containing the $Q\bar{Q}$ axis. In particular, the $\Lambda = 0$ (Σ) representations can be distinguished by their behavior under $R_{\rm light}$, with its ± 1 eigenvalue denoted by ϵ, the third BO quantum number. But the $\Lambda > 0$ configurations $|\lambda, \eta; \boldsymbol{r}\rangle$ can also be combined into eigenstates of $R_{\rm light}$ with eigenvalue ϵ: Noting that the light d.o.f. spatial-inversion parity operator $P_{\rm light}$ is simply given by $R_{\rm light}$ multiplied by a rotation by π radians about an axis normal to the plane defining $R_{\rm light}$, one sees for arbitrary λ that $R_{\rm light}\,|\lambda, \eta; \boldsymbol{r}\rangle = (-1)^\lambda \zeta\,|-\lambda, \eta; \boldsymbol{r}\rangle$, where ζ is the intrinsic parity of the light d.o.f. The eigenstate of $R_{\rm light}$ with eigenvalue ϵ for $\Lambda > 0$ is then constructed as

$$|\Lambda, \eta, \epsilon; \boldsymbol{r}\rangle \equiv \frac{1}{\sqrt{2}} \left[|\Lambda, \eta; \boldsymbol{r}\rangle + \epsilon\,(-1)^\Lambda \zeta\,|-\Lambda, \eta; \boldsymbol{r}\rangle \right]\,, \tag{11}$$

and the eigenvalue of $P_{\rm light}$ is deduced to be $\epsilon\,(-1)^\Lambda$.

With the quantum numbers Γ in hand, one then solves the Schrödinger equation of the $Q\bar{Q}$ pair in the BO potential $V_\Gamma(r)$, which produces eigenvalues and eigenfunctions labeled by a principal quantum number n. The full physical states are then completely specified by the kets

$$\left| n, L, s_{Q\bar{Q}}, J m_J; \Lambda, \eta, \epsilon \right\rangle\,, \tag{12}$$

with $J_{\rm light}$ and $L_{Q\bar{Q}}$ eigenvalues implicit. In the multiquark case, the light-quark spin quantum number $s_{q\bar{q}}$ is also implicit, providing in the notation of Ref. [43] a contribution to $\boldsymbol{J}_{\rm light}$.

The overall discrete quantum numbers for the physical state depend upon both the heavy and light d.o.f. Those for the heavy d.o.f. $Q\bar{Q}$ are obtained exactly as for ordinary mesons, while those for the light d.o.f. depend upon whether a $q\bar{q}$ pair is present, which contributes an extra factor (-1) to P and $(-1)^{s_{q\bar{q}}}$ to C. In particular, for hybrids,

$$P = \epsilon\,(-1)^{\Lambda+L+1}\,, \quad (13)$$
$$C = \eta\epsilon\,(-1)^{\Lambda+L+s_{Q\bar{Q}}}\,, \quad (14)$$

while for tetraquarks,

$$P = \epsilon\,(-1)^{\Lambda+L}\,, \quad (15)$$
$$C = \eta\epsilon\,(-1)^{\Lambda+L+s_{q\bar{q}}+s_{Q\bar{Q}}}\,. \quad (16)$$

The C eigenvalue, as before, refers to that of the neutral state of an isospin multiplet; G parity is then given by $G = C(-1)^I$. Significantly, the expressions Eqs. (15)–(16) differ from those in Ref. [43], which are the same for both hybrids and tetraquarks. Even though the light-quark pair has its spin angular momentum $s_{q\bar{q}}$ folded into the total $J_{\rm light}$ in Ref. [43], including its distinct dependence in P and C is necessary to reflect the differing symmetry of the wave functions, especially for differing values of $s_{q\bar{q}}$, which already suggests difficulties for the choice of including $s_{q\bar{q}}$ in $J_{\rm light}$. In particular, one expects states that are identical except for a relative spin flip of the light quarks, $s_{q\bar{q}}\!=\!0 \leftrightarrow s_{q\bar{q}}\!=\!1$, to belong to the same BO potential (fixed $\Gamma = \Lambda^\epsilon_\eta$), but also to have opposite C eigenvalues. This effect is particularly evident in the ground-state band Σ^+_g ($\Lambda = 0$, $\epsilon = \eta = +1$), where one may use simple quark-model reasoning as in Eq. (3). As an explicit example, in the case of $b\bar{b}c\bar{c}$ tetraquarks, for which $m_b \gg m_c \gg \Lambda_{\rm QCD}$, one expects flipping the spin of $\bar{c}$ relative to that of c (using the definition in which $s_{q\bar{q}}$ is a part of $J_{\rm light}$) to affect the value of $J_{\rm light}$ and hence Λ, which would spread these two configurations over different BO potentials. But the energy cost of this spin flip is small, $O(\Lambda^2_{\rm QCD}/m_c)$, suggesting that the BO potentials in the two configurations are the same.

We therefore adopt a more traditional definition of quantum numbers for BO potentials [45]: The angular momentum $\boldsymbol{J}_{\rm light}$ in Eqs. (9) and (10) is understood to exclude intrinsic light-quark spin $\boldsymbol{s}_{q\bar{q}}$, and the BO potential notation becomes $\Gamma \equiv {}^{2s_{q\bar{q}}+1}\Lambda^\epsilon_\eta$, the new superscript indicating the multiplicity of $(s_{q\bar{q}})_z$ eigenstates. From the above example, one also expects the configurations ${}^3\Lambda^\epsilon_\eta(nP)$ and ${}^1\Lambda^\epsilon_\eta(nP)$ to lie fairly close in energy, ignoring possible light-quark spin-dependent interactions such as those correlated with isospin. We therefore suppress the $2s_{q\bar{q}}\!+\!1$ superscript whenever possible.

Also of interest is the possibility of Λ-*doubling* [45], which occurs when two BO potentials produce the same spectrum of states, and therefore can mix. For given eigenvalues of L and Λ satisfying $L \geq 1$ and $L > \Lambda$, the states obtained from the BO potentials $\Lambda^\epsilon_\eta(nL)$ and $(\Lambda+1)^{-\epsilon}_\eta(nL)$ [*e.g.*, $\Sigma^-_u(1P)$ and $\Pi^+_u(1P)$] produce the same spectrum, and potentially can mix. The naive degeneracy between two BO potentials of opposite parity, $\Lambda^\pm_\eta(nL)$ [*e.g.*, $\Pi^\pm_u(1P)$], is thereby lifted. This effect for hybrids was first discussed in Ref. [44].

V. DIQUARK-ANTIDIQUARK BO POTENTIALS

The configuration of the tetraquark state in the dynamical diquark picture is essentially the same as for hybrid heavy-quark mesons: a spatially extended colored field connecting a heavy color-$\mathbf{3}$ ($\bar{\delta}$) source and a heavy color-$\bar{\mathbf{3}}$ source (δ). The sources themselves differ in the two cases: $Q, \bar{Q}$ carry spin $\frac{1}{2}$ and isospin 0, and are essentially pointlike, while δ, $\bar{\delta}$ carry spin 0 or 1 and isospin $\frac{1}{2}$, and are expected to be compact due to the presence of the heavy quark but still be of finite spatial extent ($\lesssim 0.5$ fm for charm [9]).

The static potentials for $Q\bar{Q}$ pairs were first calculated on the lattice some time ago, with the first high-quality results presented in Refs. [47–49], while the first unquenched simulations were carried out in Ref. [50]. A summary of the important landmarks in lattice simulations relevant to heavy-quark hybrids is presented in Ref. [44] (see also [51]). Simulations representing the state of the art for $c\bar{c}$ hybrid mesons are presented by the Hadron Spectrum Collaboration in Ref. [19]. The essential result relevant to the present analysis is that all authors agree the lowest BO potentials are determined to be the ground state Σ^+_g, followed by Π_u and Σ^-_u. The mixing of $\Pi^+_u(1P)$ and $\Sigma^-_u(1P)$ states has been noted in the previous section; but additionally, the Π_u and Σ^-_u BO potentials are seen to become degenerate in the $r\!\to\!0$ limit, giving a single color-adjoint source configuration as $r\!\to\!0$ (in this case, with $J^{PC} = 1^{+-}$) called a *gluelump*.

We therefore suppose that the lowest BO potentials producing tetraquarks in the dynamical diquark picture are the ground-state potentials Σ^+_g, whose S-, P-, and D-wave states have already been enumerated in Sec. III, followed by $\Pi^+_u(1P)$ mixed with $\Sigma^-_u(1P)$, $\Pi^-_u(1P)$, $\Sigma^-_u(1S)$, and $\Pi^+_u(1D)$. This ordering follows the results of the lattice simulations of Ref. [19], with the states identified as originating within specific BO potentials in Refs. [43, 44].

Before listing the spectra associated with these BO potentials, let us make one final modification to the notation $(X_0, X_0', X_1, Z, Z', X_2)^{(J)}_L$ introduced for the Σ^+_g states. Noting from Eqs. (15)–(16) that the P, C eigenvalues for nontrivial BO potentials Λ^ϵ_η differ from those of Σ^+_g only by

$$\rho \equiv \epsilon\,(-1)^\Lambda,\ \ \kappa \equiv \eta\epsilon\,(-1)^\Lambda = \eta\rho\,, \quad (17)$$

we adopt the final notation $(X_0, X_0', X_1, Z, Z', X_2)^{(J)\rho\kappa}_L$ for the tetraquark states. That is, Eqs. (15)–(16) are replaced by

$$P = \rho\,(-1)^L,\ \ C = \kappa\,(-1)^{L+s_{q\bar{q}}+s_{Q\bar{Q}}}\,, \quad (18)$$

which identifies ρ, κ as the "intrinsic" P, C eigenvalues of each particular BO potential. In this notation, one appends a superscript $\rho\kappa\!=\!++$ to all the states obtained from Σ^+_g, and taking $\epsilon \to -\epsilon$ or $\Lambda \to \Lambda \pm 1$ changes both $\rho \to -\rho$ and $\kappa \to -\kappa$, while taking $\eta \to -\eta$ changes $\kappa \to -\kappa$

alone. Taking $\epsilon \to -\epsilon$ or $\Lambda \to \Lambda \pm 1$ or $L \to L \pm 1$ changes both $P \to -P$ and $C \to -C$ for each state in the BO potential, while taking $\eta \to -\eta$ changes only $C \to -C$ for each state.

To see that this notation is easily interpreted, let us consider one specific example: $X_{2\,F}^{(4)\,-+}$. Here, the total component spin $S = 2$ state X_2 is defined in Eq. (4) with $s_{q\bar{q}} = 1$, $s_{Q\bar{Q}} = 1$; in addition, $L = 3$, $J = 4$, $\rho = \epsilon\,(-1)^{\Lambda} = -1$ so that $P = +$, and $\kappa = \eta\rho = +1$ so that $\eta = -1$ and $C = -$: $X_{2\,F}^{(4)\,-+}$ is a $J^{PC} = 4^{+-}$ state. The only ambiguity lies in the combination $\rho = \epsilon\,(-1)^{\Lambda} = -1$; since $\Lambda \leq L = 3$, the BO potentials $[\Sigma_u^-, \Pi_u^+, \Delta_u^-, \Phi_u^+](nF)$ can contribute.

In Table I we list the lowest multiplets of tetraquark states expected in the dynamical diquark picture, both by J^{PC} eigenvalues and the BO potential from which they emerge. States that for $q\bar{q}$ mesons have exotic J^{PC} quantum numbers (specifically, 0^{--} and the series $0^{+-}, 1^{-+}, 2^{+-}, \ldots$) are indicated with boldface. We also use the $(q\bar{q}), (Q\bar{Q})$ basis [Eq. (6)] in order to facilitate comparison in Sec. VII with the expectations of heavy-quark spin symmetry.

VI. PENTAQUARK BO POTENTIALS

The central difference between diquark-antidiquark (δ-$\bar{\delta}$) and triquark-diquark ($\bar{\theta}$-δ) BO potentials is that the latter case is analogous to heteronuclear diatomic molecules: The $\bar{\theta}$ and δ components are in no sense the same, so that the reflection symmetry leading to the $(CP)_{\rm light}$ quantum number η is lost. In addition, the (anti)triquark $\bar{\theta}$ is formed of a *light* diquark δ' in a color-$\bar{\mathbf{3}}$ bound to a heavy $\bar{Q}$ to form an overall color-$\mathbf{3}$. For the purpose of this work, we limit to the case of a $\delta' = (ud)$ diquark in a spin-0, isospin-0 configuration (a "good" diquark [46]), such as those naturally appearing in Λ_Q baryons, $Q = s, c, b$. Indeed, the pentaquark candidates $P_c(4380)$, $P_c(4450)$ were observed in Λ_b decays [52], a fact used in the construction of the diquark-triquark picture [10]. Assuming (as for the diquark $\delta = Qq$) no internal orbital angular momentum, the antitriquark $\bar{\theta} \equiv [\bar{Q}(ud)]$ carries the unique quantum numbers $s_{\bar{\theta}}^{P_{\bar{\theta}}} = \frac{1}{2}^-$. The intrinsic parity of the $Q\bar{Q}qud$ pentaquark state is -1, due to the presence of the single antiquark $\bar{Q}$; its isospin $I = \frac{1}{2}$ is determined entirely by the light quark q in δ.

As noted in Ref. [10], the Pauli exclusion principle must be taken into account if δ' contains identical quarks (in which case it would cease to be a "good" diquark). But even if δ' *is* a good diquark, then the light quark in δ is identical to one of those in δ', and possible constraints on the overall state due to antisymmetrization between these quarks must be considered. Inasmuch as δ' (as a part of the antitriquark $\bar{\theta}$) and δ are expected to achieve substantial spatial separation in the dynamical picture, the effect of antisymmetrization on matrix elements of observables should be significantly muted.

The construction of the lowest pentaquark states uses the same principles as used for the tetraquark states in Secs. III–V, so we present explicitly in this section only the most important intermediate results. Since heavy-quark spin symmetry remains of interest in this system, we begin by exhibiting the relation between the $\bar{\theta}$-δ basis and the $(q\delta')(Q\bar{Q})$ basis, in which the light quark q in δ is instead coupled with δ' to form an all-light baryonic system $B \equiv (q\delta')$. Analogous to Eq. (2), it reads

$$\begin{aligned} &\big\langle (s_q\, s_{\delta'}) s_B, (s_Q\, s_{\bar{Q}}) s_{Q\bar{Q}}, S \,\big|\, (s_q\, s_Q) s_\delta, (s_{\delta'}\, s_{\bar{Q}}) s_{\bar{\theta}}, S \big\rangle \\ &= \big([s_B][s_{Q\bar{Q}}][s_\delta][s_{\bar{\theta}}] \big)^{1/2} \begin{Bmatrix} s_q & s_{\delta'} & s_B \\ s_Q & s_{\bar{Q}} & s_{Q\bar{Q}} \\ s_\delta & s_{\bar{\theta}} & S \end{Bmatrix} . \quad (19) \end{aligned}$$

At this point, the diquark δ' spin has not yet been fixed to 0. Making this restriction, however, one finds only 3 basis states:

$$\begin{aligned} J^{PC} = \frac{1}{2}^- &: \quad P_{\frac{1}{2}} \equiv \big|\tfrac{1}{2}_{\bar{\theta}}, 0_\delta\big\rangle_{\frac{1}{2}} , \quad P'_{\frac{1}{2}} \equiv \big|\tfrac{1}{2}_{\bar{\theta}}, 1_\delta\big\rangle_{\frac{1}{2}} , \\ J^{PC} = \frac{3}{2}^- &: \quad P_{\frac{3}{2}} \equiv \big|\tfrac{1}{2}_{\bar{\theta}}, 1_\delta\big\rangle_{\frac{3}{2}} . \end{aligned} \quad (20)$$

The corresponding list that includes both these states and also allows $s_{\delta'} = 1$ (giving 6 additional states) appears in Ref. [53], albeit using a different notation.

In terms of Eq. (20) and using Eq. (19), the states of definite heavy-quark spin can then be written:

$$\begin{aligned} \tilde{P}_{\frac{1}{2}} &\equiv \big|\tfrac{1}{2}_B, 0_{Q\bar{Q}}\big\rangle_{\frac{1}{2}} = -\frac{1}{2} P_{\frac{1}{2}} + \frac{\sqrt{3}}{2} P'_{\frac{1}{2}} , \\ \tilde{P}'_{\frac{1}{2}} &\equiv \big|\tfrac{1}{2}_B, 1_{Q\bar{Q}}\big\rangle_{\frac{1}{2}} = +\frac{\sqrt{3}}{2} P_{\frac{1}{2}} + \frac{1}{2} P'_{\frac{1}{2}} , \\ P_{\frac{3}{2}} &= \big|\tfrac{1}{2}_B, 1_{Q\bar{Q}}\big\rangle_{\frac{3}{2}} . \end{aligned} \quad (21)$$

The generalization of these states to $L > 0$, analogous to Eq. (7), reads

$$\frac{J^P = (L-\frac{1}{2},\, L+\frac{1}{2})^{(-1)^{L+1}}}{P_{\frac{1}{2}\,L}^{(L-\frac{1}{2})},\ P_{\frac{1}{2}\,L}^{\prime\,(L-\frac{1}{2})};\ P_{\frac{1}{2}\,L}^{(L+\frac{1}{2})}, P_{\frac{1}{2}\,L}^{\prime\,(L+\frac{1}{2})},} \quad (22)$$

$$\frac{J^P = (L-\frac{3}{2},\, L-\frac{1}{2},\, L+\frac{1}{2},\, L+\frac{3}{2})^{(-1)^{L+1}}}{P_{\frac{3}{2}\,L}^{(L-\frac{3}{2})},\ P_{\frac{3}{2}\,L}^{(L-\frac{1}{2})},\ P_{\frac{3}{2}\,L}^{(L+\frac{1}{2})}, P_{\frac{3}{2}\,L}^{(L+\frac{3}{2})}.} \quad (23)$$

Of course, any states in this list with J disallowed by the triangle rule $|L - S| \leq J \leq L + S$ are excluded.

Since the $\bar{\theta}$-δ states are not eigenstates of $(CP)_{\rm light}$ and hence lack η (and consequently C) eigenvalues, their nontrivial BO potentials are simply labeled by Λ^ϵ, and their states carry the parity eigenvalues

$$P = \epsilon\,(-1)^{\Lambda+L+1} \equiv \rho\,(-1)^{L+1} . \quad (24)$$

The final addition to the notation of Eq. (23) is to append the superscript ρ defined in Eq. (24) to the state symbol.

TABLE I: Quantum numbers of the lowest tetraquark states expected in the dynamical diquark picture. For each of the expected lowest Born-Oppenheimer potentials, the full multiplet is presented, using both the state notation developed in this work and the corresponding J^{PC} eigenvalues. States with J^{PC} not allowed for conventional $q\bar{q}$ mesons are indicated in boldface.

BO potential	State notation State J^{PC}		
$\Sigma_g^+(1S)$	$\tilde{X}_{0\,S}^{(0)++}$ 0^{++}	$\tilde{Z}_S^{(1)++}$, $\tilde{Z}_S^{\prime\,(1)++}$ $2 \times 1^{+-}$	$\tilde{X}_{0\,S}^{\prime\,(0)++}$, $X_{1\,S}^{(1)++}$, $X_{2\,S}^{(2)++}$ $[0,1,2]^{++}$
$\Sigma_g^+(1P)$	$\tilde{X}_{0\,P}^{(1)++}$ 1^{--}	$[\tilde{Z}_P^{(0),\mathbf{(1)},(2)}]^{++}$, $[\tilde{Z}_P^{\prime\,(0),\mathbf{(1)},(2)}]^{++}$ $2 \times (0,\mathbf{1},2)^{-+}$	$\tilde{X}_{0\,P}^{\prime\,(1)++}$, $[X_{1\,P}^{\mathbf{(0)},(1),(2)}]^{++}$, $[X_{2\,P}^{(1),(2),(3)}]^{++}$ $[1,\ (\mathbf{0},1,2),\ (1,2,3)]^{--}$
$\Sigma_g^+(1D)$	$\tilde{X}_{0\,D}^{(2)++}$ 2^{++}	$[\tilde{Z}_D^{(1),\mathbf{(2)},(3)}]^{++}$, $[\tilde{Z}_D^{\prime\,(1),\mathbf{(2)},(3)}]^{++}$ $2 \times (1,\mathbf{2},3)^{+-}$	$\tilde{X}_{0\,D}^{\prime\,(2)++}$, $[X_{1\,D}^{(1),(2),(3)}]^{++}$, $[X_{2\,D}^{(0),(1),(2),(3),(4)}]^{++}$ $[2,\ (1,2,3),\ (0,1,2,3,4)]^{++}$
$\Pi_u^+(1P)$ & $\Sigma_u^-(1P)$	$\tilde{X}_{0\,P}^{(1)-+}$ 1^{+-}	$[\tilde{Z}_P^{(0),(1),(2)}]^{-+}$, $[\tilde{Z}_P^{\prime\,(0),(1),(2)}]^{-+}$ $2 \times (0,1,2)^{++}$	$\tilde{X}_{0\,P}^{\prime\,(1)-+}$, $[X_{1\,P}^{\mathbf{(0)},(1),\mathbf{(2)}}]^{-+}$, $[X_{2\,P}^{(1),\mathbf{(2)},(3)}]^{-+}$ $[1,\ (\mathbf{0},1,\mathbf{2}),\ (1,\mathbf{2},3)]^{+-}$
$\Pi_u^-(1P)$	$\tilde{X}_{0\,P}^{\mathbf{(1)}+-}$ $\mathbf{1}^{-+}$	$[\tilde{Z}_P^{\mathbf{(0)},(1),(2)}]^{+-}$, $[\tilde{Z}_P^{\prime\,\mathbf{(0)},(1),(2)}]^{+-}$ $2 \times (\mathbf{0},1,2)^{--}$	$\tilde{X}_{0\,P}^{\prime\,\mathbf{(1)}+-}$, $[X_{1\,P}^{(0),\mathbf{(1)},(2)}]^{+-}$, $[X_{2\,P}^{\mathbf{(1)},(2),\mathbf{(3)}}]^{+-}$ $[\mathbf{1},\ (0,\mathbf{1},2),\ (\mathbf{1},2,\mathbf{3})]^{-+}$
$\Sigma_u^-(1S)$	$\tilde{X}_{0\,S}^{(0)-+}$ 0^{-+}	$\tilde{Z}_S^{(1)-+}$, $\tilde{Z}_S^{\prime\,(1)-+}$ $2 \times 1^{--}$	$\tilde{X}_{0\,S}^{\prime\,(0)-+}$, $X_{1\,S}^{\mathbf{(1)}-+}$, $X_{2\,S}^{(2)-+}$ $[0,\mathbf{1},2]^{-+}$
$\Pi_u^+(1D)$	$\tilde{X}_{0\,D}^{(2)-+}$ 2^{-+}	$[\tilde{Z}_D^{(1),(2),(3)}]^{-+}$, $[\tilde{Z}_D^{\prime\,(1),(2),(3)}]^{-+}$ $2 \times (1,2,3)^{--}$	$\tilde{X}_{0\,D}^{\prime\,(2)-+}$, $[X_{1\,D}^{\mathbf{(1)},(2),\mathbf{(3)}}]^{-+}$, $[X_{2\,D}^{(0),\mathbf{(1)},(2),\mathbf{(3)},(4)}]^{-+}$ $[2,\ (\mathbf{1},2,\mathbf{3}),\ (0,\mathbf{1},2,\mathbf{3},4)]^{-+}$

The analogue to Table I for $\bar{\theta}$-δ states with $s_{\delta'} = 0$, representing the lowest expected pentaquark states in the triquark-diquark picture, is presented as Table II. Again, the notation of Eq. (21) is employed, to enable comparisons with expectations from heavy-quark spin symmetry. A state such as, *e.g.*, $\tilde{P}_{\frac{1}{2}\,D}^{\prime\,(\frac{3}{2})+}$ means $s_B = \frac{1}{2}$, $s_{Q\bar{Q}} = 1$, $S = \frac{1}{2}$, $L = 2$, $J = \frac{3}{2}$, $\rho = +$, and $P = \rho\,(-1)^{L+1} = -$: As indicated in Table II, it has $J^P = \frac{3}{2}^-$.

VII. COMPARISON TO EXPERIMENT

A. Exotic Candidates of Known Quantum Numbers

Despite the large number of exotic candidates observed, rather few have experimentally well-determined J^{PC} (or J^{PG}) values [1]. Moreover, none of those yet seen carry exotic $q\bar{q}$-meson or qqq-baryon quantum numbers,[2] so that the known candidates can actually be described as "cryptoexotic." In large part, this self-selection of quantum numbers arises from constraints imposed by the production modes and decay channels most easily accessible to experiment. For example, the 1^{--} channel is especially well studied because the initial-state radiation (ISR) process $e^+e^- \to \gamma_{\rm ISR} Y$ produces only states Y with $J^{PC} = 1^{--}$. The exotic candidates with measured quantum numbers (including "favored" values) are listed in Table III. The J^{PC} quantum numbers are also assumed known for the yet-unseen neutral isospin partners of observed charged states such as $Z_c^+(4430)$.

In comparison, Table I exhibits 5 0^{++} states, 3 0^{--} states, 10 1^{--} states, 5 1^{++} states, and 8 1^{+-} states. Table II exhibits multiple spin-$\frac{3}{2}$ and spin-$\frac{5}{2}$ states of either parity. The known exotic candidates do not exhaust the lowest multiplets ($n = 1$ and $L \leq 2$). Nor does this counting take into account the likely possibility that exotics like $Y(4140)$ decaying into $J/\psi\,\phi$ are $c\bar{c}s\bar{s}$ states, which frees up even more possible $c\bar{c}q\bar{q}$ states from Table I for identification with the observed exotic candidates. To proceed further, we next address whether heavy-quark spin symmetry, or whether selection rules (either exact or obtained from the BO potentials), can be used to constrain the possible identifications of states.

B. Heavy-Quark Spin Symmetry

Evidence for whether heavy-quark spin symmetry imposes strong constraints on the exotic candidates is not without ambiguity. If $s_{Q\bar{Q}}$ is a good quantum number for the exotics, then they should decay exclusively to $\psi(\Upsilon)$ or χ_Q if $s_{Q\bar{Q}} = 1$, and exclusively to η_Q or h_Q if $s_{Q\bar{Q}} = 0$.

No exotic candidate has yet been observed to decay to η_Q. In the case of $X(3872) \to \eta_c$, the Particle Data Group [55] presents an upper bound. However, the reconstruction of η_Q states tends to be more difficult than that for $\psi(\Upsilon)$, χ_Q, or even h_Q states, so it is difficult to draw any definite conclusion in this case.

In the $c\bar{c}$ sector, the charmonium decays of most of the exotic candidates proceed exclusively through J/ψ or $\psi(2S)$, while a few (such as $Z_c^+(4250)$ [56]) have been seen only with χ_c decays. The charmonium decays of the charged $Z_c^+(3900)$ have so far only been seen in the J/ψ channel, while those of the $Z_c^+(4020)$ have only been seen in the h_c channel [57, 58], suggesting strong support for

[2] A possible exception is the yet-unobserved neutral partner to the $Z_c^+(4240)$ [54], which would have $J^{PC} = 0^{--}$.

TABLE II: Quantum numbers of the lowest pentaquark states expected in the dynamical triquark-diquark picture. For each of the expected lowest Born-Oppenheimer potentials, the full multiplet is presented, using both the state notation developed in this work and the corresponding J^P eigenvalues.

BO potential	State notation State J^P	
$\Sigma^+(1S)$	$\tilde{P}^{(\frac{1}{2})+}_{\frac{1}{2}S}$, $\tilde{P}'^{(\frac{1}{2})+}_{\frac{1}{2}S}$	$P^{(\frac{3}{2})+}_{\frac{3}{2}S}$
	$2\times\frac{1}{2}^-$	$\frac{3}{2}^-$
$\Sigma^+(1P)$	$\left[\tilde{P}^{(\frac{1}{2}),(\frac{3}{2})}_{\frac{1}{2}P}\right]^+$, $\left[\tilde{P}'^{(\frac{1}{2}),(\frac{3}{2})}_{\frac{1}{2}P}\right]^+$	$\left[P^{(\frac{1}{2}),(\frac{3}{2}),(\frac{5}{2})}_{\frac{3}{2}P}\right]^+$
	$2\times\left(\frac{1}{2},\frac{3}{2}\right)^+$	$\left(\frac{1}{2},\frac{3}{2},\frac{5}{2}\right)^+$
$\Sigma^+(1D)$	$\left[\tilde{P}^{(\frac{3}{2}),(\frac{5}{2})}_{\frac{1}{2}D}\right]^+$, $\left[\tilde{P}'^{(\frac{3}{2}),(\frac{5}{2})}_{\frac{1}{2}D}\right]^+$	$\left[P^{(\frac{1}{2}),(\frac{3}{2}),(\frac{5}{2}),(\frac{7}{2})}_{\frac{3}{2}D}\right]^+$
	$2\times\left(\frac{3}{2},\frac{5}{2}\right)^-$	$\left(\frac{1}{2},\frac{3}{2},\frac{5}{2},\frac{7}{2}\right)^-$
$\Pi^+(1P)$ &	$\left[\tilde{P}^{(\frac{1}{2}),(\frac{3}{2})}_{\frac{1}{2}P}\right]^-$, $\left[\tilde{P}'^{(\frac{1}{2}),(\frac{3}{2})}_{\frac{1}{2}P}\right]^-$	$\left[P^{(\frac{1}{2}),(\frac{3}{2}),(\frac{5}{2})}_{\frac{3}{2}P}\right]^-$
$\Sigma^-(1P)$	$2\times\left(\frac{1}{2},\frac{3}{2}\right)^-$	$\left(\frac{1}{2},\frac{3}{2},\frac{5}{2}\right)^-$
$\Pi^-(1P)$	Same as $\Sigma^+(1P)$	
$\Sigma^-(1S)$	$\tilde{P}^{(\frac{1}{2})-}_{\frac{1}{2}S}$, $\tilde{P}'^{(\frac{1}{2})-}_{\frac{1}{2}S}$	$P^{(\frac{3}{2})-}_{\frac{3}{2}S}$
	$2\times\frac{1}{2}^+$	$\frac{3}{2}^+$
$\Pi^+(1D)$	$\left[\tilde{P}^{(\frac{3}{2}),(\frac{5}{2})}_{\frac{1}{2}D}\right]^-$, $\left[\tilde{P}'^{(\frac{3}{2}),(\frac{5}{2})}_{\frac{1}{2}D}\right]^-$	$\left[P^{(\frac{1}{2}),(\frac{3}{2}),(\frac{5}{2}),(\frac{7}{2})}_{\frac{3}{2}D}\right]^-$
	$2\times\left(\frac{3}{2},\frac{5}{2}\right)^+$	$\left(\frac{1}{2},\frac{3}{2},\frac{5}{2},\frac{7}{2}\right)^+$

TABLE III: Exotic candidates with experimentally determined J^{PC} quantum numbers (both umabiguous and "favored"). All are $c\bar{c}$-containing states, except for those carrying a b subscript, which are $b\bar{b}$-containing states.

0^{++}	$X(3915)$, $X(4500)$, $X(4700)$
0^{--}	$Z_c^0(4240)$
1^{--}	$Y(4008)$, $Y(4220)$, $Y(4260)$, $Y(4360)$, $Y(4390)$, $X(4630)$, $Y(4660)$, $Y_b(10888)$
1^{++}	$X(3872)$, $Y(4140)$, $Y(4274)$
1^{+-}	$Z_c^0(3900)$, $Z_c^0(4200)$, $Z_c^0(4430)$, $Z_b^0(10610)$, $Z_b^0(10650)$
$\frac{3}{2}^\pm$, $\frac{5}{2}^\mp$	$P_c(4380)$, $P_c(4450)$

the exotic candidates appearing in eigenstates of heavy-quark spin.

However, interesting conflicting signals occur in the region of the $Y(4260)$, which increasingly appears to be not a single state but several closely spaced ones [59, 60]. At a bare minimum, these states appear to be the $Y(4360)$ decaying to ψ, the $Y(4390)$ decaying to h_c, and the $Y(4220)$,[3] originally seen to decay to $\chi_{c0}\,\omega$ [61], but also appearing in $h_c\pi\pi$. The first two of these states are of course consistent with being $s_{Q\bar{Q}}$ eigenstates, but the latter, should it persist as a single state, is not.

The evidence for heavy-quark spin symmetry in the $b\bar{b}$ sector is much more ambiguous. There, all the known candidate exotics [$Y_b(10888)$, $Z_b(10610)$, $Z_b(10650)$] possess substantial decay branching fractions into both Υ and h_b,[4] suggesting either that heavy-quark spin symmetry is actually strongly violated in the decays,[5] or simply that the resonances produced are mixtures of heavy-quark spin eigenstates. For example, $Y_b(10888)$ might not be the state $\tilde{X}^{(0)++}_{0\,P}$ or $\tilde{X}'^{(0)++}_{0\,P}$, which are pure $s_{Q\bar{Q}}=0$ and $s_{Q\bar{Q}}=1$, respectively, but rather a pure diquark-spin eigenstate $X^{(0)++}_{0\,P}$ or $X'^{(0)++}_{0\,P}$ [Eqs. (5)–(6)]. The latter possibility appears perhaps more plausible since the diquarks are more compact due to the presence

[3] Called $Y(4230)$ in Ref. [1] and elsewhere.

[4] See [1] for collected experimental references.

[5] Such strong violations seem unlikely, particularly in the b system, since their amplitudes are suppressed by $\Lambda_{\rm QCD}/m_Q$.

of the heavier b quarks, but drawing such a conclusion must await a more detailed dynamical study.

A similar situation of a given exotic state not corresponding to an eigenstate of a single $s_{Q\bar{Q}}$ value arises if the heavy exotics are molecules of hadrons in their separate spin eigenstates (*e.g.*, a 1^{-+} $\bar{B}^* B^*$ state with no admixture of $\bar{B}B^* + \bar{B}^* B$, where B, B^* has $J^P = 0^-, 1^-$, respectively), a fact that is very well appreciated in the construction of such models [5, 40]. In either the molecular or the diquark-antidiquark spin-eigenstate limit, the spins of the heavy quarks are correlated not to each other (except in the composition of the overall state J^P), but to the spins of the corresponding light quarks with which they form hadron subunits, either $(qQ)+(\bar{q}\bar{Q})$ diquarks or $(\bar{q}Q)+(\bar{Q}q)$ hadrons. In both cases, the full state need not be an eigenstate of a single $s_{Q\bar{Q}}$ eigenvalue.

Inasmuch as heavy-quark spin symmetry does in fact hold for the exotics, Tables I–II are presented in a manner conducive to enumerating them. Specifically, with reference to Eqs. (4), (6), and (21), the leftmost entries on each line (Table I: $\tilde{X}_0, \tilde{Z}$; Table II: $\tilde{P}_{\frac{1}{2}}$) have $s_{Q\bar{Q}} = 0$, while the rightmost entries (Table I: $\tilde{Z}', X_1, X_2$; Table II: $\tilde{P}'_{\frac{1}{2}}, P_{\frac{3}{2}}$) have $s_{Q\bar{Q}} = 1$. Then, resolving into the categories $(\{s_{Q\bar{Q}} = 0\} + \{s_{Q\bar{Q}} = 1\})$, Table I exhibits (2+3) 0^{++} states, (2+1) 0^{--} states, (4+6) 1^{--} states, (1+4) 1^{++} states, and (3+5) 1^{+-} states. Table II exhibits (3+7) $\frac{3}{2}^+$, (2+5) $\frac{3}{2}^-$, (1+4) $\frac{5}{2}^+$, and (1+3) $\frac{5}{2}^-$ states.

Interestingly, the pentaquark candidates $P_c(4380)$ and $P_c(4450)$ both decay to $J/\psi\, p$, and their close spacing in mass combined with the large width for $P_c(4380)$ and small width for $P_c(4450)$ suggests—at this stage—that $P_c(4380)$ is the highest $\Sigma^+(1S)$ state $P^{(\frac{3}{2})+}_{\frac{3}{2}\,S}$ $(J^P = \frac{3}{2}^-)$, while $P_c(4450)$ is the unique $\frac{5}{2}^+$ state in the lowest multiplets,[6] namely, the $\Sigma^+(1P)$ state $P^{(\frac{5}{2})+}_{\frac{3}{2}\,P}$.

C. Selection Rules

Selection rules for strong decays of exotics in the BO approach were first developed in Ref. [62] and applied systematically in Ref. [43]. These selection rules fall into three types. The first is overall conservation of J^{PC} for the process, which is exact in strong interactions. Assuming that the initial and final $Q\bar{Q}$-containing states have quantum numbers $J_i^{P_i C_i}$ and $J_f^{P_f C_f}$, respectively, and a single hadron with quantum numbers j^{pc} is emitted with orbital angular momentum ℓ relative to the final heavy state, one immediately has the selection rules

$$\begin{aligned} P_i &= P_f\, p\, (-1)^\ell \,, \\ C_i &= C_f\, c \\ \boldsymbol{J}_i &= \boldsymbol{J}_f + \boldsymbol{j} + \boldsymbol{\ell} \,. \end{aligned} \tag{25}$$

The C eigenvalues here refer of course only to the neutral members of each isospin multiplet; if transitions involving charged states are considered, then G-parity conservation, with $G = C(-1)^I$ for each state, must be imposed. Selection rules in this class are never violated, assuming only that the decays are pure QCD processes.

The second type of selection rule in Ref. [43] references the approximate conservation of heavy-quark spin symmetry. We have already discussed in the previous subsection how well this symmetry is upheld in observed processes.

The third type of selection rule in Ref. [43] uses the BO approximation in a fundamental way: Under the assumption that the light d.o.f. adjust much more quickly in a physical process than the heavy $Q\bar{Q}$ pair, the decay to a conventional $Q\bar{Q}$ state occurs through a rapid transition from the initial BO configuration to the final one plus a light hadron, leaving the separation and orientation of the $Q\bar{Q}$ pair nearly unchanged. In fact, the heavy-quark spin-symmetry limit is implicit in this approximation. The conservation of angular momentum then reads

$$\boldsymbol{J}_{\text{light},i} = \boldsymbol{J}_{\text{light},f} + \boldsymbol{j} + \boldsymbol{\ell} \,, \tag{26}$$

and since $\boldsymbol{J}_{\text{light},i} = \boldsymbol{L}_i + \boldsymbol{s}_{q\bar{q}}$ (with the replacement $\boldsymbol{s}_{q\bar{q}} \to \boldsymbol{s}_B$ for pentaquarks) while $\boldsymbol{J}_{\text{light},f}$ (being the light d.o.f. angular momentum of a conventional $Q\bar{Q}$ state) contains no valence light quarks and hence equals $\boldsymbol{L}_f$,[7] we have

$$\boldsymbol{L}_i = \boldsymbol{L}_f + \boldsymbol{j} - \boldsymbol{s}_{q\bar{q}} + \boldsymbol{\ell} \,. \tag{27}$$

Dotting with $\hat{\boldsymbol{r}}$ gives

$$\lambda_i = \lambda_f + \hat{\boldsymbol{r}} \cdot (\boldsymbol{j} - \boldsymbol{s}_{q\bar{q}} + \boldsymbol{\ell}) \,. \tag{28}$$

This expression differs from Eq. (28) in Ref. [43] by the extra factor $-\boldsymbol{s}_{q\bar{q}}$ on the right-hand side, and arises as the result of our choice not to include $\boldsymbol{s}_{q\bar{q}}$ in $\boldsymbol{J}_{\text{light},i}$. The triangle rule for the transition in the BO approximation reads

$$|\lambda_i - \lambda_f| \le j + s_{q\bar{q}} + \ell \,. \tag{29}$$

Again, since the final state is taken to be a conventional $Q\bar{Q}$ state with BO potential Σ_g^+, then $\lambda_f = 0$, and thus:

$$\Lambda_i \le j + s_{q\bar{q}} + \ell \,, \tag{30}$$

[6] Similar reasoning in a Hamiltonian formalism led to the same J^P identification of the P_c states in the triquark-diquark model of Ref. [53].

[7] A distinct $\boldsymbol{s}_{q\bar{q}}$ factor appears as a component of $\boldsymbol{J}_{\text{light},f}$ if it is also a multiquark state.

with $s_{q\bar{q}} \to s_B$ in the pentaquark case. It is interesting to consider the limit discussed in Sec. IV in which the light-quark d.o.f. spin $\boldsymbol{s}_{q\bar{q}}$ also remains fixed (*i.e.*, for $b\bar{b}c\bar{c}$ tetraquarks, since $m_b \gg m_c \gg \Lambda_{\rm QCD}$). Then, assuming the light final-state hadron contains no internal orbital excitation, one has $\boldsymbol{j} = \boldsymbol{s}_{q\bar{q}}$, and hence from Eq. (28) follows the simple result $\Lambda_i \leq \ell$: States in $\Sigma, \Pi, \ldots$ BO potentials in this limit only decay to light hadrons in at least $S, P, \ldots$ relative partial waves, respectively. In the light-quark case, however, only the looser constraint Eq. (30) applies.

The discrete BO eigenvalues also provide approximate selection rules. Following the analysis in Sec. IV, the reflection parity $R_{\rm light}$ through any plane containing the $Q\bar{Q}$ axis $\hat{\boldsymbol{r}}$ acts upon the light hadron as a product of $P_{\rm light}$ (which introduces a factor of its intrinsic parity p as well as a factor $(-1)^\ell$ from its relative motion with respect to the final heavy state) and a rotation by π radians about the normal to the reflection plane (which introduces an extra phase $\exp(i\pi\hat{\boldsymbol{r}}\cdot\boldsymbol{s}_{q\bar{q}})$ in the initial state and $\exp[i\pi\hat{\boldsymbol{r}}\cdot(\boldsymbol{j}+\boldsymbol{\ell})]$ in the final state). According to Eq. (28), the difference of these phases is just $\exp[i\pi(\lambda_i - \lambda_f)]$, which can be written as $(-1)^{\Lambda_i - \Lambda_f}$ since both λ's are integers. In total, we have

$$\epsilon_i = \epsilon_f \, p \, (-1)^\ell (-1)^{\Lambda_i - \Lambda_f} \, , \ \text{ or } \rho_i = \rho_f \, p \, (-1)^\ell \, . \tag{31}$$

We note that no restriction to $\Lambda_i = \Lambda_f = 0$ is required, in contrast to Eq. (31) of Ref. [43].

Lastly in the tetraquark case, for which charge conjugation symmetry is relevant, the BO approximation $(CP)_{\rm light}$ quantum number η provides a selection rule (Eq. (30) of [43]):

$$\eta_i = \eta_f \, cp \, (-1)^\ell \, , \ \text{ or } \kappa_i = \kappa_f c \, . \tag{32}$$

The most incisive phenomenological tests of the exact selection rules Eqs. (25) and the BO approximation selection rules Eqs. (30), (31), (32) are decays to conventional $Q\bar{Q}$ states ($\epsilon_f = \eta_f = +, \ \lambda_f = 0$) that produce a single light hadron. The decays of this type thus far observed are the emission of a single light vector particle ($j^{pc} = 1^{--}, \ \ell = 0$) such as ρ, ω, or ϕ, and the emission of a single charged pion ($j^{pg} = 0^{--}, \ \ell = 0$). In the latter case, since the conventional $Q\bar{Q}$ states are isosinglets, one has an isotriplet exotic decaying to a single pion, in which case the $(-1)^I = -1$ factors in the definition of G parity cancel between the initial and final state, thus reducing G-parity conservation condition to the C-parity conservation condition [Eq. (25)] for the corresponding π^0 process, $c = +$ and $C_i = C_f$.

Let us first consider the pionic decay. The role of π as a Nambu-Goldstone boson of chiral symmetry breaking suggests it to be emitted predominantly in a P-wave ($\ell = 1$). However, $Z_c^+(3900)$ has been experimentally determined to be a 1^+ state [58], and therefore the observed decay $Z_c^+(3900) \to J/\psi \, \pi^+$ to the 1^{--} J/ψ requires ℓ to be even for this decay. Presumably, the S wave must dominate this particular process; if this result remains true for the other single-pion emission processes, then the selection rules reduce to

$$\begin{aligned} & P_i = -P_f, \ C_i = C_f, \ J_i = J_f, \\ & \Lambda_i \leq s_{q\bar{q}}, \ \epsilon_i = (-1)^{\Lambda_i + 1}, \ \eta_i = - \, , \\ & [\, \rho_i \kappa_i = -+ \,] \, . \end{aligned} \tag{33}$$

In particular, only u BO potentials for the initial states are represented. Since J/ψ is 1^{--} and π^0 is 0^{-+}, $Z_c^0(3900)$ is therefore 1^{+-}, and a glance at Table I shows 3 $s_{Q\bar{Q}} = 1$ candidates in a $\rho_i \kappa_i = -+$ BO potential with these quantum numbers, namely, $\tilde{X}_{0\,P}^{\prime\,(1)-+}$, $X_{1\,P}^{(1)-+}$, and $X_{2\,P}^{(1)-+}$ in the mixed $\Pi_u^+(1P)$-$\Sigma_u^-(1P)$ BO potential. Should the $Z_c^+(4020)$ (which decays to h_c) also be confirmed as a 1^+ state, its natural identification would be as the $s_{Q\bar{Q}} = 0$ state $\tilde{X}_{0\,P}^{(1)-+}$ in the same BO potential. The $Z_c^+(4200)$ and $Z_c^+(4430)$ can be analyzed similarly, but whether they are the other two $\Pi_u^+(1P)$-$\Sigma_u^-(1P)$ states, or belong to either a higher $n = 1$ BO potential or the $n = 2$ band, requires a more detailed study. The $Z_c^0(4240)$, should its 0^{--} quantum numbers be unambiguously confirmed, is more problematic because it does not fit into the $\Pi_u^+(1P)$-$\Sigma_u^-(1P)$ BO potential with an S-wave pion coupling, but with a P-wave decay it could be the state $X_{1\,P}^{(0)++}$ in $\Sigma_g^+(1P)$.

Turning now to the single light-vector decays and assuming S-wave decays, the selection rules reduce to

$$\begin{aligned} & P_i = -P_f, \ C_i = -C_f, \ J_i \in \{J_f, \, J_f \pm 1\}, \\ & \Lambda_i \leq 1 + s_{q\bar{q}}, \ \epsilon_i = (-1)^{\Lambda_i + 1}, \ \eta_i = + \, , \\ & [\, \rho_i \kappa_i = -- \,] \, . \end{aligned} \tag{34}$$

In particular, only g BO potentials for the initial states are represented. A quick glance at Table I shows that no $\rho_i \kappa_i = --$ potentials are expected among the lowest multiplets, which creates a real problem for this classification. It could be resolved in several ways: First, the BO approximation for exotic states might simply not work because the physical values $m_Q = m_c, m_b$ are not large enough; however, inasmuch as the approximation becomes exact for $m_Q \to \infty$, it would be peculiar for the classification to fail for *every* light-vector decay mode but still work for the single-pion decay modes. Second, the BO approximation is expected to fail in for states in the vicinity of two-hadron thresholds, at which point avoided energy-level crossings must be taken into account by means of a coupled-channel analysis, as discussed in [43] or implemented via the Feshbach mechanism in [63]; while this observation is certainly true and will have to be implemented in a fully complete model, not every exotic candidate (even restricting to ones decaying to light vectors) is especially close to such a threshold. In either of these first two scenarios, the conservation of the BO quantum numbers can be violated in decay transitions. Third, the light vectors might (for unknown reasons) couple predominantly to a P wave, in which case one finds $\rho_i \kappa_i = +-$; while the $\Pi_u^-(1P)$ potential fits this category and indeed produces 1^{--} states, it produces neither 0^{++} nor 1^{++} states.

A fourth option is that the listing of the lowest BO potentials for δ-$\bar{\delta}$ given in Table I is incomplete. One particularly economical solution is to suppose that the potentials $\Pi_g^+(1P)$ and $\Pi_g^-(1P)$ are among the lowest. Following the comments below Eq. (18), the listing of states for $\Pi_g^+(1P)$ looks exactly like that for $\Pi_u^+(1P)$-$\Sigma_u^-(1P)$, except that all final superscripts, κ and C, flip sign. Then several 0^{++} and 1^{++} states naturally appear [and, according to our previous discussion, it matches the quantum numbers of states in—and potentially mixes with—$\Sigma_g^-(1P)$]. Likewise, the listing of states for $\Pi_g^-(1P)$ [which may mix with $\Sigma_g^+(1P)$] looks exactly like that for $\Pi_u^-(1P)$ except for the flip of κ and C, which naturally produces multiple 1^{--} states (as well as another option for a 0^{--} state). For completeness, these additional multiplets are listed in Table IV. Whether this resolution is reasonable of course depends upon the true ordering of δ-$\bar{\delta}$ BO potentials, which presumably can be decided by lattice simulations. For example, simulations such as those described for $b\bar{b}\bar{u}\bar{d}$ in Ref. [64] will be quite valuable.

Finally, the $\bar{\theta}$-δ BO states have the same selection rules, excluding those for C and η [Eq. (32)], while $s_{q\bar{q}}$ is replaced by s_B, which we take to have its minimal value, $\frac{1}{2}$. Assuming for now that we are only interested to decays into J/ψ ($J_f^{P_f} = 1^-$) and nucleons ($j^p = \frac{1}{2}^+$), the selection rules read

$$P_i = \rho_i(-1)^{L+1} = P_f\, p\, (-1)^\ell = (-1)^{\ell+1},\ J_i \leq \tfrac{3}{2} + \ell\,,$$
$$\Lambda_i \leq 1 + \ell,\ \epsilon_i = (-1)^{\Lambda_i + \ell}\,,\ [\,\rho_i = (-1)^\ell\,]\,. \tag{35}$$

Since the two observed P_c states are found to have opposite parities, the first string of equations in (35) shows that opposite parities of ℓ are required to accommodate them. In addition, the J_i triangle rule requires $\ell \geq 1$ for the spin-$\frac{5}{2}$ state. Moreover, substituting the final equality in (35) into its first equation shows that L must be even for the potentials producing each state. The $J_i^{P_i}$ option $\frac{3}{2}^-$, $\frac{5}{2}^+$ for the P_c states [which suits the broad width of the $P_c(4380)$ and narrow width of the $P_c(4450)$, as discussed in the previous subsection] corresponds to $\ell = 0, 1$, respectively, and is accommodated most naturally by the pair $\Sigma^+(1S)$: $P_{\frac{3}{2}S}^{(\frac{3}{2})+}$, and $\Pi^+(1D)$ [or $\Sigma^-(1D)$, if also present]: $\tilde{P}'^{(\frac{5}{2})-}_{\frac{1}{2}D}$ or $P^{(\frac{5}{2})-}_{\frac{3}{2}D}$. Alternately, if the P_c states are found to be $\frac{3}{2}^+$, $\frac{5}{2}^-$, then the pair $\Sigma^-(1S)$: $P^{(\frac{3}{2})-}_{\frac{3}{2}S}$, and $\Sigma^+(1D)$ [or $\Pi^-(1D)$, if also present]: $\tilde{P}^{(\frac{5}{2})+}_{\frac{1}{2}D}$ or $P^{(\frac{5}{2})+}_{\frac{3}{2}D}$ works.

In summary, the most natural BO potentials for accommodating known tetraquark candidates appear to be $\Pi_u^+(1P)$-$\Sigma_u^-(1P)$ for those appearing in single-pion decays, $\Pi_g^+(1P)$-$\Sigma_g^-(1P)$ and $\Pi_g^-(1P)$-$\Sigma_g^+(1P)$ for those appearing in single-vector decays. The most natural BO potentials for accommodating the known pentaquark candidates, depending upon the final parity assignments, are $\Sigma^+(1S)$ and $\Pi^+(1D)$, or $\Sigma^-(1S)$ and $\Sigma^+(1D)$.

VIII. CONCLUSIONS

In this paper we developed the spectroscopy of states in the dynamical diquark picture, both for diquark-antidiquark tetraquark states and for triquark-diquark pentaquark states, within the context of the Born-Oppenheimer (BO) approximation. The first step was the group-theoretical exercise of relating the diquark-spin basis to the basis of states of well-defined heavy-quark spin, in which P and C quantum numbers are most easily determined.

Next, the BO approximation and potentials were briefly reviewed, the quantum numbers of states within these potentials were determined, and a compact notation for the states was introduced. The lowest BO potentials were identified by supposing that the lowest potentials obtained in lattice QCD simulations for hybrid mesons in the BO approximation hold also for diquark-antidiquark and triquark-diquark systems. Then the lowest expected multiplets for states in the diquark-antidiquark system were collected in Table I, and in the triquark-diquark system in Table II.

We then turned to the question of comparison with the set of exotic candidates with experimentally observed quantum numbers, and found that all of these states could be accommodated, taking into account just their J^{PC} quantum numbers. We then developed selection rules—some exact and some relying upon the BO approximation—for decays of the exotic states, and carefully examined the constraints thus obtained using known decay channels (via a single pion or a single light vector meson). We found that the observed tetraquark states decaying through pions and the pentaquark states could still be accommodated, but if the BO selection rules must hold in their strictest form, then the single-vector decays appear to demand the introduction of additional low-lying BO potentials [Table IV] beyond the ones appearing in hybrid lattice calculations.

The next steps of this study point in many different directions. First, it is important to keep track of the latest discoveries in the exotic sector, to see whether newly discovered states or old states with newly determined quantum numbers continue to fit into the BO paradigm. Second, lattice simulations of the lowest BO potentials that include nontrivial light-quark spin or isospin will be essential in firming up the identification of the known exotics with particular states and determining whether the BO selection rules actually hold in all instances. Third, particular functional forms for the potentials inspired by lattice results or models can be introduced, and the corresponding Schrödinger equations solved, in order to obtain predictions for the specific mass spectrum of the states. Fourth, the very interesting question of how coupled-channel effects with hadronic thresholds

TABLE IV: Quantum numbers for possible additional low-lying tetraquark states in the dynamical diquark picture, as suggested by the Born-Oppenheimer selection rules for light-vector decays. The notation is the same as in Table I.

BO potential	State notation State J^{PC}		
$\Pi_g^+(1P)$	$\tilde{X}_{0\,P}^{(1)--}$ 1^{++}	$[\tilde{Z}_P^{(\mathbf{0}),(1),(\mathbf{2})}]^{--}$, $[\tilde{Z}_P^{\prime\,(\mathbf{0}),(1),(\mathbf{2})}]^{--}$ $2\times(\mathbf{0},1,\mathbf{2})^{+-}$	$\tilde{X}_{0\,P}^{\prime\,(1)--}$, $[X_{1\,P}^{(0),(1),(2)}]^{--}$, $[X_{2\,P}^{(1),(2),(3)}]^{--}$ $[1,\ (0,1,2),\ (1,2,3)]^{++}$
$\Pi_g^-(1P)$	$\tilde{X}_{0\,P}^{(1)++}$ 1^{--}	$[\tilde{Z}_P^{(0),(\mathbf{1}),(2)}]^{++}$, $[\tilde{Z}_P^{\prime\,(0),(\mathbf{1}),(2)}]^{++}$ $2\times(0,\mathbf{1},2)^{-+}$	$\tilde{X}_{0\,P}^{\prime\,(1)++}$, $[X_{1\,P}^{(\mathbf{0}),(1),(2)}]^{++}$, $[X_{2\,P}^{(1),(2),(3)}]^{++}$ $[1,\ (\mathbf{0},1,2),\ (1,2,3)]^{--}$

modify these predictions must be addressed, as it cannot simply be an accident that many of the exotic candidates lie so close in mass to such thresholds (especially $m_{X(3872)} - m_{D^{*0}} - m_{D^0} = +0.01 \pm 0.18$ MeV [55]).

An ambitious program of calculations within not only the diquark model but molecular models as well, combined with the steady rate of new experimental and lattice simulation developments, will lead to a much richer and clearer understanding of these novel hadrons.

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