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Jonathan Hudson and Peter Schweitzer

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D-term and structure of point-like and composed spin-0 particles

Jonathan Hudson¹ and Peter Schweitzer¹

¹ *Department of Physics, University of Connecticut, Storrs, CT 06269-3046, U.S.A.*

This work deals with form factors of the energy-momentum tensor (EMT) of spin-0 particles and the unknown particle property *D*-term related to the EMT, and is divided into three parts.

The first part explores free, weakly and strongly interacting theories to study EMT form factors with the following findings. (i) The free Klein-Gordon theory predicts for the *D*-term $D = -1$. (ii) Even infinitesimally small interactions can drastically impact *D*. (iii) In strongly interacting theories one can encounter large negative *D* though notable exceptions exist, which includes Goldstone bosons of chiral symmetry breaking. (iv) Contrary to common belief one cannot arbitrarily add “total derivatives” to the EMT. Rather the EMT must be defined in an unambiguous way.

The second part deals with the interpretation of the information content of EMT form factors in terms of 3D-densities with following results. (i) The 3D-density formalism is internally consistent. (ii) The description is subject to relativistic corrections but those are acceptably small in phenomenologically relevant situations including nucleon and nuclei. (iii) The free field result $D = -1$ persists when a spin-0 boson is not point-like but “heuristically given some internal structure.”

The third part investigates the question, whether such “giving of an extended structure” can be implemented dynamically, and has the following insights. (i) We construct a consistent microscopic theory which, in a certain parametric limit, interpolates between extended and point-like solutions. (ii) This theory is exactly solvable which is rare in $3+1$ dimensions, admits non-topological solitons of *Q*-ball-type, and has a Gaussian field amplitude. (iii) The interaction of this theory belongs to a class of logarithmic potentials which were discussed in literature, albeit in different contexts including beyond standard model phenomenology, cosmology, and Higgs physics.

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I. INTRODUCTION

The energy momentum tensor operator (EMT) is at the heart of the field theoretical description of particles. Through it matter and gauge fields couple to gravity, and its matrix elements define fundamental properties like mass, spin and the experimentally unknown *D*-term. The latter, despite being among the most fundamental particle properties and although its presence was established in the 1960s when Pagels introduced EMT form factors [1], has received little attention for a long time as no practical process was known how to measure EMT form factors.

The situation changed in 1990s with the advent of generalized parton distribution functions (GPDs) accessible in hard-exclusive reactions [2–5]. The second Mellin moments of unpolarized GPDs are related to EMT form factors, allowing us to access information about the spin decomposition of the nucleon [3], the *D*-term [6], and mechanical properties [7]. The relation of the *D*-term to GPDs was further clarified in [8]. The potential of GPD studies as a rich source of new information about nucleon structure goes much further [9–13].

Similarly to electric form factors providing information on the electric charge distribution [14], the EMT form factors offer insights on the spatial energy density, orbital angular momentum density, and the stress tensor [7]. The EMT densities not only provide a unique way to gain insights on the particle stability and mechanical properties, but also have important practical applications [15]. For a recent review on the *D*-term we refer to [16].

The purpose of this work is to provide a comprehensive discussion of the EMT and the *D*-term in spin-0 systems. The goal, besides establishing a benchmark for further studies, is to focus on clarifying what the *D*-term is and means, undistracted by technical details associated with non-zero spin which will be addressed elsewhere.

The first part of our work is devoted to EMT form factors and the *D*-term. We explore free, weakly and strongly interacting theories. We first study the free field theory case which yields $D = -1$ and provides a point of reference. We then discuss how interactions can affect the *D*-term. We explore the Φ^4 theory as an example of a weakly interacting case and show that interactions, even if infinitesimally small, have drastic impact on the *D*-term. Hereby we show that in general it is not permissible to add total derivatives to the EMT, contrary to common belief. Rather such “improvement terms” to the EMT operator, if they are needed, must be chosen with care and require a unique, unambiguous definition. In strongly interacting theories, where we consider Goldstone bosons of chiral symmetry breaking and nuclei in QCD and *Q*-balls as examples, we show that the *D*-terms can have large magnitudes but are always negative. The Goldstone bosons are a notable exception in this context: chiral symmetry dictates $D = -1$ modulo chiral corrections which are modest for pions, and somewhat larger for kaons and η -mesons. This part contains

original results but has partly also review character. This is intentional not only to make this work self-contained and place our insights in a wider context. It is also necessary as relevant results from earlier literature were rarely (or not at all) discussed in the context of the D -term in more recent works.

The second part of our work is focused on the interpretation of EMT form factors as 3D densities and presents throughout original results. We first introduce the 3D-density formalism for the spin-0 case following the work on spin- $\frac{1}{2}$ systems [7], demonstrate the consistency, and discuss the limitations of the approach. Starting from the notion of a point-like particle, we investigate how the EMT properties are affected when the point-like particle “is given some internal structure” and “acquires a finite size.” These concepts put us in the position to quantify the “relativistic corrections” associated with 3D-densities. The presence of these corrections is well-known, but the way we quantify them is novel and we find them acceptably small for phenomenologically relevant cases including nucleon and nuclei (though derived in spin-0 case, these findings are valid for any spin). When “giving” a particle “some internal structure” we initially proceed heuristically with the remarkable result that the free field theory result $D = -1$ is preserved when the particle “acquires” a finite size. We demonstrate that this heuristic picture is fully consistent with EMT conservation and other general principles.

In the third part, we address the question whether it is possible to construct a microscopic theory where such an internal structure arises from dynamics with $D = -1$ and the EMT densities corresponding to what one would heuristically expect for a “smeared out” point-like particle. We show that a Lagrangian can be constructed with an interaction known from different contexts in literature. We demonstrate that this theory describes stable non-topological solitons of Q -ball type, and show that it can be solved analytically. This by itself is a remarkable result, as it is rare to find analytically solvable theories in $3 + 1$ dimensions.

The outline of this work is as follows. The first part in Sec. II is focused on EMT form factors, which we define in Sec. II A, and evaluate in Klein-Gordon theory in Sec. II B. We discuss the weakly interacting case in Sec. II C, consider strongly interacting theories in Secs. II D and II E, and briefly review also higher spin systems in Sec. II F. The second part in Sec. III deals with the EMT densities. We introduce the formalism in Sec. III A, compute the EMT densities of a point-like particle in Sec. III B, and discuss limitations of the approach in Sec. III C. We show that the property $D = -1$ persists when a point-like particle is heuristically given an internal structure in Sec. III D. The third part in Sec. IV is devoted to the study of a dynamical theory which describes a particle whose internal structure corresponds naturally to the notion of a smeared-out point-particle with $D = -1$. After a brief review of the EMT of Q -balls in Sec. IV A which provides the setting, we construct and solve the theory in Sec. IV B, before addressing important technical aspects of this theory in Sec. IV C, and indicating potential applications in Sec. IV D. In Sec. V we present our conclusions. The Appendices contain remarks on notation and technical details.

II. EMT FORM FACTORS OF SPIN-0 PARTICLES

In this section we define the EMT form factors of a spin-0 particle, and calculate the EMT form factors and the D -term of an elementary free spin-0 boson as described by the free Klein-Gordon theory. We then discuss what happens to the D -term when interactions are present and consider both the weak- as well as strong-coupling regime.

A. Formalism and definitions

For a spin-0 particle with mass m the EMT matrix elements are described in terms of two form factors [1],

$$\langle \vec{p}' | \hat{T}^{\mu\nu}(0) | \vec{p} \rangle = \frac{P^\mu P^\nu}{2} A(t) + \frac{\Delta^\mu \Delta^\nu - g^{\mu\nu} \Delta^2}{2} D(t), \quad (1)$$

where $\hat{T}^{\mu\nu}(0)$ denotes the EMT operator at space-time position zero. The kinematic variables are defined as

$$P^\mu = p^{\mu'} + p^\mu, \quad \Delta^\mu = p^{\mu'} - p^\mu, \quad t = \Delta^2. \quad (2)$$

The convention for the covariant normalization of one-particle states is

$$\langle \vec{p}' | \vec{p} \rangle = 2E (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{p}'), \quad E = \sqrt{\vec{p}^2 + m^2}. \quad (3)$$

Performing the analytic continuation of the form factors to zero-momentum transfer yields

$$\lim_{t \rightarrow 0} A(t) = A(0) = 1, \quad (4a)$$

$$\lim_{t \rightarrow 0} D(t) = D(0) \equiv D. \quad (4b)$$

The constraint (4a) is explained by recalling that for $\vec{p} \rightarrow 0$ and $\vec{p}' \rightarrow 0$ only the 00-component remains in Eq. (1), and $H = \int d^3x \hat{T}_{00}(x)$ is the Hamiltonian of the system with $H|\vec{p}\rangle = m|\vec{p}\rangle$ for $\vec{p} \rightarrow 0$. With the conventions (1, 3) (see Appendix A for other notations) one obtains the constraint $A(0) = 1$ in (4a). It is important to stress that no such constraint exists for the form factor $D(t)$ such that the D -term $D \equiv D(0)$ must be determined from experiment.

For later convenience, let us disentangle the contributions of the 2 form factors in Eq. (1). For that we contract the EMT with the symmetric tensors $g^{\mu\nu}$ and $a^{\mu\nu}$ defined as

$$a^{\mu\nu} = \frac{P^\mu P^\nu}{P^2}, \quad P^2 = 4m^2 - t. \quad (5)$$

Notice that the only other symmetric tensors available in this case are proportional to $(P^\mu \Delta^\nu + P^\nu \Delta^\mu)$ or $\Delta^\mu \Delta^\nu$, and both are of no use for our purposes since $\Delta^\mu \langle \vec{p}' | \hat{T}_{\mu\nu}(0) | \vec{p} \rangle = 0$ due to EMT conservation.

With $n = g^{\mu\nu} g_{\mu\nu} = 4$ denoting the number of space-time dimensions we obtain

$$\left[(n-1) a^{\mu\nu} - g^{\mu\nu} \right] \langle \vec{p}' | \hat{T}_{\mu\nu}(0) | \vec{p} \rangle = \frac{n-2}{2} P^2 A(t), \quad (6a)$$

$$\left[a^{\mu\nu} - g^{\mu\nu} \right] \langle \vec{p}' | \hat{T}_{\mu\nu}(0) | \vec{p} \rangle = \frac{n-2}{2} \Delta^2 D(t). \quad (6b)$$

Specifically for $n = 3 + 1$ space-time dimensions we have

$$A(t) = \frac{1}{P^2} (3 a^{\mu\nu} - g^{\mu\nu}) \langle \vec{p}' | \hat{T}_{\mu\nu}(0) | \vec{p} \rangle, \quad (7a)$$

$$D(t) = \frac{1}{\Delta^2} (a^{\mu\nu} - g^{\mu\nu}) \langle \vec{p}' | \hat{T}_{\mu\nu}(0) | \vec{p} \rangle. \quad (7b)$$

B. Free field theory case

It is instructive to start with the free field case. We consider the Lagrangian of a non-interacting real spin-0 field

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \Phi)(\partial^\mu \Phi) - V_0(\Phi), \quad V_0(\Phi) = \frac{1}{2} m^2 \Phi^2 \quad (8)$$

which describes a free spin-0 boson of mass m obeying the Klein-Gordon equation

$$(\square + m^2) \Phi(x) = 0. \quad (9)$$

If under parity transformations the field transforms as $\Pi \Phi(x) \Pi^{-1} = \pm \Phi(x)$ then the theory describes scalars (for $+$) or pseudoscalars (for $-$). In theories like (8) the conserved canonical EMT operator is symmetric, and given by

$$\hat{T}^{\mu\nu}(x) = (\partial^\mu \Phi)(\partial^\nu \Phi) - g^{\mu\nu} \mathcal{L}, \quad (10)$$

where normal ordering is implied. To evaluate the matrix elements of the EMT we recall that the free field solutions to the equation of motion (9) are given by

$$\Phi(x) = \int \frac{d^3k}{2\omega_k(2\pi)^3} \left(\hat{a}(\vec{k}) e^{-ikx} + \hat{a}^\dagger(\vec{k}) e^{ikx} \right), \quad \omega_k = \sqrt{\vec{k}^2 + m^2} \quad (11)$$

with creation and annihilation operators satisfying $[\hat{a}(k), \hat{a}^\dagger(k')] = 2\omega_k (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}')$ in canonical equal-time quantization. The free one-particle states are defined as $|\vec{p}_{\text{free}}\rangle = \hat{a}^\dagger(\vec{p})|0\rangle$, and are normalized covariantly according to Eq. (3) with the trivial vacuum state normalized as $\langle 0|0\rangle = 1$. The EMT matrix elements can be readily evaluated

$$\langle \vec{p}'_{\text{free}} | \hat{T}^{\mu\nu}(x) | \vec{p}_{\text{free}} \rangle = e^{i(p'-p)x} \times \left\{ p'^\mu p^\nu + p^\mu p'^\nu - g^{\mu\nu} (p' \cdot p - m^2) \right\}. \quad (12)$$

In the notation of Eq. (2) one has $p' \cdot p - m^2 = -\frac{1}{2} \Delta^2$ and $p'^\mu p^\nu + p^\mu p'^\nu = \frac{1}{2} (P^\mu P^\nu - \Delta^\mu \Delta^\nu)$ such that

$$\langle \vec{p}'_{\text{free}} | \hat{T}^{\mu\nu}(x) | \vec{p}_{\text{free}} \rangle = e^{i(p'-p)x} \frac{1}{2} \left\{ P^\mu P^\nu - \Delta^\mu \Delta^\nu + g^{\mu\nu} \Delta^2 \right\}. \quad (13)$$

The trivial dependence on the coordinate x is due to translational invariance $\hat{T}^{\mu\nu}(x) = \exp(i\hat{P}x) \hat{T}^{\mu\nu}(0) \exp(-i\hat{P}x)$ where $\hat{P}^\mu = \int d^3x \hat{T}^{0\mu}$ denotes the momentum operator. In most definitions one therefore quotes $\hat{T}^{\mu\nu}(0)$ as in Eq. (1).

Comparing the result (13) with Eq. (1) we see that

$$A(t) = 1, \quad D(t) = -1. \quad (14)$$

Several remarks are in order. First, the form factors are constant functions of t as expected for a free point-like particle. Second, the constraint $A(0) = 1$ in (4a) is of course satisfied. Third, the free Klein-Gordon theory makes the unambiguous prediction $D = -1$ and the negative sign is in line with studies in other theoretical frameworks. Fourth, repeating the calculation with a complex Klein-Gordon field reveals that a spin-0 particle and its anti-particle have the same D -term.

It seems to have been largely overlooked in more recent literature that in Ref. [1] not only the notion of EMT form factors was introduced for spin-0 and spin- $\frac{1}{2}$ hadrons and applications were discussed. In addition to that in Ref. [1] also the form factors of a free Klein-Gordon particle were quoted. Our result in Eq. (14) agrees with Ref. [1].

The free Klein-Gordon prediction for the D -term of a spin-0 particle sets a reference point for further studies. It is instructive to examine what happens if one switches on interactions or the particle is not point-like but extended. We will investigate these topics in the following.

C. Weakly interacting case

Let us introduce in (8) a generic interaction, $V(\Phi) = \frac{1}{2} m^2 \Phi^2 + \mathcal{O}(\lambda)$, characterized by a small coupling constant $\lambda \ll 1$ such that it is justified to use perturbative methods to solve the theory. In such a situation, one could naively think the D -term would be $D_{\text{interacting naive}} = -1 + \mathcal{O}(\lambda)$ and reduce to the free theory value (14) for $\lambda \rightarrow 0$. However, this is not the case for two reasons. (i) As a conserved current, the EMT is a renormalization scale invariant operator so its matrix elements cannot not depend on the renormalization scale μ . But λ acquires in an interacting quantum field theory a dependence on μ governed by the β -function of the theory. Therefore the D -term must not receive an $\mathcal{O}(\lambda)$ -contribution in a perturbative treatment of an interacting theory. (ii) As no $\mathcal{O}(\lambda)$ -contribution is allowed, one could then naively think that $D_{\text{interacting naive}} = -1$. However, in general also this is not the case. We illustrate this point considering a specific interacting scalar theory, the Φ^4 theory.

The EMT of the Φ^4 theory was studied in detail in Ref. [17]. In our context it is instructive to review here the findings from Ref. [17], see also the works [18–23]. The theory is defined by

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \Phi)(\partial^\mu \Phi) - V(\Phi), \quad V(\Phi) = \frac{1}{2} m^2 \Phi^2 + \frac{\lambda}{4!} \Phi^4. \quad (15)$$

According to the general understanding one can add to the EMT operator (10) “any quantity whose divergence is zero and which does not contribute to the Ward identities” [23]. (Below we shall see that this general statement has to be formulated more carefully.) Among possible choices the following “improvement term” plays a special role [17],

$$T_{\text{improve}}^{\mu\nu} = T_{\text{Eq.(10)}}^{\mu\nu} + \theta_{\text{improve}}^{\mu\nu}, \quad \theta_{\text{improve}}^{\mu\nu} = -h(\partial^\mu \partial^\nu - g^{\mu\nu} \square) \phi(x)^2, \quad h = \frac{1}{4} \left(\frac{n-2}{n-1} \right), \quad (16)$$

where n denotes the number of space-time dimensions. To motivate the improvement term (16) we recall that the coupling of spin-0 fields like (8, 15) to gravity is given by an effective action

$$S_{\text{grav}} = \int d^4x \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} (\partial_\mu \Phi)(\partial_\nu \Phi) - V(\Phi) - \frac{1}{2} h R \Phi^2 \right) \quad (17)$$

where $-\frac{1}{2} h R \Phi^2$ is a non-minimal coupling term, R is the Riemann scalar, g denotes the determinant of the metric, and it is understood that gravity is treated to lowest order. From (17) one obtains the EMT operator via

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_{\text{grav}}}{\delta g^{\mu\nu}}. \quad (18)$$

Omitting the non-minimal term in (17) yields a correct description of a scalar field theory (minimally) coupled to a gravitational background field, and one recovers from (18) the canonical EMT operator (10). Keeping the non-minimal term yields the improved EMT (16). (In flat space the Riemann scalar R vanishes, but its variation with respect to the metric is nevertheless non-zero.)

In classical theory, the improvement term with the particular value for h in (16) is fixed by requiring the kinetic energy in (17) to be conformally invariant: with this improvement term the trace $T^\mu_\mu = m^2 \Phi(x)^2$ which preserves conformal symmetry of the classical theory in the limit where m vanishes. On quantum level, the conformal symmetry is broken, but the improvement term is required to make Greens functions of the renormalized fields with an insertion of the improved EMT (16) finite. More precisely, the value for h in (16) removes UV divergences up to three-loops in dimensional regularization. The four-loop expression for h would acquire in addition to the result quoted in (16) a contribution proportional to $(n-4)^3$ needed to cancel pole contributions in dimensional regularization. However, the overall shape of the improvement term and the independence of h on the renormalized coupling, mass, renormalization scale μ remain to all orders [23].

To compute the D -term in Φ^4 theory it is therefore sufficient to investigate the effect of the improvement term at tree-level: loop corrections produce UV divergences which the improvement term (16) removes [17–23], and due to the renormalization scale invariance of the EMT operator the final result must not be altered by $\mathcal{O}(\lambda)$ -corrections. Evaluating the improvement operator at tree-level yields

$$\langle \vec{p}'_{\text{free}} | \hat{\theta}_{\text{improve}}^{\mu\nu}(x) | \vec{p}_{\text{free}} \rangle = 2h e^{i(p'-p)x} \left\{ \Delta^\mu \Delta^\nu - g^{\mu\nu} \Delta^2 \right\}. \quad (19)$$

There is no effect on $A(t)$. This is expected because $A(0) = 1$ is fixed from general principles and one obtains this result already without including any improvement term, see Sec. II B. The inclusion of the improvement term therefore must not, and does not, spoil the general constraint (4a).

The situation is different for the D -term which interestingly is affected. From Eq. (19) we obtain

$$D_{\text{interacting improved}} = -1 + 4h. \quad (20)$$

With $h = \frac{1}{6}$ in $n = 3 + 1$ space-time dimensions we obtain $D_{\text{interacting improved}} = -\frac{1}{3}$. This is a remarkable result. Even infinitesimally weak interactions can have a drastic effect on the value of the D -term. This insightful observation deserves several comments.

First, adding total derivatives to the EMT leaves $P^\mu \equiv \int d^3x T^{0\mu}$ and other Poincaré group generators unaffected, i.e. it does not impact the particle mass or spin. But we see that D in general is sensitive to adding total derivatives: the improvement term is one such total derivative. The D -term is a measurable quantity, even though challenging to infer from experiment. This means in general one cannot add total derivatives to the EMT at will, contrary to common belief. When this happens to be necessary (Belifante procedure in Dirac case, Φ^4 theory) it is crucial to establish a unique definition for improvement term(s) as dictated by the general properties of the theory, in order to ensure a uniquely defined D -term.

Second, when dealing with a free massive field theory case, there is no criterion to motivate and uniquely define a specific improvement term. In lack of such a criterion we conclude that in free scalar theory $D = -1$, Eq. (14). This is an unambiguous prediction of the free Klein-Gordon theory (minimally coupled to gravity), analog to the anomalous magnetic moment $g = 2$ predicted from free Dirac theory (minimally coupled to an electromagnetic background field).

Third, in Φ^4 theory we deal with an interacting quantum field theory which has to be renormalized. In this case the unique improvement term (16) ensures that Greens functions with an insertion of the improved EMT are finite. This guarantees the “renormalizability of the combined theory of gravity and matter, with gravity treated to lowest order and the self-interactions of matter [in Φ^4 theory] to all orders” [23]. The inclusion of the improvement term has a drastic effect on the D -term. Assuming even an infinitesimally small coupling constant $\lambda \lll 1$ (such that calculations to three or fewer loops are sufficient) we have $D_{\text{interacting improved}} = -\frac{1}{3}$ instead of the value -1 in the free theory.¹ This clearly demonstrates that the D -term is highly sensitive to interactions and the dynamics.

Fourth, the renormalizability of the Φ^4 theory has been studied in weak curved gravitational background fields, and the same improvement term (16) is required [24], which means $D = -\frac{1}{3}$ in weakly interacting Φ^4 theory in presence of gravity. As no quantum theory of gravity is known, it is of course also not known whether (16) would ensure renormalizability if quantum gravity effects were included. At this point one might be tempted to think that gravity

¹ For completeness we remark that in the conformally invariant massless free scalar theory, one also has to introduce the improvement term (16) to restore $T^\mu_\mu = m^2 \Phi(x)^2 \rightarrow 0$ and recover a divergenceless (conserved) conformal current. Thus, in the massless free case we also have $D = -\frac{1}{3}$. At this point one may wonder whether the improvement term (16) should also be added in the massive free Klein-Gordon theory. Then the D -term would exhibit a smooth behavior when m goes to zero. This would certainly be a legitimate step, though there is in general no reason to expect necessarily a smooth behavior of particle properties in a limit such as $m \rightarrow 0$. However, one may also invoke arguments which support that $D = -1$ is a consistent result in the massive free case, see App. B. At the end we shall briefly review which definition of the EMT is appropriate in App. C.

is far too weak to be of relevance in particle physics. However, the lesson we learned is that even infinitesimally small interactions in Φ^4 theory can impact the D -term. So why not infinitesimally small gravitational interactions?

Fifth, the D -term emerges to be strongly sensitive to interactions. One must consistently include all forces, perhaps even gravity, to determine the true improvement term and the “true” value of the D -term. These issues are beyond the scope of our work as is the very question whether a non-trivial Φ^4 theory actually exists [25].

The above arguments certainly do not apply to theories which have to be solved in non-perturbative regime. At this point one may therefore wonder how the D -terms of spin-0 particles are affected in strongly interacting theories. We shall discuss two examples in the next sections, QCD and Q -balls.

D. Strongly interacting theory, QCD

It is not possible to tell what the D -term would be in a strongly interacting Φ^4 -theory, where the perturbative expansion indicated in Eq. (20) would be inappropriate. Fortunately, the D -terms can be computed for a special class of spin-0 particles in a much more relevant and realistic strongly interacting theory, QCD. This is possible for pions, kaons and η -meson, the Goldstone bosons of chiral symmetry breaking by exploring low energy theorems. The results were already obtained in 1980, but have not been discussed in the context of the physics of the D -term. It is therefore of interest to review them here.

In Refs. [26, 27] the charmonium decays $\psi' \rightarrow J/\psi \pi \pi$ were studied. The description of these decays requires the matrix elements $\langle \pi' \pi | \hat{T}^{\mu\nu}(0) | 0 \rangle$, or $\langle \pi' | \hat{T}^{\mu\nu}(0) | \pi \rangle$ after applying crossing symmetry. Similar matrix elements enter also the description of a hypothetical light Higgs boson decay [28] into two pions which was discussed at some point in the past in literature [29].

Chiral symmetry uniquely determines the interactions of soft pions. In Refs. [26, 27] the following low energy theorem was derived which, in our notation, is given by

$$\langle \pi(\vec{p}') | \hat{T}^{\mu\nu}(0) | \pi(\vec{p}) \rangle = \frac{1}{2} \left(P^\mu P^\nu - \Delta^\mu \Delta^\nu + g^{\mu\nu} \Delta^2 \right) + \mathcal{O}(E^4). \quad (21)$$

Here E is the soft scale associated with the soft momenta of the Goldstone bosons or their masses, i.e. generically $E \sim \mathcal{O}(p, p', m_\pi)$. From (21) we read off (notice the first term on the right-hand side of (21) is already E^2)

$$D_h = -1 + \mathcal{O}(E^2), \quad h = \pi, K, \eta, \quad (22)$$

where we added that the same result is obtained also for kaons and the η -meson. This is a remarkable result. In the soft pion limit chiral symmetry dictates that the form factors of the EMT and the D -term of the light octet mesons coincide (at small values of $-t \sim m_\pi^2 \sim E^2$) with the free-field case in Eq. (14), despite the fact that we deal with strongly interacting particles. Notice, however that the Goldstone bosons have no internal structure to the considered order in the soft scale in Eqs. (21, 22), which makes it plausible why the free field value (14) is naturally recovered. In particular, this implies that

$$\lim_{E \rightarrow 0} D_h = -1, \quad h = \pi, K, \eta. \quad (23)$$

This result was derived independently from a soft-pion theorem for pion GPDs in Ref. [6]. At this point one may wonder why no improvement term analog to (16) was added, which would be relevant in massless case, see footnote 1. However, the answer is that such an improvement term is forbidden as it violates chiral symmetry [30, 31].

The chiral properties of the EMT form factors $A_i(t)$ and $D_i(t)$ for $i = \pi, K, \eta$ were studied beyond the chiral limit and evaluated in chiral perturbation theory to order $\mathcal{O}(E^4)$ in Ref. [32]. We quote here only the results for the D -terms [32] which are given by

$$D_\pi = -1 + 16a \frac{m_\pi^2}{F^2} + \frac{m_\pi^2}{F^2} I_\pi - \frac{m_\pi^2}{3F^2} I_\eta + \mathcal{O}(E^4) \quad (24a)$$

$$D_K = -1 + 16a \frac{m_K^2}{F^2} + \frac{2m_K^2}{3F^2} I_\eta + \mathcal{O}(E^4) \quad (24b)$$

$$D_\eta = -1 + 16a \frac{m_\eta^2}{F^2} - \frac{m_\pi^2}{F^2} I_\pi + \frac{8m_K^2}{3F^2} I_K + \frac{4m_\eta^2 - m_\pi^2}{3F^2} I_\eta + \mathcal{O}(E^4) \quad (24c)$$

where

$$a = L_{11}(\mu) - L_{13}(\mu), \quad I_i = \frac{1}{48\pi^2} \left(\log \frac{\mu^2}{m_i^2} - 1 \right), \quad i = \pi, K, \eta, \quad (24d)$$

and F denotes the pion decay constant $F \simeq 93$ MeV. The expansion parameter in chiral perturbation theory is often associated with the dimensionless ratio $E^2/(4\pi F)^2$ where $(4\pi F)^2 \sim 1$ GeV². In Eq. (24d) the renormalization scale μ appears, which is arbitrary because changes in μ are absorbed by appropriate redefinitions of the low energy constants L_{11} and L_{13} . This reflects the fact that the EMT is a renormalization scale invariant operator. Notice also that to the order considered in (24a–24d) which corresponds to $\mathcal{O}(E^6)$ in Eq. (21) the form factors exhibit a t -dependence, which signals that the Goldstone bosons acquire an internal structure.

This allows one to make more realistic predictions for the D -terms than the chiral limit prediction (23). The values of the low energy constants were estimated [32] as $L_{11}(1 \text{ GeV}) = (1.4\text{--}1.6) \times 10^{-3}$ and $L_{13}(1 \text{ GeV}) = (0.9\text{--}1.1) \times 10^{-3}$ using the meson dominance model (lower values) and dispersion relation technics (higher values). This yields

$$D_\pi = -0.97 \pm 0.01, \quad (25a)$$

$$D_K = -0.77 \pm 0.15, \quad (25b)$$

$$D_\eta = -0.69 \pm 0.19, \quad (25c)$$

where the uncertainties are due to $\delta L_{11} = \delta L_{13} = 0.2 \times 10^{-3}$, the use of the physical value of the pion decay constant $F = 93$ MeV [32] vs chiral limit value $F = 88$ MeV [33], and a heuristic estimate of higher order chiral corrections proportional to $E^4/(4\pi F)^4$ with E the respective meson mass. All these uncertainties are added in quadrature. Chiral interactions alter the soft theorem result $D = -1$, and are not unexpectedly more sizable for heavier mesons. However, the D -terms remain negative.

For completeness we remark that the effects of the electromagnetic interaction on the EMT form factors of charged and neutral pions were investigated in [33]. More recently pion EMT form factors were studied in chiral quark models, where definite predictions for the low energy constants can be made [34].

The quark contribution to pion EMT form factors was also studied in lattice QCD for pion masses in the range $550 \text{ MeV} \leq m_\pi \leq 1090 \text{ MeV}$ for lattice spacings $0.07\text{--}0.12$ fm and spatial lattice sizes $1.6\text{--}2.2$ fm [35, 36]. The quark contribution to the D -term was found to be, see Table 7.3 in [36],

$$D_\pi^Q = -(0.264 \pm 0.032) \quad (26)$$

at a renormalization scale of 2 GeV in $\overline{\text{MS}}$ scheme. The error includes the statistical accuracy of the lattice simulations combined with an estimate of uncertainties due to the extrapolation procedure (to physical pion masses and $t = 0$). Finite volume effects were noticed but could not be quantified as systematic uncertainties [35, 36]. It is not possible to confront this result with the prediction (25a) from chiral perturbation theory because $D_\pi^Q = -(0.264 \pm 0.032)$ is only a partial result (currently no information from lattice QCD is available on the gluonic contribution to the D -term of the pion or any other hadron). In addition it is difficult to reliably quantify the uncertainty due to extrapolation from the pion mass region above 550 MeV to the physical point. It will be interesting to see new lattice calculations on present-day stage-of-the-art lattices where physical pion masses can be handled.

The light pseudoscalar octet mesons are an exception, since they are Goldstone bosons of chiral symmetry breaking. For other hadrons no low energy theorems exist which would allow to predict their D -terms, and one may in general obtain much different numerical values for D . This is nicely illustrated by studies of nuclei. In general the description of nuclei in QCD is rather complex, and certainly no easier than that of Goldstone bosons and any other hadron. However, the saturation property and short range of the “residual” nuclear forces make it possible to predict gross features of nuclear D -terms.

Both properties are well-captured in the liquid drop model which was explored to study nuclear D -terms [7]. Of course only ground states of even-even nuclei (even number of protons Z and even number of neutrons N) are “guaranteed” to have spin zero. But spin effects play no role in the liquid drop model. Interestingly, nuclear radii grow as $A^{1/3}$ and nuclear masses as A with the mass number $A = N + Z$. But nuclear D -terms, due to the surface tension in the liquid drop model, are negative and show a far stronger dependence $D \propto A^{7/3}$ [7]. Numerical calculations in the Walecka model for selected $J^\pi = 0^+$ isotopes (^{12}C , ^{16}O , ^{40}Ca , ^{90}Zr , ^{208}Pb) were presented in [37]. The D -terms were found negative. For nuclei heavier than ^{12}C it was found $D \propto A^{2.26}$ in good agreement with [7]. For completeness we remark that in Ref. [38] a different A -behavior was found.

Let us summarize what we know about the D -terms of spin-0 hadrons. For the Goldstone bosons of chiral symmetry breaking in strong interactions one can explore low energy theorems and chiral perturbation theory to predict that $D = -1$ modulo chiral corrections which make the D -term less negative, but do not change its sign. D -terms of nuclei are also negative, much more sizable than those of the light pseudoscalar mesons and strongly grow with the mass number as $D \propto A^{7/3}$ which can be tested in experiments on hard exclusive reactions off nuclei [7].

E. Strongly interacting theory, Q -balls

Another example of a strongly interacting theory of scalar particles is the Q -ball system [39], see also [40, 41]. In this section we briefly review the Q -ball theory and quote some results regarding the D -term from [43–45]. More details about Q -balls will be provided in Sec. IV B where we will explore the Q -ball framework for further applications.

Q -balls are solitons in scalar theories with a global symmetry where a “suitable potential” satisfies certain conditions. The theory can be formulated in terms of one complex scalar field, or equivalently in terms of two real scalar fields which we shall choose to do here. The Lagrangian and the equations of motion are given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi_1)(\partial^\mu \Phi_1) + \frac{1}{2}(\partial_\mu \Phi_2)(\partial^\mu \Phi_2) - V, \quad \square \Phi_i(x) + \frac{\partial V}{\partial \Phi_i} = 0, \quad i = 1, 2, \quad (27)$$

with a potential V such that the theory is invariant under global continuous $\text{SO}(2)$ symmetry transformations ($\beta \in \mathbb{R}$)

$$\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \rightarrow \begin{pmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}. \quad (28)$$

The global symmetry implies a conserved Noether current $J^\mu = \Phi_1 \partial^\mu \Phi_2 - \Phi_2 \partial^\mu \Phi_1$. The associated conserved charge $Q = \int d^3x J^0(x)$ is instrumental for the existence of the soliton solutions which are, in their rest frames, of the type

$$\begin{pmatrix} \Phi_1(\vec{x}, t) \\ \Phi_2(\vec{x}, t) \end{pmatrix} = \begin{pmatrix} \cos(\omega t) \\ \sin(\omega t) \end{pmatrix} \phi(r), \quad (29)$$

where $r = |\vec{x}|$ and ω is bound by $\omega_{\min}^2 < \omega^2 < \omega_{\max}^2$. The limiting frequencies are defined in terms of the properties of the potential V , with V understood as a function of the radial field $\phi(r)$, as follows

$$0 < \omega_{\min}^2 \equiv \min_{\phi} \left[\frac{2V(\phi)}{\phi^2} \right] < \omega_{\max}^2 = V''(\phi) \Big|_{\phi=0}. \quad (30)$$

Notice that $m = \omega_{\max}$ defines the mass of the elementary quanta of the fields Φ_1 and Φ_2 . The solutions satisfying (not satisfying) the equivalent conditions

$$\frac{d}{d\omega} \left(\frac{M}{Q} \right) \geq 0 \quad \Leftrightarrow \quad \frac{dQ}{d\omega} \leq 0 \quad \Leftrightarrow \quad \frac{d^2 M}{d\omega^2} \leq 0, \quad (31)$$

are stable (unstable) with respect to small fluctuations [40, 41]. The point where the inequalities in (31) become equalities defines the critical frequency ω_c , i.e. for instance $Q'(\omega) = 0$ at $\omega = \omega_c$. The solutions are absolutely stable if $M < mQ$ where m denotes the mass of the elementary fields [41].

In the Q -ball system a general analytical proof was formulated that $D < 0$ for any suitable potential [43]. It was also shown that the numerical values of the D -terms can span orders of magnitude. For that the suitable, often studied (non-renormalizable, effective) theory was used with the sextic potential $V_6 = A\phi^2 - B\phi^4 + C\phi^6$ with $\phi^2 = \Phi_1^2 + \Phi_2^2$ and positive A, B, C satisfying $0 < \zeta \equiv B^2/(4AC) < 1$ [39]. For this potential $\omega_{\min}^2 = 2A(1 - \zeta)$ and $\omega_{\max}^2 = 2A$. For the parameters $A = 1.1$, $B = 2.0$, $C = 1.0$ it was found $|D| \geq |D_c|$ with $D_c = -113.4$ numerically close to the critical frequency $\omega_c = 1.38$ [43]. For ω not in the vicinity of ω_c the D -terms are becoming quickly more and more negative.

In the “ Q -ball limit” $\varepsilon_{\min} \equiv \sqrt{\omega^2 - \omega_{\min}^2} \rightarrow 0$ one deals with absolutely stable well-localized solitons [39] characterized by constant charge density in their interior, whose sizes grow as ε_{\min}^{-4} , and the masses and charges grow as ε_{\min}^{-6} . The most spectacular growth, however, is exhibited by the D -term which behaves as $D \propto \varepsilon_{\min}^{-14}$ in this limit [43].

In the opposite “ Q -cloud limit” $\varepsilon_{\max} \equiv \sqrt{\omega_{\max}^2 - \omega^2} \rightarrow 0$ [42] the solutions become infinitely dilute, diffuse and disintegrate into a cloud of free Q -quanta. In this limit mass, charge, and mean radii of the solutions diverge as ε_{\max}^{-1} . Again, the D -term is the property exhibiting the strongest pattern of divergence with $D \propto \varepsilon_{\max}^{-2}$ [45]. Interestingly, in the Q -cloud limit the sextic term in V_6 becomes irrelevant (in the sense of critical phenomena), and after a suitable rescaling one deals with a (complex) Φ^4 theory continued analytically to a negative coupling constant λ [45].

Q -balls can have also excited states (all with spin zero and positive parity as the ground state) which are unstable and have also negative D -terms. The solution $\phi(r)$ of the N^{th} excitation exhibits N nodes (ground state has no node). For a fixed frequency ω the mass and charge of the N^{th} excitation scale as N^3 , while the D -term scales as N^8 [44].

The Q -ball system confirms that D -terms of spin-0 particles can deviate significantly from the free-field theory result $D = -1$ though the negative sign of the D -term is preserved. The Q -ball results also strongly support the observation that the D -term is the particle property which is most sensitive to the details of the dynamics of a theory.

F. Particles with higher spins

We remark that also the D -terms of particles with non-zero spin were investigated in a variety of theoretical frameworks and models. In all cases the D -terms were found negative, including nucleon (spin $\frac{1}{2}$) [46–54], photon (spin 1) [55], and Δ -resonance (spin $\frac{3}{2}$) [56]. Notice that no analog of the low energy theorem (21) exists for hadrons other than Goldstone bosons. Therefore, chiral perturbation theory cannot predict the D -term of e.g. the nucleon, though it can make predictions on the small- t dependence of the EMT form factors [57].

III. 3D EMT DENSITIES

In this section we introduce the notion of 3D-densities of the EMT, apply it to the case of a free point-like particle, and demonstrate its consistency. We show that the description is physically well-formulated and justified in the heavy mass limit. We then “give” the particle a finite size. Hereby we initially proceed in a heuristic way. The finite size naturally introduces an additional scale in the theory, which is required to formulate adequately the heavy mass limit. We show that the property $D = -1$ is then still preserved. Finally we demonstrate that it is possible to construct dynamical microscopic theories which describe extended particles where the free field property $D = -1$ is preserved.

A. Static EMT and definitions

The information content associated with EMT form factors can be interpreted in analogy to the electromagnetic form factors [14] in the Breit frame which is characterized by $\Delta^0 = E' - E = 0$. In this frame, one defines the static energy-momentum tensor² as [7]

$$T_{\mu\nu}(\vec{r}) = \int \frac{d^3\Delta}{2E(2\pi)^3} \exp(i\vec{\Delta}\vec{r}) \langle \vec{p}' | \hat{T}_{\mu\nu}(0) | \vec{p} \rangle, \quad (32)$$

where $E = E' = \sqrt{m^2 + \vec{\Delta}^2}/4$. This provides information on the energy density $T_{00}(\vec{r})$ and the stress tensor $T_{ik}(\vec{r})$. The $T_{0k}(\vec{r})$ components vanish in the spin-0 case.

The energy density yields the particle mass according to $m = \int d^3r T_{00}(\vec{r})$, which implies the constraint (4a). The stress tensor is described in terms of two functions, the distribution of shear forces $s(r)$ and pressure $p(r)$,

$$T_{ij}(\vec{r}) = s(r) \left(\vec{e}_r^i \vec{e}_r^j - \frac{1}{3} \delta_{ij} \right) + p(r) \delta_{ij}, \quad (33)$$

where $\vec{e}_r = \vec{r}/r$ denotes the radial unit vector and $r = |\vec{r}|$. The EMT conservation, $\partial^\mu \hat{T}_{\mu\nu} = 0$, implies for the static stress tensor $\nabla^i T^{ij}(\vec{r}) = 0$ from which one can derive two helpful relations. First, $p(r)$ and $s(r)$ are connected by

$$\frac{2}{3} \frac{\partial s(r)}{\partial r} + \frac{2s(r)}{r} + \frac{\partial p(r)}{\partial r} = 0. \quad (34)$$

Second, the pressure $p(r)$ must satisfy the von Laue condition [59, 60], which is a necessary (but not sufficient) condition for stability,

$$\int_0^\infty dr r^2 p(r) = 0. \quad (35)$$

Owing to Eq. (34) the D -term can be expressed in two different ways in terms of shear and pressure distributions as

$$D \stackrel{(a)}{=} -\frac{4m}{15} \int d^3r r^2 s(r) \stackrel{(b)}{=} m \int d^3r r^2 p(r). \quad (36)$$

² Notice that the factor $1/(2E)$ in Eq. (5) of [7] should appear under the integral [58].

The concepts of “mechanical stability” [7] impose stability criteria on the densities in the classical theory which can be introduced also in quantum field theory and imply for the EMT densities [56]

$$(a) \quad T_{00}(r) \geq 0, \quad (b) \quad \frac{2}{3} s(r) + p(r) \geq 0. \quad (37)$$

For practical applications it is helpful to derive the explicit expressions for the densities in terms of the form factors and demonstrate their consistency. For that we recall that in the Breit frame $P^\mu = (P^0, 0, 0, 0)$ and $\Delta^\mu = (0, \vec{\Delta})$. With this we obtain from (32) for the energy density and the stress tensor the results

$$T_{00}(r) = m^2 \int \frac{d^3\Delta}{E(2\pi)^3} e^{i\vec{\Delta}\vec{r}} \left[A(t) - \frac{t}{4m^2} (A(t) + D(t)) \right] \quad (38a)$$

$$T_{ij}(\vec{r}) = \frac{1}{2} \int \frac{d^3\Delta}{2E(2\pi)^3} e^{i\vec{\Delta}\vec{r}} \left[\Delta_i \Delta_j - \delta_{ij} \vec{\Delta}^2 \right] D(t). \quad (38b)$$

From Eq. (38b) we can project out the expressions for the pressure and shear forces, namely

$$p(r) = \frac{1}{3} \int \frac{d^3\Delta}{2E(2\pi)^3} e^{i\vec{\Delta}\vec{r}} D(t) \left(-\vec{\Delta}^2 \right), \quad (39a)$$

$$s(r) = \frac{1}{4} \int \frac{d^3\Delta}{2E(2\pi)^3} e^{i\vec{\Delta}\vec{r}} D(t) \left(-\vec{\Delta}^2 + 3(\vec{e}_r \vec{\Delta})^2 \right). \quad (39b)$$

If we choose the coordinates in the Δ -integration such that \vec{r} points along the direction of the Δ_z -axis and define $\vec{e}_r \vec{\Delta} = \cos \theta_\Delta |\vec{\Delta}|$ then, recalling that $t = -\vec{\Delta}^2$ in Breit frame,

$$p(r) = \frac{1}{3} \int \frac{d^3\Delta}{2E(2\pi)^3} e^{i\vec{\Delta}\vec{r}} P_0(\cos \theta_\Delta) \left(t D(t) \right), \quad (40a)$$

$$s(r) = \frac{3}{4} \int \frac{d^3\Delta}{2E(2\pi)^3} e^{i\vec{\Delta}\vec{r}} P_2(\cos \theta_\Delta) \left(t D(t) \right), \quad (40b)$$

The expressions (40a, 40b) can be further simplified. Using the expansion of a plane wave in spherical Bessel functions and the orthogonality relation of Legendre polynomials,

$$e^{i\vec{\Delta}\vec{r}} = \sum_{l=0}^{\infty} i^l (2l+1) j_l(|\vec{\Delta}|r) P_l(\cos \theta_\Delta), \quad \int_{-1}^1 dx P_l(x) P_k(x) = \frac{2}{2l+1} \delta_{lk}, \quad (41)$$

yields

$$p(r) = \frac{1}{3} \int \frac{d^3\Delta}{2E(2\pi)^3} j_0(|\vec{\Delta}|r) \left(t D(t) \right), \quad (42a)$$

$$s(r) = -\frac{1}{2} \int \frac{d^3\Delta}{2E(2\pi)^3} j_2(|\vec{\Delta}|r) \left(t D(t) \right). \quad (42b)$$

It is instructive to verify the consistency of these definitions. As a first consistency check we integrate the expression for the energy density in Eq. (38a) over the volume

$$\begin{aligned} \int d^3r T_{00}(r) &= m^2 \int d^3r \int \frac{d^3\Delta}{E(2\pi)^3} e^{i\vec{\Delta}\vec{r}} \left[A(t) - \frac{t}{4m^2} (A(t) + D(t)) \right] \\ &= m^2 \int \frac{d^3\Delta}{E(2\pi)^3} \left[A(t) - \frac{t}{4m^2} (A(t) + D(t)) \right] (2\pi)^3 \delta^{(3)}(\vec{\Delta}) \\ &= \lim_{t \rightarrow 0} \frac{m^2}{E} \left[A(t) - \frac{t}{4m^2} (A(t) + D(t)) \right] = m \end{aligned} \quad (43)$$

where in the last step we used that $E = m$ for $t = -\vec{\Delta}^2 \rightarrow 0$, which yields the desired result. As a second consistency

check we integrate the pressure, as defined in Eq. (39a), over the volume. We obtain

$$\begin{aligned} \int d^3r p(r) &= \frac{1}{3} \int d^3r \int \frac{d^3\Delta}{2E(2\pi)^3} e^{i\vec{\Delta}\vec{r}} \left[t D(t) \right] \\ &= \frac{1}{3} \int \frac{d^3\Delta}{2E(2\pi)^3} \left[t D(t) \right] (2\pi)^3 \delta^{(3)}(\Delta) \\ &= \frac{1}{3} \lim_{t \rightarrow 0} \left[\frac{1}{2E} t D(t) \right] = 0 \end{aligned} \quad (44)$$

which reproduces the von Laue condition (35). As a third consistency test we verify the differential equation (34) connecting the pressure and shear forces. Inserting the expressions (42a, 42b) into Eq. (34), defining $z = |\vec{\Delta}|r$, recalling that $t = -\vec{\Delta}^2$, and using primes to denote derivatives of a function with respect to its argument, we obtain

$$\frac{2}{3} \frac{\partial s(r)}{\partial r} + \frac{2s(r)}{r} + \frac{\partial p(r)}{\partial r} = \int \frac{d^3\Delta}{2E(2\pi)^3} \left\{ \frac{2}{3} \left(-\frac{1}{2} j_2'(z) \right) + \frac{2}{z} \left(-\frac{1}{2} j_2(z) \right) + \left(\frac{1}{3} j_0'(z) \right) \right\} |\vec{\Delta}| \left[t D(t) \right] = 0 \quad (45)$$

which vanishes because the expression in the curly brackets is zero due to the identity $j_0'(z) - j_2'(z) - 3j_2(z)/z = 0$.

B. Densities of a point-like particle

Let us compute the static EMT densities of a point-like Klein-Gordon particle. With the results from Sec. II B we obtain for the energy density, pressure, and shear forces as defined in Eqs. (38a, 39a, 39b) the results

$$\begin{aligned} T_{00}(\vec{r}) &= m^2 \int \frac{d^3\Delta}{E(2\pi)^3} e^{i\vec{\Delta}\vec{r}} = \frac{m^2}{\sqrt{m^2 - \vec{\nabla}^2/4}} \delta^{(3)}(\vec{r}), \\ p(r) &= \frac{1}{3} \int \frac{d^3\Delta}{2E(2\pi)^3} \vec{\Delta}^2 e^{i\vec{\Delta}\vec{r}} = -\frac{1}{6} \frac{\vec{\nabla}^2}{\sqrt{m^2 - \vec{\nabla}^2/4}} \delta^{(3)}(\vec{r}), \\ s(r) &= -\frac{3}{4} \int \frac{d^3\Delta}{2E(2\pi)^3} e^{i\vec{\Delta}\vec{r}} \left((\vec{e}_r \vec{\Delta})^2 - \frac{1}{3} \vec{\Delta}^2 \right) = \frac{1}{8} \frac{3 \vec{e}_r^i \vec{e}_r^j \nabla^i \nabla^j - \vec{\nabla}^2}{\sqrt{m^2 - \vec{\nabla}^2/4}} \delta^{(3)}(\vec{r}). \end{aligned} \quad (46)$$

As expected, the EMT densities of a point like particle are given by singular δ -distributions or their derivatives. Notice that in Eq. (46) it is understood that the derivatives act only on the δ -functions.

The infinite tower of derivatives implicit in the square roots is a consequence of what is sometimes referred to as “relativistic corrections.” Let us first show that despite these corrections the expressions are theoretically consistent. For that we assume that the square roots in Eq. (46) can be formally expanded in terms of a series in powers of $\vec{\nabla}^2/(4m^2)$. The derivatives on the δ -functions are handled using $\int d^3r h(\vec{r}) \nabla^i \nabla^j \delta^{(3)}(\vec{r}) = [\nabla^i \nabla^j h(\vec{r})]_{\vec{r}=0}$ where $h(\vec{r})$ denotes a generic trial function. In the case of the mass $m = \int d^3r T_{00}(r)$ and the von Laue condition $\int d^3r p(r) = 0$ the trial functions are $h(\vec{r}) = 1$, and we immediately see that $T_{00}(r)$ and $p(r)$ in Eq. (46) comply with these constraints. In order to verify that the D -term as defined in Eqs. (36a, 36b) is correctly reproduced, we note that in this case the trial function is $h(\vec{r}) = r^2$ and $\nabla^i \nabla^j r^i r^j = 12$ and $\vec{\nabla}^2 r^2 = 6$ holds. This confirms the correct result $D = -1$.

While the expressions are consistent in the above sense, the presence of relativistic corrections artificially mimics an internal structure. This can be seen, for instance, by computing the moments of the energy density, which we define and normalize such that the zeroth moment is unity (it would be the mass of the particle, had we not normalized it), the first moment is the mean square radius of $T_{00}(\vec{r})$, etc. With this definition, and assuming that the expansion of the square root under the integral is allowed, we obtain for the moments of the energy density

$$M_k \equiv \frac{1}{m} \int d^3r r^{2k} T_{00}(\vec{r}) = \int d^3r r^{2k} \left[\frac{1}{\sqrt{1 - \vec{\nabla}^2/(4m^2)}} \delta^{(3)}(\vec{r}) \right] = \int d^3r r^{2k} \left[\sum_{j=0}^{\infty} c_j (\vec{\nabla}^2)^j \delta^{(3)}(\vec{r}) \right], \quad (47)$$

with $c_j = (2j-1)!!/[(4m^2)^j 2^j j!]$ where $(-1)!! = 1!! = 1$ and $(2j+1)!! = 1 \cdot 3 \cdot \dots \cdot (2j-1) \cdot (2j+1)$ for $j > 1$. Performing $2j$ partial integrations in each term of the sum over j and using $[(\vec{\nabla}^2)^j r^{2k}]_{r=0} = (2k+1)! \delta_{jk}$ yields

$$M_k = \frac{(2k+1)!! (2k-1)!!}{(4m^2)^k}. \quad (48)$$

Let's recall that for a point-like particle one naturally expects $M_k = \delta_{k0}$ and that $M_k \neq 0$ for $k > 0$ would imply an extended structure. This is a consequence of relativistic corrections, and a general limitation of the interpretation of 3D-Fourier transforms of form factors as 3D-densities. One could also define moments of $s(r)$ and $p(r)$ analog to (47) to show that relativistic corrections do not spoil the lowest moments related to von Laue conditions and the D -term, as shown in Sec. III B. However, higher moments of $s(r)$ and $p(r)$ would be altered similarly to those of the energy density and lead to unphysical results.

The presence of relativistic corrections is of course well known, and their appearance can be understood in various ways, see e.g. [11] for a review. In the next section we will discuss how (and when) one can, at least in principle, go about these relativistic corrections. It is important to notice that the relativistic corrections set limitations for the interpretation. Nevertheless formally all theoretical results remain correct and consistent as we have shown above.

C. “Switching off” relativistic corrections

In order to “switch off” such relativistic corrections and recover well-defined 3D-densities consistent with the notion of a point-like particle, let us assume from now on that we work in the heavy mass limit $m \rightarrow \infty$, and retain only the respectively leading terms. Such a description in principle applies to the (free) Higgs boson, which is the only presently known fundamental spin zero particle. In this way we obtain for a heavy boson

$$\begin{aligned} T_{00}(\vec{r}) &= m \delta^{(3)}(\vec{r}), \\ p(r) &= -\frac{\vec{\nabla}^2}{6m} \delta^{(3)}(\vec{r}), \\ s(r) &= \frac{3 e_r^i e_r^j \nabla^i \nabla^j - \vec{\nabla}^2}{8m} \delta^{(3)}(\vec{r}). \end{aligned} \quad (49)$$

One sees immediately that the expressions in (49) are consistent. The von Laue condition (35) is satisfied, one obtains the same result $D = -1$ for the D -term using its both representations in terms of $s(r)$ and $p(r)$ in Eqs. (36a, 36b), and the moments of the energy distribution defined in Eq. (47) satisfy $M_k = \delta_{k0}$ as expected for a point like particle.

An important question is: the mass m of our boson is large, but with respect to what? This question is ill-posed in a free theory where the only dimensionfull parameter is m , and the only available length scale is the Compton wave-length of the particle $\lambda_C = 1/m$. To give a meaning to heavy mass limit we must “give some internal structure” to our heavy boson. To take into consideration the effects of an internal structure, we proceed here heuristically³ and replace the δ -functions in the expressions (49) with suitably smeared-out regular and normalized functions $f(r)$,

$$\delta^{(3)}(\vec{r}) \rightarrow f(r), \quad I_0 \equiv \int d^3r f(r) = 1, \quad (50)$$

where it is understood that $f(r)$ reduces to a δ -function in some well-defined limit.

Let us investigate the effect of such an internal structure on the energy density. We choose, at this point merely for illustrative purposes, the following representation $f_R(r)$ for the δ -function

$$f_R(r) = \frac{1}{\pi^{3/2} R^3} \exp\left(-\frac{r^2}{R^2}\right) \quad (51)$$

from which we recover $f_R(r) \rightarrow \delta^{(3)}(\vec{r})$ for $R \rightarrow 0$. In the heavy mass limit using the densities in Eq. (49) the “true” first moment of the energy distribution M_1 , i.e. mean square radius of the energy density, is given by

$$\langle r_E^2 \rangle \equiv M_1 = \frac{3}{2} R^2. \quad (52)$$

Having a specific “(toy) model” for the energy density, we can equally well evaluate the mean square radius of $T_{00}(r)$ using the expression (46) which includes relativistic corrections. The result we obtain and condition required for the interpretation in terms of 3-D densities to be applicable are as follows

$$\langle r_E^2 \rangle \equiv M_1 = \frac{3 R^2}{2} \left(1 + \delta_{\text{rel}}\right), \quad \delta_{\text{rel}} \equiv \frac{1}{2m^2 R^2} \ll 1. \quad (53)$$

³ We postpone here the question how to describe such an “internal structure” in terms of a microscopic dynamical Lagrangian theory. This question will be addressed later.

Thus relativistic corrections are negligible when $m^2 R^2 \gg 1$, i.e. when the Compton wave-length is small compared to the “actual size” of the particle $\lambda_C^2 \ll R^2$. We obtained this condition here in the context of the mean square radius of the energy density, but it holds also for the other densities and can be derived from general considerations [11].

It is instructive to estimate the size of the corrections as defined in Eq. (53) for various particles, see Table I. For light mesons, like pions, kaons or η the concept of 3D-densities is clearly not applicable. However, for heavier mesons containing charmed quarks the concept makes sense: e.g. for η_c the relativistic corrections are of the order of $\mathcal{O}(4\%)$. For nuclei the concept can be safely applied: for instance for ${}^4\text{He}$, the lightest spin-0 nucleus, the corrections are merely of the order of $\mathcal{O}(0.05\%)$ and they diminish quickly for heavier nuclei. This can be understood in the liquid drop model of the nucleus, where a nucleus with mass number A has approximately the mass $\sim A \times 0.93 \text{ GeV}$ and the radius $\sim A^{1/3} \times 1.2 \text{ fm}$ which yields $\delta_{\text{rel}} \sim 1.2 A^{-8/3}$. Although they are not spin-0 particles, we have included the proton, deuteron and ${}^6\text{Li}$ in Table I for comparison. The concept of 3D-densities is applicable in all 3 cases with a reasonable accuracy of the order of $\mathcal{O}(3\%)$ for proton, $\mathcal{O}(0.1\%)$ for deuteron, and $\mathcal{O}(0.1\%)$ for ${}^6\text{Li}$.

Notice that it is customary to speak about mean square charge radii also for particles like (charged) pions and kaons, even though the concept of 3D-densities cannot be applied here. These “radii” are simply defined by the slopes of the electric form factors as, e.g. in the case of the pion

$$F_\pi(t) = 1 + \frac{\langle r_{\pi,em}^2 \rangle}{6} t + \mathcal{O}(t^2), \quad \text{or} \quad \langle r_{\pi,em}^2 \rangle = 6 F'_\pi(t) \Big|_{t=0}. \quad (54)$$

Of course, one can introduce the concept of the “spatial structure” and “size” of pions and kaons (and other particles) without relativistic corrections by working with 2D densities [66–69]. In that approach the 2D-radius of the particle is still related to the slope of the form factor, but now as $F_\pi(t) = 1 + \frac{1}{4} \langle r_{\pi,em,2D}^2 \rangle t + \mathcal{O}(t^2)$ (in each case the numerical prefactor is $1/(2d_{\text{space}})$ with d_{space} the number of space dimensions in the Fourier-transform).

But the concepts of pressures, shear forces and mechanical stability are inherently 3D. No interpretation exists for the stress tensor in terms of 2D densities. Therefore, if we wish to learn about the mechanical stability of nucleons and nuclei, we have to pay the prize of dealing with 3D densities and the associated relativistic corrections. However, the relativistic “blurring” of the 3D densities for nucleons and nuclei, about 3% for proton and much less for nuclei, seems acceptably small to carry on this program.

It is important to stress the different objectives of the 2D- vs 3D-density interpretations. The 2D-density description is exact and this is indispensable for a rigorous probabilistic partonic interpretation. The 3D-density description does not describe partonic probability densities. It describes in our context mechanical response functions of a system. These are to be understood as *correlation functions* which come with relativistic corrections. This approach is justified and gives valuable insights, as long as the corrections are acceptably small. As shown in Table I, this is the case in particular also in the phenomenologically relevant cases of the nucleon and nuclei.

D. Stress tensor of an extended spin-0 particle

In this Section we investigate the stress tensor of a point-like (heavy) boson which “is given” some “internal structure.” We continue proceeding heuristically, see Footnote 3, and replace the δ -function in the expressions for $p(r)$ and $s(r)$ in Eq. (49) with a suitable regular normalized function $f(r)$ as given in Eq. (50). We shall assume that $f(r)$ has the properties that (a) it is a radially symmetric function of \vec{r} , (b) it is at least three times continuously differentiable, (c) it satisfies $r^3 f''(r) \rightarrow 0$ and $r^2 f'(r) \rightarrow 0$ for $r \rightarrow 0$, and (d) it vanishes at large distances faster than any power of r . These restrictions will be convenient in the following, even though some of them could be relaxed (e.g. a large- r behavior $\propto 1/r^5$ would be sufficient in all physically relevant situations [49] including the chiral limit).

From Eq. (49) we obtain the results

$$\begin{aligned} p(r) &= -\frac{1}{6m} \left(f''(r) + \frac{2}{r} f'(r) \right), \\ s(r) &= \frac{1}{4m} \left(f''(r) - \frac{1}{r} f'(r) \right), \end{aligned} \quad (55)$$

where the primes denote derivatives with respect to the argument. It is important that in Eq. (55) we use the same function $f(r)$ in the expressions for $s(r)$ and $p(r)$. This is dictated by the conservation of the EMT, which imposes the differential equation (34). In fact, the relation (34) holds exactly

$$\frac{2}{3} \frac{\partial s(r)}{\partial r} + \frac{2s(r)}{r} + \frac{\partial p(r)}{\partial r} = \frac{2}{3} \left(\frac{f'''(r)}{4m} - \frac{f''(r)}{4mr} + \frac{f'(r)}{4mr^2} \right) + \frac{2}{r} \left(\frac{f''(r)}{4m} - \frac{f'(r)}{4mr} \right) + \left(-\frac{f'''(r)}{6m} - \frac{2f''(r)}{6mr} + \frac{2f'(r)}{6mr^2} \right) = 0 \quad (56)$$

particle	J^π	mass [GeV]	size [fm]	δ_{rel}
pion	0^-	0.14	0.67	2.2
kaon	0^-	0.49	0.56	2.5×10^{-1}
η -meson	0^-	0.55	0.68	1.4×10^{-1}
η_c -meson	0^-	2.98	0.26	3.8×10^{-2}
proton	$\frac{1}{2}^+$	0.94	0.89	2.8×10^{-2}
deuteron	1^+	1.88	2.14	1.2×10^{-3}
${}^6\text{Li}$	1^+	5.60	2.59	9.3×10^{-5}
${}^4\text{He}$	0^+	3.73	1.68	5.0×10^{-4}
${}^{12}\text{C}$	0^+	11.2	2.47	2.6×10^{-5}
${}^{20}\text{Ne}$	0^+	18.6	3.01	6.2×10^{-6}
${}^{32}\text{S}$	0^+	29.8	3.26	2.1×10^{-6}
${}^{56}\text{Fe}$	0^+	52.1	3.74	5.1×10^{-7}
${}^{132}\text{Xe}$	0^+	123	4.79	5.6×10^{-8}
${}^{208}\text{Pb}$	0^+	194	5.50	1.7×10^{-8}
${}^{244}\text{Pu}$	0^+	227	5.89	1.1×10^{-8}

TABLE I: Masses, radii, and the sizes of relativistic corrections δ_{rel} as defined in Eq. (53) for various spin-0 mesons and nuclei. Proton, deuteron, ${}^6\text{Li}$ are included for comparison. Masses and mean charge radii of mesons and proton are from [61] except for the radii of η taken from the estimate [62] and η_c taken from the lattice calculation [63]. Nuclear masses are from [64] and nuclear mean charge radii from [65]. The smaller δ_{rel} the more safely is applicable the 3D-density interpretation of form factors.

for every function $f(r)$ which satisfies the properties a–c. (Only here we need that $f(r)$ is 3 times continuously differentiable. For all other purposes 2 times continuously differentiable would be sufficient.) Since Eq. (56) holds for the extended particle and since it is equivalent to the conservation of the EMT, it is clear that all other properties related to the conservation of the EMT are also satisfied. Let us show this explicitly. The von Laue condition is

$$\int_0^\infty dr \, r^2 p(r) = \frac{1}{6m} \int_0^\infty dr \left(r^2 f''(r) + 2r f'(r) \right) = \frac{1}{6m} \int_0^\infty dr \, \frac{\partial}{\partial r} \left(r^2 f'(r) \right) = 0 \quad (57)$$

for every function $f(r)$ which satisfies the properties a–c. This proves Eq. (35). Finally, for the D -term of an extended particle we obtain from the shear forces and pressure in Eq. (34) the unambiguous result

$$D = m \int d^3r \, r^2 p(r) = -4\pi \int_0^\infty dr \left(r^4 \frac{f''(r)}{6} + r^3 \frac{f'(r)}{3} \right) = - \left(\frac{4 \cdot 3 I_0}{6} - \frac{3 I_0}{3} \right) = -1, \quad (58a)$$

$$= - \frac{4m}{15} \int d^3r \, r^2 s(r) = -4\pi \int_0^\infty dr \left(r^4 \frac{f''(r)}{15} - r^3 \frac{f'(r)}{15} \right) = - \left(\frac{4 \cdot 3 I_0}{15} + \frac{3 I_0}{15} \right) = -1, \quad (58b)$$

where we performed one or two partial integrations in the respective terms to express the final results in terms of the integral I_0 introduced in Eq. (50). The conclusion is that the property $D = -1$ holds also for an extended boson, and this is guaranteed by the normalization of the function $f(r)$ in Eq. (50).

At this point a comment is in order. One must choose one and the same representation for $\delta^{(3)}(\vec{r})$ when smearing out the δ -functions in the expressions for $p(r)$ and $s(r)$ in Eq. (49), because they are connected by the relations (34, 56). However, there is no reason why we should use the same regular function $f(r)$ when smearing out $T_{00}(r)$. At this point of our considerations, $T_{00}(r)$ is unrelated to $p(r)$ and $s(r)$. This is of course an unphysical feature. The expressions for all EMT densities should be derived from a Lagrangian of a dynamical theory. A non-trivial question

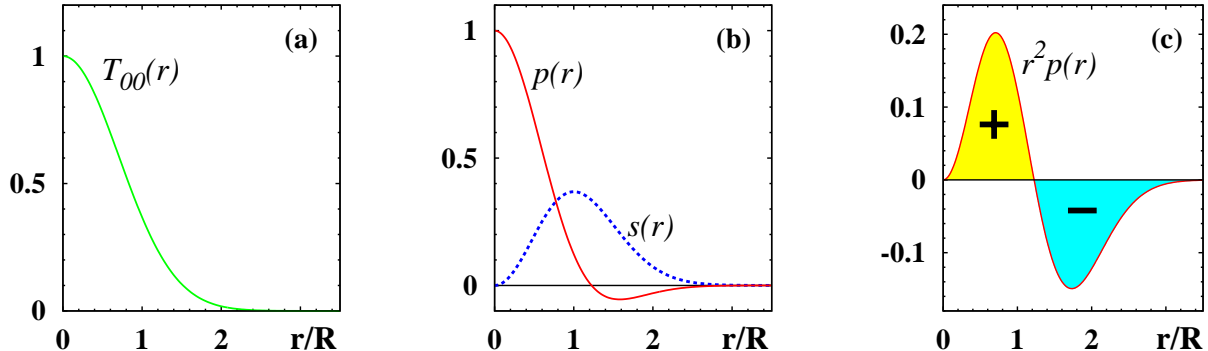


FIG. 1: (a) The energy density $T_{00}(r)$ in units of $T_{00}(0)$ as function of r in units of R for a “smeared-out” point-particle from the Gaussian representation (51) of a δ -function. (b) The same as Fig. 1a but for $s(r)$ and $p(r)$ in units of $p(0)$. (c) Visualization of the von Laue condition Eq. (35) with units as in Fig. 1b. In the limit $R \rightarrow 0$ (which implies $T_{00}(0) \rightarrow \infty$ and $p(0) \rightarrow \infty$) one recovers the original singular expressions (49). Notice that $D = -1$ holds both for finite R as well as in the limit $R \rightarrow 0$. For the 3D interpretation to be physically sound R is required to be larger than the Compton wave-length of the particle.

is whether it is possible to construct a dynamical theory where a particle has the property $D = -1$ but is extended and exhibits the EMT densities of a “smeared-out point-like” particle.

Before addressing this question in the next section, we visualize the EMT densities of such an “extended particle.” For purely illustrative purposes, we choose the representation $f_R(r)$ for the δ -function defined in Eq. (51). The results are shown in Fig. 1. It is remarkable, that in this way we effortlessly (without invoking dynamics, just by smearing out a point-like particle) recover the main features of the EMT densities calculated non-perturbatively in dynamical theories of Q -balls [43–45], chiral solitons [49, 50], or Skyrmions [51, 52].

IV. STRONGLY INTERACTING GAUSSIAN SCALAR FIELD

The previous section has shown that the free-theory result $D = -1$ persisted even if the point-like spin-0 boson was given an extended structure. Thereby we “introduced” the internal structure in a heuristic way. The emerging question is: can one construct a microscopic dynamical theory in which the spin-0 particles

- (a) have an extended structure,
- (b) have the desired property $D = -1$ of a free “point-like” particle, and
- (c) exhibit the heuristically obtained EMT densities corresponding to “smeared δ -functions” or their derivatives?

The answer is yes. In the following we will present one such theory, which can be formulated in the Q -ball system already mentioned in Sec. II E. We will begin by briefly reviewing the description of the EMT properties of Q -balls [43] in Sec. IV A, and then show that for a specific Q -ball potential one deals with exactly our “heuristically smeared-out point-like” particles from Sec. IV B. To streamline the presentation we address technical details of this theory separately in Sec. IV C, and discuss potential applications in Sec. IV D.

A. Brief review of the EMT properties of Q -balls

A brief introduction to Q -balls was already given in Sec. II E. To make this work self-contained, we review first the general Q -ball properties [39] including the expressions for the EMT densities of Q -balls derived in [43].

The theory defined in Eq. (27) of Sec. II E admits non-topological solitons for a suitable potential V [39]. In their rest frame the soliton solutions are given by the expression quoted in Eq. (29) with the radial field $\phi(r)$ obeying the equation of motion and the boundary conditions (primes denote differentiation with respect to the argument)

$$\begin{aligned} \phi''(r) + \frac{2}{r} \phi'(r) + \omega^2 \phi(r) - V'(\phi) &= 0, \\ \phi(0) \equiv \phi_0 \neq 0, \quad \phi'(0) &= 0, \quad \phi(r) \rightarrow 0 \text{ for } r \rightarrow \infty. \end{aligned} \tag{59}$$

The global U(1) symmetry implies a Noether current with the conserved charge

$$Q = \int d^3x \rho_{\text{ch}}(r), \quad \rho_{\text{ch}}(r) = \omega \phi(r)^2, \quad (60)$$

whose sign is determined by ω . Below we choose $\omega > 0$ without loss of generality. The EMT densities read

$$T_{00}(r) = \frac{1}{2} \omega^2 \phi(r)^2 + \frac{1}{2} \phi'(r)^2 + V, \quad (61a)$$

$$p(r) = \frac{1}{2} \omega^2 \phi(r)^2 - \frac{1}{6} \phi'(r)^2 - V, \quad (61b)$$

$$s(r) = \phi'(r)^2, \quad (61c)$$

The Q -ball densities satisfy the relation

$$T_{00}(r) + p(r) = \omega \rho_{\text{ch}}(r) + \frac{1}{3} s(r), \quad (62)$$

which implies the interesting Q -ball specific relation

$$D = \frac{4}{9} \left(\omega Q M \langle r_Q^2 \rangle - M^2 \langle r_E^2 \rangle \right), \quad (63)$$

with the Q -ball mass M and mean square radii of energy and charge densities defined as

$$M = \int d^3x T_{00}(r), \quad \langle r_E^2 \rangle = \frac{1}{M} \int d^3x r^2 T_{00}(r), \quad \langle r_Q^2 \rangle = \frac{1}{Q} \int d^3x r^2 \rho_{\text{ch}}(r). \quad (64)$$

B. Q -balls in logarithmic potential with $D = -1$

To find a microscopic theory of “smeared out” elementary particles, we consider Q -balls in the logarithmic potential

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \Phi_1) (\partial^\mu \Phi_1) + \frac{1}{2} (\partial_\mu \Phi_2) (\partial^\mu \Phi_2) - V_{\log}, \quad V_{\log} = A (\Phi_1^2 + \Phi_2^2) - B (\Phi_1^2 + \Phi_2^2) \log \left(C (\Phi_1^2 + \Phi_2^2) \right). \quad (65)$$

This potential is not bound from below, and understood as the limiting case of a well-defined theory, see Sec. IV C. Actually two parameters are sufficient to define this theory: we can replace $C \rightarrow 1/B$ and $A \rightarrow A - B \log(AC)$ without loss of generality which we shall do from now on. For this potential the eom for the radial field reads

$$\phi''(r) + \frac{2}{r} \phi'(r) + \left(\omega^2 - 2A + 2B \right) \phi(r) + 2B \phi(r) \log \left(\frac{\phi(r)^2}{B} \right) = 0. \quad (66)$$

The solution satisfying the boundary conditions (59) can be found analytically and is given by

$$\phi(r) = \phi_0 \exp \left(-B r^2 \right), \quad \phi_0 = \sqrt{B} \exp \left(\frac{2A + 4B - \omega^2}{4B} \right). \quad (67)$$

With the solution (67) all Q -ball properties can be evaluated analytically. In particular, we obtain for the densities

$$T_{00}(r) = (\omega^2 - 2B + 4B^2 r^2) \phi(r)^2, \quad (68a)$$

$$p(r) = \left(2B - \frac{8}{3} B^2 r^2 \right) \phi(r)^2, \quad (68b)$$

$$s(r) = 4 B^2 r^2 \phi(r)^2, \quad (68c)$$

$$\rho_{\text{ch}}(r) = \omega \phi(r)^2. \quad (68d)$$

The expressions for $s(r)$ and $p(r)$ satisfy the general differential equation (34), $p(r)$ satisfies the von Laue condition (35), and all densities comply with the Q -ball specific relation (62). For the global Q -ball properties we obtain

$$Q = N_0 \omega, \quad M = N_0 (B + \omega^2), \quad D = -N_0^2 (B + \omega^2), \quad N_0 \equiv \phi_0^2 \left(\frac{\pi}{2B} \right)^{3/2} \quad (69a)$$

$$\langle r_E^2 \rangle = \frac{3}{4B} \frac{3B + \omega^2}{B + \omega^2}, \quad \langle r_Q^2 \rangle = \frac{3}{4B} \quad (69b)$$

It is important to stress that the same result for D follows in 3 different ways, from Eqs. (36a, 36b) and (63). At this point it is also worth stressing that we obtain an analytic result for D which is manifestly negative, in agreement with all available theoretical calculations.

Next we discuss the requirements on the parameters. The conditions (37b, 31, 37a) imply (in this order):

$$\frac{2}{3}s(r) + p(r) = 2B\phi(r)^2 \geq 0 \quad \Leftrightarrow \quad B \geq 0, \quad (70a)$$

$$\frac{d}{d\omega} \left(\frac{M}{Q} \right) = \frac{d}{d\omega} \left(\omega + \frac{B}{\omega} \right) \geq 0 \quad \Leftrightarrow \quad \omega^2 \geq B, \quad (70b)$$

$$T_{00}(r) = (\omega^2 - 2B + 4B^2 r^2) \phi(r)^2 \geq 0 \quad \Leftrightarrow \quad \omega^2 \geq 2B. \quad (70c)$$

All conditions are satisfied and the solutions classically stable if $2\omega^2 \geq B > 0$ (we exclude $B = 0$ in (70a) which would reproduce free theory). For the limiting value $\omega^2 = 2B$ the energy density vanishes in the center, which is a feature not observed so far in the Q -ball literature to the best of our knowledge. For $2B < \omega^2 < 4B$ the energy density exhibits a dip in the center. Such dips occur naturally when the “surface tension” of the Q -matter is strong enough to produce a peak in $T_{00}(r)$ at the “edge” of the Q -ball [43]. Finally, for $\omega^2 \geq 4B$ we have a $T_{00}(r)$ which has no dip and is monotonically decreasing for all r .

Notice that the parameter A is completely unconstrained. We can choose \sqrt{B} to serve as unit of mass in our theory, and $1/\sqrt{B}$ as length unit. Then the role of A is to provide an overall rescaling of the fields by the factor $\exp(\frac{1}{2}AB^{-1})$, as can be seen from (67). This implies a corresponding rescaling of the properties in (69a) via $N_0 \propto \exp(AB^{-1})$. While at this point A can take any value, in Sec. IV C we shall see that certain restrictions for A exist.

Now we discuss how to fix the parameters such that $D = -1$. We notice that in general for our logarithmic Q -balls

$$\frac{(-D)}{Q^2} = 1 + \frac{B}{\omega^2} > 1, \quad (71)$$

where the inequality arises from $0 < B \leq 2\omega^2$. Clearly, parameters can be chosen such that either $D = -1$ or $Q = 1$ but not both simultaneously (unless one considered a limit like $\omega \rightarrow \infty$ for fixed B). However, Q is a conserved but not a topological quantum number and not required to be an integer. It also does not need to correspond in general to the electric charge. Notice that, if we wished to do it, we could simply redefine the unit in which the charges are measured to have integer-valued charges. Thus, there is no principle obstacle to have $D = -1$. Notice that similarly $M^2 = (-D)(\omega^2 + B)$ holds, implying the nice result $M = \sqrt{\omega^2 + B}$ for $D = -1$.

To obtain the desired value for the D -term $D = -1$ we may fix A and ω as follows,

$$\omega^2 = \alpha B, \quad A = \frac{B}{2} \left[\alpha - 4 - \log \left[\frac{\pi^3}{8} (1 + \alpha) \right] \right], \quad (72)$$

with an arbitrary positive parameter α which will be constrained shortly. In this way we obtain

$$D = -1, \quad M = \sqrt{B} \sqrt{1 + \alpha}, \quad Q = \sqrt{\frac{\alpha}{1 + \alpha}}, \quad \langle r_E^2 \rangle = \frac{3}{4B} \frac{3 + \alpha}{1 + \alpha}, \quad \langle r_Q^2 \rangle = \frac{3}{4B}. \quad (73)$$

For any value of α we have $D = -1$. Stability considerations (70a–70c) require $\alpha \geq 2$ leaving this parameter otherwise unconstrained. In order to further constrain α we consider our criterion (53) with $R^2 \rightarrow \langle r_Q^2 \rangle$. (We could equally well use $\langle r_E^2 \rangle$ for that, but due to the general relation $\langle r_Q^2 \rangle < \langle r_E^2 \rangle$ the criterion is more restrictive with $\langle r_Q^2 \rangle$.) We obtain

$$\delta_{\text{rel}} = \frac{2}{3} \frac{1}{1 + \alpha}. \quad (74)$$

At this point the parameter α is still not fixed, and we are free to choose its value to make relativistic corrections as small as we wish, for instance choosing $\alpha > 65$ guarantees $\delta_{\text{rel}} < 1\%$.

In order to close the loop and make contact with the heuristic discussion in Secs. III C and III D we remark that the densities can be rewritten in terms of the Gaussian introduced in Eq. (51) to smear out the δ -functions as follows

$$T_{00}(r) = M \left(\frac{\alpha - 2}{\alpha + 1} + \frac{2}{1 + \alpha} \frac{r^2}{R^2} \right) f(r), \quad \rho_{\text{ch}}(r) = Q f(r), \quad (75a)$$

$$p(r) = -\frac{1}{6M} \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right) f(r), \quad s(r) = \frac{1}{4M} \left(\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} \right) f(r), \quad (75b)$$

$$f(r) \equiv \frac{1}{\pi^{3/2} R^3} \exp \left(-\frac{r^2}{R^2} \right) \quad \text{with} \quad R = \frac{1}{\sqrt{2B}}. \quad (75c)$$

The smeared-out δ -function representation for $T_{00}(r)$ differs from that of the other densities (we discussed that this is in general expected). Notice that $f(r) \equiv M \phi(r)^2$ and $\int d^3r f(r)^2 = 1$. We can consider several limits.

In the large- α limit with B kept fixed in Eq. (72) the energy density can be expressed in terms of the same smeared-out function $f(r)$ which defines $p(r)$ and $s(r)$. In this interesting limit $D = -1$, $Q \rightarrow 1$ and $\langle r_i^2 \rangle \rightarrow 3/(4B)$ ($i = E, Q$) are fixed while the mass grows as $M \rightarrow \sqrt{\alpha B}$ justifying the applicability of the 3D-density description with $\delta_{\text{rel}} \rightarrow 0$.

Another way to implement limits is to keep α fixed (at a large enough value to keep δ_{rel} in (74) reasonably small) and take $B \rightarrow \infty$. Now we recover a heavy particle which becomes point-like as $\langle r_i^2 \rangle \rightarrow 0$ in this limit. Then $f(r) \rightarrow \delta^{(3)}(r)$ and we literally recover the description of a heavy point-like particle, with $D = -1$ of course, which we wrote down heuristically in Eq. (49) in Sec. III C.

We consider finally the limit that $\alpha \rightarrow \infty$ and $B \rightarrow 0$ such that the mass $M = \sqrt{\alpha + 1} \sqrt{B}$ remains fixed. Nothing prevents from choosing M to be moderately small or even light (but it must be non-zero). However, in this limit the size of our light particle grows since $\langle r_i^2 \rangle \rightarrow 3\alpha/M$ which guarantees the smallness of δ_{rel} in (74) and the applicability of the 3D-density description. We are not aware of systems of this kind in particle physics, but Rydberg atoms (fixed and moderate mass, extremely large size) provide an example from atomic physics.

It is gratifying to notice that there is no way to take a limit in which one could recover a light and small (point-like) particle, even if one were willing to pay the price of large relativistic corrections in Eq. (74). This is not surprising: our very starting point was the assumption that the 3D-density description is applicable, so our theory does not permit to take such a limit.

The Fig. 1 basically shows the EMT densities of our logarithmic Q -ball. More precisely, Fig. 1a shows $T_{00}(r)$ for $\alpha \gg 1$, while the Figs. 1b and 1c show the exact shear forces and the pressure distributions for any α . We recall that the results in Fig. 1 were initially obtained on the basis of heuristic arguments (“smearing out a point-like particle”), and now we have derived them from a dynamical theory.

Finally, let us remark that the logarithmic potential also admits excited states, which will be addressed elsewhere.

C. Proper boundary conditions for logarithmic Q -ball theory

This section is devoted to several technical, but indispensable details regarding the logarithmic potential in Eq. (65) which is not bound from below and does not constitute an “acceptable” Q -ball potential in the sense of Ref. [39]. Here we present a potential which is acceptable, bound from below, and contains our log-potential as limiting case.

Let us denote for simplicity $V = V(\phi)$ where $\phi = \phi(r)$ is the radial field. V is an “acceptable” Q -ball potential if (i) V is two times continuously differentiable with $V(0) = 0$, $V'(0) = 0$, $V''(0) = \omega_{\text{max}}^2 \equiv m_{\Phi}^2 > 0$, $V(\phi) > 0$ for $\phi \neq 0$, (ii) $V(\phi)/\phi^2$ has a minimum at some $\phi_{\text{min}} \neq 0$ which defines the lower limit $\omega_{\text{min}}^2 = 2V(\phi_{\text{min}})/\phi_{\text{min}}^2$ for frequencies, (iii) positive numbers a, b, c exist with $c > 2$ such that $\frac{1}{2}m_{\Phi}^2\phi^2 - V(\phi) \leq \min[a, b|\phi|^c]$.

To construct a potential complying with the above criteria and containing (65) as a limiting case, we introduce the dimensionless parameters $0 < \varepsilon_i \ll 1$ with $i = 1, 2$. One acceptable regular logarithmic potential V_{reg} is defined by

$$V_{\text{reg}} = A\phi^2 + \varepsilon_1\phi^4 - B\phi^2 \log\left(\varepsilon_2 + \frac{\phi^2}{B}\right). \quad (76)$$

The role of the term with $\varepsilon_1\phi^4$ is to make sure the potential is bound from below for $\varepsilon_1 > 0$. The effect of ε_2 is to ensure a regular small field expansion of the potential exists, $V_{\text{reg}} = (A - B \log \varepsilon_2)\phi^2 + \mathcal{O}(\phi^4)$, which generates a finite mass term for the fundamental field. In the limit that the ε_i are negligible we recover the log-potential (65). Below we will see how this limit is understood. We begin by considering the limiting frequencies (30) and their difference,

$$\omega_{\text{max}}^2 = m_{\Phi}^2 = [V''_{\text{reg}}(\phi)]_{\phi=0} = 2A - 2B \log \varepsilon_2, \quad (77a)$$

$$\omega_{\text{min}}^2 = \min_{\phi} \left[\frac{2V_{\text{reg}}(\phi)}{\phi^2} \right] = 2A + 2B(1 + \log \varepsilon_1 - \varepsilon_1 \varepsilon_2), \quad (77b)$$

$$\Delta\omega^2 = \omega_{\text{max}}^2 - \omega_{\text{min}}^2 = 2B f(\varepsilon_1 \varepsilon_2), \quad f(z) = z - \log z - 1. \quad (77c)$$

We first show that $\Delta\omega^2 > 0$, i.e. that there is finite ω -range for solitons to exist. This is the case because $B > 0$ holds due to (37b) (still valid for $\varepsilon_i \ll 1$) and $f(z) > 0$ for $0 < z < 1$.

Next we will show that $\omega_{\text{min}}^2 > 0$ which means that $V_{\text{reg}}(\phi)/\phi^2 > 0$ at its minimum. Notice that in the general situation the expression for ω_{min}^2 in (77b) does not need to be positive: for given A and B one cannot have arbitrarily small ε_1 . This imposes a constraint on the parameters. The general condition is

$$\omega_{\text{min}}^2 > 0 \quad \Leftrightarrow \quad \varepsilon_1 \exp(1 - \varepsilon_1 \varepsilon_2) < \exp(-A/B). \quad (78a)$$

Here we are interested in the specific situation with $D = -1$ where A, B are related to each other by Eq. (72) modulo negligible $\mathcal{O}(\varepsilon_i)$ corrections. This implies

$$\omega_{\min}^2 > 0 \quad \Leftrightarrow \quad \varepsilon_1 > c_0 \sqrt{\frac{\alpha + 1}{e^\alpha}} + \mathcal{O}(\varepsilon_i^2), \quad c_0 = e \sqrt{\frac{\pi^3}{8}}, \quad (78b)$$

i.e. ε_1 cannot be arbitrarily small. In practice, however, this is a loose bound as α must be large enough to ensure small relativistic corrections δ_{rel} , Eq. (74). For instance, if we demand $\delta_{\text{rel}} \lesssim 1\%$ then $\alpha \gtrsim 66$ and $\varepsilon_1 \gtrsim 2.1 \times 10^{-13}$. Thus ε_1 can be chosen so small that it can be neglected for practical purposes. Even the limit $\varepsilon_1 \rightarrow 0$ can be realized for $\alpha \rightarrow \infty$ in which case we deal with the heavy mass limit of a fixed-size particle, see Sec. IV B. We remark that $\omega_{\min}^2 > 0$ also guarantees $V_{\text{reg}}(\phi) > 0$ for $\phi \neq 0$, which ensures that $\phi = 0$ is the correct vacuum of the theory.

Obviously also $\omega_{\max}^2 > 0$ since $\omega_{\max}^2 = \omega_{\min}^2 + \Delta\omega^2$ and we have already proven that ω_{\min}^2 and $\Delta\omega^2$ are both positive. This is also clear from (77a) where (for $\varepsilon_2 \ll 1$) we see that ω_{\max}^2 is evidently positive and defines the mass of the Φ_i -quanta. This completes the demonstration that V_{reg} satisfies the criteria (i) and (ii) of an acceptable potential.

Finally we turn to the criterion (iii), and introduce the notation

$$U_{\text{eff}}(\phi) \equiv \frac{1}{2} m_\Phi^2 \phi^2 - V_{\text{reg}}(\phi) = \varepsilon_2 B^2 h(z), \quad h(z) = z \log(1+z) - \varepsilon z^2, \quad z = \frac{\phi^2}{\varepsilon_2 B}, \quad \varepsilon = \varepsilon_1 \varepsilon_2. \quad (79)$$

The function $h(z)$ satisfies

$$h(z) \leq z \log(1+z) \leq z^2 \quad \Leftrightarrow \quad U_{\text{eff}}(\phi) \leq b |\phi|^c, \quad b = \varepsilon_2 B^2, \quad c = 4. \quad (80)$$

This bound is useful for $\phi < \phi_{\text{eff,max}}$ where $U_{\text{eff}}(\phi)$ exhibits a maximum. For $\phi \geq \phi_{\text{eff,max}}$ a stronger bound is provided by $U_{\text{eff}}(\phi) \leq U_{\text{eff}}(\phi_{\text{eff,max}})$. To determine the latter we need the extrema of $U_{\text{eff}}(\phi)$ and consider

$$h'(z) = \log(1+z) + \frac{z}{1-z} - 2\varepsilon z \stackrel{!}{=} 0 \quad (81)$$

which has one solution at $z = 0$ corresponding to a local minimum. The second solution describes the global maximum at large $z \gg 1$ where we may approximate (81) as $h'(z) = \log(z) + 1 - 2\varepsilon z + \mathcal{O}(1/z^2) \stackrel{!}{=} 0$ which is solved by

$$z = -\frac{1}{2\varepsilon} W_{-1}\left(-\frac{2\varepsilon}{e}\right) = \frac{1}{2\varepsilon} \log\left(\frac{e}{2\varepsilon}\right) + \frac{1}{2\varepsilon} \log\left(\log\left(\frac{e}{2\varepsilon}\right)\right) + \dots \quad (82)$$

$W_{-1}(x)$ denotes the inverse function of $y = x \exp(x)$ known as Lambert W-function which is defined for $x \geq -1/e$ and multivalued at negative x . More precisely, $W_{-1}(x)$ denotes the branch with $W_{-1}(x) \leq -1$. In the second step in (82) we explored the asymptotic expansion of $W_{-1}(x)$ for small $(-x) \rightarrow 0$ [70] with the dots indicating subsubleading terms. Keeping only the leading terms we find for the position and value of the global maximum of $U_{\text{eff}}(\phi)$ the results

$$\phi_{\text{eff,max}}^2 = \frac{B}{2\varepsilon_1} \log\left(\frac{e}{2\varepsilon_1 \varepsilon_2}\right) + \dots, \quad U_{\text{eff}}(\phi_{\text{eff,max}}) = \frac{B^2}{4\varepsilon_1} \log^2\left(\frac{e}{2\varepsilon_1 \varepsilon_2}\right) + \dots \quad (83)$$

which shows that a maximum exists for $\varepsilon_i > 0$. Thus $U_{\text{eff}}(\phi) \leq \min[a, b |\phi|^c]$ where we can choose $a = U_{\text{eff}}(\phi_{\text{eff,max}})$ and b, c as shown in Eq. (80). This completes the demonstration that also the criterion (iii) is satisfied.

To end this section we briefly report the results of a numerical check with the scope to investigate the size of the deviations for D and other quantities for $\varepsilon_i \neq 0$. We have chosen the parameters $B = 2.5$, $\alpha = 65$ and a common value $\epsilon \equiv \varepsilon_1 = \varepsilon_2 = 10^{-5}$ for sake of easier comparison. Recall that other Q -ball parameters are fixed by Eq. (72) which ensures $D = -1$ for $\varepsilon_i \rightarrow 0$. Let us in the following denote the additional dependence on ϵ of the quantities as $\phi(r, \epsilon)$, $M(\epsilon)$, etc with $\phi(r, 0)$, $M(0)$, etc corresponding to $\phi(r)$, M in Sec. IV B where the ε_i were strictly zero. To measure the deviations we introduce $\delta\phi(r) = \phi(r, \epsilon) - \phi(r, 0)$, $\delta M = M(\epsilon) - M(0)$, etc. For the radial field we obtain

$$-0.6 \times 10^{-3} < \frac{\delta\phi(r)}{\phi(r)} < 0.3 \times 10^{-3} \quad (84)$$

with the largest negative deviation at small r and the largest positive deviation around $r = (1-2)$. For the integrated quantities we obtain

$$\frac{\delta Q}{Q} = -0.5 \times 10^{-3}, \quad \frac{\delta M}{M} = -0.6 \times 10^{-3}, \quad \frac{\delta D}{D} = 4 \times 10^{-3}, \quad \frac{\delta \langle r_E^2 \rangle}{\langle r_E^2 \rangle} = 3 \times 10^{-3}, \quad \frac{\delta \langle r_Q^2 \rangle}{\langle r_E^2 \rangle} = 3 \times 10^{-3}. \quad (85)$$

Let us remark that the relative accuracy of the used numerical method is of the order $\epsilon_{\text{num}} = \mathcal{O}(10^{-7})$ which we verified by reproducing within such accuracy the numerical value of D using the 3 different methods (36a, 36b, 63), and by performing other numerical tests as described in [43].

For the D -term we obtain for our chosen $\epsilon = 10^{-5}$ the value $D(\epsilon) = -0.995828$ instead on -1 . Notice that we had to chose $\epsilon \gg \epsilon_{\text{num}} = \mathcal{O}(10^{-7})$. Otherwise the effect of non-zero ϵ could not be resolved within our numerical accuracy. At the same time, for the chosen parameters α, B we have the theoretical constraint $\varepsilon_1 > 2.1 \times 10^{-13}$ (see above). Such small $\epsilon = \varepsilon_1 = \varepsilon_2$ can be truly neglected for all practical (numerical) purposes. This demonstrates how our logarithmic potential (65) can be practically understood as the limiting case of the theory (76).

D. Potential applications in Cosmology and Beyond Standard Model

We end this section with an exercise to get some feeling for the involved numbers. The only fundamental scalar particle known in the standard model is the Higgs boson. If we would choose e.g. $\alpha = 99$ and our logarithmic Q -ball to have the mass of the Higgs boson, then $m_{\text{Higgs}} = 10\sqrt{B}$ and $\langle r_{\text{Higgs}}^2 \rangle^{1/2} = 0.014 \text{ fm}$. It is not in our scope to discuss here the phenomenology of standard model extensions with composed Higgs, see e.g. [71, 72]. Let us only remark that in such extensions of the standard model the Higgs is typically considered to be composed of new particles with masses often in the TeV range, implying a much smaller size $\sim (1 \text{ TeV})^{-1} \sim 0.0002 \text{ fm}$ compared to what our logarithmic Q -ball picture would suggest. Notice, however, that this not necessarily a contradiction because the size dictated by the logarithmic Q -ball theory is not due to interactions with external (new physics) particles, but due to self-interactions and the observed Higgs boson signal [61] does not need to be incompatible with such an internal boson size. Indeed, logarithmic potentials for a Higgs self-interaction can be derived naturally from beyond standard model theories [74] whereby only the Higgs self-interaction is modified, but not the couplings to other standard model particles. The effective infra-red theory derived in [74] contains a logarithmic Higgs-mass term analog to our effective theory (65). An attractive possibility is that the Higgs could be a relatively light soliton of much heavier elementary scalar fields of a beyond-standard-model theory. Finally, let us remark that logarithmic potentials have been considered in literature, also for instance in the context of inflationary models driven by logarithmic potentials [73], or baryogenesis in minimally supersymmetric extensions of the standard model [75–77]. Such logarithmic potentials have to be understood as effective potentials which can be generated, for instance, radiatively [78, 79].

V. CONCLUSIONS

We have presented a study of the EMT form factors in spin-0 systems. Particular emphasis was put on the D -term, an interesting but so far experimentally unknown particle property [6], which plays the key role in accessing information on the internal forces inside extended particles such as nucleon and nuclei [7]. Our study has focused on free, weakly and strongly interacting theories, and revealed that the D -term is the particle property which is most strongly dependent on the dynamics of the theory.

As a starting point we studied the D -term in free field theory, and showed that the free Klein-Gordon theory makes the unambiguous prediction $D = -1$. This result, originally obtained by Pagels in 1965 [1] and largely overlooked in recent literature, is analog to the prediction $g = 2$ for the anomalous magnetic moment from the Dirac equation.

We illustrated the particular sensitivity of the D -term to the dynamics by exploring the Φ^4 theory. Neither the mass nor the spin are affected by introducing a weak Φ^4 interaction in the free theory. But the D -term is changed from its free theory value $D = -1$ to $-\frac{1}{3}$ no matter how infinitesimally weak the interaction due to renormalization [23] (assuming the mass is renormalized such that it coincides with its counterpart in the classical Lagrangian).

Interestingly in QCD the Goldstone bosons of spontaneous chiral symmetry breaking have the D -terms $D = -1$ in the soft pion limit, just as in free field theory. This is a non-trivial consequence of chiral symmetry breaking [26, 27]. On the basis of results from literature [32] we estimated the D -terms of pions, kaons, η -mesons which are numerically close to $D = -1$. In general, however, in strongly interacting theories one may encounter sizable (always negative) values $|D| \gg 1$ for the D -terms, as we have shown by reviewing results from nuclei [37] and Q -balls [43–45].

The deeper reason why the D -term is more strongly sensitive to dynamics than mass and spin is because the latter are related to operators of the Poincaré group, which imposes rigid constraints. The D -term is in spin-0 (and spin- $\frac{1}{2}$) systems the only quantity related to the EMT with no constraint due to generators of the Poincaré group. For this reason the D -term offers a unique and sensitive probe of the dynamics. Although the mass itself is of course also the result of dynamics, nevertheless the observation is that D exhibits a far stronger sensitivity to dynamics, as is exemplified by our insights from “switching on” interactions in Φ^4 theory and supported by many studies.

The second important focus of this work was the interpretation of EMT form factors in terms of 3D-densities giving insights on the stress tensor and “mechanical forces” inside composite particles [7]. Again we started from the free theory, tested the formalism by applying it to a point-like particle, and showed the internal consistency of the 3D-description. This description is justified in the heavy mass limit which requires the introduction of an additional scale, the size of a particle. We quantified the corrections to this picture and found that they are reasonably small for a particle with the mass and size of the nucleon, and safely negligible even for the lightest nuclei.

We showed that the free theory result $D = -1$ persists even when the spin-0 boson is not point-like but given “some internal structure.” For that we heuristically “smeared out” the point-like particle solution, and showed that the resulting description is consistent. We constructed a microscopic theory where the “giving” of an internal structure to a particle is implemented dynamically. This theory allows us to “interpolate” between extended and point-like particle solutions with the latter emerging in a certain parametric limit. The interaction in this microscopic theory is given by a logarithmic potential. Interactions of such type have been explored in literature in various contexts including beyond standard model phenomenology, Higgs physics and cosmology. Remarkably, this theory can be solved analytically. The solution is a non-topological soliton of Q -ball type [39] which, when formulated in its rest frame in terms of a complex scalar field, is of the type $\Phi(t, \vec{x}) = \Phi_0 \exp(i\omega t) \exp(-r^2/R^2)$, i.e. a Gaussian.

We stress that we use the 3D-density approach as a framework to interpret mechanical *response functions* of a system: the stress tensor, shear forces and pressure are inherently 3D concepts. The interpretation of such response functions in terms of 3D-densities remains to be taken with a grain of salt due to relativistic corrections. In the case of the phenomenologically interesting nucleon and nuclei such corrections are, however, acceptably small to allow us to carry on this program and gain valuable insights into internal forces.

A derivation of a 2D interpretation of the D -term in terms of lightcone densities was beyond the scope of this work. Such an interpretation, which would be free of relativistic corrections [66] and shed new light on the D -term, remains to be addressed in future studies.

This work contributes to a better understanding of the D -term, which has emerged already in the pre-QCD era as a fixed pole contribution in the angular momentum plane to the virtual Compton scattering amplitude in the framework of Regge theory [80–82] (which reflects that the D -term determines the asymptotics of GPDs in the limit of renormalization scale $\mu \rightarrow \infty$ [9, 10], see also [83, 84] for discussions). After a first vague and inevitably model-dependent glimpse on the D -term from the HERMES experiment [85] more insights are expected [86] on deeply virtual Compton scattering off nucleon [87] and nuclei [88] from Jefferson Lab, COMPASS at CERN [89], and the envisioned future Electron-Ion-Collider [90] which will allow us to test the theoretical understanding of this fascinating property.

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Appendix A: Notation

There appears to be no unique notation for EMT form factors in literature. Here are some of the used notations (on the left-hand-side of each equation) in relation to our notation (on the right-hand-side of each equation):

$$\text{Ref. [1], Eq. (8):} \quad \frac{G_1(q^2)}{2m^2} = A(t), \quad \frac{G_2(q^2)}{2m^2} = -D(t), \quad q^2 = t, \quad (\text{A1})$$

$$\text{Ref. [11], Eq. (3.152):} \quad \theta_2(\Delta^2) = A(t), \quad \theta_1(\Delta^2) = -D(t), \quad \Delta^2 = t, \quad (\text{A2})$$

$$\text{Ref. [32], Eq. (25):} \quad \theta_2(q^2) = A(t), \quad \theta_1(q^2) = -D(t), \quad q^2 = t, \quad (\text{A3})$$

$$\text{Ref. [37], Eq. (2):} \quad \frac{1}{2}M_A(t) = A(t), \quad \frac{2}{5}d_A(t) = -D(t). \quad (\text{A4})$$

Notice also that in GPD literature, e.g. [6, 9], the notion of the D -term is used in a wider sense than in this work. There the D -term is a contribution, $D^a(z, t)$ for $a = q, \bar{q}, g$ with $q = u, d, \dots$ and $z = \frac{x}{\xi}$ with support in the region $|x| \leq |\xi|$, to unpolarized GPDs. In even Mellin moments, e.g. $\int dx x^{n-1} H^a(x, \xi, t) = c_{n,0}^a(t) + c_{n,2}^a(t)\xi^2 + \dots + c_{n,n}^a(t)\xi^n$ with n even, this contribution gives rise to the generalized form factors $c_{n,n}^a(t)$. The $D^a(z, t)$ can (for the purposes of leading order evolution) be conveniently expanded in Gegenbauer polynomials with coefficients $d_1^a(t), d_3^a(t), \dots$ where the $d_{n-1}^a(t)$ are related to the $c_{n,n}^a(t)$. In contrast to this, in our work the D -term is defined more narrowly as the form factor associated with the Lorentz structure $(\Delta^\mu \Delta^\nu - g^{\mu\nu} \Delta^2)$ in the Lorentz decomposition of the matrix elements of the total EMT operator. Our $D(t)$ coincides with $\frac{4}{5}d_1(t) = \frac{4}{5}\sum_a d_1^a(t)$ in the notation of [9].

Appendix B: D -term of point like particle from 3D densities

It is instructive to “rederive” the result $D = -1$ of a free point-like particle using the concept of 3D-densities and consistency considerations. Our starting are two natural assumptions: (i) the EMT form factors of a free point-like particle are constant, (ii) the energy density of a point-like particle must be given by $T_{00}(r) = m \delta^{(3)}(\vec{r})$ if the particle is “heavy” or by the expression in Eq. (46) valid for any $m > 0$.

The constraint $A(0) = 1$ in (4a) immediately implies with assumption (i) that $A(t) = 1$ for all t . By the same argument $D(t) = D$ is of course also t -independent, but its value is a priori not known. To determine the value of D we use assumption (ii) which implies that the square bracket in the expression for $T_{00}(r)$ in Eq. (38a) must be a constant,

$$T_{00}(r) = m^2 \int \frac{d^3\Delta}{E(2\pi)^3} e^{i\vec{\Delta}\vec{r}} \underbrace{\left[A(t) - \frac{t}{4m^2}(A(t) + D(t)) \right]}_{=\text{const}}. \quad (\text{B1})$$

Clearly, we will recover the desired result if and only if $A(t) + D(t) = 0$. As we already established that $A(t) = 1$ these considerations immediately lead us to the conclusion that $D(t) = -1$, and in particular

$$D = -1 \quad (\text{B2})$$

for a point-like heavy particle. In this way, by imposing the abstract mathematical notion of a point-like particle, we recover $D = -1$ for a free point-like particle as a consistency condition of the 3D-density description. Notice that we have to explore here $T_{00}(r)$ for our purposes. Analog considerations of other EMT densities would not constrain D .

The above arguments do not apply to the massless case discussed in footnote 1 simply because our concepts require a massive particle. These arguments also do not apply to e.g. the Φ^4 -theory, because the bosons are not free there, and similarly in other interacting theories. This explains why in general we obtain different D -terms in other theories. For Goldstone bosons of chiral symmetry breaking it is $D = -1$ in the soft pion limit, but this cannot be “explained” in the above way: in this limit the Goldstone bosons are massless, and 3D-density concepts are not applicable. The result $D = -1$ for Goldstone bosons is a non-trivial consequence of chiral symmetry breaking and soft pion theorems.

Appendix C: Canonical vs conformal EMT

In this work we have seen that the D -term depends on the used EMT definition. We encountered two definitions: (i) The *canonical* EMT which is defined as the Noether current of space-time translations of the theory and symmetric in spin-0 case, Eq. (10). (ii) The *conformal* EMT which is given by (10) supplemented by the improvement term (16)

which, in the limit where all dimensionfull parameters in a Lagrangian are taken to zero, ensures conformal symmetry at classical level (which is broken in many theories by quantum corrections and renormalization).

In the free massless case it is necessary to work with the conformal EMT, because this theory is conformally invariant and the improvement term (16) is essential to preserve this property, see footnote 1. The massive Φ^4 theory is not conformally invariant, but it is appropriate to use the conformal EMT also here because adding the improvement term renders the EMT operator of that theory finite, see Sec. II C and [23]. For Goldstone bosons it is forbidden to use the conformal EMT as the improvement term would violate chiral symmetry [30, 31]. Hence in these theories it is clear, for one reason or another, whether the canonical or the conformal EMT has to be used.

In other cases, it might be less clear which definition of the EMT should be used. For instance, in the free massive theory we argued that it is appropriate to use the canonical EMT due to the lack of a unique prescription why an improvement term should be added, see Sec. II C. We have seen that this choice receives a certain support in the shape of consistency argument discussed in App. B. But one does not need to be convinced by the argument of App. B, and it is legitimate to wonder what we would obtain from a conformal EMT. In the massive free theory case the answer is just $D = -1/3$ instead on -1 , cf. footnote 1.

Also the results for the D -term in the Q -ball system, Refs. [43–45] and Sec. IV B, were obtained from the canonical EMT. At this point we are not aware of an argument why a conformal EMT should be used for these calculations. But it is instructive to explore it for the sake of obtaining an insight on how the EMT densities of an extended particle might be affected by working with one or the other EMT definition. When the improvement term (16) is included in the Q -ball theory, then the EMT densities are altered as follows:

$$T_{00}(r)_{\text{conformal}} = T_{00}(r)_{\text{canonical, Eq. (61a)}} + \delta_h T_{00}(r) \quad (\text{C1a})$$

$$p(r)_{\text{conformal}} = p(r)_{\text{canonical, Eq. (61b)}} + \delta_h p(r) \quad (\text{C1b})$$

$$s(r)_{\text{conformal}} = s(r)_{\text{canonical, Eq. (61c)}} + \delta_h s(r) \quad (\text{C1c})$$

with the additional terms given, in any Q -ball theory with an acceptable (in the sense of Sec. IV C) potential, by

$$\delta_h T_{00}(r) = -h \frac{1}{r} (r\phi(r)^2)'' \quad (\text{C2a})$$

$$\delta_h p(r) = -h \left(\frac{1}{3} (\phi(r)^2)'' + \frac{2}{3} \frac{1}{r} (\phi(r)^2)' - \frac{1}{r} (r\phi(r)^2)'' \right). \quad (\text{C2b})$$

$$\delta_h s(r) = -h \left((\phi(r)^2)'' - \frac{1}{r} (\phi(r)^2)' \right). \quad (\text{C2c})$$

We see that the conformal energy density differs from the canonical one. But due to

$$\int d^3 \delta_h T_{00}(r) = -4\pi h \int_0^\infty dr r (\phi(r)^2)'' = -4\pi h \left[r (\phi(r)^2)' - (r\phi(r)^2) \right]_0^\infty = 0 \quad (\text{C3})$$

one obtains the *same* mass from the conformal and canonical EMT for every Q -ball theory which is of course expected. Similarly the conformal pressure differs from the canonical one, but it preserves the von Laue condition since

$$\begin{aligned} \int_0^\infty dr r^2 \delta_h p(r) &= -h \int_0^\infty dr r^2 \left(\frac{1}{3} (\phi(r)^2)'' + \frac{2}{3} \frac{1}{r} (\phi(r)^2)' - \frac{1}{r} (r\phi(r)^2)'' \right) \\ &= -h \left[\frac{r^2}{3} (\phi(r)^2)' - r(r\phi(r)^2)' + (r\phi(r)^2) \right]_0^\infty = 0. \end{aligned} \quad (\text{C4})$$

Thus, independently of whether we use the conformal or canonical EMT (for the latter the proof was given in [43]) to describe the internal forces, the necessary condition for stability is satisfied in the same way.

The conformal expressions for $s(r)$ and $p(r)$ yield the same D -term via Eqs. (36a, b). This can be seen by taking the difference of the expressions for D from pressure and shear forces, which yields

$$m \int d^3 r r^2 \delta_h p(r) + \frac{4m}{15} \int d^3 r r^2 \delta_h s(r) = \frac{4}{5} m h 4\pi \int_0^\infty dr (r^4 \phi(r) \phi'(r))' = 0, \quad (\text{C5})$$

Again this is a result valid for any Q -ball theory. However, the canonical and the conformal D -term differ, which is not surprising, see Sec. II. We obtain

$$D_{\text{conformal}} = D_{\text{canonical, Eqs. (36a,b)}} + \delta_h D, \quad \delta_h D = -\frac{4}{3} h 4\pi m \int_0^\infty dr r^3 (\phi(r)^2)' > 0, \quad (\text{C6})$$

where in the last step we conclude that $\delta_h D$ is positive, because $\phi(r)^2$ is a monotonically decreasing function of r , i.e. $(\phi(r)^2)' < 0$ making the integrand in Eq. (C6) negative.

So far we considered a general Q -ball theory. It is insightful to look at our analytically solvable logarithmic Q -ball theory from Sec. IV where all results can be obtained analytically. The modification of the conformal as compared to canonical densities are particularly lucid in this theory, namely

$$T_{00}(r)_{\text{conformal}} = T_{00}(r)_{\text{canonical, Eq. (61a)}} + 6h p(r)_{\text{canonical, Eq. (61b)}}, \quad (\text{C7a})$$

$$p(r)_{\text{conformal}} = p(r)_{\text{canonical, Eq. (61b)}} \times (1 - 4h), \quad (\text{C7b})$$

$$s(r)_{\text{conformal}} = s(r)_{\text{canonical, Eq. (61c)}} \times (1 - 4h). \quad (\text{C7c})$$

Thus for logarithmic Q -balls, the modification of the energy density is proportional to the pressure which (conformal or not) integrates to zero as we have seen above. This illustrates how the modified energy density can still yield the same Q -ball mass. The modifications of pressure and shear forces result in a simple overall prefactor $1 - 4h = \frac{1}{3}$ (with $h = \frac{1}{6}$ in $3 + 1$ space-time dimensions). This explains how in the conformal case the Laue condition is satisfied, and why we still get the same D -term from pressure and shear forces. The value of D is, however, reduced by the factor $1 - 4h = \frac{1}{3}$. In particular, with the parameters (72) which ensure $D_{\text{canonical}} = -1$ we obtain $D_{\text{conformal}} = -\frac{1}{3}$.

Thus, in the logarithmic Q -ball theory the conformal EMT yields an equally satisfactory description of EMT densities as the canonical EMT. We checked that this really is a general feature in the Q -ball system. For instance in the Q -ball theory with the sextic potential explored in Refs. [43–45] and reviewed in Sec. II E one obtains qualitatively the same picture, although the relation $D_{\text{conformal}} : D_{\text{canonical}} = 1 : 3$ is specific to the logarithmic Q -ball theory. Thus, if it became clear that a more consistent description of EMT densities would be provided by the conformal (instead of canonical) EMT, one could switch to that description without sacrificing any of the insights obtained in prior works. As mentioned at this point we have no argument why the use of the conformal EMT could be more appropriate than the use of the canonical EMT. One possible situation to revise this point could occur when considering quantum corrections to the classical Q -ball solution [91], which was beyond the scope of this work.

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