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Andrzej Czarnecki and William J. Marciano

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Top-quark loops and the muon anomalous magnetic moment

Andrzej Czarnecki

Department of Physics, University of Alberta, Edmonton, Alberta, Canada T6G 2E1

William J. Marciano*

Department of Physics, Brookhaven National Laboratory, Upton, NY 11973, USA

The current status of electroweak radiative corrections to the muon anomalous magnetic moment is discussed. Asymptotic expansions for some important electroweak two loop top quark triangle diagrams are illustrated and extended to higher order. Results are compared with the more general integral representation solution for generic fermion triangle loops coupled to pseudoscalar and scalar bosons of arbitrary mass. Excellent agreement is found for a broader than expected range of mass parameters.

I. INTRODUCTION

The muon anomalous magnetic moment, $a_\mu = (g_\mu - 2)/2$, has been both precisely measured [1] and very accurately computed for the Standard Model (SM) [2, 3].

Currently, there exists a provocative 3.5 sigma difference between experiment and SM theory [4]:

$$a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = 268 (63)_{\text{exp}} (43)_{\text{SM}} \times 10^{-11} = 268 (76) \times 10^{-11} \quad (1)$$

which may indicate problems with the experiment and/or theory. A more exciting possibility is that the discrepancy may be a harbinger of “New Physics” [5], beyond SM expectations. To clarify the situation, a more sensitive experiment at Fermilab [6] is getting underway with the goal of reducing the experimental uncertainty by a factor of 4. Also, a distinctly low energy approach is being pursued at JPARC [7]. Meanwhile, theoretical uncertainties, primarily from hadronic loops, are expected to be further reduced (by perhaps a factor of 2) from a combination of dispersion relations involving $e^+e^- \rightarrow \text{hadrons}$ data and lattice gauge theory calculations [8]. If “New Physics” is responsible for the current deviation, it should be fully exposed with a solid $> 5\sigma$ discovery during the next few years.

Given the importance of the theory calculations behind $a_\mu^{\text{SM}} = a_\mu^{\text{QED}} + a_\mu^{\text{hadronic}} + a_\mu^{\text{EW}}$, it is important to scrutinize all of their underlying properties, including the reliability of the computational methodology. Electroweak (EW) Feynman loop diagrams contributing to a_μ^{SM} typically involve at least two mass scales: the muon mass and the boson mass. At two loops, exact expressions for the resulting integrals are complicated. In many cases they are not known. It is useful to exploit the wide separation of these mass scales and expand the integrals in their ratio. Such expansions have a long tradition in mathematical physics [9]. In quantum field theory, they are especially powerful when combined with dimensional regularization that does not introduce additional scales (unlike for example Pauli-Villars regularization). The crucial property is the vanishing of diagrams that do not involve any mass scales, so called *massless tadpoles*, analogous to the vanishing of scaleless integrals in the theory of distributions [10].

In this paper, we provide a check on the asymptotic expansion method used in the calculation [11, 12] of the two-loop electroweak (EW) contributions to a_μ^{EW} . Those are the corrections to the well known one loop contribution [13–16]:

$$a_\mu^{\text{EW}} (1 \text{ loop}) = \frac{5}{3} \frac{G_\mu m_\mu^2}{8\sqrt{2}\pi^2} \left[1 + \frac{1}{5} (1 - 4 \sin^2 \theta_W)^2 + \mathcal{O} \left(\frac{m_\mu^2}{M^2} \right) \right] = 194.8 \times 10^{-11}, \quad (2)$$

where $G_\mu = 1.1663787(6) \times 10^{-5} \text{ GeV}^{-2}$ is the Fermi coupling constant [17, 18], m_μ is the muon mass, M represents the mass of electroweak gauge or Higgs bosons, and θ_W is the weak mixing angle, $\sin^2 \theta_W = 1 - \frac{m_W^2}{m_Z^2} \simeq 0.223$. Two loop corrections are of the form:

$$a_\mu^{\text{EW}} (2 \text{ loop}) = \frac{5}{3} \frac{G_\mu m_\mu^2}{8\sqrt{2}\pi^2} \sum_i C_i \frac{\alpha}{\pi}. \quad (3)$$

* marciano@bnl.gov

where in the 't Hooft-Feynman gauge the C_i represent contributions from about 240 Feynman diagrams as well as one loop induced counterterms. Three loop leading log EW effects were shown to be negligible using the renormalization group [19, 20]. Collectively, for $m_H = 125$ GeV, higher orders reduce a_μ^{EW} to [21]

$$a_\mu^{\text{EW}} = a_\mu^{\text{EW}} (1 \text{ loop}) + a_\mu^{\text{EW}} (2 \text{ loop}) + a_\mu^{\text{EW}} (3 \text{ loop leading logs}) = 154 (1) \times 10^{-11} \quad (4)$$

where the uncertainty stems mainly from light quark two loop triangle diagrams and non-leading-log three loop effects. The central value in that result has been rounded off from 153.7×10^{-11} , after individual contributions were computed up to $\mathcal{O}(0.1 \times 10^{-11})$.

Figure 1. Top-loop connected to the muon line by a photon and a Z boson (a), neutral Goldstone boson (b), Higgs boson (c). These diagrams have additional versions: left-right reflected and with the external magnetic field coupling to the other top line.

To examine the convergence properties of the asymptotic expansions used in eq. (3), we consider two particularly interesting examples containing top quark triangle loops. The first case, illustrated in Figs. 1(a,b) represents a part of the fermion anomaly diagrams. Together with contributions from the τ lepton and b quark they provide an anomaly-free subset of two loop contributions. Light fermion effects are easily calculated; but the top quark loop was originally computed as an asymptotic expansion in $m_Z^2/m_t^2 \simeq 0.28$. We illustrate that prescription below. Fig. 1(c) represents a Higgs scalar contribution for which the expansion parameter $m_H^2/m_t^2 \simeq 0.52$ is more of a concern, because of its relatively large value in comparison with the underlying assumption $m_H^2/m_t^2 \ll 1$. Although, as we will show, if several terms are included, the expansion remains valid even for considerably larger values of m_H^2/m_t^2 than 0.52. The diagrams in Figs. 1(b) and (c) are examples of what are called Barr-Zee diagrams in the literature [22]. They often occur for heavy fermion triangle loops coupled to heavy or light pseudoscalar or scalar particles. After discussing top SM effects, we address the more general case of arbitrary “new physics” mass scales in similar types of diagrams.

II. TOP LOOP DIAGRAMS

As a concrete illustration of the asymptotic expansion method, we begin by computing the diagrams shown in Fig. 1(a,b) using the 't Hooft-Feynman gauge. These diagrams involve three masses whose squares we can consider widely-separated,

$$m_t^2 \gg m_Z^2 \gg m_\mu^2. \quad (5)$$

Indeed, with $m_Z = 91.2$ GeV and $m_t = 173$ GeV, the largest ratio is $\frac{m_Z^2}{m_t^2} \simeq 0.28$. Of course, the muon with $m_\mu \simeq 0.106$ GeV appears to be almost massless in comparison with these heavy particles. However, its mass must be retained since the electroweak correction Δa_μ vanishes in the massless muon limit. Also, having three mass scales better illustrates the power of the method. We thus have two small parameters, $\frac{m_\mu^2}{m_Z^2} \simeq 10^{-6}$ and $\frac{m_Z^2}{m_t^2}$ in which the diagrams will be expanded. We use the asymptotic expansion approach [23] to identify relevant regions of loop momenta. The top-quark diagrams are an interesting application of this approach with a relatively rich structure of the hierarchy of momenta.

First, consider the case when both loop momenta are on the order of the largest mass, $k_1 \sim k_2 \sim m_t$. Then the muon and Z propagators can be expanded around their massless limits and the external muon momentum can also be

treated as small. The resulting integration is simplified because it depends only on the top-quark mass, so the actual integrals that have to be computed are dimensionless numbers. Their structure is shown in Fig. 2(i): the top-quark loop remains as in Fig. 1(a), but the Z mass, the muon mass, and the external momentum are no longer present (except as factors multiplying the integrand; they can be taken out of the integration). Thus, the Z propagator and the muon propagator, together with the photon propagator, all form some power of a massless propagator indicated in Fig. 2(i) by a dashed line. There are no external legs in that Figure because external momenta are taken out of the integral.

Figure 2. Asymptotic expansion of the diagrams in Fig. 1. The momentum in the top quark loop is always on the order of m_t . The other loop momentum is $\mathcal{O}(m_t)$ in (a), $\mathcal{O}(m_Z)$ in (b), and $\mathcal{O}(m_\mu)$ in (c). Solid (dashed) lines denote massive (massless) propagators.

After the two-loop integration in this part, we find a divergent result. Using dimensional regularization with $D = 4 - 2\epsilon$,

$$\Delta C_i = \frac{m_Z^2}{m_t^2} \left[\frac{17}{15} - \frac{2}{5} \left(\frac{1}{\epsilon} - 2 \ln m_t^2 \right) \right]. \quad (6)$$

Throughout this calculation we keep only results suppressed at most by two powers of $\frac{1}{m_t}$. In the 't Hooft-Feynman gauge, the Z diagrams are subleading, i.e. suppressed by the inverse top-quark mass. The leading contribution arises from the Goldstone boson diagram depicted in Fig. 1(b).

The logarithm of the top mass in (6) arises because m_t is the only scale in this part of the integration. Since the integration element is $d^D k_1 d^D k_2$, the fractional power of mass must be $m_t^{-4\epsilon} = 1 - 2\epsilon \ln m_t^2 + \mathcal{O}(\epsilon^2)$. Multiplying the divergence, it gives $-\frac{2}{5\epsilon} (1 - 2\epsilon \ln m_t^2) = -\frac{2}{5\epsilon} + \frac{4}{5} \ln m_t^2$. We anticipate that the divergences cancel in the sum of the three regions shown in Fig. 2. A top-quark loop is present in all these regions, so a factor $m_t^{-2\epsilon}$ is present in all these three contributions. Thus we expect only half of the m_t logarithm in (6) to survive in the final result. Indeed, in eq. (18) in [11] the logarithmic part of the relevant diagram, denoted there by $\Delta C_{1(d)}^{\text{ferm}}(t)$, is just $\frac{2}{5} \frac{m_Z^2}{m_t^2} \ln \frac{m_t^2}{m_Z^2}$. We now proceed to show how the logarithm of m_Z arises.

To this end, consider the region where the integration momentum in the lower loop in Fig. 1(a) is $\mathcal{O}(m_Z)$. Then we can still expand in the external muon momentum but must keep the momentum dependence of the Z propagator. On the other hand, we can expand the top-quark propagators in the Z momentum. The result is a product of two one-loop integrals shown in Fig. 2(ii): for the top-loop integration, all external momenta including the other loop momentum can be pulled out of the integral. The other loop depends on the Z boson mass as the only scale, so the muon propagator is expanded in the external momentum and becomes again a part of the massless line. We find that this region contributes

$$\Delta C_{ii} = \frac{m_Z^2}{m_t^2} \left[-\frac{7}{15} + \frac{2}{5} \left(\frac{1}{\epsilon} - \ln m_Z^2 - \ln m_t^2 \right) \right] + \frac{m_\mu^2}{m_t^2} \left[-\frac{34}{135} + \frac{4}{9} \left(\frac{1}{\epsilon} - \ln m_Z^2 - \ln m_t^2 \right) \right] \quad (7)$$

In this region, we observe a more complicated result, due to the dependence on two mass scales. In the first term, the anticipated logarithm of m_Z appears, as well as the logarithm of the top mass with a sign opposite than in (6). In the sum of (6) and (7), the divergences multiplying $\frac{m_Z^2}{m_t^2}$ cancel and the logarithm $\frac{2}{5} \frac{m_Z^2}{m_t^2} \ln \frac{m_t^2}{m_Z^2}$ of eq. (18) in [11] is reproduced. In addition, in (7) there is a term of order $\frac{m_\mu^2}{m_t^2}$. It is beyond our intended accuracy but we keep it

to further illustrate the generation of logarithms of ratios of various scales. With this in mind, we proceed to the remaining, third region (iii) in Fig. 2.

In this region, the momentum in the lower loop in Fig. 1(a) is $\mathcal{O}(m_\mu)$. We can expand in it both the top-quark loop and the Z propagator, but we must retain the exact dependence of the muon propagator on the external momentum. We find

$$\Delta C_{\text{iii}} = \frac{m_\mu^2}{m_t^2} \left[\frac{68}{135} - \frac{4}{9} \left(\frac{1}{\epsilon} - \ln m_\mu^2 - \ln m_t^2 \right) \right], \quad (8)$$

a contribution that cancels the divergence in the second term in (7). Summing $\Delta C_{\text{i,ii,iii}}$ we obtain

$$\Delta C_Z = \Delta C_{\text{i}} + \Delta C_{\text{ii}} + \Delta C_{\text{iii}} = \frac{m_Z^2}{m_t^2} \left(\frac{2}{3} + \frac{2}{5} \ln \frac{m_t^2}{m_Z^2} \right) + \frac{m_\mu^2}{m_t^2} \left(\frac{34}{135} - \frac{4}{9} \ln \frac{m_Z^2}{m_\mu^2} \right), \quad (9)$$

a finite result whose leading term reproduces eq. (18) in [11]. While the divergences canceled in the sum of the three regions, the differences of logarithms containing various mass scales combined to form logs of dimensionless ratios.

The result in (9) must be supplemented by the contribution of the neutral Goldstone boson, Fig. 1(b). The same three regions contribute and we find, neglecting terms $\mathcal{O}\left(\frac{m_\mu^2}{m_t^2}\right)$, the first two expansion terms in m_Z^2/m_t^2 ,

$$\Delta C_G = -\frac{16}{5} - \frac{8}{5} \ln \frac{m_t^2}{m_Z^2} + \frac{m_Z^2}{m_t^2} \left(-\frac{4}{9} - \frac{4}{15} \ln \frac{m_t^2}{m_Z^2} \right), \quad (10)$$

where the leading terms reproduce eq. (22) in [11], while the subleading $\mathcal{O}(m_Z^2/m_t^2)$ terms are small and were previously dropped. Here, we retain them and find, adding eqs. (9) and (10)

$$\Delta C_Z + \Delta C_G = -\frac{16}{5} - \frac{8}{5} \ln \frac{m_t^2}{m_Z^2} + \frac{m_Z^2}{m_t^2} \left(\frac{2}{9} + \frac{2}{15} \ln \frac{m_t^2}{m_Z^2} \right) + \mathcal{O}\left(\frac{m_Z^4}{m_t^4}, \frac{m_\mu^2}{m_t^2}\right). \quad (11)$$

Adding the b and τ loops (taken from eq. (17) in ref. [11]) gives the contribution of the third generation [20],

$$\Delta a_\mu^{\text{EW}}[\tau, b, t] = -\frac{\alpha}{\pi} \frac{G_\mu m_\mu^2}{8\pi^2 \sqrt{2}} \left[\frac{8}{3} \ln \frac{m_t^2}{m_Z^2} - \frac{2}{9} \frac{m_Z^2}{m_t^2} \left(\ln \frac{m_t^2}{m_Z^2} + \frac{5}{3} \right) + 4 \ln \frac{m_Z^2}{m_b^2} + 3 \ln \frac{m_b^2}{m_\tau^2} - \frac{8}{3} \right] = -8.21 \cdot 10^{-11}, \quad (12)$$

in agreement with the result first published in [20] and recently checked in an automated calculation [24]. Note, although they play an insignificant role, contributing less than 10^{-12} to eq. (12), we have retained terms of order m_Z^2/m_t^2 in eq. (7) to illustrate their effect on the asymptotic expansion.

For the analogous Higgs boson diagram, Fig. 1(c), we find after expanding to rather high order in m_H^2/m_t^2 ,

$$\begin{aligned} \Delta C_H = & -\frac{104}{45} - \frac{16}{15} \ln \frac{m_t^2}{m_H^2} - \frac{m_H^2}{25m_t^2} \left(\frac{104}{15} + 4 \ln \frac{m_t^2}{m_H^2} \right) - \frac{m_H^4}{525m_t^4} \left(\frac{2692}{105} + 16 \ln \frac{m_t^2}{m_H^2} \right) \\ & - \frac{m_H^6}{315m_t^6} \left(\frac{971}{315} + 2 \ln \frac{m_t^2}{m_H^2} \right) - \frac{m_H^8}{5775m_t^8} \left(\frac{41758}{3465} + 8 \ln \frac{m_t^2}{m_H^2} \right) - \frac{m_H^{10}}{6435m_t^{10}} \left(\frac{267401}{90090} + 2 \ln \frac{m_t^2}{m_H^2} \right) - \dots \end{aligned} \quad (13)$$

$$= -3.2 \text{ for } m_H^2/m_t^2 = 0.52. \quad (14)$$

Although $m_H^2/m_t^2 = 0.52$ is relatively large, higher order terms in that expansion are suppressed by small coefficients. As a result, the terms beyond the leading order contribute of order 10^{-12} to a_μ^{EW} which is covered by the uncertainty in eq. (4). Expression (13) agrees with the integral representation [22, 25, 26],

$$\Delta C_H \left(z = \frac{m_t^2}{m_H^2} \right) = -\frac{8z}{5} \int_0^1 \frac{1-2x(1-x)}{x(1-x)-z} \ln \frac{x(1-x)}{z} dx. \quad (15)$$

Fig. 3(a) shows that the asymptotic expansion is valid in a remarkably broad range even for $m_H \simeq 1.4m_t \simeq 240$ GeV. Convergence of the expansion is illustrated with Fig. 4. For $m_t > m_H$, already the leading part of (13), without $\frac{m_H^2}{m_t^2}$ corrections, differs from the integral representation by only about 0.2. This corresponds to a contribution to Δa_μ of about 10^{-12} which is in the noise but included in eq. (4) before roundoff.

Figure 3. (a) Comparison of the asymptotic expansion in eq. (13), valid for $m_t > m_H$, (the dotted curve shows $-\Delta C_H$) with the integral representation in eq. (15) (solid). (b) Similar comparison for the Higgs boson replaced by a pseudoscalar particle with mass m_P , the asymptotic expansion given by eq. (17) and the integral representation by (16).

If the scalar Higgs is replaced by a generic pseudo scalar with mass m_P and the same couplings as the G^0 in Fig. 1(b), the integral representation becomes [22, 25, 26]

$$\Delta C_P \left(z = \frac{m_t^2}{m_P^2} \right) = -\frac{8z}{5} \int_0^1 \frac{1}{x(1-x)-z} \ln \frac{x(1-x)}{z} dx, \quad (16)$$

while our asymptotic expansion gives

$$\Delta C_P(z) = -\frac{16}{5} - \frac{8}{5} \ln z - \frac{\frac{20}{3} + 4 \ln z}{15z} - \frac{\frac{94}{15} + 4 \ln z}{75z^2} - \frac{\frac{319}{105} + 2 \ln z}{175z^3} - \frac{\frac{1879}{315} + 4 \ln z}{1575z^4} - \frac{\frac{20417}{6930} + 2 \ln z}{3465z^5} - \dots, \quad (17)$$

The leading and next to leading terms agree with eq. (10). Again, as illustrated in Fig. 3(b), the asymptotic expansion agrees remarkably well with the integral representation over a much broader range of parameters than one might have expected. The quality of even the truncated asymptotic expansion is nicely demonstrated in Fig. 4 where for $m_t^2/m_H^2 > 0.5$, effects of higher order terms in m_H^2/m_t^2 are shown to be relatively small.

Figure 4. Convergence of the asymptotic expansion of $-C_H$ on the basis of eq. (13): solid line: all six powers of m_H^2/m_t^2 ; dashed: just the first two powers; dotted: the first two terms only (no terms m_H^2/m_t^2). The dot-dashed line shows the integral representation in eq. (15).

The integral representations employed above have proved very useful for confirming the validity of our asymptotic expansion for the muon anomalous magnetic moment. In addition, they can be easily applied to a broad range of single parameter ratios that may be required for some “New Physics” scenarios. However, they are strictly valid in lowest order of the muon mass. Asymptotic expansions can be extended to include powers and logarithms of the muon mass, analogous to our example in eq. (9), where region (iii) was necessary to complete the determination of the muon mass dependence. In addition, for multi loop diagrams involving several mass ratios, the asymptotic expansion method is generally applicable while relatively simple integral representations of the type we used may be difficult to attain.

In summary, we have checked the numerical validity of the EW two loop top quark triangle diagrams originally evaluated using asymptotic expansions in m_Z^2/m_t^2 and m_H^2/m_t^2 , by comparing them with values obtained using the integral representation. Agreement is excellent, even when retaining only one or two terms in the expansion. Indeed, any truncation error is well below the uncertainty budget of $\pm 1 \times 10^{-11}$ assigned to a_μ^{EW} in eq. (4). Extending the expansion to six terms for a generic scalar or pseudo scalar coupled to a heavy fermion of arbitrary mass allowed us to compare the asymptotic expansion approach with the integral representation for Barr-Zee diagrams that have been used for both anomalous magnetic moment and electric dipole moment calculations. We found asymptotic expansion agreement with the integral representation for expansion parameters as large as 2 due to small coefficients in the mass ratio squared expansion. That seems to indicate that the real expansion parameter is actually much smaller than the mass ratio squared and the asymptotic expansion method is very robust.

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