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Scheme-Independent Calculations of Physical Quantities in an $\mathcal{N}=1$ Supersymmetric Gauge Theory

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We consider an asymptotically free, vectorial, $\mathcal{N}=1$ supersymmetric gauge theory with gauge group G and N_f pairs of chiral superfields in the respective representations $\mathcal{R}$ and $\bar{\mathcal{R}}$ of G, having an infrared fixed point (IRFP) of the renormalization group at α_{IR}. We present exact results for the anomalous dimensions of various (gauge-invariant) composite chiral superfields $\gamma_{\Phi_{\rm prod}}$ at the IRFP and prove that these increase monotonically with decreasing N_f in the non-Abelian Coulomb phase of the theory and that scheme-independent expansions for these anomalous dimensions as powers of an N_f-dependent variable, Δ_f, exhibit monotonic and rapid convergence to the exact $\gamma_{\Phi_{\rm prod}}$ throughout this phase. We also present a scheme-independent calculation of the derivative of the beta function, $d\beta/d\alpha|_{\alpha=\alpha_{IR}}$, denoted β'_{IR}, up to $O(\Delta_f^3)$ for general G and $\mathcal{R}$, and, for the case $G={\rm SU}(N_c)$, $\mathcal{R}=F$, we give an analysis of the properties of β'_{IR} calculated to $O(\Delta_f^4)$.

I. INTRODUCTION

An important fact about quantum field theories is that their properties depend on the Euclidean energy/momentum scale μ at which these properties are measured. The change in these properties as a function of μ is described by the renormalization group (RG). Asymptotically free gauge theories are particularly amenable to renormalization-group analysis because the running gauge coupling, $g(\mu)$, goes to zero in the limit of large μ in the deep ultraviolet (UV), so that in this regime one can describe the theory accurately using perturbative methods. The dependence of $g(\mu)$, or equivalently, $\alpha(\mu)=g(\mu)^2/(4\pi)$, on μ, is described by the beta function,

$$\beta = \frac{d\alpha}{dt} \ , \tag{1.1}$$

where $dt = d\ln\mu$.

Here we consider an asymptotically free, vectorial, $\mathcal{N}=1$ supersymmetric gauge theory with gauge group G and N_f pairs of massless[1] chiral superfields Φ_i and $\tilde{\Phi}_i$ transforming according to the respective representations $\mathcal{R}$ and $\bar{\mathcal{R}}$ of G. In an asymptotically free theory of this type, as μ decreases from large values in the UV toward $\mu=0$ in the infrared, $\alpha(\mu)$ increases. There are several possible types of infrared behavior, depending on the gauge group and matter content of the theory. We focus on the case in which the beta function has a zero at a certain value $\alpha=\alpha_{IR}$, which is an IR fixed point (IRFP) of the renormalization group. Thus, as μ decreases from the UV to the IR, $\alpha(\mu)$ increases (monotonically) from 0 to the limiting value α_{IR}. In this IR limit, the theory is scale-invariant, and is inferred to be conformally invariant [1]. The combination of this conformal invariance with the supersymmetry means that the theory is invariant under a superconformal algebra. We denote the full operator dimension of a physical (gauge-invariant) operator $\mathcal{O}$ as $D_{\mathcal{O}}$. In general, this can be written as

$$D_{\mathcal{O}} = D_{\mathcal{O},{\rm free}} - \gamma_{\mathcal{O}} \ , \tag{1.2}$$

where $D_{\mathcal{O},{\rm free}}$ is the Maxwellian dimension that the operator would have in a free theory and $\gamma_{\mathcal{O}}$ is the anomalous dimension[2] of $\mathcal{O}$.

In this paper we present new scheme-independent results on the values of physical quantities at this superconformal IR fixed point. These quantities include anomalous dimensions of gauge-invariant operators, $\gamma_{\mathcal{O}}$ and the derivative of the beta function, $\beta' \equiv d\beta/d\alpha$, evaluated at $\alpha=\alpha_{IR}$ and thus denoted $\gamma_{\mathcal{O},IR}$ and β'_{IR}. Specifically, we present exact results for anomalous dimensions of various (gauge-invariant) composite chiral superfield operator products $\Phi_{\rm prod}$ and study the properties of scheme-independent expansions of these operators as power series in Δ_f, where Δ_f is an N_f-dependent expansion variable given in Eq. (2.27) below [2–5]. We prove that these anomalous dimensions increase monotonically with decreasing N_f in the non-Abelian Coulomb phase of the theory and that scheme-independent expansions for these anomalous dimensions as powers of Δ_f exhibit monotonic and rapid convergence to the exact $\gamma_{\Phi_{\rm prod}}$ throughout this phase. We also present a scheme-independent calculation of β'_{IR} up to $O(\Delta_f^3)$ for general G and $\mathcal{R}$ and analyze the properties of this expansion up to $O(\Delta_f^4)$ for $G={\rm SU}(N_c)$ and $\mathcal{R}=F$, the fundamental representation. Previously, we have presented results for the anomalous di-

[1] The assumption of massless Φ incurs no loss of generality, since if Φ had a nonzero mass m_0, it would be integrated out of the effective field theory at scales $\mu < m_0$, and hence would not affect the IR limit $\mu \to 0$.

[2] Some authors use the opposite sign convention for the anomalous dimension, writing $D_{\mathcal{O}} = D_{\mathcal{O},{\rm free}} + \gamma_{\mathcal{O}}$.

mension $\gamma_{M,IR}$ of a meson-type chiral superfield using n-loop series expansions and scheme-independent series expansions [6]-[15]. The current paper substantially extends our earlier results.

This paper is organized as follows. Some relevant background and methods are discussed in Section II. In Section III we prove several theorems on anomalous dimensions of (gauge-invariant) chiral superfields. In Sections IV-VI we present exact results on anomalous dimensions of various composite chiral superfield operators. These are generalized to theories with higher-dimension matter chiral superfields in Section VII. Section VIII contains our results on β'_{IR}. For the case $G = \mathrm{SU}(N_c)$ and $\mathcal{R} = F$, Section IX contains an analysis of properties in the limit $N_c \to \infty$ and $N_f \to \infty$ with the ratio N_f/N_c fixed and finite. Our conclusions are given in Section X.

II. BACKGROUND AND METHODS

In this section we review some background and methods that we will use in our calculations. We consider an asymptotically free $\mathcal{N} = 1$ supersymmetric vectorial gauge theory with gauge group G and N_f copies (flavors) of matter chiral superfields Φ^i and $\tilde{\Phi}_i$, $1 \le i \le N_f$, transforming as the $\mathcal{R}$ and $\bar{\mathcal{R}}$ representations of G, respectively. We write the decomposition of the matter chiral superfield Φ in terms of component fields (with group and flavor indices suppressed here) as

$$\Phi = \phi + \sqrt{2}\,\theta\psi + \theta\theta F \ , \tag{2.1}$$

where ϕ, ψ, and F are, respectively, the scalar, fermionic, and auxiliary component fields, and θ is an anticommuting Grassmann variable. The chiral superfield W_α contains the gluino λ_α and the field-strength tensor $F^a_{\mu\nu}$, where here α and a are spinor and gauge indices, respectively.

The beta function of this theory has the series expansion

$$\beta = -2\alpha \sum_{\ell=1}^{\infty} b_\ell \left(\frac{\alpha}{4\pi}\right)^\ell = -2\alpha \sum_{\ell=1}^{\infty} \bar{b}_\ell \, \alpha^\ell \ , \tag{2.2}$$

where b_ℓ is the ℓ-loop coefficient and $\bar{b}_\ell = b_\ell/(4\pi)^\ell$. Here, we restrict our analysis to mass-independent regularization/renormalization schemes. Statements concerning gauge and scheme independence also assume that scheme transformations do not depend on a gauge parameter [16]. For recent discussions dealing with other schemes, see, e.g., [5]. The first two coefficients, which are scheme-independent [16], are [17][3]

$$b_1 = 3C_A - 2T_f N_f \tag{2.3}$$

and [18]

$$b_2 = 6C_A^2 - 4(C_A + 2C_f)T_f N_f \ . \tag{2.4}$$

The requirement of asymptotic freedom restricts N_f to be less than an upper (u) bound N_u, i.e.,

$$N_f < N_u \ , \tag{2.5}$$

where

$$N_u = \frac{3C_A}{2T_f} \ . \tag{2.6}$$

Note that N_u is not necessarily an integer.[4]

The anomalous dimension of a (gauge-invariant) operator $\mathcal{O}$ has a series expansion in powers of the coupling of the form

$$\gamma_{\mathcal{O}} = \sum_{\ell=1}^{\infty} c_{\mathcal{O},\ell} \left(\frac{\alpha}{4\pi}\right)^\ell \ , \tag{2.7}$$

where $c_{\mathcal{O},\ell}$ is the ℓ-loop coefficient. In particular, for a chiral superfield Φ, one may write

$$\gamma_\Phi = \sum_{\ell=1}^{\infty} c_\ell \left(\frac{\alpha}{4\pi}\right)^\ell \ . \tag{2.8}$$

From a calculation of the contribution of instantons to the action, Novikov, Shifman, Vainshtein, and Zakharov (NSVZ) derived a closed-form expression for the beta function [19]:

$$\beta_{NSVZ} = -\frac{\alpha^2}{2\pi}\left[\frac{b_1 - 2N_f T_f\, \gamma_M}{1 - \frac{C_A \alpha}{2\pi}}\right] \ , \tag{2.9}$$

where γ_M is the anomalous dimension of the fermion bilinear that occurs in the (gauge-invariant) quadratic chiral superfield operator product. We focus here on the IR non-Abelian Coulomb phase (NACP), to be discussed further below, in which the nonanomalous global chiral symmetry of the theory is exact. Although we will analyze meson and baryon operators, as well as other gauge-invariant products of chiral superfields later in the paper, it should be kept in mind that there is no confinement in this NACP, and hence no physical mesons or baryons.

[3] C_A and C_f are the quadratic Casimir invariants for the adjoint representation and the fermion representation $\mathcal{R}$, and T_f is the trace invariant. We use the standard normalizations for these, so that for $G = \mathrm{SU}(N_c)$, $C_A = N_c^2 - 1$ and for $\mathcal{R} = F$, $C_f = (N_c^2 - 1)/(2N_c)$ and $T_f = 1/2$.

[4] Thus, here and below, if an expression for N_f formally evaluates to a non-integral real value, it is understood implicitly that one infers an appropriate integral value from it.

The reason that we restrict to gauge-singlet operators is so that the corresponding anomalous dimensions are gauge-invariant and hence physical.

In the NACP, a quadratic chiral superfield operator transforms according to an (irreducible) representation of this global chiral symmetry. Since the anomalous dimensions are the same for these different representations (see, e.g., [20]), we denote the common anomalous dimension simply as that for the singlet representation, corresponding to the quadratic operator product $\tilde{\Phi}\Phi = \sum_{i=1}^{N_f} \tilde{\Phi}_i \Phi^i$. Since this corresponds to the (gauge-invariant) fermion bilinear $\bar{\psi}\psi$ in a non-supersymmetric vectorial gauge theory, the anomalous dimension γ_M has often been denoted as $\gamma_{\bar{\psi}\psi}$ in our previous papers [6, 11–15].

A number of exact results have been established about the (zero-temperature) IR phase structure of the theory [19, 21, 22]. In the IR limit $\mu \to 0$, $\alpha(\mu)$ approaches the limiting value α_{IR}. In particular, the theory flows from the UV to a non-Abelian Coulomb phase (NACP) in the IR if

$$\text{NACP}: \quad N_\ell < N_f < N_u \ , \tag{2.10}$$

where

$$N_\ell = \frac{3C_A}{4T_f} = \frac{N_u}{2} \ . \tag{2.11}$$

As with N_u, note that N_ℓ is not necessarily an integer; it is the actual physical lower end of the NACP if and only if it is an integer. In particular, we note the important special case

$$G = \text{SU}(N_c), \quad \mathcal{R} = F \implies N_\ell = \frac{3}{2}N_c, \quad N_u = 3N_c, \tag{2.12}$$

so that in this special case, N_ℓ is only physical if and only if N_c is even. This is to be understood implicitly below, when N_ℓ is referred to as the lower end of the non-Abelian Coulomb phase.[5] Throughout the paper we will often consider a formal generalization in which N_f is analytically continued from the non-negative integers to the (non-negative) real numbers, with the understanding that physical values of N_f are positive integers. One reason for doing this is to study the behavior of various quantities as N_f approaches N_u from below and N_ℓ from above in the non-Abelian phase.

The two-loop beta function has an IR zero if N_f is in the interval $N_{f,b2z} < N_f < N_u$, where

$$N_{f,b2z} = \frac{3C_A^2}{2T_f(C_A + 2C_f)} \ . \tag{2.13}$$

[5] This complication with N_ℓ being unphysical for odd N_c is avoided in the LNN limit (9.1), in which one takes $N_c \to \infty$ and $N_f \to \infty$ with the ratio $r = N_f/N_c$ fixed and finite. In this LNN limit, physical quantities are functions of the real variable r instead of the integer variables N_c and N_f and N_ℓ is replaced by the quantity r_ℓ defined below in Eq. (9.4), which is always physical.

As we discussed in [6], $N_{f,b2z}$ may be larger than or smaller than N_ℓ, depending on the chiral superfield representation $\mathcal{R}$. One has

$$N_{f,b2z} - N_\ell = \frac{3C_A(C_A - 2C_f)}{4T_f(C_A + 2C_f)} \ . \tag{2.14}$$

This difference can be positive or negative. For the fundamental representation, $\mathcal{R} = F$,

$$\mathcal{R} = F \implies N_{f,b2z} - N_\ell = \frac{3N_c}{2(2N_c^2 - 1)} \ , \tag{2.15}$$

which is positive. However, for example, for the adjoint representation, $R = adj$, this difference is negative:

$$\mathcal{R} = adj : \implies N_{f,b2z} - N_\ell = -\frac{1}{4} \ . \tag{2.16}$$

For general G, the supersymmetric theory under consideration here is invariant under a classical continuous global (gb) symmetry

$$\begin{aligned} G_{cgb} &= \text{U}(N_f) \otimes \text{U}(N_f) \otimes \text{U}(1)_R \\ &= \text{SU}(N_f) \otimes \text{SU}(N_f) \otimes \text{U}(1)_V \otimes \text{U}(1)_A \otimes \text{U}(1)_R \ , \end{aligned} \tag{2.17}$$

where the first and second $\text{U}(N_f)$ groups consist of operators acting on $\Phi = (\Phi^1, ..., \Phi^{N_f})$ and $\tilde{\Phi} = (\tilde{\Phi}_1, ..., \tilde{\Phi}_{N_f})$, respectively, and the $\text{U}(1)_R$ group is defined by the commutation relations

$$[Q_\alpha, R] = Q_\alpha \ , \quad [Q_\alpha^\dagger, R] = -Q_\alpha^\dagger \ , \tag{2.18}$$

where the Q_α and $Q_\alpha^\dagger$ are the generators of the supersymmetry transformations (with α a spinor index here). The $\text{U}(1)_A$ symmetry is anomalous, due to instantons, so the actual nonanomalous continuous global symmetry of the theory is

$$G_{gb} = \text{SU}(N_f) \otimes \text{SU}(N_f) \otimes \text{U}(1)_V \otimes \text{U}(1)_R \ . \tag{2.19}$$

This symmetry is exact at a superconformal IRFP in the non-Abelian Coulomb phase. Usually, for a U(1) (global or gauge) symmetry, the physics is invariant under a multiplication of the charges of all fields by a nonzero real constant. However, the situation is different for the $\text{U}(1)_R$ symmetry in a superconformal field theory; in this case, the R charges of chiral superfields under the (global) $\text{U}(1)_R$ symmetry are uniquely determined [22–26].

The representations of the matter chiral superfields under the gauge and global symmetry groups are listed in Table I for the generic case in which the representation $\mathcal{R}$ is complex. The case of (real) $\mathcal{R}$ will be discussed below.) We recall the derivation of the R-charge assignment to Φ and $\tilde{\Phi}$ (noting also that one can take $R_\Phi = R_{\tilde{\Phi}}$). This assignment can be determined by the condition that the $\text{U}(1)_R$ symmetry should not have a triangle anomaly determines the R charges of Φ (where the gauge and flavor indices are suppressed in the notation). The R charge of the fermionic component ψ in Φ is $R_\psi = R_\Phi - 1$. Given that $R_\lambda = 1$ for the gluino, λ, the sum of the contributions to the triangle anomaly from the gluino, and the Φ and $\tilde{\Phi}$ matter superfields are $C_A + 2(R_\Phi - 1)T_f N_f$. The condition that this sum must be zero yields

$$R_\Phi = R_{\tilde{\Phi}} = 1 - \frac{C_A}{2T_f N_f} \ . \tag{2.20}$$

For the $U(1)_R$ symmetry to be non-anomalous, it is also necessary that, similarly to the situation in non-supersymmetric theories, the one-loop contribution is not modified by higher-order contributions, and this requisite property holds [27].

One can construct gauge-invariant quadratic operator products of the "meson"-type, namely

$$M_i^j = \tilde{\Phi}_i \Phi^j \ , \tag{2.21}$$

where, as above, i and j are flavor indices and the group indices are implicit, with it being understood that they are contracted in such a way as to yield a singlet under the gauge group G. As a holomorphic product of chiral superfields, M_i^j is again a chiral superfield. The fermionic bilinear operator product in M_i^j is $\tilde{\psi}_i \psi^j \equiv \tilde{\psi}_{i,L}^T C \psi_L^j$, where C is the conjugation Dirac matrix and we follow the usual convention of writing the holomorphic chiral superfields as left-handed. Because the global symmetry (2.19) is exact in the NACP, the meson-type quadratic chiral superfields transform according to (irreducible) representations of the group G_{gb}. We focus on the anomalous dimension of the diagonal operator $\tilde{\Phi}\Phi = \sum_{i=1}^{N_f} \tilde{\Phi}_i \Phi^i$ evaluated at the IRFP α_{IR}, which we denote as γ_M. [6]

Consider next the case where $G = SU(N_c)$ and $\mathcal{R} = F$. The transformation properties of the matter chiral superfields in this theory under the global symmetry group G_{gb} are listed in Table II. Since we focus on the non-Abelian Coulomb NACP, where an IRFP is exact, N_f must lie in the interval $(3/2)N_c < N_f < 3N_c$. Therefore, N_f automatically satisfies the requirement $N_f \ge N_c$ to construct the baryonic composite chiral superfield operator

$$B^{i_1 \ldots i_{N_c}} = \epsilon_{a_1 \ldots a_{N_c}} \Phi^{a_1, i_1} \Phi^{a_2, i_2} \cdots \Phi^{a_{N_c}, i_{N_c}} \tag{2.22}$$

and the corresponding operator involving the $\tilde{\Phi}$ chiral superfields,

$$\tilde{B}_{i_1 \ldots i_{N_c}} = \epsilon_{a_1 \ldots a_{N_c}} \tilde{\Phi}^{a_1}_{i_1} \tilde{\Phi}^{a_2}_{i_2} \cdots \tilde{\Phi}^{a_{N_c}}_{i_{N_c}} \ , \tag{2.23}$$

where here the a_k and the i_ℓ are group and flavor indices, respectively and $\epsilon_{a_1 \ldots a_{N_c}}$ is the totally antisymmetric tensor density for the $SU(N_c)$ gauge group. (If $N_f < N_c$, the operator products (2.22) and (2.23) vanish identically.). Since the flavors are equivalent with respect to the gauge interaction, we will henceforth suppress the flavor dependence in the notation. The full scaling dimensions of Φ and $\tilde{\Phi}$ are equal, and the same is true for the full scaling dimensions of B and $\tilde{B}$, i.e., $D_{B,F} = D_{\tilde{B},F}$ (where the subscript indicates that $\mathcal{R} = F$, the fundamental representation), so that the anomalous dimensions of these baryonic operators, denoted $\gamma_{B,F}$ and $\gamma_{\tilde{B},F}$, are also equal. We thus have

$$\begin{aligned} D_{B,F} = D_{\tilde{B},F} &= D_{B,F,\text{free}} - \gamma_{B,F} \\ &= N_c - \gamma_{B,F} \ . \end{aligned} \tag{2.24}$$

We shall discuss baryonic chiral superfield operator products for the case where $\mathcal{R}$ is a higher-dimensional representation of G later in the paper.

In general (suppressing flavor indices), from M, B, and $\tilde{B}$, one can construct a number of different composite gauge-invariant chiral superfields. We denote such a generic composite chiral superfield consisting of a (holomorphic) product of n_M factors of a meson-type chiral superfield M times n_B factors of B and $n_{\tilde{B}}$ factors of $\tilde{B}$ chiral superfields as Φ_{prod}:

$$\Phi_{\text{prod}} = M^{n_M} B^{n_B} \tilde{B}^{n_{\tilde{B}}} \ . \tag{2.25}$$

Here, to avoid cumbersome notation, the values of n_M, n_B, and $n_{\tilde{B}}$ are kept implicit in $\Phi_{\text{prod}} \equiv \Phi_{\text{prod};n_M,n_B,n_{\tilde{B}}}$. One could also include a factor $(W_\alpha W^\alpha)^{n_W}$, but (2.25) will be sufficient for our present analysis.

There are several important quantities that characterize the properties of the superconformal field theory at the IRFP at α_{IR}. These include the derivative

$$\beta'_{IR} \equiv \frac{d\beta}{d\alpha}\bigg|_{\alpha=\alpha_{IR}} \tag{2.26}$$

and the anomalous dimensions of various gauge-invariant composite chiral superfield operators evaluated at $\alpha = \alpha_{IR}$ such as γ_M, γ_B, $\gamma_{\tilde{B}}$, and $\gamma_{\Phi_{\text{prod}}}$. (Here and below, we will often leave the dependence on $\mathcal{R}$ implicit in the notation.)

As (gauge-invariant) physical quantities, β'_{IR} and these anomalous dimensions are scheme-independent. However, the series expansions of these quantities in powers of α, calculated to a finite order, do not maintain this scheme-independence beyond the lowest orders. Hence, it is quite useful to calculate and analyze series expansions for these quantities that are scheme-independent at each order. An important property is that $\alpha_{IR} \searrow 0$ as $N_f \nearrow N_u$. This property is also shared by a quantity that is manifestly scheme-independent, namely

$$\Delta_f = N_u - N_f \ , \tag{2.27}$$

[6] Concerning notation, in earlier works in which we dealt with series expansions for anomalous dimensions as powers of α, we included a subscript IR when discussing the values at a conformal or superconformal IRFP. Here, since we will always be discussing the properties at a superconformal theory at an IRFP, it will not be necessary to include this subscript. Therefore, although we retain the IR subscript in β'_{IR}, we will usually omit it in the anomalous dimensions to simplify the notation.

where N_u was defined in Eq. (2.6). The maximal value of Δ_f in the NACP is

$$(\Delta_f)_{max,NACP} = N_u - N_\ell = \frac{N_u}{2} = \frac{3C_A}{4T_f} \ . \qquad (2.28)$$

As was observed by Banks and Zaks [2] (for a non-supersymmetric vectorial gauge theory, in which $N_u = 11C_A/(4T_f)$), Δ_f is a natural scheme-independent expansion variable. In addition to [2], some early work with the Δ_f expansion was carried out in [3, 4]. In addition to our previous works on scheme-independent series expansions [11–15], see also [28].

One may write a scheme-independent series expansions of β'_{IR} in powers of Δ_f as

$$\beta'_{IR} = \sum_{j=2}^{\infty} d_j \, \Delta_f^j \ . \qquad (2.29)$$

In general, the calculation of d_j requires, as inputs, the values of b_ℓ with $1 \le \ell \le j$.

The property that $d_1 = 0$, so that β'_{IR} vanishes like Δ_f^2 as $\Delta_f \to 0$, was derived in [13]. This property is general and does not depend on whether the theory is supersymmetric or non-supersymmetric. A simple way to understand this result is to note that for either type of theory, the one-loop coefficient in the beta function has the form $b_1 = b_{1,0} + b_{1,1}N_f$ (where $b_{1,0} > 0$ and $b_{1,1} < 0$), so that $N_u = -b_{1,0}/b_{1,1}$. Then, since $\Delta_f = N_u - N_f = -b_1/b_{1,1}$, it follows that

$$\Delta_f \propto b_1 \ . \qquad (2.30)$$

From Eq. (8.1) below, $\beta'_{IR} = -2a_{IR}\sum_{\ell=1}^{\infty}(\ell+1)b_\ell \, a_{IR}^{\ell-1}$, where $a_{IR} = \alpha_{IR}/(4\pi)$. As $N_f \to N_u$, α_{IR} vanishes linearly in Δ_f, so in this limit, $\beta'_{IR} \propto \alpha_{IR} b_1 \propto \Delta_f^2$.

One may write the scheme-independent series expansion of γ_M at the superconformal IRFP in powers of Δ_f for a meson superfield operator:

$$\gamma_M = \sum_{j=1}^{\infty} \kappa_j \, \Delta_f^j \ . \qquad (2.31)$$

The calculation of κ_j requires, as inputs, the values of b_ℓ with $1 \le \ell \le j+1$ and c_ℓ with $1 \le \ell \le j$. Similarly, the scheme-independent series expansion of $\gamma_B = \gamma_{\tilde{B},IR}$ at the IRFP in powers of Δ_f can be written as

$$\gamma_B = \gamma_{\tilde{B}} = \sum_{j=2}^{\infty} f_{B,j} \Delta_f^j \ . \qquad (2.32)$$

More generally, the scheme-independent expansion for a gauge-invariant composite chiral superfield $\Phi_{\rm prod}$ consisting of a (holomorphic) product of an arbitrary number of mesonic, baryonic, and conjugate baryonic superfields, evaluated at the IRFP, can be written as

$$\gamma_{\Phi_{\rm prod}} = \sum_{j=2}^{\infty} f_{\Phi_{\rm prod},j} \Delta_f^j \ . \qquad (2.33)$$

These are thus series expansions extending downward below N_u in the non-Abelian Coulomb phase. The truncations of these infinite series to order $j = p$ inclusive are denoted $\beta'_{IR,\Delta_f^p} \equiv \beta'_{\Delta_f^p}$, γ_{M,Δ_f^p}, γ_{B,Δ_f^p}, and $\gamma_{\Phi_{\rm prod},\Delta_f^p}$, respectively.

For a scalar operator (other than the identity), the condition of unitarity in a conformal field theory implies the lower bound [23, 24, 29]

$$D_{\mathcal{O}} \ge 1 \ . \qquad (2.34)$$

This bound holds regardless of whether the theory is supersymmetric or not.

In a supersymmetric conformal (i.e., superconformal) theory, one can take advantage of additional information about the operator dimensions. First, if a (composite or fundamental) chiral superfield $\mathcal{O}$ has R charge $R_{\mathcal{O}}$, then [22–25, 29, 30]

$$D_{\mathcal{O}} = \frac{3}{2} R_{\mathcal{O}} \ . \qquad (2.35)$$

We recall that since $D_{\mathcal{O}}$ is a physical quantity, the meaningfulness of this relation depends on the fact that in a superconformal theory, the R charges are uniquely determined. Since the $\mathrm{U}(1)_R$ symmetry is exact in the non-Abelian Coulomb phase considered here, the R charge of an operator is a conserved quantity. The R charge of a holomorphic product of chiral superfields is the sum of the R charges of each of the chiral superfields in the product:

$$R_{\Phi_{\rm prod}} = \sum_{k=1}^{p} R_{\Phi_k} \ . \qquad (2.36)$$

Hence, the full dimension of a holomorphic product $\Phi_{\rm prod}$ of chiral superfields Φ_k, $k = 1, ..., p$, $\Phi_{\rm prod} = \prod_{k=1}^{p} \Phi_k$, is the sum of the full dimensions of each chiral superfield in the product (e.g., [25]):

$$D_{\Phi_{\rm prod}} = \sum_{k=1}^{p} D_{\Phi_k} \ . \qquad (2.37)$$

Furthermore, the anomalous dimension of $\Phi_{\rm prod}$ is the sum of the anomalous dimensions of the individual Φ_k superfields:

$$\gamma_{\Phi_{\rm prod}} = \sum_{k=1}^{p} \gamma_{\Phi_k} \ . \qquad (2.38)$$

III. THEOREMS ON PROPERTIES OF THE ANOMALOUS DIMENSIONS OF COMPOSITE CHIRAL SUPERFIELDS

In this section we prove some theorems on the properties of the anomalous dimension $\gamma_{\Phi_{\rm prod}}$ of a gauge-invariant composite chiral superfield consisting of a (holomorphic) product of powers of Φ and/or $\tilde{\Phi}$ (where flavor indices are suppressed). Our results for the anomalous dimensions $\gamma_{\Phi_{\rm prod}}$ of various particular composite chiral superfields given later in the paper will illustrate these general theorems.

The properties of the R charge (2.20) form the basis of the resultant properties of the anomalous dimensions of the various composite chiral superfields that we will consider. We first use these properties to prove a general monotonicity theorem concerning the anomalous dimension of a chiral superfield operator containing products of Φ and/or $\tilde{\Phi}$. This theorem applies for an arbitrary gauge group G and fermion representation. We recall that $N_\ell = N_u/2$, as is evident in Eqs. (2.6) and (2.11). For the following discussion, we implicitly use the above-mentioned generalization of N_f from non-negative integers to real numbers. As N_f decreases from N_u to N_ℓ in the NACP, R_Φ decreases from 0 to -1. Since the full scaling dimension of a chiral superfield operator containing products of Φ and/or $\tilde{\Phi}$ satisfies (2.35) and since this full dimension is related to the anomalous dimension of the operator according to (1.2), it follows that the anomalous dimension $\gamma_{\mathcal{O}}$ is a monotonically increasing function of decreasing N_f in the NACP, which increases from $\gamma_{\mathcal{O}} = 0$ at the upper end of the NACP to a maximal value at the lower end of the NACP.

We next prove a theorem on the structure the anomalous dimension of a general composite chiral superfield containing products of Φ and/or $\tilde{\Phi}$, and the coefficients $f_{\Phi_{\rm prod},j}$ in (2.33). To do this, we first express $R_\Phi = R_{\tilde{\Phi}}$ as a function of Δ_f, obtaining

$$R_\Phi = 1 - \frac{1}{1-\frac{\Delta_f}{N_u}} \ . \tag{3.1}$$

Combining this with Eqs. (2.35) and (1.2), it follows, as a second theorem, that the anomalous dimension of a general composite chiral superfield containing products of Φ and/or $\tilde{\Phi}$, evaluated at the superconformal IRFP, is of the form

$$\begin{aligned} \gamma_{\Phi_{prod}} &= C\Big[1 - \frac{1}{1-\frac{\Delta_f}{N_u}}\Big] \\ &= C\sum_{j=1}^{\infty}\Big(\frac{\Delta_f}{N_u}\Big)^j \ , \end{aligned} \tag{3.2}$$

where C is a Δ_f-independent constant depending on G, the fermion representation, and the structure of Φ_{prod}. Hence, as a corollary to this theorem, we find that the coefficient $f_{\Phi_{\rm prod},j}$ of the $O(\Delta_f^j)$ term in the expansion (2.33) is given by

$$f_{\Phi_{\rm prod},j} = \frac{1}{N_u^j} \ . \tag{3.3}$$

That is, up to an overall multiplicative factor C, $\gamma_{\Phi_{\rm prod}}$ is a geometric series in powers of Δ_f, with the coefficients given in Eq. (3.3). As is evident in Eq. (3.3) is positive, this coefficient $f_{\Phi_{\rm prod},j}$ is positive. This leads to two further monotonicity theorems. Define $\gamma_{\Phi_{rmprod},\Delta_f^p}$ as equal to the right-hand side of Eq. (2.33) with the upper limit $j=\infty$ replaced by $j=p$, i.e., the truncation of this infinite series to order $O(\Delta_f^p)$. Then the positivity of the coefficients $f_{\Phi_{\rm prod},j}$ implies, as the third and fourth theorems, that (i) for fixed p, the $O(\Delta_f^p)$ approximation, $\gamma_{\Phi_{rmprod},\Delta_f^p}$, to the exact $\gamma_{\Phi_{\rm prod}}$, is a monotonically increasing function of Δ_f, i.e., of decreasing N_f, and (ii) for fixed N_f and thus Δ_f, $\gamma_{\Phi_{rmprod},\Delta_f^p}$ is a monotonically increasing function of the truncation order, p. We had noted these monotonicity results in our earlier work for γ_M [11–15], and here we prove them in general.

A fifth theorem concerns the region of analyticity of the expression for $\gamma_{\Phi_{prod}}$ in (3.2) and the corresponding radius of convergence of the series expansion (2.33) in powers of Δ_f. As is evident in Eq. (3.2), this exact explicit expression for $\gamma_{\Phi_{prod}}$ is an analytic function of Δ_f in the complex Δ_f plane within a disk defined by

$$|\Delta_f| < N_u \tag{3.4}$$

and, correspondingly, the infinite series (2.33) converges for all Δ_f in this disk. This region of convergence covers the entire non-Abelian Coulomb phase because the maximal value of Δ_f in this phase, as given by Eq. (2.28), is $(\Delta_f)_{max,NACP} = N_u/2$.

IV. ANOMALOUS DIMENSION γ_M

In this section we discuss some results on γ_M at a superconformal IRFP that will be used in the paper. Since

$$R_M = R_\Phi + R_{\tilde{\Phi}} = 2\Big(1 - \frac{C_A}{2T_fN_f}\Big) \ , \tag{4.1}$$

the full dimension of the quadratic chiral superfield operator M (at the superconformal IRFP) is

$$\begin{aligned} D_M &= \frac{3}{2}R_M = 3\Big(1 - \frac{C_A}{2T_fN_f}\Big) \\ &= 2 - \gamma_M \ , \end{aligned} \tag{4.2}$$

and hence

$$\gamma_M = \frac{3C_A}{2T_fN_f} - 1 = \frac{N_u}{N_f} - 1 \ . \tag{4.3}$$

where N_u depends on $\mathcal{R}$. Expressing this anomalous dimension in terms of Δ_f, we have

$$\gamma_M = \frac{1}{1-\frac{\Delta_f}{N_u}} - 1 = \sum_{j=1}^{\infty}\Big(\frac{\Delta_f}{N_u}\Big)^j \ , \tag{4.4}$$

so the coefficient κ_j in Eq. (2.31) is

$$\kappa_j = \frac{1}{N_u^j} = \Big(\frac{2T_f}{3C_A}\Big)^j \ . \tag{4.5}$$

One sees that this general derivation is consistent with the NSVZ beta function. This can be seen from the fact that at the IRFP, $\beta_{NSVZ} = 0$; solving this equation yields the result (4.3). Expressing γ_M as a function of Δ_f, we obtain the same results as in Eqs. (4.4) and (4.5).

For an $\mathcal{N} = 1$ supersymmetric gauge theories with general G and $\mathcal{R}$, γ_M was calculated up to three-loop order in [6] and studied further in [7]-[10]. Concerning the scheme-independent series expansion (2.31), for general G and $\mathcal{R}$, κ_1 and κ_2 were calculated in [11], while for $G = \mathrm{SU}(N_c)$ and $\mathcal{R} = F$, κ_3 was computed in [14]. These calculations used the beta function coefficients b_1-b_4 and the anomalous dimension coefficients c_1-c_3 from [17, 18, 31]. Importantly, we found that the results of our scheme-independent calculations of the κ_j for this supersymmetric gauge theory agreed perfectly with the Taylor series expansion of the exact expression (4.4).

Furthermore, as is evident from the exact result (4.4), the small-Δ_f expansion of the exact result is (absolutely) convergent for $|\Delta_f| < N_u$, i.e.,

$$|\Delta_f| < \frac{3C_A}{2T_f} \ . \tag{4.6}$$

This covers all of the non-Abelian Coulomb phase, which extends from $N_u = 3C_A/(2T_f)$ down to $N_\ell = N_u/2 = 3C_A/(4T_f)$, i.e., from $\Delta_f = 0$ to $\Delta_f = 3C_A/(4T_f)$.

We next discuss the limiting values of γ_M at a superconformal IRFP at the upper and lower end of the NACP. If one formally generalizes N_f from the positive integers to real numbers and lets N_f decrease from N_u to N_ℓ in the NACP, γ_M increases monotonically from 0 to 1, saturating the upper bound allowed by conformal invariance at the lower end of the NACP. This behavior holds for general matter chiral superfield representation R and is a consequence of the fact that $N_\ell = N_u/2$. As stated, this is formal, because, in general, neither N_u nor N_ℓ is an integer, so the physical N_f, restricted as it is to integer values, cannot necessarily take on either the value N_u at which $\gamma_M = 0$ or the value N_ℓ at which $\gamma_M \nearrow 1$, saturating the upper bound from conformality. In order for N_f to be able to reach N_ℓ, it is necessary that N_ℓ be an integer. In the case $G = \mathrm{SU}(N_c)$ with $\mathcal{R} = F$, (i) N_u is always an integer, but (ii) since $N_\ell = (3/2)N_c$, it follows that N_ℓ is an integer if and only if N_c is even. If, on the other hand, N_c is odd, then as N_c decreases from $N_u = 3N_c$ in the NACP, it cannot actually reach N_ℓ since the latter is half-integral. In this case, γ_M does not saturate its conformality upper bound at the lower end of the NACP. In this case where the matter chiral superfield representation is $\mathcal{R} = F$, one may avoid this complication by taking the limit $N_c \to \infty$, $N_f \to \infty$ with the ratio $r = N_f/N_c$ fixed and finite. As will be discussed below, in this limit, r is a real number and can always reach the lower end of the non-Abelian Coulomb phase, so that γ_M always saturates its upper bound from conformal invariance.

It should be noted that the Δ_f expansion avoids a problem in which an IRFP may not be manifest as a physical IR zero of the n-loop beta function for some n. Indeed, although the two-loop beta function, $\beta_{2\ell}$, and the three-loop $\beta_{3\ell}$, calculated in the $\overline{DR}$ scheme [32], have physical $\alpha_{IR,n\ell}$ zeros for $N_{f,b2z} < N_f < N_u$ in this supersymmetric theory [6], we find that the four-loop beta function, $\beta_{4\ell}$ (calculated in the $\overline{DR}$ scheme), does not exhibit a physical IR zero, $\alpha_{IR,4\ell}$, for a substantial range of N_f in this interval. This is similar to what we found for $\alpha_{IR,5\ell}$ in the non-supersymmetric gauge theory [33]. In both cases, the Δ_f expansions (2.31) and (2.29) circumvent this problem of a possible unphysical $\alpha_{IR,n\ell}$ that one may encounter in using the conventional expansions (2.2).

V. ANOMALOUS DIMENSION $\gamma_B = \gamma_{\tilde{B}}$ FOR $\mathcal{R} = F$

In this section we specialize to the theory with gauge group $G = \mathrm{SU}(N_c)$ and N_f pairs of chiral superfields $\Phi^{a,i}$ and $\tilde{\Phi}^a_i$ (where a and j are group and flavor indices) in the fundamental and conjugate fundamental representations, denoted F and $\bar{F}$, with Young tableaux $\square$ and $\overline{\square}$, respectively. The matter content of this theory is summarized in Table II

The R charges of the basic chiral superfields are given in Table II. From Eq. (2.36), it follows that

$$R_{B,F} = R_{\tilde{B},F} = N_c R_\Phi = N_c\Big(1 - \frac{N_c}{N_f}\Big) \ . \tag{5.1}$$

Combining this with Eq. (2.35), one has the known exact result

$$D_{B,F} = D_{\tilde{B},F} = \frac{3}{2} R_{B,F} = \frac{3}{2} N_c\Big(1 - \frac{N_c}{N_f}\Big) \ . \tag{5.2}$$

where we indicate $\mathcal{R} = F$ explicitly. Hence, the (equal) anomalous dimensions of B and $\tilde{B}$ at the superconformal IRFP are

$$\gamma_{B,F} = \gamma_{\tilde{B},F} = \frac{N_c}{2}\Big(\frac{3N_c}{N_f} - 1\Big) \ . \tag{5.3}$$

In Fig. 1 we plot the the value of $\gamma_{B,F}$ at the IRFP calculated to order $O(\Delta_f^p)$ with $1 \le p \le 3$, in comparison with the exact value, Eq. (5.3), for the illustrative value $N_c = 3$. As was true of γ_M, we see that these $O(\Delta_f^p)$ truncations of the infinite series converge rapidly to the exact result.

Expressed as a function of $\Delta_f = 3N_c - N_f$, $\gamma_{B,F}$ is

$$\gamma_{B,F} = \gamma_{\tilde{B},F} = \frac{N_c}{2}\left(\frac{\frac{\Delta_f}{3N_c}}{1 - \frac{\Delta_f}{3N_c}}\right) \ . \tag{5.4}$$

From Eqs. (4.4) and (5.3), one sees that $\gamma_{B,F}$ is simply proportional to $\gamma_{M,F}$:

$$\gamma_{B,F} = \frac{N_c}{2}\,\gamma_{M,F} \ . \tag{5.5}$$

As $N_f \nearrow 3N_c$, i.e., $\Delta_f \searrow 0$, the common anomalous dimension $\gamma_{B,F} = \gamma_{\tilde{B},F}$ vanishes, and as $N_f \searrow (3/2)N_c$, i.e., $\Delta_f \nearrow (3/2)N_c$, it approaches the value

$$\lim_{N_f \searrow (3/2)N_c} \gamma_{(B,\tilde{B})} = \frac{N_c}{2} \qquad (5.6)$$

from below.

These baryonic composite chiral superfields have spin 0 (and are not equal to the identity), so their respective full dimensions are bounded by the unitarity constraint from conformality, $D_B \ge 1$ and $D_{\tilde{B}} \ge 1$. This implies the upper bounds

$$\gamma_{B,F} \le N_c - 1 \ , \qquad (5.7)$$

and thus also $\gamma_{\tilde{B},F} \le N_c - 1$. Except for the case $N_c = 2$, where, owing to the reality of the representations of SU(2), the baryonic and mesonic composite chiral superfield operators are equivalent, the anomalous dimensions of the B and $\tilde{B}$ operators at the IRFP do not saturate their unitarity upper bound. This is true, in particular, for the infinite set of even values of N_c, for which N_ℓ is an integer and hence is physical. This behavior is in contrast to the situation that we found for the anomalous dimension $\gamma_{M,F}$, which does saturate its upper bound of 1 as $N_f \searrow N_\ell$ (assuming that N_c is even so that N_ℓ is an integer).

VI. ANOMALOUS DIMENSIONS OF COMPOSITE CHIRAL SUPERFIELDS

In this section we derive exact expressions for the full dimensions and hence also the anomalous dimensions of a variety of composite chiral superfields. We first discuss a SU(N_c) theory with N_f pairs of matter chiral superfields Φ^i and $\tilde{\Phi}_i$, $i = 1, ..., N_f$, transforming as the F and $\bar{F}$ representations, respectively. Our explicit results illustrate the general theorems that we have proven above concerning these anomalous dimensions. We consider the composite chiral superfield $\Phi_{\rm prod}$ in Eq. (2.25). Using Eqs. (2.36), we have

$$R_{\Phi_{\rm prod}} = \Big[2n_M + (n_B + n_{\tilde{B}})N_c\Big]\Big(1 - \frac{N_c}{N_f}\Big) \ . \qquad (6.1)$$

Using Eq. (2.35), we have

$$D_{\Phi_{\rm prod}} = \frac{3}{2}\Big[2n_M + (n_B + n_{\tilde{B}})N_c\Big]\Big(1 - \frac{N_c}{N_f}\Big) \ . \qquad (6.2)$$

Hence,

$$\gamma_{\Phi_{\rm prod}} = \Big[n_M + \frac{(n_B + n_{\tilde{B}})}{2}\,N_c\Big]\Big(\frac{3N_c}{N_f} - 1\Big) \ . \qquad (6.3)$$

One sees that for the special case $(n_M, n_B, n_{\tilde{B}}) = (1,0,0)$, the general result (6.3) reduces to Eq. (4.3), while for the special cases $(n_M, n_B, n_{\tilde{B}}) = (0,1,0)$ and $(n_M, n_B, n_{\tilde{B}}) = (0,0,1)$, Eq. (6.3) reduces to Eq. (5.3). Expressing Eq. (6.3) as a function of Δ_f yields the result

$$\begin{aligned}\gamma_{\Phi_{\rm prod}} &= \Big[n_M + \frac{(n_B + n_{\tilde{B}})}{2}\,N_c\Big]\Big[\frac{1}{1 - \frac{\Delta_f}{3N_c}} - 1\Big] \\ &= \Big[n_M + \frac{(n_B + n_{\tilde{B}})}{2}\,N_c\Big]\sum_{j=1}^{\infty}\Big(\frac{\Delta_f}{3N_c}\Big)^j \ .\end{aligned} \qquad (6.4)$$

In agreement with our general monotonicity theorem proved above, this anomalous dimension $\gamma_{\Phi_{\rm prod}}$ increases monotonically as a function of Δ_f or equivalently decreasing N_f in the NACP. As N_f decreases below N_u, $\gamma_{\Phi_{\rm prod}}$ increases monotonically from 0 to a maximum of

$$\lim_{N_f \searrow (3/2)N_c} \gamma_{\Phi_{\rm prod}} = n_M + \Big(\frac{n_B + n_{\tilde{B}}}{2}\Big)N_c \ . \qquad (6.5)$$

From the conformality lower bound on the full dimension, $D_{\Phi_{\rm prod}} \ge 1$, one obtains the corresponding upper bound

$$\gamma_{\Phi_{\rm prod}} \le 2n_M + (n_B + n_{\tilde{B}})N_c - 1 \ . \qquad (6.6)$$

Expanding the exact expression in Taylor series, we read off the coefficient $f_{\Phi_{\rm prod},j}$ as

$$f_{\Phi_{\rm prod},j} = \Big[n_M + \frac{(n_B + n_{\tilde{B}})}{2}\,N_c\Big]\Big(\frac{1}{3N_c}\Big)^j \ . \qquad (6.7)$$

As is evident from Eq. (6.4), this series converges if

$$|\Delta_f| < 3N_c \ . \qquad (6.8)$$

This includes all of the NACP for this theory.

VII. BARYONIC OPERATORS WITH CHIRAL SUPERFIELDS IN HIGHER-DIMENSIONAL REPRESENTATIONS

A. General

Here we derive corresponding exact results for anomalous dimensions of (gauge-invariant) composite chiral superfield operators in (a vectorial, asymptotically free, $\mathcal{N} = 1$ supersymmetric) SU(N_c) gauge theory containing N_f pairs of matter chiral superfields transforming according to respective higher-dimensional representations $\mathcal{R}$ and $\bar{\mathcal{R}}$ of the gauge group. As part of our analysis, we consider cases in which the representation is real (or pseudoreal).[7] For a given type of higher-dimensional representation $\mathcal{R}$, the value of N_f is subject to the constraints

[7] Let T^a denote a (hermitian) generator of a Lie algebra in the representation $\mathcal{R}$, satisfying $[T^a, T^b] = if^{abc}T_c$. It follows that $-(T^a)^*$ is also a generator and defines the complex conjugate of $\mathcal{R}$, denoted $\bar{\mathcal{R}}$. A representation $\mathcal{R}$ is real if $T^a = -(T^a)^* \ \forall \ a$ and pseudoreal if for some a, $T^a \ne -(T^a)^*$, but there exists a unitary transformation U such that $UT^aU^{-1} = -(T^a)^* \ \forall \ a$, i.e., $\mathcal{R}$ and $\bar{\mathcal{R}}$ are equivalent representations.

that (i) the theory is asymptotically free, so $N_f < N_u$, where N_u was given in Eq. (2.6), and $N_f \geq N_\ell$, where N_ℓ was given in Eq. (2.11), since we focus here on an exact IRFP in the non-Abelian Coulomb phase.

For a general representation $\mathcal{R}$ of the matter chiral superfield, the representations (charges) under the (anomaly-free) global symmetry can be read from Table I. If the gauge representation is real or pseudoreal, then the global symmetry is enhanced, and the matter chiral superfield has the representations given in Table III. Real representations include the (i) all representations of SU(2), (ii) the adjoint representation of a general group G, and (iii) the antisymmetric rank-k representation of SU($2k$).

B. Adjoint Representation

If $\mathcal{R}$ is the adjoint representation, then $N_u = 3/2$ and $N_\ell = 3/4$, which allows just one Dirac value of N_f, namely $N_f = 1$. Since the adjoint representation is real, this is equivalent to $N_f = 2$ Majorana chiral superfields. Furthermore, owing to the reality of the adjoint representation, composite superfields of baryon and meson type are equivalent. We denote these by M_i^j, and they are written as

$$M_{ij} = \Phi^{a_1}{}_{a_2,i}\Phi^{a_2}{}_{a_1 j} = \mathrm{Tr}(\Phi_i \Phi_j) \ , \tag{7.1}$$

where the trace is over color indices, and i, j are flavor indices. The full scaling dimension of this operator is

$$D_{M,adj} = \frac{3}{2}R_M = 3\Big(1 - \frac{1}{2N_f}\Big) \ , \tag{7.2}$$

and therefore the anomalous dimension is

$$\gamma_{M,adj} = 2 - 3\Big(1 - \frac{1}{2N_f}\Big) = \frac{3}{2N_f} - 1 \ . \tag{7.3}$$

Thus, $\gamma_{M,adj}$ takes the values $1/2$ and $-1/4$ for the cases $N_f = 1, 2$, respectively. Note that these values are independent of N_c. Expressed as a function of $\Delta_f = N_u - N_f = (3/2) - N_f$, this anomalous dimension is

$$\gamma_{M,adj} = \frac{1}{1 - \frac{2}{3}\Delta_f} - 1 = \sum_{j=1}^{\infty}\Big(\frac{2\Delta_f}{3}\Big)^j \ . \tag{7.4}$$

We thus identify the coefficient κ_j for this case as

$$\kappa_{j,adj} = \Big(\frac{2}{3}\Big)^j \ . \tag{7.5}$$

As before, formally continuing N_f from its allowed integral values to real values, we may study the properties of the small-Δ_f expansion to the exact result. In Fig. 3 we plot $O(\Delta_f^p)$ approximations to $\gamma_{M,adj}$, together with the exact result. As is evident from this figure and from Eq. (7.3), finite truncations of this series converge rapidly to the exact result in the NACP. As we will see, this rapid convergence is also true of the other anomalous dimensions that we calculate below.

C. Rank-2 Symmetric Tensor Representation

Here we consider the case in which $G = \mathrm{SU}(N_c)$ and $\mathcal{R} = S_2$, the rank-2 symmetric tensor representation. If $N_c = 2$, then the S_2 representation is the adjoint representation, which we have already discussed. Therefore, we take $N_c \geq 3$. Here,

$$N_{u,S_2} = \frac{3N_c}{N_c + 2} \tag{7.6}$$

and

$$N_{\ell,S_2} = \frac{3N_c}{2(N_c + 2)} \ , \tag{7.7}$$

so that the non-Abelian Coulomb phase is comprised of the integer values of N_f in the formal interval $N_\ell \leq N_f < N_u$, i.e.,

$$\mathrm{NACP}_{S_2}: \quad \frac{3N_c}{2(N_c + 2)} \leq N_f < \frac{3N_c}{N_c + 2} \ . \tag{7.8}$$

The condition that N_f should be in the NACP restricts N_f. For example, for the values $N_c = 3$ and $N_c = 4$ the inequality (7.8) reads $9/10 < N_f < 9/5$ and $1 \leq N_f < 2$, respectively, allowing only the integer value $N_f = 1$. For $N_c = 5$, the inequality (7.21) reads $15/14 < N_f < 15/7$, allowing only the integer value $N_f = 2$, and more generally, for $N_c \geq 5$, the inequality (7.8) only allows the value $N_f = 2$. As $N_c \to \infty$, the inequality (7.8) approaches the limiting form $3/2 < N_f < 3$, with only the solution $N_f = 2$.

For $N_c \geq 3$, the S_2 representation is complex, so we consider both meson and baryon chiral superfield operator products. The meson product is

$$M_i^j = \tilde{\Phi}_{a_1 a_2, i}\Phi^{a_1 a_2, j} = \mathrm{Tr}(\tilde{\Phi}_i \Phi^j) \ , \tag{7.9}$$

where the trace is over the color indices and $\Phi^{a_1 a_2, i} = \Phi^{a_2 a_1, i}$. The full scaling dimension of this operator is

$$D_{M,S_2} = \frac{3}{2}R_{M,S_2} = 3\Big[1 - \frac{N_c}{N_f(N_c + 2)}\Big] \ , \tag{7.10}$$

and the anomalous dimension is

$$\begin{aligned} \gamma_{M,S_2} &= 2 - 3\Big[1 - \frac{N_c}{N_f(N_c + 2)}\Big] \\ &= \frac{3N_c}{N_f(N_c + 2)} - 1 = \frac{N_{u,S_2}}{N_f} - 1 \ . \end{aligned} \tag{7.11}$$

As is clear from Eq. (7.11), this is of the form (4.3) with $N_u = N_{n,S_2}$. Expressed in terms of $\Delta_f = N_u - N_f$, one obtains the special case of (4.4) for the present theory with $N_u = N_{u,S_2}$ given by (7.6). As was the case with $\mathcal{R} = F$, since N_ℓ is not, in general, an integer, N_f cannot actually decrease all the way to be equal to N_ℓ, so γ_{M,S_2} does not actually saturate its upper bound $\gamma_{M,S_2} \leq 1$ from conformal invariance. However, if one formally analytically continues N_f from integers to real numbers, then this N_f can decrease all the way to N_ℓ at the lower boundary of the NACP, so γ_{M,S_2} does saturate this upper bound. In Fig. 4 we plot $O(\Delta_f^p)$ approximations to γ_{M,S_2}, together with the exact result, for the case $N_c = 3$. We see again that finite truncations of this series converge rapidly to the exact result throughout the NACP.

The baryon and antibaryon operators in this case are

$$B^{i_1,...,i_{N_c}} = \frac{1}{N_c!}\epsilon_{a_1,...,a_{N_c}}\epsilon_{a'_1,...,a'_{N_c}}\Phi^{a_1 a'_1,i_1}\cdots\Phi^{a_{N_c}a'_{N_c},i_{N_c}} \tag{7.12}$$

and

$$\tilde{B}_{i_1,...,i_{N_c}} = \frac{1}{N_c!}\epsilon^{a_1,...,a_{N_c}}\epsilon^{a'_1,...,a'_{N_c}}\tilde{\Phi}_{a_1 a'_1,i_1}\cdots\tilde{\Phi}_{a_{N_c}a'_{N_c},i_{N_c}} \ . \tag{7.13}$$

The way in which the color indices are contracted is similar to the determinant of a matrix. This is the reason we have included the $1/(N_c!)$ normalization factor. These operators have R charge

$$R_{B,S_2} = R_{\tilde{B},S_2} = N_c\Big[1 - \frac{N_c}{N_f(N_c+2)}\Big] \ . \tag{7.14}$$

Hence, the full scaling dimensions of these operators are

$$\begin{aligned} D_{B,S_2} = D_{\tilde{B},S_2} &= \frac{3}{2}R_B \\ &= \frac{3}{2}N_c\Big[1 - \frac{N_c}{N_f(N_c+2)}\Big] \ . \end{aligned} \tag{7.15}$$

and the anomalous dimensions are

$$\gamma_{B,S_2} = \gamma_{\tilde{B},S_2} = \frac{N_c}{2}\Big[\frac{3N_c}{N_f(N_c+2)} - 1\Big] \ . \tag{7.16}$$

In Fig. 5 we plot the $O(\Delta_f^p)$ approximations to γ_{B,S_2} for $G = \mathrm{SU}(3)$, together with the exact result.

The unitarity constraint for the baryons is the lower bound $D_B \ge 1$, and since $D_{B,S_2} = 2N_c - \gamma_B$, this implies the upper bound

$$\gamma_{B,S_2} < N_c - 1 \ . \tag{7.17}$$

Formally continuing N_f to real numbers and evaluating γ_B at $N_f = N_\ell$, we find

$$\gamma_{B,S_2} = \frac{N_c}{2} \quad \text{at } N_f = N_\ell \ . \tag{7.18}$$

For all $N_c \ge 3$, this does not saturate the upper bound (7.17). Furthermore, for most values of N_c, N_ℓ is not an integer, so the physical values of N_f do not allow N_f to actually decrease all the way to N_ℓ, and hence the largest value of γ_{B,S_2} is actually smaller than $N_c/2$.

D. Rank-2 Antisymmetric Tensor Representation

We next consider the case in which $G = \mathrm{SU}(N_c)$ and $\mathcal{R} = A_2$, the rank-2 antisymmetric tensor representation. We restrict to $N_c \ge 4$, since for $N_c = 2$, then A_2 is the singlet and if $N_c = 3$, then $A_2 = \bar{F}$, the conjugate fundamental. We have

$$N_{u,A_2} = \frac{3N_c}{N_c - 2} \tag{7.19}$$

and

$$N_{\ell,A_2} = \frac{3N_c}{2(N_c-2)} \ , \tag{7.20}$$

so that the non-Abelian Coulomb phase is comprised of the integer values of N_f in the formal interval $N_\ell \le N_f < N_u$, i.e.,

$$\mathrm{NACP}_{A_2}: \quad \frac{3N_c}{2(N_c-2)} \le N_f < \frac{3N_c}{N_c-2} \ . \tag{7.21}$$

As with the adjoint and S_2 representations, here also, the condition that N_f should be in the NACP restricts N_f. For example, for the values $N_c = 4$ and $N_c = 5$ the inequality (7.21) reads $3 \le N_f < 6$ and $5/2 \le N_f < 5$, allowing only the integer values $N_f = 3, 4$. For $N_f = 8$, the inequality (7.21) is $2 \le N_f < 4$, allowing only the values $N_f = 2,\ 3$. As $N_c \to \infty$, the inequality (7.21) approaches the same limiting form as for $R = S_2$, namely, $3/2 < N_f < 3$, with only the solution $N_f = 2$.

Here the meson-type chiral superfield product M_i^j has the same form as (7.9), but with $\Phi^{a_1 a_2,i} = -\Phi^{a_2 a_1,i}$. The full scaling dimension of this operator is

$$D_{M,A_2} = \frac{3}{2}R_{M,A_2} = 3\Big[1 - \frac{N_c}{N_f(N_c-2)}\Big] \ , \tag{7.22}$$

and the anomalous dimension is

$$\begin{aligned} \gamma_{M,A_2} &= 2 - 3\Big[1 - \frac{N_c}{N_f(N_c-2)}\Big] \\ &= \frac{3N_c}{N_f(N_c-2)} - 1 = \frac{N_{u,A_2}}{N_f} - 1 \ . \end{aligned} \tag{7.23}$$

Again, this is in accord with our general result (7.11) with $N_u = N_{u,A_2}$, and again, this can be expressed as a function of $\Delta_f = N_u - N_f$, as in Eq. (4.4), with $N_u = N_{u,A_2}$. The same comments that were made above apply here, namely that if one formally continues N_f from the integers to the real numbers, so that N_f can decrease all the way to N_ℓ, then γ_{M,A_2} saturates its upper bound of 1. However, since N_ℓ is not, in general, an integer, so that N_f, restricted to physical, integral values, cannot actually reach N_ℓ, then, just as was true with γ_{M,S_2}, γ_{M,A_2} does not saturate its upper bound from conformal invariance at the lower end of the NACP.

In Figs. 6 and 7 we plot the anomalous dimension γ_{M,A_2} to first, second and third order in Δ_f for $N_c = 4$ and $N_c = 5$, together with the respective exact results. Note that for $N_c = 4$ the A_2 representation is real, so the meson and baryon operators are equivalent.

For the baryons and antibaryons, we need to distinguish between even and odd values of N_c. For even $N_c = 2k$, these are

$$B^{i_1\cdots i_k} = \frac{1}{2^k k!}\epsilon_{a_1,\ldots,a_{2k}}\Phi^{a_1a_2,i_1}\cdots\Phi^{a_{2k-1}a_{2k},i_k} \quad (7.24)$$

and

$$\tilde{B}_{i_1\cdots i_k} = \frac{1}{2^k k!}\epsilon^{a_1,\ldots,a_{2k}}\tilde{\Phi}_{a_1a_2,i_1}\cdots\tilde{\Phi}_{a_{2k-1}a_{2k},i_k} \quad (7.25)$$

while for odd N_c, they are

$$B^{i_1,\ldots,i_{N_c}} = \frac{1}{N_c!}\epsilon_{a_1,\ldots,a_{N_c}}\epsilon_{a'_1,\ldots,a'_{N_c}}\Phi^{a_1a'_1,i_1}\cdots\Phi^{a_{N_c}a'_{N_c},i_{N_c}} \quad (7.26)$$

and

$$\tilde{B}_{i_1,\ldots,i_{N_c}} = \frac{1}{N_c!}\epsilon^{a_1,\ldots,a_{N_c}}\epsilon^{a'_1,\ldots,a'_{N_c}}\tilde{\Phi}_{a_1a'_1,i_1}\cdots\tilde{\Phi}_{a_{N_c}a'_{N_c},i_{N_c}} \quad (7.27)$$

Thus, for even and odd values of N_c, the respective baryon operators involves $N_c/2 = k$ and N_c A_2 chiral superfields. Correspondingly, for even and odd N_c, the contractions of the color indices are analogous to a Pfaffian and a determinant, respectively.

For even N_c (denoted Nce), the full scaling dimension of the baryon and antibaryon operators is

$$D_{B,A_2,Nce} = D_{\tilde{B},A_2,Nce} = \frac{3N_c}{4}\Big[1 - \frac{N_c}{N_f(N_c-2)}\Big] , \quad (7.28)$$

so the anomalous dimension is

$$\gamma_{B,A_2,Nce} = \gamma_{\tilde{B},A_2,Nce} = \frac{N_c}{4}\Big[\frac{3N_c}{N_f(N_c-2)} - 1\Big] . \quad (7.29)$$

We plot $\gamma_{B,A_2,Nce}$ for $N_c = 6$ in Fig. 8.

The unitarity constraint from conformal invariance is again $D_B > 1$, and since $D_B = (N_c/2) - \gamma_B$, this implies the upper bound

$$\gamma_{B,A_2,Nce} < \frac{N_c}{2} - 1 . \quad (7.30)$$

If one formally analytically continues N_f to the real numbers, as discussed above, so that N_f can decrease all the way to N_ℓ in the NACP, then the maximal value of γ_{B,A_2} is

$$\gamma_{B,A_2,Nce} = \frac{N_c}{4} \quad \text{at } N_f = N_{\ell,A_2} . \quad (7.31)$$

If $N_c = 4$, then at $N_\ell = 1$, $\gamma_{B,A_2,Nce}$ reaches a maximum value of 1, saturating the unitarity upper bound $\gamma_{B,A_2,Nce} \le 1$ from conformal invariance. For even $N_c \ge 6$, the maximum value of $\gamma_{B,A_2,Nce}$ as N_f formally decreases to N_ℓ does not saturate the unitarity upper bound, since $N_c/4 < (N_c/2) - 1$ for $N_c \ge 6$. As $N_c \to \infty$ through even values, the ratio of the maximum value of $\gamma_{B,A_2,Nce}$ evaluated at the formal (non-integral) value of N_ℓ divided by the unitarity upper bound from conformal invariance approaches 1/2.

For odd N_c (denoted Nco), the full scaling dimension of the baryon is

$$D_{B,A_2,Nco} = D_{\tilde{B},A_2,Nco} = \frac{3N_c}{2}\Big[1 - \frac{N_c}{N_f(N_c-2)}\Big] , \quad (7.32)$$

so the corresponding anomalous dimension is

$$\gamma_{B,A_2,Nco} = \gamma_{\tilde{B},A_2,Nco} = \frac{N_c}{2}\Big[\frac{3N_c}{N_f(N_c-2)} - 1\Big] . \quad (7.33)$$

We plot γ_B for $N_c = 5$ in Fig. 9.

The unitarity constraint from conformal invariance is again $D_{B,A_2} \ge 1$, and since $D_{B,A_2,Nco} = (N_c/2) - \gamma_{B,A_2,Nco}$, this implies the upper bound

$$\gamma_{B,A_2,Nco} < N_c - 1 . \quad (7.34)$$

With the same analytic continuation as above,

$$\gamma_{N,A_2,Nco} = \frac{N_c}{2} \quad \text{at } N_f = N_{\ell,A_2} . \quad (7.35)$$

Even with an analytic continuation of N_f from the integers to the real numbers so that N_f can actually reach down to N_{ℓ,A_2}, this never saturates the unitarity upper bound from conformal invariance at the lower end of the NACP, since $(N_c/2) < N_c - 1$ for $N_c \ge 3$.

VIII. SCHEME-INDEPENDENT CALCULATION AND ANALYSIS OF β'_{IR}

A. General

In this section we study the scheme-independent expansion for the derivative of the beta function evaluated at the superconformal IR fixed point, denoted β'_{IR}, in the non-Abelian Coulomb phase. Specifically, we present our calculations of the scheme-independent coefficients d_2 and d_3 for general G and $\mathcal{R}$ and analyze the properties of d_4 and β'_{IR} calculated to $O(\Delta_f^4)$ for the case $G = \mathrm{SU}(N_c)$ and $\mathcal{R} = F$. For this special case $G = \mathrm{SU}(N_c)$ and $\mathcal{R} = F$, quantities equivalent to the d_j were calculated in [4] for $2 \le j \le 4$. Our new contributions here are calculations of d_2 and d_3 for general G and $\mathcal{R}$ and also a different analysis of β'_{IR} in the lower part of the non-Abelian Coulomb phase. One of the reasons for interest in this derivative is that β'_{IR} is equivalent [34] to the anomalous dimension of the Konishi supercurrent [35].

B. Calculation via Series Expansion in α

It is useful first to review the calculation of β'_{IR} in [7, 8] using a conventional series expansion in powers of α up to three-loop order. In general, from Eq. (2.2), it follows that

$$\beta' = -2\sum_{\ell=1}^{\infty}(\ell+1)b_\ell\, a^\ell \ . \tag{8.1}$$

where $a = \alpha/(4\pi) = g^2/(16\pi^2)$. Evaluating the n-loop truncation of this series at the IR zero in the n-loop beta function, $\alpha_{IR,n\ell}$ yields the n-loop value of the derivative, $\beta'_{IR,n\ell}$. Since b_1 and b_2 are scheme-independent, this is also true of $\beta'_{IR,2\ell}$, for which one finds [7]

$$\begin{aligned}\beta'_{IR,2\ell} &= -\frac{2b_1^2}{b_2} \\ &= \frac{(3C_A - 2T_fN_f)^2}{2(C_A+2C_f)T_fN_f - 3C_A^2} \ . \end{aligned}\tag{8.2}$$

For general G and $\mathcal{R}$, $\beta'_{IR,2\ell}$ increases monotonically as N_f decreases from N_u in the NACP. At the three-loop level, the condition for the IR zero is the quadratic equation $b_1 + b_2a + b_3a^2 = 0$, whence, $a^2 = -(b_1+b_2a)/b_3$. Substituting this into Eq. (8.1), one has

$$\beta'_{IR,3\ell} = 2a_{IR,3\ell}(2b_1 + b_2a_{IR,3\ell}) \ , \tag{8.3}$$

where $a_{IR,3\ell}$ is the physical root of the quadratic equation above. The three-loop calculation in [7] used the value of b_3 in the $\overline{\rm DR}$ scheme [31]. As mentioned above, we have found that the four-loop beta function does not exhibit a physical IR zero over a substantial interval of N_f in the NACP. That is, extracting the prefactor of a^2 in $\beta_{4\ell}$, we have found that the cubic equation $b_1+b_2a+b_3a^2+b_4a^3 = 0$ has no real positive zero in this range of N_f. We will discuss this further in the subsection on the LNN limit.

In the special case $G = {\rm SU}(N_c)$ and $\mathcal{R} = F$, Eq. (8.2) reduces to

$$\beta'_{IR,2\ell} = \frac{N_c(3N_c - N_f)^2}{(2N_c^2-1)N_f - 3N_c^2} \ . \tag{8.4}$$

To write an expression for the three-loop derivative, $\beta'_{IR,3\ell}$, it is convenient first to define two auxiliary polynomials in N_c and N_f:

$$\begin{aligned} D_s &= -21N_c^5 + 21N_c^4N_f - 4N_c^3N_f^2 - 9N_c^2N_f \\ &+ 3N_cN_f^2 - 2N_f \end{aligned}\tag{8.5}$$

and

$$\begin{aligned} C_s &= -54N_c^6 + 72N_c^5N_f - 29N_c^4N_f^2 + N_c^3N_f(4N_f^2-21) \\ &+ 14N_c^2N_f^2 - 3N_cN_f(N_f^2+2) + 3N_f^2 \ . \end{aligned}\tag{8.6}$$

Then

$$\beta'_{IR,3\ell} = \frac{N_c}{D_s}\Big[3N_c^3 - 2N_c^2N_f + N_f + \sqrt{C_s}\ \Big] \ . \tag{8.7}$$

We will discuss these n-loop calculations further in the LNN limit (9.1) below.

C. Calculation via Series Expansion in Δ_f

Proceeding with the scheme-independent Δ_f expansion, we calculate, for general G and $\mathcal{R}$,

$$d_2 = \frac{2T_f^2}{3C_AC_f} \tag{8.8}$$

and

$$d_3 = \frac{2T_f^3(C_A+2C_f)}{(3C_AC_f)^2} \ . \tag{8.9}$$

To our knowledge, these results are new. If $G = {\rm SU}(N_c)$ and $\mathcal{R} = F$, then these take the form

$${\rm SU}(N_c), \ R = F: \quad d_{2,F} = \frac{1}{3(N_c^2-1)} \tag{8.10}$$

and

$$d_{3,F} = \frac{2N_c^2-1}{9N_c(N_c^2-1)^2} \ . \tag{8.11}$$

For this case of $G = {\rm SU}(N_c)$ and $\mathcal{R} = F$, the next-higher-order coefficient is

$$d_{4,F} = \frac{(N_c^4 - 2N_c^2 + 5) - 18N_c^2(N_c^2+1)\zeta_3}{108N_c^2(N_c^2-1)^3} \ , \tag{8.12}$$

where $\zeta_s = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function. These results for d_j, $2 \le j \le 4$ for $G = {\rm SU}(N_c)$ and $\mathcal{R} = F$ agree with equivalent quantities given in [4]. From these results for d_j, $2 \le j \le 4$, it is evident that the coefficients d_j in expansion (2.29) for β'_{IR} does not have the form of a geometric series. This is in contrast to our theorem above and the resultant Eq. (3.3) for the coefficient $f_{\Phi_{\rm prod},j}$ in expansion of the anomalous dimension of a composite chiral superfield $\Phi_{\rm prod}$ in powers of Δ_f, which showed that the latter series is a geometric series. This is completely consistent with our theorem, since the Konishi supercurrent is not a (composite) chiral superfield.

The coefficients d_2 and d_3 are manifestly positive for any G and R. We find that d_4 is negative for all physical $N_c \ge 2$. These are qualitatively the same results that we found in [13] for non-supersymmetric theories, namely that for arbitrary G and R, d_2 and d_3 are positive and in the case $G = {\rm SU}(N_c)$ and $\mathcal{R} = F$, d_4 is negative for all $N_c \ge 2$.

The perfect agreement that we have found between the κ_j that we have calculated and the exact result (4.4) suggests that the same agreement could hold for the d_j with $1 \le j \le 3$ that we have calculated. That is, these should also agree with the d_j coefficients obtained from the expansion of the exact β'_{IR} as a series in powers of Δ_f as expressed in Eq. (2.29). The only difference is that in contrast to γ_M, one does not have an exact closed-form expression for β'_{IR} with which to compare in this $\mathcal{N} = 1$ supersymmetric gauge theory.

In Table IV we list the (scheme-independent) values that we calculate for β'_{IR,F,Δ_f^p} with $2 \le p \le 4$ for the illustrative gauge groups $G =$ SU(2), SU(3), and SU(4), as functions of N_f in the respective non-Abelian Coulomb phase intervals given in Eq. (2.10). Numerically,

$$\text{SU}(2): \quad \beta'_{IR,F,\Delta_f^4} = \Delta_f^2 \Big[0.11111 + (4.3210 \times 10^{-2})\Delta_f - (3.5986 \times 10^{-2})\Delta_f^2 \Big] \tag{8.13}$$

$$\text{SU}(3): \quad \beta'_{IR,F,\Delta_f^4} = \Delta_f^2 \Big[4.1667 \times 10^{-2} + (0.98380 \times 10^{-2})\Delta_f - (3.7763 \times 10^{-3})\Delta_f^2 \Big] \tag{8.14}$$

and

$$\text{SU}(4): \quad \beta'_{IR,F,\Delta_f^4} = \Delta_f^2 \Big[2.2222 \times 10^{-2} + (3.8272 \times 10^{-3})\Delta_f - (0.96987 \times 10^{-3})\Delta_f^2 \Big] . \tag{8.15}$$

where the numerical coefficients are listed to the given floating-point accuracy.

In Figs. 10-12 we show plots of β'_{IR,F,Δ_f^p} with $2 \le p \le 4$ for these three theories for N_f in the respective non-Abelian Coulomb phase interval, $(3/2)N_c < N_f < 3N_c$. (The plots also show the behavior for N_f values slightly below the lower end of the NACP.)

We next address the question of how well, for a given G, $\mathcal{R}$, and N_f, the Δ_f expansion for β'_{IR} converges in this $\mathcal{N} = 1$ supersymmetric gauge theory. We had carried out a similar analysis for the Δ_f expansions for γ_M and β'_{IR} in our previous work [11]-[15]. The Δ_f expansion is a series expansion about $\Delta_f = 0$, i.e., $N_f = N_u$, at the upper end of the non-Abelian Coulomb phase. As Δ_f increases, i.e., as N_f decreases below N_u, one needs progressively more terms in this expansion to obtain an accurate estimate of a given quantity. In general, if $f(z)$ is an analytic function at $z = 0$, then it has a Taylor series expansion

$$f(z) = \sum_{j=1}^{\infty} f_j z^j . \tag{8.16}$$

The radius of convergence of this series, z_c, can be determined by the ratio test as

$$z_c = \lim_{j \to \infty} \frac{|f_{j-1}|}{|f_j|} . \tag{8.17}$$

With the Δ_f expansion for β'_{IR} considered as a Taylor series expansion, one could, in principle, calculate the radius of convergence (cv), $|\Delta_{f,\text{cnv}}|$ as

$$\Delta_{f,\text{cnv}} = \lim_{j \to \infty} \frac{|d_j|}{|d_{j+1}|} . \tag{8.18}$$

Clearly, it is not possible to apply this test precisely here for β'_{IR} as a series in powers of Δ_f, since one does not know the d_j for $j \to \infty$. Nevertheless, one can obtain a rough estimate of the radius of convergence by calculating the ratios of adjacent coefficients for the first few d_j. We define the estimate of the radius of convergence given by the ratio $|d_j/d_{j+1}|$ as

$$\Delta_{f,\text{cnv},(j,j+1)} = \frac{|d_j|}{|d_{j+1}|} . \tag{8.19}$$

Correspondingly, for a given G and R, the minimum value of N_f to which the small-Δ_f expansion would be estimated to be convergent (denoted mc for "minimum (N_f) for convergence") is

$$N_{f,\text{mc},(j,j+1)} = N_u - \Delta_{f,\text{cnv},(j,j+1)} , \tag{8.20}$$

where N_u was given in Eq. (2.6). For general G and $\mathcal{R}$, we have

$$\Delta_{f,\text{cnv},(j,j+1)} = \frac{d_2}{d_3} = \frac{3C_A C_f}{T_f(C_A + 2C_f)} , \tag{8.21}$$

and hence

$$N_{f,\text{mc},(2,3)} = \frac{3C_A^2}{2T_f(C_A + 2C_f)} . \tag{8.22}$$

This may lie above or below the lower end of the non-Abelian Coulomb phase at N_ℓ, as determined by the difference

$$N_{f,mc,(2,3)} - N_\ell = \frac{3C_A(C_A - 2C_f)}{4T_f(C_A + 2C_f)} . \tag{8.23}$$

For example, for $G = \mathrm{SU}(N_c)$, this difference is positive for the fundamental representation, but negative for the adjoint representation.

We now focus on the case of main interest here, namely $G = \mathrm{SU}(N_c)$ and $\mathcal{R} = F$. For this case,

$$\frac{d_{2,F}}{d_{3,F}} = \frac{3N_c(N_c^2-1)}{2N_c^2-1} \ , \tag{8.24}$$

so that

$$N_{f,\mathrm{mc},(2,3)} = \frac{3N_c^3}{2N_c^2-1} \ . \tag{8.25}$$

Parenthetically, we observe that this difference is equal to the special case of $N_{f,b2z}$ (given in general in Eq. (2.13)) for $G = \mathrm{SU}(N_c)$ and $\mathcal{R} = F$. The value $N_{f,mc,(2,3)}$ lies above the lower end of the non-Abelian Coulomb phase, as is evident from the difference

$$N_{f,\mathrm{mc},(2,3)} - N_\ell = \frac{3N_c^2}{2(2N_c^2-1)} \ . \tag{8.26}$$

As $N_c \to \infty$, this difference approaches zero.

For the ratio of the next higher-order coefficients, we find

$$\frac{d_{3,F}}{|d_{4,F}|} = \frac{12N_c(N_c^2-1)(2N_c^2-1)}{18N_c^2(N_c^2+1)\zeta_3 - (N_c^4-2N_c^2+5)} \ , \tag{8.27}$$

so

$$N_{f,\mathrm{mc},(3,4)} = \frac{3N_c[18N_c^2(N_c^2+1)\zeta_3 - 9N_c^4 + 14N_c^2 - 9]}{18N_c^2(N_c^2+1)\zeta_3 - (N_c^4-2N_c^2+5)} \ . \tag{8.28}$$

This value lies above the lower end of the NACP, as is evident from the difference

$$N_{f,\mathrm{mc},(3,4)} - N_\ell = \frac{3N_c[18N_c^2(N_c^2+1)\zeta_3 - 17N_c^4 + 26N_c^2 - 13]}{18N_c^2(N_c^2+1)\zeta_3 - (N_c^4-2N_c^2+5)} \ . \tag{8.29}$$

In Table V we list values of N_ℓ, N_u, $N_{f,\mathrm{mc},(2,3)}$, $N_{f,\mathrm{mc},(2,3)} - N_\ell$, $N_{f,\mathrm{mc},(3,4)}$, and $N_{f,\mathrm{mc},(3,4)} - N_\ell$ for the illustrative cases $N_c = 2, 3, 4$. Thus, our analysis of the first two ratios of coefficients in the small-Δ_f series expansion for β'_{IR} suggests that the small-Δ_f expansion for β'_{IR} may be reliable over a substantial portion of the non-Abelian Coulomb phase, including, in particular, the upper and middle parts. In general, one would not expect the small-Δ_f expansion to apply reliably for small values N_f, where the properties of the theory are qualitatively different from the properties in the non-Abelian Coulomb phase.

These results on the convergence of the small-Δ_f expansion (2.29) for β'_{IR} may be compared with our results for the convergence of the corresponding expansion (2.31) for γ_M. As recalled above, we found from our calculation of the coefficients κ_j in the latter expansion that they agreed perfectly with the Taylor series expansion of the exact result (4.4). This Taylor series expansion of (4.4) converges throughout the entire non-Abelian Coulomb phase. Superficially, from the analysis of the coefficients d_j with $j = 2, 3, 4$ in the small-Δ_f series expansion of β'_{IR}, one might infer that this series expansion might not converge as rapidly as the small-Δ_f expansion for γ_M [4]. However, one would need more terms to get a better estimate of the actual region of convergence of the series expansion of β'_{IR} in powers of Δ_f. Especially in view of our proof above that the series expansion in powers of Δ_f of the anomalous dimension $\gamma_{\Phi_{\rm prod}}$ converges throughout the entirety of the non-Abelian Coulomb phase, we believe that it is plausible that the same is true of the corresponding series for β'_{IR}.

For general G and $\mathcal{R}$, since d_2 and d_3 are positive, β'_{IR} increases (initially quadratically) from 0 as Δ_f increases from 0, i.e., as N_f decreases below its upper bound from asymptotic freedom, N_u. In the class of theories with $G = \mathrm{SU}(N_c)$ and $\mathcal{R} = F$, we have calculated the next higher-order coefficient, $d_{4,F}$ and have shown that it is negative for all physical N_c. It is of interest to investigate the consequences of the fact that $d_{4,F}$ is negative, bearing in mind the cautionary remarks concerning the range in N_f in which the small-Δ_f may be reasonably reliable. Because $d_{4,F}$ is negative, as Δ increases from 0, i.e., as N_f decreases from N_u, the $d_4\Delta_f^4$ term in β'_{IR} eventually stops the initial increase in β'_{IR,Δ_f^4} and, for smaller N_f, causes β'_{IR,Δ_f^4} to decrease. If one were to use the Δ_f expansion for sufficiently small values of N_f, then the series for β'_{IR} calculated to $O(\Delta_f^4)$, i.e., β'_{IR,Δ_f^4}, would pass through zero to negative values. We first determine the value of Δ_f, or equivalently, N_f, at which β'_{IR,Δ_f^4} vanishes. The condition that $\beta'_{IR,\Delta_f^4} = 0$ is the equation

$$\Delta_f^2(d_2 + d_3\,\Delta_f + d_4\,\Delta_f^2) = 0 \ . \tag{8.30}$$

Aside from the solution $\Delta_f = 0$, i.e., $N_f = N_u$, this equation has two solutions, corresponding to the roots of the quadratic factor. Of these, we denote the relevant one as $\Delta_0 = N_u - N_{f,0}$. We calculate

$$N_{f,0} = \frac{3N_c\Big[N_c^4(-5+18\zeta_3) + 2N_c^2(4+9\zeta_3) - 7 - 2(N_c^2-1)\sqrt{S_0}}{N_c^4(18\zeta_3-1) + 2N_c^2(1+9\zeta_3) - 5} \ , \tag{8.31}$$

where

$$S_0 = 3N_c^4(1+6\zeta_3) + 2N_c^2(-1+9\zeta_3) - 4 \ . \qquad (8.32)$$

(The other root of the quadratic factor, with the opposite sign in front of the square root, is greater than N_u and hence is not relevant here, since we restrict $N_f < N_u$ for asymptotic freedom.) Numerically, for the illustrative values $N_c = 2,\ 3,\ 4$, our expression for $N_{f,0}$ (understood to be continued from the positive integers to the positive real numbers) takes the respective values 3.5427, 4.1294, and 4.8496. In these three cases, as is evident from Table V, N_ℓ has the respective values =3, 4.5, 6, so that for $\mathcal{R} = F$ and $G = {\rm SU}(2)$, $N_{f,0} > N_\ell$, while for SU(3) and SU(4), $N_{f,0} < N_\ell$.

Using electric-magnetic duality, it has been concluded that for $G = {\rm SU}(N_c)$ and $\mathcal{R} = F$, β'_{IR} vanishes quadratically at the lower end of the non-Abelian Coulomb phase at $N_f = N_\ell = (3/2)N_c$ [34]:

$$\beta'_{IR} = \frac{28}{3}\Big(\frac{N_f}{N_c} - \frac{3}{2}\Big)^2 \quad \text{as } N_f \searrow \frac{3N_c}{2} \ . \qquad (8.33)$$

Given the fact that our Δ_f expansion starts from the other (i.e., the upper) end of the non-Abelian Coulomb phase, we would not expect our calculations of β'_{IR} to $O(\Delta_f^4)$ for this theory to precisely reproduce this behavior at $N_f = (3/2)N_c$. Taking this into account, our numerical results on $N_{f,0}$ are consistent with the behavior in (8.33). It should be noted that the three values listed above for $N_{f,0}$ actually lie below the minimum values where where our estimates indicate that the small-Δ_f series is reliable, namely the values 4.8, 6.4, and 8.05 for $N_c = 2,\ 3,\ 4$, respectively, as listed in Table V. A general statement is that our calculations of series expansions for β'_{IR} in both the nonsupersymmetric gauge theory [13–15] and the results present here for the supersymmetric gauge theory show qualitatively quite different behavior than we have found for both γ_M and γ_B. In the latter two cases, all of the coefficients in the small-Δ_f expansion are positive, leading to the two monotonicity theorems mentioned above.

IX. RESULTS IN THE LIMIT OF LARGE N_c AND N_f WITH N_f/N_c FIXED

A. General

For this class of theories with $G = {\rm SU}(N_c)$ and $\mathcal{R} = F$, an interesting limit is

$$LNN: \quad N_c \to \infty \ , \quad N_f \to \infty$$
$$\text{with } r \equiv \frac{N_f}{N_c} \text{ fixed and finite}$$
$$\text{and } \ \xi(\mu) \equiv \alpha(\mu) N_c \text{ is a finite function of } \mu \ . \qquad (9.1)$$

We will use the symbol $\lim_{LNN}$ for this limit, where "LNN" stands for "large N_c and N_f" with the constraints in Eq. (9.1) imposed. This is sometimes called the 't Hooft-Veneziano limit.

We define the following quantities in this limit:

$$\xi = 4\pi x = \lim_{LNN} \alpha N_c \ , \qquad (9.2)$$

$$r_u = \lim_{LNN} \frac{N_u}{N_c} \ , \qquad (9.3)$$

and

$$r_\ell = \lim_{LNN} \frac{N_\ell}{N_c} \ , \qquad (9.4)$$

with values

$$r_u = 3 \ , \qquad r_\ell = \frac{3}{2} \ . \qquad (9.5)$$

These quantities are listed in Table VI. Thus, the non-Abelian Coulomb phase occurs for r in the interval

$$\text{LNN, NACP}: \quad \frac{3}{2} < r < 3 \ . \qquad (9.6)$$

We define the scaled scheme-independent expansion parameter for the LNN limit

$$\Delta_r \equiv \frac{\Delta_f}{N_c} = r_u - r = 3 - r \ . \qquad (9.7)$$

As r decreases from r_u to r_ℓ in the non-Abelian Coulomb phase, Δ_r increases from 0 to a maximal value

$$(\Delta_r)_{\max} = r_u - r_\ell = \frac{3}{2} \quad \text{for } r \in \text{NACP} \ . \qquad (9.8)$$

B. γ_M in the LNN Limit

For the analysis of γ_M at the superconformal IRFP, we define rescaled coefficients $\hat\kappa_{j,F}$

$$\hat\kappa_{j,F} \equiv \lim_{N_c \to \infty} N_c^j \, \kappa_{j,F} \qquad (9.9)$$

that are finite in this LNN limit. The anomalous dimension γ_{IR} is also finite in this limit and is given by

$$\mathcal{R} = F: \quad \lim_{LNN} \gamma_{M,LNN} = \sum_{j=1}^\infty \hat\kappa_{j,F} \Delta_r^j \ . \qquad (9.10)$$

In this LNN limit, the exact result for $\gamma_{M,LNN}$ (4.4) takes the form

$$\gamma_{M,LNN} = \frac{\frac{\Delta_r}{3}}{1 - \frac{\Delta_r}{3}} \ , \qquad (9.11)$$

so that

$$\hat\kappa_{j,F} = \frac{1}{3} \quad \forall \ j \ . \qquad (9.12)$$

C. Rescaled γ_B in the LNN Limit

To construct a rescaled anomalous dimension at the superconformal IRFP that is finite in the LNN limit, we define

$$\hat{\gamma}_B \equiv \lim_{LNN} \frac{\gamma_B}{N_c} \ , \tag{9.13}$$

and similarly with $\hat{\gamma}_{\tilde{B}} = \hat{\gamma}_B$. By construction, these rescaled baryon anomalous dimensions are finite in the LNN limit and have the common value

$$\lim_{LNN} \hat{\gamma}_B = \frac{1}{2}\Big(\frac{3}{r} - 1\Big) \ . \tag{9.14}$$

D. β'_{IR} in the LNN Limit

The rescaled beta function that is finite and nontrivial in the LNN limit is

$$\beta_\xi \equiv \frac{d\xi}{dt} = \lim_{LNN} \beta_\alpha N_c \ , \tag{9.15}$$

with the series expansion

$$\beta_\xi \equiv \frac{d\xi}{dt} = -8\pi x \sum_{\ell=1}^{\infty} \hat{b}_\ell x^\ell = -2\xi \sum_{\ell=1}^{\infty} \tilde{b}_\ell \xi^\ell \ , \tag{9.16}$$

where

$$\hat{b}_\ell = \lim_{LNN} \frac{b_\ell}{N_c^\ell} \ , \quad \tilde{b}_\ell = \lim_{LNN} \frac{\bar{b}_\ell}{N_c^\ell} \ . \tag{9.17}$$

The first two rescaled coefficients of the beta function, which are scheme-independent, are

$$\hat{b}_1 = 3 - r \tag{9.18}$$

and

$$\hat{b}_2 = 2(3 - 2r) \ . \tag{9.19}$$

In the $\overline{\rm DR}$ scheme,

$$\hat{b}_3 = 21 - 21r + 4r^2 \tag{9.20}$$

and

$$\hat{b}_4 = 2[51 - 66r + 3(7 + 2\zeta_3)r^2] \ . \tag{9.21}$$

In the LNN limit, one has the scheme-independent two-loop result

$$x_{IR,2\ell} = -\frac{\hat{b}_1}{\hat{b}_2} = \frac{3-r}{2(2r-3)} \ . \tag{9.22}$$

At the three-loop level, $x_{IR,3\ell}$ is the physical root among the two roots of the quadratic equation $\hat{b}_1 + \hat{b}_2 x + \hat{b}_3 x^2 = 0$. It is convenient to define two auxiliary polynomials:

$$C_r = \lim_{LNN} \frac{C_s}{N_c^6} = -54 + 72r - 29r^2 + 4r^3 \tag{9.23}$$

and

$$D_r = \lim_{LNN} \frac{D_s}{N_c^5} = -21 + 21r - 4r^2 \ . \tag{9.24}$$

Then

$$x_{IR,3\ell} = \frac{2[-(2r-3) + \sqrt{C_s}\ ]}{D_s} \ . \tag{9.25}$$

These inputs were used to calculate β'_{IR} in the LNN limit [7, 8]. At two-loop order, one has the scheme-independent result,

$$\beta'_{\xi,IR,2\ell} = \frac{(3-r)^2}{2r-3} \ . \tag{9.26}$$

At the three-loop order [7, 8]

$$\begin{aligned} \beta'_{\xi,IR,3\ell} &= 2x_{IR,3\ell}(2b_1 + b_2 x_{IR,3\ell}) \\ &= 2(3-r)\Big[-(2r-3) + \sqrt{C_r}\ \Big] \times \\ &\times \ \left[1 + \frac{2\Big[-(2r-3) + \sqrt{C_r}\ \Big]}{D_s} \right] \ . \end{aligned} \tag{9.27}$$

We list the values of $\beta'_{IR,2\ell}$ and $\beta'_{IR,3\ell}$ in Table VII.

We find that the four-loop beta function does not exhibit a physical (i.e., real, positive) IR zero over a substantial portion of the NACP interval $3/2 < r < 3$. Specifically, extracting the prefactor proportional to x^2 in $\beta_{\xi,4\ell}$, we find that, as r decreases from its upper bound of $r = 3$ in the NACP, the equation $\hat{b}_1 + \hat{b}_2 x + \hat{b}_3 x^2 + \hat{b}_4 x^3 = 0$ ceases to exhibit a physical zero as r decreases below $r_0 = 2.6165$. We recall that we found that although the n-loop beta function had a physical IR zero for $n = 2$, 3, and 4 loops in the corresponding nonsupersymmetric SU(N_c) theory with N_f fermions with $\mathcal{R} = F$, this was not the case at the five-loop level [33], and in the LNN limit, as r decreased below its upper limit of 5.5, the five-loop beta beta function ceased to exhibit a physical IR zero as r decreased through the value $r_{cx} = 4.3226$ (as given in Eq. (5.3) of [15]). Thus, this complication appears at one loop lower (i.e. at the four-loop level) in the present supersymmetric theory, as compared with the case of the nonsupersymmetric theory with the same G and R. This shows again the advantage of the scheme-independent expansion method, since it does circumvents the explicit extraction of $\alpha_{IR,n\ell}$ (here, $x_{n,\ell}$ in the LNN limit) in order to calculate values of physical quantities at the IRFP.

For the scheme-independent expansion, in addition to the rescaled quantity Δ_r defined in Eq. (9.7), we define the rescaled coefficient

$$\hat{d}_{j,F} = \lim_{LNN} N_c^j \, d_{j,F} \ , \tag{9.28}$$

which is finite. Then each term

$$\lim_{LNN} d_{j,F}\Delta_r^j = \lim_{LNN} (N_c^j d_{j,F})\Big(\frac{\Delta_r}{N_c}\Big)^j = \hat d_{j,F}\Delta_r^j \ . \quad (9.29)$$

is finite in this limit. Thus, writing $\lim_{LNN}\beta'_{IR}$ as $\beta'_{IR,LNN}$ for this $\mathcal{R}=F$ case, we have

$$\beta'_{IR,LNN} = \sum_{j=1}^{\infty} \hat d_{j,F}\Delta_r^j \ . \quad (9.30)$$

From our results (8.10), (8.11), and (8.12), it follows that

$$\hat d_{2,F} = \frac{1}{3} \ , \quad (9.31)$$

$$\hat d_{3,F} = \frac{2}{9} = 0.22222 \ , \quad (9.32)$$

and

$$\hat d_{4,F} = -\frac{(18\zeta_3-1)}{108} = -0.19108 \ . \quad (9.33)$$

Thus, in this LNN limit, to $O(\Delta_r^4)$,

$$\beta'_{IR,LNN,\Delta_r^4} = \Delta_r^2\Big[\frac{1}{3} + \frac{2}{9}\Delta_r - \Big(\frac{18\zeta_3-1}{108}\Big)\Delta_r^2\Big] \ . \quad (9.34)$$

In Table VII we list the (scheme-independent) values that we calculate for $\beta'_{IR,LNN,\Delta_r^p}$ and in Fig. 13, we plot $\beta'_{IR,LNN,\Delta_r^p}$ with $2 \le p \le 4$, as functions of r in the non-Abelian Coulomb phase interval $3/2 < r < 3$. (The plot also shows the behavior slightly below the lower end of the NACP.)

To obtain a rough estimate of the interval in r in which this small-Δ_r expansion is reliable, we follow the same procedure as before for finite N_c and N_f. Analogously to Eqs. (8.19) and (8.20), we define

$$(\Delta_r)_{\rm cnv,(j,j+1)} = \frac{|\hat d_j|}{|\hat d_{j+1}|} \quad (9.35)$$

and

$$r_{\rm mc,(j,j+1)} = r_u - \Delta_{r,{\rm cnv},(j,j+1)} \ . \quad (9.36)$$

We calculate

$$(\Delta_r)_{\rm cnv,(2,3)} = \frac{3}{2} \quad (9.37)$$

and

$$(\Delta_r)_{cnv,(3,4)} = \frac{24}{18\zeta_3-1} = 1.16296 \ , \quad (9.38)$$

so that

$$r_{\rm mc,(2,3)} = \frac{3}{2} \quad (9.39)$$

and

$$r_{\rm mc,(3,4)} = \frac{27(2\zeta_3-1)}{18\zeta_3-1} = 1.8370 \quad (9.40)$$

Since the lower end of the non-Abelian Coulomb phase occurs at $r_\ell = 3/2$, this analysis suggests that the small-Δ_r expansion may be reasonably reliable for a substantial part of this phase, extending down from $r = 3$ to around $r \sim 1.8$.

In the present LNN limit, the condition that $\beta'_{IR,LNN} = 0$ is satisfied at $\Delta_r = 0$, i.e., $r = 3$, and at the relevant solution of the quadratic factor in Eq. (9.34). We define

$$\Delta_{r,0} = 3 - r_0 \ , \quad (9.41)$$

with

$$r_0 = \lim_{LNN} \frac{N_{f,0}}{N_c} \ . \quad (9.42)$$

We calculate

$$r_0 = \frac{3\Big[18\zeta_3 - 5 - 2\sqrt{3(1+6\zeta_3)}\ \Big]}{18\zeta_3-1} = 0.975415 \ , \quad (9.43)$$

where the numerical value is given to the indicated floating-point accuracy. (The other root of the quadratic, with the opposite sign in front of the square root, is $r = 3.861627$, which is greater than $r_u = 3$ and hence is not relevant.) Evidently, r_0 is less $r_\ell = 1.5$, i.e., it lies below the lower end of the non-Abelian Coulomb phase and well below the region in r where the small-Δ_r expansion is expected to be reliable, based on our analysis of ratios of $\hat d_j$ above.

In the LNN limit, the result (8.33) from [34] is

$$\beta'_{IR} = \frac{28}{3}\Big(r - \frac{3}{2}\Big)^2 \quad \text{as } r \searrow \frac{3}{2} \ . \quad (9.44)$$

Analogously to our discussion above for finite N_c and N_f, here in the LNN limit, given that the Δ_r series expansion for β'_{IR} starts from the other end of the NACP at $r = 3$, i.e., $\Delta_r = 0$, we would not anticipate that our series expansion to $O(\Delta_r^4)$ would closely reproduce this property of β'_{IR}. With the calculation of β'_{IR} to $O(\Delta_r^4)$, we may observe at least that as r decreases toward the lower end of the NACP, β'_{IR} curves over and decreases, approaching the zero at r_0. As was true of our analysis for finite N_c and N_f, given the limited order in the Δ_r series expansion to which we have calculated β'_{IR} and our estimate of the region over which this expansion may be used reliably, we consider that our results are consistent with the behavior (9.44).

In view of (9.44), it is of interest to study a structural form for $\beta'_{IR,LNN}$ that incorporates a double zero at $r = 3/2$, via the factor $[1-(2/3)\Delta_r]^2$ as well as the double zero at $r = 3$, as embodied in the factor $(3-r)^2 = \Delta_r^2$. We thus write

$$\beta'_{IR} = \Delta_r^2\,[1-(2/3)\Delta_r]^2\,\Big[\hat h_2 + \hat h_3\Delta_r + \hat h_4\Delta_r^2 + O(\Delta_r^3)\ \Big] \quad (9.45)$$

The coefficients $\hat{h}_j$ are related to the $\hat{d}_j$ that we have calculated as follows:

$$\hat{h}_2 = \hat{d}_2 = \frac{1}{3} \tag{9.46}$$

$$\hat{h}_3 = \hat{d}_3 + \frac{4}{3}\hat{h}_2 = \frac{2}{3} \tag{9.47}$$

$$\begin{aligned} \hat{h}_4 &= \hat{d}_4 + \frac{4}{3}\hat{h}_3 - \frac{4}{9}\hat{h}_2 \\ &= \frac{9-2\zeta_3}{12} \ . \end{aligned} \tag{9.48}$$

Calculations to higher order in Δ_r would be necessary in order to reproduce the coefficient (28/3) in Eq. (9.44).

E. Padé Approximants for β'_{IR} in the LNN Limit

It is also of interest to calculate and analyze Padé approximants. For this purpose, it is convenient to define a reduced (red.) function normalized to be equal to unity at $\Delta_r = 0$:

$$\beta'_{IR,LNN,red.} = \frac{\beta'_{IR,LNN}}{\hat{d}_2\Delta_r^2} = 1 + \frac{1}{\hat{d}_2}\sum_{j=3}^{\infty} \hat{d}_j \Delta_r^{j-2} \ . \tag{9.49}$$

Thus, from $\beta'_{IR,LNN,\Delta_r^5}$, we have

$$\begin{aligned} \beta'_{IR,LNN,red.} &= 1 + \frac{2}{3}\Delta_r - \frac{(18\zeta_3-1)}{36}\Delta_r^2 + O(\Delta_r^3) \\ &= 1 + 0.66667\Delta_r - 0.57325\Delta_r^2 + O(\Delta_r^3) \ . \end{aligned} \tag{9.50}$$

We recall that the $[p,q]$ Padé approximant to a finite Taylor series $f(x) = 1 + \sum_{n=1}^{m} x^n$ is the rational function

$$f_{[p,q]} = \frac{1 + \sum_{j=1}^{p} n_j x^j}{1 + \sum_{k=1}^{q} d_k\, x^k} \tag{9.51}$$

with $p+q = m$, where the coefficients n_j and d_k are independent of x. Thus, in the present case, with $x = \Delta_r$ and $f(\Delta_r) = \beta'_{IR,LNN,red.}$, calculated to $O(\Delta_r^2)$ (corresponding to the calculation of $\beta'_{IR,LNN}$ to $O(\Delta_r^4)$), it follows that, aside from the Padé approximant [2,0], which is the function $\beta'_{IR,LNN,red.}$, itself, there are two Padé approximants to the series, namely [1,1] and [0,2]. For the first of these, we calculate

$$\beta'_{IR,LNN,red.,[1,1]} = \frac{1 + \frac{1}{8}(5+6\zeta_3)\Delta_r}{1 + \frac{1}{24}(18\zeta_3-1)\Delta_r} \ . \tag{9.52}$$

The pole in this [1,1] Padé approximant occurs at

$$(\Delta_r)_{[1,1],\mathrm{pole}} = -\frac{24}{18\zeta_3 - 1} = -1.162958 \ , \tag{9.53}$$

where the numerical value is given to the indicated floating-point accuracy. Hence, this Padé approximant converges in a disk centered at $\Delta_r = 0$ in the complex Δ_r plane of radius 1.162958. This does not cover all of the non-Abelian Coulomb phase, but does extend down to $r = 1.8370$, close to the lower boundary of the NACP at $r = 1.5$. This [1,1] Padé approximant does not have any zero in the NACP; its zero occurs at $\Delta_r = -8/(5+6\zeta_3)$, or equivalently, in terms of r, at

$$r_{[1,1],\mathrm{zero}} = \frac{23 + 18\zeta_3}{5 + 6\zeta_3} = 3.6551 \ . \tag{9.54}$$

Evidently, this zero lies above the upper end of the NACP at $r_u = 3$ (but within the radius of convergence of the approximant).

For the [0,2] Padé approximant to $\beta'_{IR,LNN,red.}$, we calculate

$$\beta'_{IR,LNN,red.,[0,2]} = \frac{1}{1 - \frac{2}{3}\Delta_r + \frac{1}{12}(5+6\zeta_3)\Delta_r^2} \ . \tag{9.55}$$

The poles of the approximant occur at the complex-conjugate points

$$(\Delta_r)_{[0,2],\mathrm{pole}} = \frac{2(2 \pm i\sqrt{11+18\zeta_3}\ )}{5+6\zeta_3} \ . \tag{9.56}$$

These have magnitude

$$|(\Delta_r)_{[0,2],\mathrm{pole}}| = \frac{2\sqrt{3}}{\sqrt{5+6\zeta_3}} = 0.991268 \ , \tag{9.57}$$

so that this [0,2] Padé approximant converges for Δ_r in the disk defined by $|\Delta_r| < 0.991268$ in the complex Δ_r plane. On the real axis, this disk of convergence extends down to $r = 2.0087$ and hence covers about 2/3 of the non-Abelian Coulomb phase interval $3/2 < r < 3$.

Although a $[p,q]$ Padé approximant only contains information about a function up to the highest-order term that has been calculated, namely the $O(\Delta_r^2)$ term in $\beta'_{IR,LNN,red.}$ (equivalently, the $O(\Delta_r^4)$ term in $\beta'_{IR,LNN}$), it is of interest to investigate the series expansion of such an approximant with $q \neq 0$, going to higher order. This can sometimes give a hint about the next-higher order term in the Taylor series expansion for the original function. In the present case, we calculate the expansions

$$\begin{aligned} \beta'_{IR,LNN,red.,[1,1]} &= 1 + \frac{2}{3}\Delta_r - \Big(\frac{18\zeta_3-1}{2^2\cdot 3^2}\Big)\Delta_r^2 \\ &+ \frac{(18\zeta_3-1)^2}{2^5\cdot 3^3}\Delta_r^3 + O(\Delta_r^4) \\ &= 1 + 0.66667\Delta_r - 0.57325\Delta_r^2 + 0.49292\Delta_r^3 + O(\Delta_r^4) \end{aligned} \tag{9.58}$$

and

$$\beta'_{IR,LNN,red.,[0,2]} = 1 + \frac{2}{3}\Delta_r - \Big(\frac{18\zeta_3-1}{2^2\cdot 3^2}\Big)\Delta_r^2$$

$$-\Big(\frac{18\zeta_3+7}{3^3}\Big)\,\Delta_r^3+O(\Delta_r^3)$$
$$= 1+0.66667\Delta_r-0.57325\Delta_r^2-1.06063\Delta_r^3+O(\Delta_r^4)\ . \tag{9.59}$$

Since the sign of the $O(\Delta_r^3)$ term of $\beta'_{IR,LNN,red.}$ (equivalent to the sign of d_5, since $\mathrm{sgn}(d_1) > 0$) predicted by the Taylor series expansion of $\beta'_{IR,LNN,red.,[1,1]}$ is positive, which is opposite to the negative-sign prediction of the Taylor series expansion of $\beta'_{IR,LNN,red.,[0,2]}$, these expansions do not give any consistent hint of the sign of $\hat{d}_5$.

In this context, one may ask what the analogous calculations would have yielded in the case of a nonsupersymmetric SU(N_c) gauge theory in the same LNN limit. In our previous analyses [14, 15] we had already gone beyond this stage and calculated the actual d_5 coefficient and thus β'_{IR} to $O(\Delta_f^5)$. However, since we do not have b_5 available in the supersymmetric theory, in contrast to the nonsupersymmetric theory, there is a motivation here to see what the Taylor series expansions of the Padé approximants to β'_{IR}, calculated to $O(\Delta_r^4)$ would have suggested about the possible sign of the next-higher-order coefficient, $\hat{d}_5$. Thus, we calculate Padé approximants to the reduced function defined in Eq. (9.49) defined to be unit-normalized at $\Delta_r = 0$. From our results in [13–15] we have

$$\begin{aligned}\beta'_{IR,LNN,red.,ns.} &= 1+\frac{26}{3\cdot 5^2}\Delta_r+\Big(\frac{366782}{3^3\cdot 5^8}-\frac{352}{3^2\cdot 5^4}\zeta_3\Big)\Delta_r^2 \\ &+ \Big(-\frac{596389102}{3^4\cdot 5^{12}}-\frac{90304}{3^3\cdot 5^7}\zeta_3+\frac{22528}{3^3\cdot 5^6}\zeta_5\Big)\Delta_r^3+O(\Delta_r^4) \\ &= 1+0.34667\Delta_r-0.040446\Delta_r^2-0.0262475\Delta_r^3+O(\Delta_r^4)\ ,\end{aligned} \tag{9.60}$$

where the subscript $ns.$ stands for "nonsupersymmetric". Our format here and below is to indicate the simple factorizations of the denominators of the various terms. In general, the numerators do not have such simple factorizations; for example, $366782 = 2\cdot 13\cdot 14107$, etc. Now we calculate the [1,1] and [0,2] Padé approximants to the truncation of $\beta'_{IR,LNN,red.,ns.}$ to $O(\Delta_r^2)$. These are

$$\beta'_{IR,LNN,red.,ns.,[1,1]} = \frac{1+\Big(\frac{34643}{3^2\cdot 5^6}+\frac{176}{3\cdot 5^2\cdot 13}\zeta_3\Big)\Delta_r}{1+\Big(-\frac{14107}{3^2\cdot 5^6}+\frac{176}{3\cdot 5^2\cdot 13}\zeta_3\Big)\Delta_r} \tag{9.61}$$

and

$$\beta'_{IR,LNN,red.,ns.,[0,2]} = \frac{1}{1-\frac{26}{3\cdot 5^2}\Delta_r+\Big(\frac{900718}{3^3\cdot 5^8}+\frac{352}{3^2\cdot 5^4}\zeta_3\Big)\Delta_r^2}\ . \tag{9.62}$$

Next, we expand these in Taylor series around $\Delta_r = 0$ to see what they predict for the $O(\Delta_r^3)$ term $(\hat{d}_5/\hat{d}_2)\Delta_r^3$ in $\beta'_{IR,LNN,red.,ns.}$, or equivalently, the $O(\Delta_r^5)$ term in $\beta'_{IR,LNN,ns.}$. We thus ascertain how these predictions compare with the actual $O(\Delta_r^5)$ term that we have calculated in $\beta'_{IR,LNN,ns.}$ in [14, 15]. We have

$$\begin{aligned}\beta'_{IR,LNN,red.,ns.,[1,1]} &= 1+\frac{26}{3\cdot 5^2}\Delta_r+\Big(\frac{366782}{3^3\cdot 5^8}-\frac{352}{3^2\cdot 5^4}\zeta_3\Big)\Delta_r^2+\Big(\frac{366782}{3^3\cdot 5^8}-\frac{352}{3^2\cdot 5^4}\zeta_3\Big)\Big(\frac{14107}{3^2\cdot 5^6}-\frac{176}{3\cdot 5^2\cdot 13}\zeta_3\Big)\Delta_r^3+O(\Delta_r^4) \\ &= 1+0.34667\Delta_r-0.040446\Delta_r^2+0.0047188\Delta_r^3+O(\Delta_r^4)\end{aligned} \tag{9.63}$$

and

$$\begin{aligned}\beta'_{IR,LNN,red.,ns.,[0,2]} &= 1+\frac{26}{3\cdot 5^2}\Delta_r+\Big(\frac{366782}{3^3\cdot 5^8}-\frac{352}{3^2\cdot 5^4}\zeta_3\Big)\Delta_r^2-\Big(\frac{13882336}{3^4\cdot 5^{10}}+\frac{18304}{3^3\cdot 5^6}\zeta_3\Big)\Delta_r^3+O(\Delta_r^4) \\ &= 1+0.34667\Delta_r-0.040446\Delta_r^2-0.0697040\Delta_r^3+O(\Delta_r^4)\ .\end{aligned} \tag{9.64}$$

The terms up to $O(\Delta_r^2)$, must, of course, coincide with the corresponding terms in $\beta'_{IR,LNN,red.,ns.}$. We find that the Taylor series expansions of $\beta'_{IR,LNN,red.,ns.,[1,1]}$ and $\beta'_{IR,LNN,red.,ns.,[0,2]}$ yield respective $O(\Delta_r^3)$ terms with signs that are opposite to, and the same as, the actual $O(\Delta_r^3)$ term in $\beta'_{IR,LNN,red.,ns.}$ that we calculated in [14, 15], shown above in Eq. (9.60). Hence, if this nonsupersymmetric case is a guide to the situation in the supersymmetric theory considered here, then our Taylor series expansion of the $\beta'_{IR,LNN,red.,[0,2]}$ in the supersymmetric theory (Eq. (9.59) may be expected to yield the correct sign of the $O(\Delta_r^3)$ term in $\beta'_{IR,LNN,red.}$, or equivalently, the $O(\Delta_r^5)$ term in $\beta'_{IR,LNN}$, i.e., the sign of $\hat{d}_5$. Thus, this predicts that the sign of $\hat{d}_5$ is negative. We emphasize, however, that this procedure is obviously nonrigorous, since these Padé approximants in the supersymmetric theory only contain information from the $\hat{d}_j$ with $j = 2,\ 3,\ 4$.

X. CONCLUSIONS

In this paper, we have presented several new results on an asymptotically free, vectorial, $\mathcal{N} = 1$ supersymmetric gauge theory with gauge group G and N_f pairs of chiral superfields in the respective representations $\mathcal{R}$ and $\bar{\mathcal{R}}$ of G, having an infrared fixed point of the renormalization group at α_{IR} in the non-Abelian Coulomb phase. At this point, the theory has superconformal invariance. We have derived exact expressions for the anomalous dimension, $\gamma_{\Phi_{\rm prod}}$, of a composite chiral superfield consisting of a (holomorphic) product of an arbitrary number of meson, baryon, and conjugate baryon superfields M, B, and $\tilde{B}$, evaluated at a superconformal IR fixed point of the renormalization group. We have proved that $\gamma_{\Phi_{\rm prod}}$, increases monotonically with decreasing N_f in the non-Abelian Coulomb phase of the theory and that scheme-independent expansions for these anomalous dimensions as powers of an N_f-dependent variable, Δ_f, exhibit monotonic and rapid convergence to the exact $\gamma_{\Phi_{\rm prod}}$ throughout this phase. However, in contrast to the behavior of γ_M, which saturates its upper bound at the lower end of the NACP, this is not, in general, the case for either γ_B or $\gamma_{\Phi_{\rm prod}}$. In particular, γ_B saturates is conformal upper bound of 1 if and only if $N_c = 2$, in which case, the operator B is equivalent to M. Finally, we have presented and analyzed scheme-independent calculations of the derivative of the beta function, β'_{IR} at the superconformal IR fixed point, up to $O(\Delta_f^3)$ for general G and $\mathcal{R}$, and have given an analysis of the properties of β'_{IR} up to $O(\Delta_f^4)$ for $G = {\rm SU}(N_c)$ and $\mathcal{R} = F$. We believe that these new results are useful additions to the knowledge of superconformal gauge theories.

Acknowledgments

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TABLE I: Matter content of an $\mathcal{N} = 1$ supersymmetric gauge theory with a general complex matter representation.

	$SU(N_c)$	$SU(N_f)$	$SU(N_f)$	$U(1)$	$U(1)_R$
Φ	$\mathcal{R}$	$\square$	1	1	$1-[C_A/(2T_fN_f)]$
$\tilde{\Phi}$	$\overline{\mathcal{R}}$	1	$\overline{\square}$	-1	$1-[C_A/(2T_fN_f)]$

TABLE II: Matter content of the $\mathcal{N} = 1$ supersymmetric gauge theory with gauge group $SU(N_c)$ and N_f pairs of chiral superfields in the fundamental and conjugate fundamental representations.

	$SU(N_c)$	$SU(N_f)$	$SU(N_f)$	$U(1)_V$	$U(1)_R$
Φ	$\square$	$\square$	1	1	$1-(N_c/N_f)$
$\tilde{\Phi}$	$\overline{\square}$	1	$\overline{\square}$	-1	$1-(N_c/N_f)$

TABLE III: Matter content of an $\mathcal{N} = 1$ supersymmetric gauge theory with an arbitrary real or pseudoreal matter representation.

	$SU(N_c)$	$SU(2N_f)$	$U(1)_R$
Φ	$\mathcal{R}$	$\square$	$1-[C_A/(2T_fN_f)]$

TABLE IV: Scheme-independent values of β'_{IR,F,Δ_f^p} with $2 \le p \le 4$ for $G = \mathrm{SU}(2)$, SU(3), and SU(4) with N_f pairs of chiral superfields in the fundamental and conjugate fundamental representations, as functions of N_f, in the respective non-Abelian Coulomb phase intervals, $(3/2)N_c < N_f < 3N_c$. Here, $\Delta_f = 3N_c - N_f$.

N_c	N_f	β'_{IR,F,Δ_f^2}	β'_{IR,F,Δ_f^3}	β'_{IR,F,Δ_f^4}
2	3	1.000	2.167	−0.7482
2	4	0.444	0.790	0.214
2	5	0.111	0.154	0.118
3	5	0.667	1.296	0.330
3	6	0.375	0.641	0.335
3	7	0.167	0.245	0.185
3	8	0.0417	0.0515	0.0477
4	6	0.800	1.627	0.370
4	7	0.555	1.034	0.428
4	8	0.355	0.6005	0.352
4	9	0.200	0.303	0.225
4	10	0.0889	0.1195	0.104
4	11	0.0222	0.02605	0.0251

TABLE V: Values of N_ℓ, $N_{f,b2z}$, N_u, $N_{f,\mathrm{mc},(2,3)}$, $N_{f,\mathrm{mc},(2,3)} - N_\ell$, $N_{f,\mathrm{mc},(3,4)}$, and $N_{f,\mathrm{mc},(3,4)} - N_\ell$ for the illustrative cases $2 \le N_c \le 4$. For notational brevity, the subscripts mc are suppressed.

N_c	N_ℓ	$N_{f,b2z}$	N_u	$N_{f,(2,3)}$	$N_{f,(2,3)} - N_\ell$	$N_{f,(3,4)}$	$N_{f,(2,3)} - N_\ell$
2	3	3.429	6	3.428	0.429	4.799	1.799
3	4.5	4.765	9	4.765	0.265	6.395	1.895
4	6	6.1936	12	6.1935	0.1935	8.054	2.054

TABLE VI: Values of r_ℓ, $r_{f,b2z}$, r_u, $r_{\mathrm{mc},(2,3)}$, $r_{\mathrm{mc},(2,3)} - r_\ell$, $r_{\mathrm{mc},(3,4)}$, and $r_{\mathrm{mc},(3,4)} - r_\ell$ in the LNN limit.

r_ℓ	r_{b2z}	r_u	$r_{\mathrm{mc},(2,3)}$	$r_{\mathrm{mc},(2,3)} - r_\ell$	$r_{\mathrm{mc},(3,4)}$	$r_{\mathrm{mc},(2,3)} - r_\ell$
3/2	3/2	3	3/2	0	1.8370	0.3370

TABLE VII: Scheme-independent values of $\beta'_{IR,LNN,\Delta_r^p}$ with $2 \le p \le 4$ as functions of r for r in the non-Abelian Coulomb phase interval, $3/2 < r < 3$. For comparison, we also list $\beta'_{IR,2\ell}$ (which is scheme-independent) and $\beta'_{IR,3\ell}$, as calculated in the $\overline{\rm DR}$ scheme. See text for further discussion.

r	$\beta'_{IR,2\ell}$	$\beta'_{IR,3\ell}$	$\beta'_{IR,LNN,\Delta_r^2}$	$\beta'_{IR,LNN,\Delta_r^3}$	$\beta'_{IR,LNN,\Delta_r^4}$
1.5	u	6.000	0.750	1.500	0.533
1.6	9.800	3.484	0.653	1.263	0.529
1.7	4.225	2.301	0.563	1.052	0.506
1.8	2.400	1.604	0.480	0.864	0.468
1.9	1.5125	1.145	0.403	0.699	0.419
2.0	1.000	0.823	0.333	0.5556	0.365
2.1	0.675	0.590	0.270	0.432	0.307
2.2	0.457	0.417	0.213	0.327	0.249
2.3	0.306	0.288	0.163	0.240	0.194
2.4	0.200	0.193	0.120	0.168	0.143
2.5	0.125	0.122	0.0833	0.111	0.0992
2.6	0.0727	0.0719	0.0533	0.0676	0.0627
2.7	0.0375	0.0373	0.0300	0.0360	0.03445
2.8	0.01538	0.01536	0.0133	0.0151	0.0148
2.9	0.003571	0.003570	0.00333	0.003556	0.00354
3.0	0	0	0	0	0

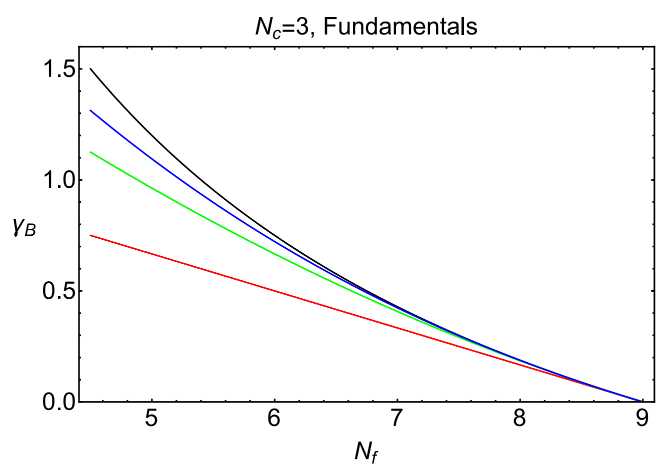

FIG. 1: Plot of $\gamma_{B,F,\Delta_f^p} = \gamma_{\tilde{B},F,\Delta_f^p}$ with $1 \le p \le 3$, together with the exact $\gamma_{B,F}$, for $G = {\rm SU}(3)$ and $\mathcal{R} = F$, as a function of N_f, at an IRFP in the non-Abelian Coulomb phase for this theory. In this and the later figures, we consider N_f to be generalized from integers in the NACP to real numbers. For notational simplicity, the vertical axis is labeled simply as γ_B. At $N_f = 8$, from bottom to top, the curves (with colors online) refer to γ_{B,F,Δ_f} (red), γ_{B,F,Δ_f^2} (green), γ_{B,F,Δ_f^3} (blue), and the exact $\gamma_{B,F}$ (black).

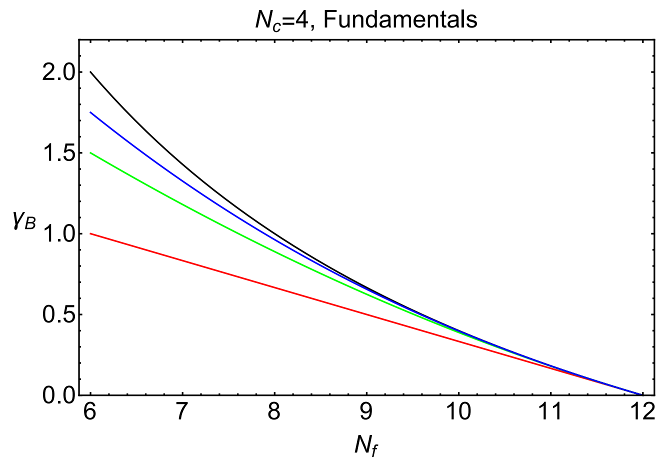

FIG. 2: Plot of $\gamma_{B,F,\Delta_f^p} = \gamma_{\tilde{B},\Delta_f^p} =$ with $1 \le p \le 3$, together with the exact $\gamma_{B,F}$, for $G = \mathrm{SU}(4)$ and $\mathcal{R} = F$, as a function of N_f, at an IRFP in the non-Abelian Coulomb phase for this theory. For notational simplicity, the vertical axis is labeled simply as γ_B. At $N_f = 8$, from bottom to top, the curves (with colors online) refer to $\gamma_{B,F,\Delta}$ (red), γ_{B,F,Δ_f^2} (green), γ_{B,F,Δ_f^3} (blue), and the exact $\gamma_{B,F}$ (black).

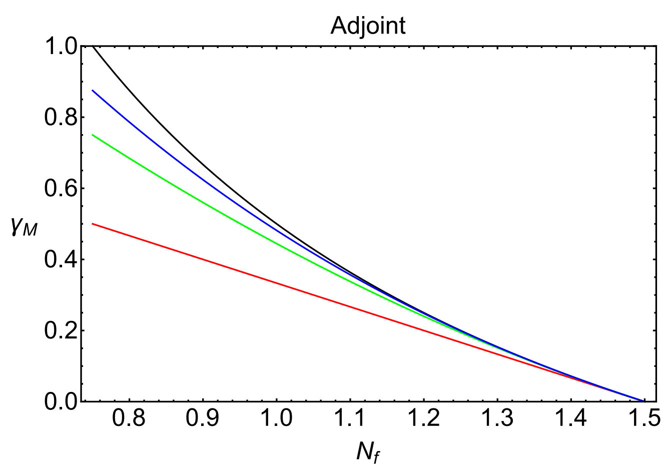

FIG. 3: Plot of the exact $\gamma_{M,adj}$ at an IRFP in the non-Abelian Coulomb phase, together with the $O(\Delta_f^p)$ approximations to this result with $1 \le p \le 3$, for $G = \mathrm{SU}(4)$ and $\mathcal{R}$ equal to the adjoint representation. From bottom to top, the curves (with colors online) refer to γ_{M,adj,Δ_f} (red), $\gamma_{M,adj,\Delta_f^2}$ (green), $\gamma_{M,adj,\Delta_f^3}$ (blue), and the exact $\gamma_{M,adj}$ (black).

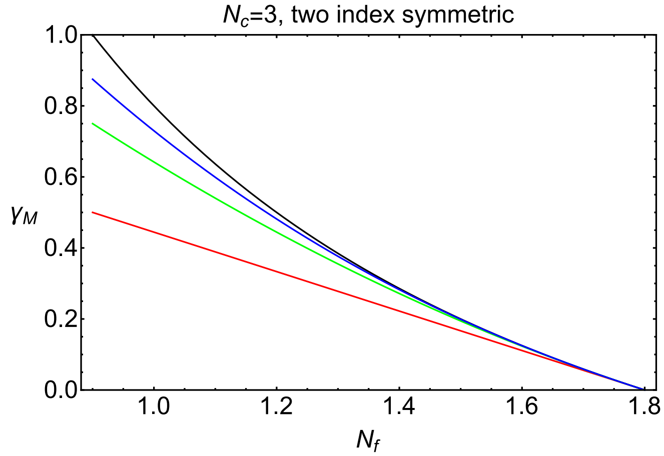

FIG. 4: Plot of the exact γ_{M,S_2} at an IRFP in the non-Abelian Coulomb phase, together with the $O(\Delta_f^p)$ approximations to this result with $1 \le p \le 3$, for $G = \mathrm{SU}(3)$ and $\mathcal{R} = S_2$, the symmetric rank-2 tensor representation. From bottom to top, the curves (with colors online) refer to γ_{M,S_2,Δ_f} (red), $\gamma_{M,S_2,\Delta_f^2}$ (green), $\gamma_{M,S_2,\Delta_f^3}$ (blue), and the exact γ_{M,S_2} (black).

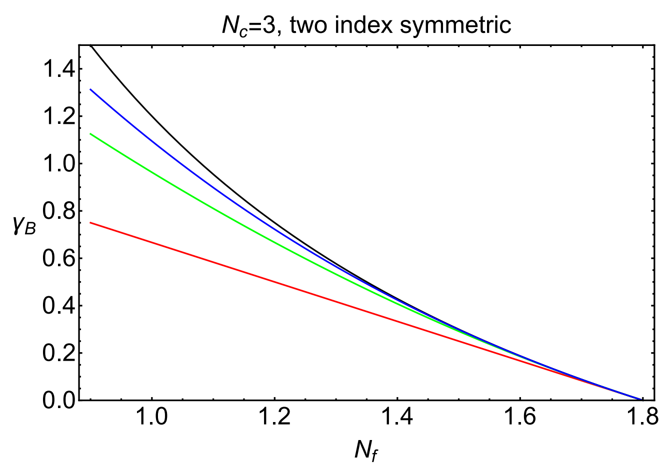

FIG. 5: Plot of the exact γ_{B,S_2} at an IRFP point in the non-Abelian Coulomb phase, together with the $O(\Delta_f^p)$ approximations to this result with $1 \le p \le 3$, for $G = \mathrm{SU}(3)$ and $\mathcal{R} = S_2$. From bottom to top, the curves (with colors online) refer to γ_{B,S_2,Δ_f} (red), $\gamma_{B,S_2,\Delta_f^2}$ (green), $\gamma_{B,S_2,\Delta_f^3}$ (blue), and the exact γ_{B,S_2} (black).

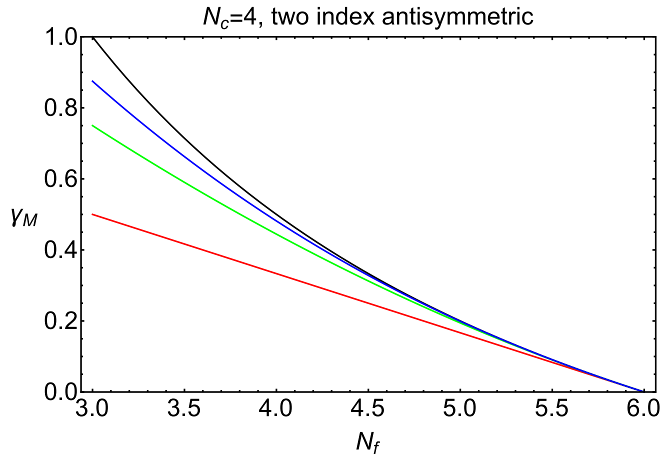

FIG. 6: Plot of the exact γ_{M,A_2} at an IRFP in the non-Abelian Coulomb phase, together with the $O(\Delta_f^p)$ approximations to this result with $1 \le p \le 3$, for $G = \mathrm{SU}(4)$ and $\mathcal{R} = A_2$, the rank-2 antisymmetric tensor representation. From bottom to top, the curves (with colors online) refer to γ_{M,A_2,Δ_f} (red), $\gamma_{M,A_2,\Delta_f^2}$ (green), $\gamma_{M,A_2,\Delta_f^3}$ (blue), and the exact γ_{M,A_2} (black).

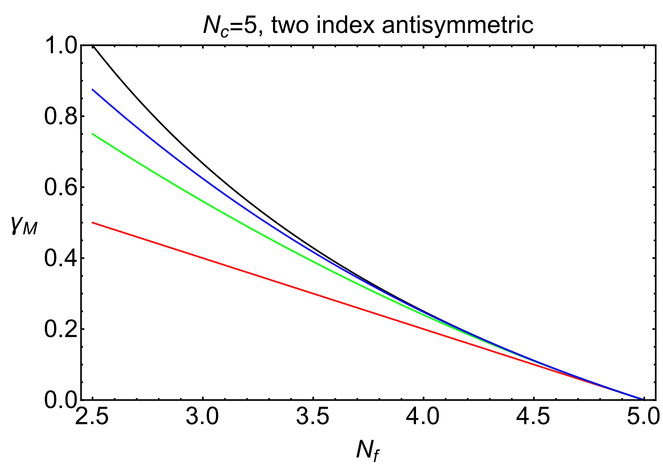

FIG. 7: Plot of the exact γ_{M,A_2} at an IRFP in the non-Abelian Coulomb phase, together with the $O(\Delta_f^p)$ approximations to this result with $1 \le p \le 3$, for $G = \mathrm{SU}(5)$ and $\mathcal{R} = A_2$. From bottom to top, the curves (with colors online) refer to γ_{M,A_2,Δ_f} (red), $\gamma_{M,A_2,\Delta_f^2}$ (green), $\gamma_{M,A_2,\Delta_f^3}$ (blue), and the exact γ_{M,A_2} (black).

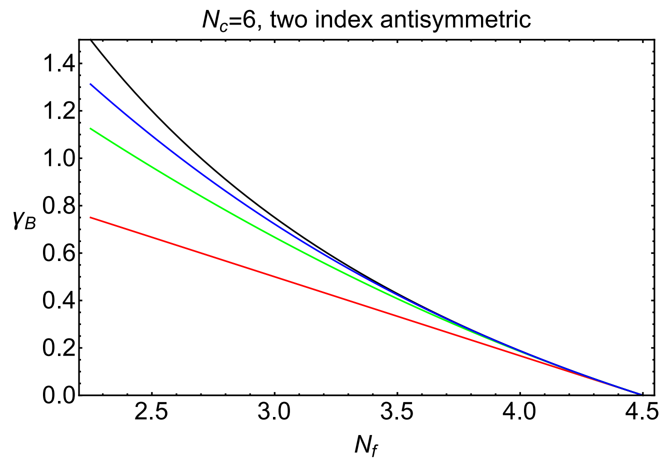

FIG. 8: Plot of the exact γ_{B,A_2} at an IRFP in the non-Abelian Coulomb phase, together with the $O(\Delta_f^p)$ approximations to this result with $1 \le p \le 3$, for $G = \mathrm{SU}(6)$ and $\mathcal{R} = A_2$. This is the special case of $\gamma_{B,A_2,Nce}$ in the text for $N_c = 6$, where the subscript Nce denotes even N_c. From bottom to top, the curves (with colors online) refer to γ_{B,A_2,Δ_f} (red), $\gamma_{B,A_2,\Delta_f^2}$ (green), $\gamma_{B,A_2,\Delta_f^3}$ (blue), and the exact γ_{B,A_2} (black).

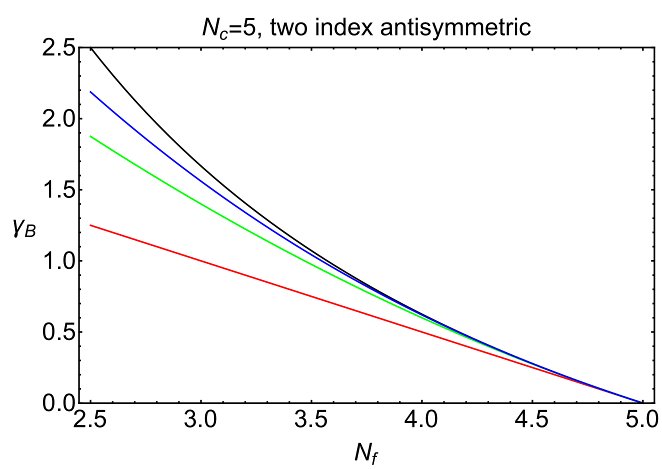

FIG. 9: Plot of the exact γ_{B,A_2} at an IRFP in the non-Abelian Coulomb phase, together with the $O(\Delta_f^p)$ approximations to this result with $1 \le p \le 3$, for $G = \mathrm{SU}(5)$ and $\mathcal{R} = A_2$. This is the special case of $\gamma_{B,A_2,Nco}$ in the text for $N_c = 5$, where the subscript Nco denotes odd N_c. From bottom to top, the curves (with colors online) refer to γ_{B,A_2,Δ_f} (red), $\gamma_{B,A_2,\Delta_f^2}$ (green), $\gamma_{B,A_2,\Delta_f^3}$ (blue), and the exact γ_{B,A_2} (black).

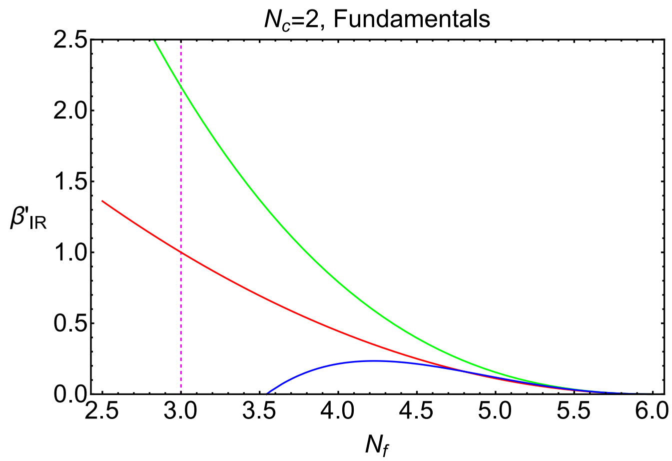

FIG. 10: Plot of β'_{IR,Δ_f^p} with $2 \le p \le 4$ for $G = \mathrm{SU}(2)$ and $\mathcal{R} = F$, as a function of N_f at an IRFP in the non-Abelian Coulomb phase for this theory. At $N_f = 4$, from bottom to top, the curves (with colors online) refer to β'_{IR,Δ_f^4} (blue), β'_{IR,Δ_f^2} (red), and β'_{IR,Δ_f^3} (green).

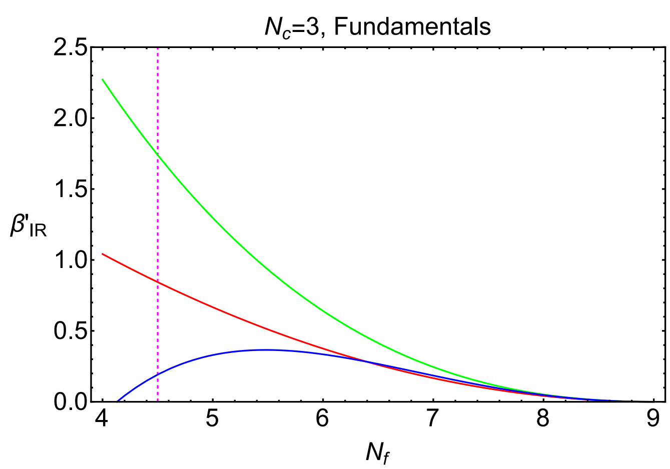

FIG. 11: Plot of β'_{IR,Δ_f^p} with $2 \le p \le 4$ for $G = \mathrm{SU}(3)$ and $\mathcal{R} = F$, as a function of N_f at an IRFP in the non-Abelian Coulomb phase for this theory. At $N_f = 5$, from bottom to top, the curves (with colors online) refer to β'_{IR,Δ_f^4} (blue), β'_{IR,Δ_f^2} (red), and β'_{IR,Δ_f^3} (green).

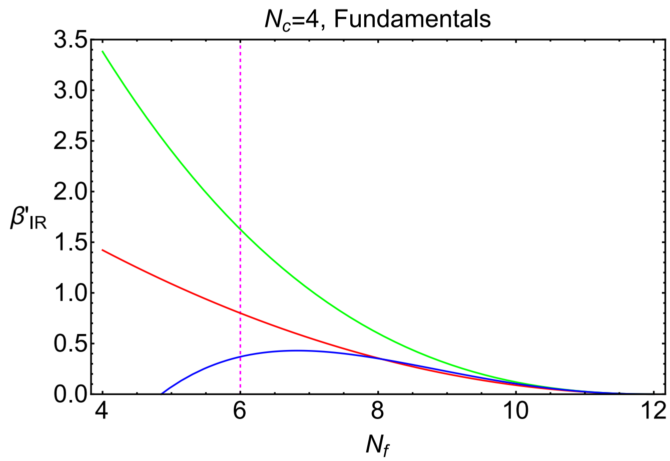

FIG. 12: Plot of β'_{IR,Δ_f^p} with $2 \le p \le 4$ for $G = \mathrm{SU}(4)$ and $\mathcal{R} = F$, as a function of N_f at an IRFP in the non-Abelian Coulomb phase for this theory. At $N_f = 6$, from bottom to top, the curves (with colors online) refer to β'_{IR,Δ_f^4} (blue), β'_{IR,Δ_f^2} (red), and β'_{IR,Δ_f^3} (green).

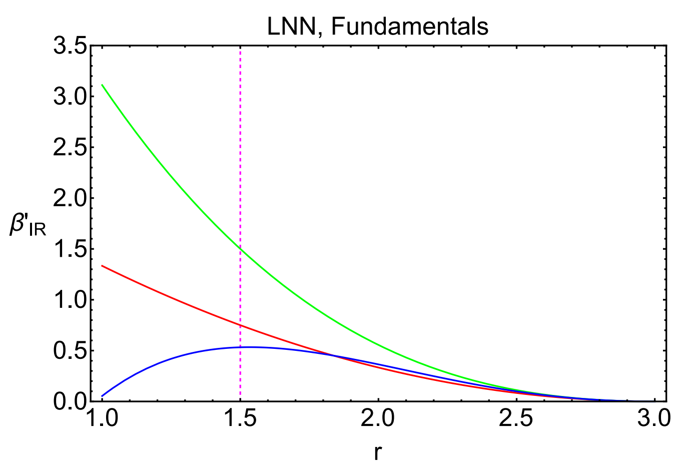

FIG. 13: Plot of $\beta'_{IR,LNN,\Delta_r^p}$ with $2 \le p \le 4$ as a function of r in the LNN limit (9.1), for r at an IRFP in the non-Abelian Coulomb phase. At $r = 1.6$, from bottom to top, the curves (with colors online) refer to $\beta'_{IR,LNN,\Delta_r^4}$ (blue), $\beta'_{IR,LNN,\Delta_r^2}$ (red), and $\beta'_{IR,LNN,\Delta_r^3}$ (green).