Dual conformal symmetry, integration-by-parts reduction, differential equations, and the nonplanar sector

Zvi Bern, Michael Enciso, Harald Ita, and Mao Zeng

Phys. Rev. D **96**, 096017 — Published 21 November 2017

DOI: 10.1103/PhysRevD.96.096017

UCLA/17/TEP/106 FR-PHENO-2017-016

Dual Conformal Symmetry, Integration-by-Parts Reduction, Differential Equations and the Nonplanar Sector

Zvi Bern[a], Michael Enciso[a], Harald Ita[b] and Mao Zeng[a]

[a] *Mani L. Bhaumik Institute for Theoretical Physics*
Department of Physics and Astronomy
University of California at Los Angeles
Los Angeles, CA 90095, USA
[b] *Physikalisches Institut*
Albert-Ludwigs-Universität Freiburg
D79104 Freiburg, Germany

We show that dual conformal symmetry, mainly studied in planar $\mathcal{N} = 4$ super-Yang–Mills theory, has interesting consequences for Feynman integrals in nonsupersymmetric theories such as QCD, including the nonplanar sector. A simple observation is that dual conformal transformations preserve unitarity cut conditions for any planar integrals, including those without dual conformal symmetry. Such transformations generate differential equations without raised propagator powers, often with the right hand side of the system proportional to the dimensional regularization parameter ϵ. A nontrivial subgroup of dual conformal transformations, which leaves all external momenta invariant, generates integration-by-parts relations without raised propagator powers, reproducing, in a simpler form, previous results from computational algebraic geometry for several examples with up to two loops and five legs. By opening up the two-loop three- and four-point nonplanar diagrams into planar ones, we find a nonplanar analog of dual conformal symmetry. As for the planar case this is used to generate integration-by-parts relations and differential equations. This implies that the symmetry is tied to the analytic properties of the nonplanar sector of the two-loop four-point amplitude of $\mathcal{N} = 4$ super-Yang–Mills theory.

Contents

I. INTRODUCTION

Dual conformal symmetry is a hidden symmetry of planar $\mathcal{N}=4$ super-Yang-Mills theory [1, 2] which puts strong constraints on the analytic structure of its scattering amplitudes. In this paper we will discuss applications of this symmetry towards questions of practical interest in generic theories, such as finding useful and compact integration-by-parts (IBP) relations and differential equations (DEs) for loop integrals. We also use these ideas to extend the symmetry to the nonplanar sector by explicitly constructing it for the full two-loop four-point amplitude of $\mathcal{N}=4$ super-Yang–Mills theory. As for the planar case, the symmetry leads to useful IBP relations and DEs.

An important feature of the IBP relations and DEs generated by dual conformal transformations is that they are naturally compatible with generalized unitarity [3], which is a powerful method for computing multi-loop scattering amplitudes. Generalized unitarity helps to overcome the fast growth of complexity as the loop order and the number of legs increase. At one loop, unitarity-compatible integrand-based reduction [4–6] simplifies loop amplitudes to a linear combination of master integrals, with coefficients determined from generalized unitarity cuts. This has led to tremendous progress, including the "NLO revolution" for computing NLO QCD corrections for collider processes (see e.g. Refs. [7]). To extend the reach of generalized unitarity to generic theories at higher loops, it is natural to retain the following two important properties: (i) the parameterization is minimal without redundant parameters, leading to invertible linear systems which can be solved to determine the integrand; (ii) the integrand is decomposed into master integrands and spurious integrands that vanish upon integration, so only the coefficients of the master integrands are needed to evaluate the amplitudes.

These methods for evaluating scattering amplitudes offer great promise to tackle general problems at two loops and beyond (see e.g. Ref. [8]). For dimensionally regularized integrals beyond one loop, it is in fact easy to write down a parameterization that satisfies property (i) by identifying a minimal set of "irreducible numerators" that cannot be expressed as linear combinations of inverse propagators. For integrals in integer (most often four) dimensions, the problem is more intricate, as Gram determinant identities further reduce the number of independent terms in the integrand. But a complete and computationally efficient solution has been found using polynomial division algorithms [9, 10]. To construct a parameterization to satisfy the above property (ii), a first step has been developed in the mentioned papers exploiting the rotation symmetry in the "transverse" directions orthogonal to all external momenta. This is in direct analogy with the one-loop case [11]. A second step, which is substantially more nontrivial, is to identify all remaining contributions that integrate to zero. At higher loops the only known practical means to accomplish this [12] is to exploit IBP relations [13] without increasing propagator powers [14], to not only simplify the problem, but to make it naturally compatible with generalized unitarity. Our approach based on exploiting dual conformal transformations automatically generates IBP relations with these properties.

Geometrically, the special IBP relations which do not lead to higher propagator powers are generated by polynomial vector fields that are tangent to the unitarity cut surface [12]. This is related to the tangent algebra studied in the mathematics literature [15], as pointed out in Ref. [16]. A key problem for generating unitarity-compatible IBP relations is finding these special IBP-generating vectors. One solution is to solve "syzygy equations" using computational algebraic geometry [8, 14, 17, 18]. This is often time-consuming for the more complicated multi-loop integrals, and produces lengthy and unenlightening results. Analytic insights into the IBP-generating vectors from Ref. [12] shows that for generic two-loop integrals with massive external legs and internal propagators, a complete set of IBP-generating vectors comes from simple combinations of one-loop rotation vectors. These vectors will be referred to as "generic" vectors, and can be constructed as minors of matrices in Section VIII. For Feynman integrals involving vanishing or degenerate mass configurations, however, "exceptional" IBP-generating vectors appear, in addition to the generic vectors found by the aforementioned reference. This leads to extra IBP relations, e.g. relations between one-loop triangle integrals and bubble integrals. A full analytic understanding of these exceptional vectors is still missing in the literature, though a connection with singularities of unitarity cut surfaces has been explored [16]. We will show that important missing insights, at least for the integral topologies covered in this paper, come from $\mathcal{N}=4$ super-Yang–Mills theory.

In the study of scattering amplitudes, theories with more symmetries have often led to unexpected simplifications for theories with fewer symmetries. For example tree-level gluon amplitudes in pure Yang-Mills have hidden supersymmetry because they coincide with the same amplitudes in super-Yang–Mills theory [19]. A one-loop example is that supersymmetric decompositions can be applied to nonsupersymmetric theories [3, 20]. Following this philosophy, we aim to develop a relatively simple analytic understanding of IBP-generating vectors for a variety of one- and two-loop Feynman integrals with vanishing or degenerate masses, using dual conformal symmetry of planar $\mathcal{N}=4$ super-Yang–Mills theory as a guiding principle. The use of dual conformal symmetry also extends to a large class of planar Feynman integrals in even integer dimensions, with an appropriate number of propagators [1, 21, 22]. This is easiest to implement for planar diagrams where dual conformal symmetry is defined, but as we shall see by opening up nonplanar diagrams into planar diagrams [23], we identify a symmetry that is analogous to dual conformal symmetry.

When we consider integrals in arbitrary dimensions, generic numerators or integrals with too few propagators, the symmetries are lost because the numerators cannot balance the conformal weights from the denominators and the integration measure. However, for our purpose of finding IBP-generating vectors, only the geometry of the unitarity cut surface, fixed by the propagators not the numerators, is relevant. Therefore we can still find insights from dual conformal symmetry in order to analyze the loop integrals of any theory more generally. It turns out that a subgroup of dual conformal transformations, which leaves external momenta unchanged, generates infinitesimal shifts in the loop momenta to produce IBP relations without higher-power propagators. This is connected to the fact that under dual conformal transformations and their nonplanar generalization, the infinitesimal variations of inverse propagators are proportional to the inverse propagators themselves. It turns out that IBP-generating vectors obtained from conformal transformations contain exceptional vectors which we seek to understand. In the process we also find that at one loop the exceptional vectors relate directly to Landau equations [24, 25].

To illustrate the ideas in a simple context, we first present a number of one-loop examples. As a toy example we illustrate the case of the one-loop triangle diagram with a single external mass. While standard integral reductions [4, 5, 26] reduce tensor triangle integrals to the scalar triangle integral, we show that dual conformal transformations can be directly applied to reduce the scalar triangle integral to bubble integrals. Then we use this example to illustrate the embedding formalism [22, 27] which reduces conformal transformations in an $\mathrm{SO}(d-1,1)$ dual spacetime to linear Lorentz transformations in an $\mathrm{SO}(d,2)$ embedding space. The latter treatment will involve a general algorithm that can be applied to all one-loop integrals. Finally, we turn to two-loop examples, including nonplanar cases. We adopt a level-by-level approach to IBP reduction. For each topology, we only identify IBP relations which reduce all tensor integrals to top-level master integrals and lower-level integrals with fewer propagators. One can descend into the lower-level topologies recursively to accomplish the complete IBP reduction.

We also use dual conformal symmetries to generate DEs for integrals [28]. This has proven to be a powerful means for evaluating integrals. The DEs we generate are in terms of integrals without propagators raised to higher powers, along the lines of Ref. [29]. For the integrands that would be invariant in four dimensions under dual conformal transformations or their nonplanar analogs, the right hand side of the DEs are automatically proportional to the dimensional regularization parameter $\epsilon=(4-d)/2$. If there were no infrared singularities, we could take $\epsilon\to 0$, and the right side of the DEs would vanish. This property is already known for such integrals, after reducing to a carefully chosen basis of integrals [30, 31]. In our case, it follows from the existence of a symmetry.

Besides the practical utility of IBP relations and DEs, our considerations point to a nontrivial generalization of dual conformal symmetry to the full nonplanar sector of $\mathcal{N}=4$ super-Yang–Mills theory. Refs. [32, 33] found in a variety of nontrivial examples that the analytic properties implied by dual conformal symmetry such as having only logarithmic singularities, no poles at infinity and other properties carry over to the nonplanar sector. What symmetries might be behind this? In this paper we take initial steps towards understanding the symmetries behind these properties, by building on the connection between dual conformal transformations and polynomial tangent vectors of unitarity cut surfaces. For the case of the nonplanar sector of the two-loop four-point amplitude [34] based on our analysis of symmetries of integrals we show that there is indeed a symmetry analogous to dual conformal symmetry.

This paper is organized as follows. In Section II, we review unitarity-compatible IBP relations, dual conformal transformations and the embedding formalism which linearizes the transformations. In Section III, we illustrate the application of dual conformal transformations, starting from the simple toy example of the one-loop triangle with massless propagators and one massive external leg. Two parallel treatments are presented, one based directly on dual conformal transformations in d dimensions and the other on the $\mathrm{SO}(d,2)$ embedding space. The latter part of the section will present two more complicated examples at one loop, namely the triangle diagram with two external masses, and the massive triangle diagram involved in QCD corrections of the $H\to b\bar{b}$ decay. Section IV gives two-loop planar examples, reproducing nontrivial IBP-generating vectors previously obtained from computational algebraic geometry. Section V outlines applications to obtain DEs using transformations that act nontrivially on the external momenta. Section VI formulates a nonplanar analog of dual conformal symmetry. Applications to IBP and differential equations for dimensionally-regularized nonplanar integrals are also worked out. In Section VII, we show the invariance of the two-loop four-point amplitude of $\mathcal{N}=4$ super-Yang–Mills theory under this symmetry. Finally in Section VIII we describe the interesting connection between the exceptional IBP vectors and Landau equations. Our conclusions and outlook are presented in Section IX. An appendix giving matrices describing the dual conformal transformations of the two-loop pentabox integrals is also included.

II. BASIC CONCEPTS

In this section we give an overview of basic concepts that will be useful for the remainder of the paper. We first review the notion of unitarity-compatible IBP relations that do not increase the propagator powers, which generically occurs whenever derivatives hit propagators. Then we discuss using dual conformal transformations as a means for generating IBP relations that are compatible with unitarity cuts and do not increase the powers of the propagators. We will also review the embedding formalism for dual conformal transformations. This will be useful in subsequent sections, since it reduces conformal transformations to simpler Lorentz transformations in two higher dimensions.

A. Unitarity-compatible IBP relations

Consider an L-loop Feynman integral with L independent loop momenta, $l_1, l_2, \ldots, l_L$, M external legs with momenta p_i, $1 \le i \le M$, and N propagators, $1/\rho_j$, $1 \le j \le N$,

$$\int \prod_{A=1}^{L} d^d l_A \frac{\mathcal{N}}{\prod_j \rho_j} \,. \tag{2.1}$$

where $\mathcal{N}$ is a numerator that has polynomial dependence on all possible Lorentz-invariant dot products amongst loop and external momenta.

Integration-by-parts relations [13] arise because total derivatives integrate to zero in dimensional regularization,

$$0 = \int \prod_{A=1}^{L} d^d l_A \, \frac{\partial}{\partial l_B^\mu} \frac{v_B^\mu \mathcal{N}}{\prod_j \rho_j} \,, \tag{2.2}$$

where there is implicit summation over the loop momentum label B, and v_B^μ is built out of all possible Lorentz vectors p_i^μ and l_A^μ, each multiplied by polynomials in Lorentz-invariant dot products. The identity amongst integrals comes from explicitly applying the derivative. We will refer to

$$v_B^\mu \frac{\partial}{\partial l_B^\mu} \,, \tag{2.3}$$

as an IBP-generating vector or IBP vector.

If the vector satisfies the condition [14],

$$v_B^\mu \frac{\partial}{\partial l_B^\mu} \rho_j = \mathcal{W}_j \rho_j \,, \tag{2.4}$$

where there is an implicit sum over B and μ, for each $1 \le j \le N$, with the $\mathcal{W}_j$ being polynomials in Lorentz-invariant dot products, then the IBP relation Eq. (2.2) will not lead to propagators raised to two or more powers. More generally speaking, if we start with some propagator raised to a power, the power of that propagator will not be increased further in the IBP relation [8]. This will be called a "unitarity-compatible" IBP relation, as unitarity cut conditions are easily imposed when there are no raised propagator powers. The standard ways to find IBP vectors that satisfy Eq. (2.4) are based on solving syzygy equations [8, 14, 17, 18], often using software for computational algebraic geometry [35].

This is natural with the unitarity approach. If a certain inverse propagator ρ_j is set to zero by a unitarity cut, then for that case the right hand side of Eq. (2.4) is zero, which means the IBP-generating vector is a tangent vector to the unitarity cut surface of *any* cut, maximal or non-maximal [12]. It should be emphasized that it is a polynomial (rather than rational) tangent vector.

B. Unitarity-compatible differential equations

A powerful method for evaluating Feynman integrals is differential equations with respect to external momenta [28]. In this method, one computes derivatives

$$\chi_i^\mu \frac{\partial}{\partial p_i^\mu} \int \prod_{A=1}^{L} d^d l_A \frac{\mathcal{N}}{\prod_j \rho_j}\,, \tag{2.5}$$

where there is implicit summation of i over every external momentum, and χ_i^μ generates an infinitesimal change in the kinematic invariants (i.e. Lorentz-invariant dot products between external momenta). We require χ_i^μ to have no dependence on loop momenta. Since total derivatives vanish upon integration, Eq. (2.5) is equivalent to

$$\begin{aligned}\int \prod_{A=1}^{L} d^d l_A \left[\chi_i^\mu \frac{\partial}{\partial p_i^\mu} \frac{\mathcal{N}}{\prod_j \rho_j} + \frac{\partial}{\partial l_B^\mu} \frac{v_B^\mu \mathcal{N}}{\prod_j \rho_j}\right] \\ = \int \prod_{A=1}^{L} d^d l_A \left[\frac{\partial v_B^\mu}{\partial l_B^\mu} + \left(\chi_i^\mu \frac{\partial}{\partial p_i^\mu} + v_B^\mu \frac{\partial}{\partial l_B^\mu}\right)\right] \frac{\mathcal{N}}{\prod_j \rho_j}\,.\end{aligned} \tag{2.6}$$

We will refer to

$$\chi_i^\mu \frac{\partial}{\partial p_i^\mu} + v_B^\mu \frac{\partial}{\partial l_B^\mu} \tag{2.7}$$

as the DE-generating vector. Under the condition [29]

$$\left(\chi_i^\mu \frac{\partial}{\partial p_i^\mu} + v_B^\mu \frac{\partial}{\partial l_B^\mu}\right) \rho_j = \mathcal{W}_j \rho_j\,, \tag{2.8}$$

for some polynomial $\mathcal{W}_j$ for each $1 \le j \le N$, Eq. (2.6) has no propagators raised to higher powers, i.e. is unitarity-compatible. In our framework, IBP-generating vectors are special cases of DE-generating vectors without external momentum derivatives. Similarly, IBP relations are regarded as special cases of differential equations whose left hand side is zero rather than an external momentum derivative of the integral. Similar to the interpretation of Eq. (2.4), Eq. (2.8) implies that the DE-generating vector is a tangent vector to unitarity cut surfaces, considered as solutions to unitarity cut conditions in the space of *both* external and loop momenta.

We will refer to $\mathcal{W}_j$ as the weight of the inverse propagator ρ_j under the infinitesimal transformation of p_i and l_B generated by the vector (2.7). The total divergence term $\partial v_B^\mu / \partial l_B^\mu$ in Eq. (2.6) may be regarded as the weight $\mathcal{W}_{\text{measure}}$ of the integration measure, coming from an infinitesimal deviation of the Jacobian from unity (see a later discussion around Eq. (2.23)), under the same transformation. In addition, in some cases of interest, the numerator $\mathcal{N}$ also has a well-defined weight $\mathcal{W}_\mathcal{N}$ with polynomial dependence on external and loop momenta. In this case Eq. (2.6) is rewritten as

$$\begin{aligned}\int \prod_{A=1}^{L} d^d l_A \left[\frac{\partial v_B^\mu}{\partial l_B^\mu} + \left(\chi_i^\mu \frac{\partial}{\partial p_i^\mu} + v_B^\mu \frac{\partial}{\partial l_B^\mu}\right)\right] \frac{\mathcal{N}}{\prod_j \rho_j} \\ = \int \prod_{A=1}^{L} d^d l_A \left(\mathcal{W}_{\text{measure}} + \mathcal{W}_\mathcal{N} - \sum_k \mathcal{W}_k\right) \frac{\mathcal{N}}{\prod_j \rho_j}\,.\end{aligned} \tag{2.9}$$

If in the above equation,

$$\mathcal{W}_{\text{measure}} + \mathcal{W}_\mathcal{N} - \sum_k \mathcal{W}_k = 0\,, \tag{2.10}$$

then the integral is formally invariant under the infinitesimal transformation generated by the vector (2.7). A trivial example is a Lorentz transformation (in both external and loop momenta), under which the integration measure, propagators, and the numerator are separately invariant. In most cases the integrals are infrared singular and an infrared regulator is needed. This shifts the weight of the measure factor by terms proportional to ϵ, making the symmetry anomalous.

C. Properties of IBP- and DE-generating vectors

IBP-generating vectors defined by Eq. (2.4) and DE-generating vectors defined by Eq. (2.8) satisfy the following properties:

First, if an IBP-generating vector (or DE-generating vector) is multiplied by a polynomial in Lorentz-invariant dot products of external and loop momenta, it is still a valid IBP-generating vector (or DE-generating vector). Furthermore, the linear combination of two IBP-generating vectors (or DE-generating vectors) is still a valid vector. Therefore, IBP- and DE-generating vectors form *modules* over the ring of polynomials.

Second, by applying Eq. (2.8) twice, it can be seen that the composition of two DE-generating vectors still does not raise the power of any propagator. Furthermore, the components χ_i^μ remain independent of the loop momenta. This can be used to compute higher-order differential equations [36, 37] without generating doubled propagators.

Third, it follows from the second property above that IBP- and DE-generating vectors form a closed Lie algebra. The action of the DE vector Eq. (2.7) in Eq. (2.6) is, in the language of differential geometry, the Lie derivative action on the form

$$\prod_{A=1}^{L} d^d l_A \, \frac{\mathcal{N}}{\prod_j \rho_j} \,. \tag{2.11}$$

It is well known that the Lie derivative action of vectors commutes with the Lie bracket of vectors, i.e. the Lie algebra structure extends to the action of IBP- and DE-generating vectors. This is essentially the observation of Ref. [38] in the slightly different context of IBP reduction with doubled propagators. As in the aforementioned reference, the Lie algebra structure allows us to reduce the redundancy of IBP relations—all the necessary IBP relations arise from the action of a minimal generating set of IBP vectors on the possible tensor integrals.

Fourth, given the unitarity-compatible conditions Eq. (2.4) and (2.8), the IBP- and DE-generating vectors are valid on unitarity cuts and can be used to generate relations between cut integrals [39].

D. Dual conformal symmetry

If the Feynman integral Eq. (2.1) is planar and only has massless propagators, we can write each inverse propagator as either

$$(y_A - y_B)^2, \qquad (A \neq B)\,, \tag{2.12}$$

or

$$(y_A - x_j)^2 \,, \tag{2.13}$$

where A and B are loop-momentum labels, and x_j are the vertices of a coordinate-space polygon whose edge $(x_{i+1} - x_i)$ is equal to the external momenta p_i. We will refer to x_j as external momentum points and y_A as loop-momentum points. This is known as the dual-space version of planar Feynman integrals, as each y_A and x_j may be considered as coordinate-space points in a dual SO$(d-1,1)$ "spacetime" (not to be confused with ordinary spacetime).

As a simple example, consider the two-loop planar double-box integral,

$$I^{(2)} = \int d^d l_1 \, d^d l_2 \frac{1}{l_1^2 (l_1 - p_1)^2 (l_1 - p_1 - p_2)^2 l_2^2 (l_2 + p_4)^2 (l_2 + p_3 + p_4)^2 (l_1 - l_2)^2} \,, \tag{2.14}$$

illustrated in Fig. 1. We define the dual points implicitly, via

$$\begin{aligned} &p_1 = x_2 - x_1 \,, \qquad p_2 = x_3 - x_2 \,, \qquad p_3 = x_4 - x_3 \,, \qquad p_4 = x_1 - x_4 \,, \\ &l_1 = y_1 - x_1 \,, \qquad l_2 = y_2 - x_1 \,. \end{aligned} \tag{2.15}$$

These variables automatically enforce momentum conservation on the p_i. Performing the change of variables (2.15) in the double box, gives,

$$I^{(2)} = \int d^d y_1 \, d^d y_2 \, \frac{1}{(y_1 - x_1)^2 (y_1 - x_2)^2 (y_1 - x_3)^2 (y_1 - y_2)^2 (y_2 - x_1)^2 (y_2 - x_3)^2 (y_2 - x_4)^2} \,. \tag{2.16}$$

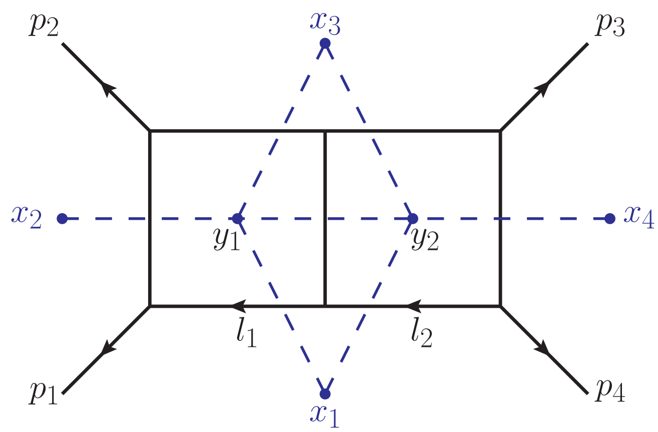

FIG. 1: The double box integrals. Differences of the dual points give momenta flowing in the diagram. The x_i and y_i are dual coordinates the double box. The dual diagram is given by the dashed (blue) diagram.

The dual conformal transformations include scalings $z_i \to a z_i$ and inversions $z_i^\mu \to z_i^\mu / z_i^2$, where z_i may be either an external x_i or internal y_A dual point. Under the inversion, we have

$$(x_i - x_j)^2 \to \frac{(x_i - x_j)^2}{x_i^2 x_j^2}, \qquad (y_A - x_j)^2 \to \frac{(y_A - x_j)^2}{y_A^2 x_j^2}, \qquad (y_A - y_B)^2 \to \frac{(y_A - y_B)^2}{y_A^2 y_B^2}. \tag{2.17}$$

From the perspective of planar $\mathcal{N} = 4$ super-Yang-Mills theory, dual conformal transformations are interesting because they formally leave the amplitude invariant, ignoring regulator issues. From our perspective, what makes them interesting is that they leave the unitarity cut surface $(y_A - x_j)^2 = 0$ invariant. These considerations suggest that we can generate IBP relations and differential equations that are automatically compatible with unitarity. This is true whether or not the integrals are invariant. Indeed, the noninvariance is precisely what we will use to generate nontrivial IBP relations and differential equations.

To generate IBP relations and differential equations we should phrase the conformal transformation as infinitesimal transformations. Under an infinitesimal conformal boost defined by an $\mathrm{SO}(d-1,1)$ vector b^μ, a dual coordinate z^μ transforms as

$$\Delta z^\mu = \frac{1}{2} b^\mu z^2 - (b \cdot z) z^\mu . \tag{2.18}$$

Under an infinitesimal scaling (i.e. dilatation) transformation with parameter β,

$$\Delta z^\mu = \beta z^\mu . \tag{2.19}$$

(Throughout this paper, Δ will be understood as a differential operator or a symmetry generator, so the right hand side of the above equation is not multiplied by an explicit infinitesimal parameter.) Finally, under Lorentz transformations parametrized by an antisymmetric $\Omega^{\mu\nu}$,

$$\Delta z^\mu = \Omega^{\mu\rho} \eta_{\rho\nu} z^\nu = \Omega^\mu{}_\nu \, z^\nu , \tag{2.20}$$

where $\eta_{\rho\nu}$ is the metric. Combining the conformal boost, scaling, and Lorentz transformations, we have

$$\Delta z^\mu = \frac{1}{2} b^\mu z^2 + (\beta - b \cdot z) z^\mu + \Omega^\mu{}_\nu \, z^\nu . \tag{2.21}$$

In terms of the infinitesimal transformations, if two points z_1^μ and z_2^μ both transform according to Eq. (2.21), then a simple calculation gives

$$\Delta (z_1 - z_2)^2 = \left[2\beta - b \cdot (z_1 + z_2) \right] (z_1 - z_2)^2 , \tag{2.22}$$

which is proportional to $(z_1 - z_2)^2$. Therefore, under an infinitesimal dual conformal transformation for y_A^μ and x_j^μ, the variation of any inverse propagator is proportional to the inverse propagator itself. This immediately echoes the condition Eq. (2.8) for the lack of propagators raised to higher powers, and implies that dual conformal transformations generate unitarity-compatible differential equations [29]. The dual-spacetime integration measure transforms as the trace of the infinitesimal deviation of the Jacobian matrix from the identity matrix,

$$\Delta\left(d^d z\right) = d^d z \frac{\partial \Delta z^\mu}{\partial z^\mu} = d^d z\,(\beta - b\cdot z)\,d\,. \tag{2.23}$$

As discussed in Subsection II B, IBP-generating vectors arise if we impose the further condition that the infinitesimal dual conformal transformations do not shift the external points,

$$\Delta x_j = 0\,, \tag{2.24}$$

for each external point x_j. We will give examples in subsequent sections for explicitly solving this constraint.

E. Embedding formalism

A convenient means for carrying out conformal transformations is via the embedding formalism of Refs. [22, 27]. In this construction, the system is embedded in a space with two extra dimensions. This allows us to reformulate dual conformal transformations as Lorentz transformations in the higher-dimensional space.

The embedding formalism maps each dual point z^μ in the $\mathrm{SO}(d-1,1)$ dual space to a point in $\mathrm{SO}(d,2)$ invariant space. Following the conventions of Ref. [22], we introduce

$$Z^a = \begin{pmatrix} Z^\mu \\ Z^- \\ Z^+ \end{pmatrix} = \begin{pmatrix} z^\mu \\ -z^2 \\ 1 \end{pmatrix}. \tag{2.25}$$

These vectors are defined modulo the identification

$$Z \cong \alpha Z, \quad \alpha \neq 0\,, \tag{2.26}$$

which is referred to as a GL(1) "gauge freedom". The inverse map is

$$z^\mu = \frac{Z^\mu}{Z^+}\,. \tag{2.27}$$

The $\mathrm{SO}(d,2)$ invariant contraction is defined by the inner product

$$(XY) = X^a X_a \equiv 2X^\mu Y_\mu + X^+ Y^- + X^- Y^+\,. \tag{2.28}$$

Thus the point defined in Eq. (2.25) is on the lightcone,

$$(ZZ) = 0\,. \tag{2.29}$$

We introduce the point at infinity, I, which is the limit of Eq. (2.25) with all components of x^μ uniformly tending to infinity, with an appropriate scaling using the gauge freedom in Eq. (2.26),

$$I^a = \lim_{|z|\to\infty} \left(-\frac{1}{z^2}\right) \begin{pmatrix} z^\mu \\ -z^2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{2.30}$$

This has the effect of compactifying the loop-momentum space [40]. Using Eq. (2.25), we map the loop-momentum points y_A^μ to

$$Y_A^a = \begin{pmatrix} y_A^\mu \\ -y_A^2 \\ 1 \end{pmatrix}, \tag{2.31}$$

and map the dual kinematic points x_j^μ to

$$X_j^a = \begin{pmatrix} x_j^\mu \\ -x_j^2 \\ 1 \end{pmatrix}. \tag{2.32}$$

The inverse propagators are now represented by SO$(d,2)$ inner products between these points,

$$(y_A - y_B)^2 = -\frac{(Y_A Y_B)}{(Y_A I)(Y_B I)}\,, \tag{2.33}$$

$$(y_A - x_j)^2 = -\frac{(Y_A X_j)}{(Y_A I)(X_j I)}\,, \tag{2.34}$$

where GL(1) invariance is ensured by the denominators involving the point at infinity. The denominators are unity in the gauge of Eq. (2.31). The factor $(X_j I)$ in the denominator of the right hand side of the second line can be omitted, because we will always choose the gauge $(X_j I) = 1$, as in Eq. (2.32).

The integration measure for each loop becomes, suppressing the loop label,

$$d^d y \to \frac{d^{d+2}Y\, \delta(Y^2/2)}{(YI)^d\, \mathrm{Vol(GL(1))}}\,, \tag{2.35}$$

where Y^2 is a shorthand for $(YY) = Y^a Y_a$ and the expression is formally divided by the volume of the GL(1) gauge orbit.

We define SO$(d,2)$ Lorentz transformations acting on some function $f(Z)$ using two reference vectors Z_i and Z_j, as

$$\begin{aligned}\Delta f(Z) &= (Z_{[i}\, Z)\left(Z_{j]}\, \frac{\partial}{\partial Z}\right) f(Z) = Z^a_{[i}\, Z_a\, Z^b_{j]}\, \frac{\partial}{\partial Z^b} f(Z) \\ &= \left(Z^a_i\, Z_a\, Z^b_j\, \frac{\partial}{\partial Z^b} - Z^a_j\, Z_a\, Z^b_i\, \frac{\partial}{\partial Z^b}\right) f(Z)\,,\end{aligned} \tag{2.36}$$

where a and b are SO$(d,2)$ indices. Notice that the factor $\delta(Y^2/2)$ in Eq. (2.35) is invariant under these transformations. The square-bracket notation in the first line indicates antisymmetrization over i and j, as explicitly implemented in the second line.

Integration-by-parts relations follow from Lorentz invariance identities [22],

$$0 = \int \frac{d^{d+2}Y\, \delta(Y^2/2)}{\mathrm{Vol(GL(1))}}\, u(Z_i, Z_j)\mathcal{I}\,, \tag{2.37}$$

where

$$u(Z_i, Z_j) \equiv (Z_{[i}\, Y)\left(Z_{j]}\, \frac{\partial}{\partial Y}\right) = (Z^b_i Y_b Z^a_j - Z^b_j Y_b Z^a_i)\frac{\partial}{\partial Y^a}\,, \tag{2.38}$$

is a one-loop IBP-generating vector. In Eq. (2.37) it acts on some general loop integrand $\mathcal{I}$. The factor $1/(YI)^d$ from the integration measure in Eq. (2.35) is absorbed into $\mathcal{I}$. Concrete examples of such IBP relations will be given in subsequent sections.

The SO$(d,2)$ Lorentz transformations exactly correspond to conformal transformations in Minkowski space with SO$(d-1,1)$ invariant metric, which can be checked using the inverse map formula Eq. (2.27). For example, in Eq. (2.38), a d-dimensional translation $\Delta z^\mu = e^\mu$ is equivalent to setting

$$Z_i = I = \begin{pmatrix}0\\1\\0\end{pmatrix}, \qquad Z_j = \begin{pmatrix}e^\mu\\0\\0\end{pmatrix}. \tag{2.39}$$

A d-dimensional conformal boost Eq. (2.18) with parameter b^μ is equivalent to setting

$$Z_i = -\frac{1}{2}\begin{pmatrix}0\\0\\1\end{pmatrix}, \qquad Z_j = \begin{pmatrix}b^\mu\\0\\0\end{pmatrix}. \tag{2.40}$$

Finally, a scaling transformation Eq. (2.19) is equivalent to setting

$$Z_i = I = \begin{pmatrix}0\\1\\0\end{pmatrix}, \qquad Z_j = -\begin{pmatrix}0\\0\\\beta\end{pmatrix}. \tag{2.41}$$

Therefore the IBP relations from SO$(d,2)$ Lorentz invariance arise from infinitesimal conformal transformations of the d-dimensional loop momenta. Following the logic of the previous subsection, such IBP relations will not have propagators raised to higher powers if the SO$(d,2)$ Lorentz transformations in Eq. (2.36) leaves the external momenta invariant, i.e. leaves the X_j points invariant up to $GL(1)$ gauge scaling.

More generally, we can consider any IBP-generating vector in the embedding space,

$$V^a \frac{\partial}{\partial Y^a} \,. \tag{2.42}$$

The above expression can be identified with an IBP-generating vector $v^\mu \partial_\mu$ in ordinary SO$(d-1,1)$ space if it satisfies the following two conditions: (i) it must be GL(1) gauge-invariant, and (ii) it must commute with the measure factor $\delta(Y^2/2)$, i.e.,

$$V^a Y_a = 0 \,. \tag{2.43}$$

The resulting IBP relation is, again showing the one-loop case for illustration,

$$\begin{aligned} 0 &= \int \frac{d^{d+2}Y\, \delta(Y^2/2)}{\mathrm{Vol(GL(1))}} \frac{\partial}{\partial Y^a} (V^a \mathcal{I}) \\ &= \int \frac{d^{d+2}Y\, \delta(Y^2/2)}{\mathrm{Vol(GL(1))}} \left(\mathcal{I} \frac{\partial V^a}{\partial Y^a} + V^a \frac{\partial \mathcal{I}}{\partial Y^a} \right), \end{aligned} \tag{2.44}$$

consisting of a divergence term proportional to an integrand $\mathcal{I}$ and a second term involving derivatives of $\mathcal{I}$. For an IBP-generating vector as in Eq. (2.38) from Lorentz invariance, the divergence term vanishes, so Eq. (2.37) only involves derivatives of $\mathcal{I}$.

We can extend the above discussion to include internal masses [40, 41] by modifying Eq. (2.32) to map the external momentum point x_j^μ to

$$X_j^a = \begin{pmatrix} x_j^\mu \\ -x_j^2 + m_j^2 \\ 1 \end{pmatrix} . \tag{2.45}$$

This changes Eq. (2.34) to

$$(x_j - y_A)^2 - m_j^2 = -\frac{(Y_A X_j)}{(Y_A I)} \,. \tag{2.46}$$

Since Eq. (2.46) contains a mass m_j that is independent of the loop label A, the formula only allows arbitrary masses at the one-loop level, and at higher loops, the masses of some propagators must be correlated or vanishing.

III. IBP FOR ONE-LOOP TRIANGLE INTEGRALS

To illustrate the ideas of the previous section, we present some simple one-loop examples. It is well known that by Passarino-Veltman or OPP reduction[4, 5, 26], triangle tensor integrals can all be reduced to triangle scalar integrals and daughter integrals (i.e. bubble and tadpole integrals from collapsing certain propagators of the triangle diagram). In the language of unitarity-compatible IBP reduction, this is accomplished by IBP-generating vectors which are rotation generators in the spacetime directions orthogonal to all external momenta [12]. However, under special kinematic configurations, scalar triangle integrals can be further reduced to bubble integrals using IBP reduction. As will be shown in Section VIII near the end of the paper, these special kinematic configurations are exactly those which allow leading Landau singularities. The necessary IBP-generating vectors will be the main topic of this section.

First we show directly how dual conformal transformations can be used to generate unitarity-compatible IBP relations without higher-power propagators. We then streamline the procedure using the embedding formalism [22, 27] that reduces conformal transformations to simpler Lorentz transformations in higher dimensions.

A. One-external-mass triangle: direct treatment

Consider the one-loop triangle shown in Fig. 2. For illustrative purposes, we specialize to the simple case of all internal and external legs being massless, with the exception of the right-most leg of the figure. The three inverse

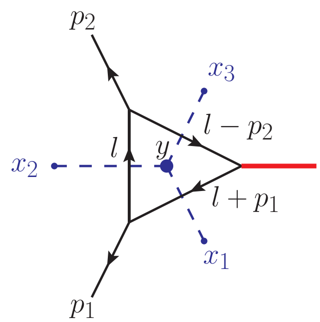

FIG. 2: The one-loop triangle with outgoing external momenta p_1, p_2, $-p_1 - p_2$ and dual points x_1, x_2, x_3. All internal propagators are massless, and the single massive external leg has mass $(p_1 + p_2)^2 = s$, shown as a thick (red) line. The dashed (blue) lines indicate the dual diagram.

propagators are

$$\rho_1 = (l + p_1)^2, \qquad \rho_2 = l^2, \qquad \rho_3 = (l - p_2)^2 \,. \tag{3.1}$$

The external kinematic invariants are

$$p_1^2 = p_2^2 = 0, \qquad (p_1 + p_2)^2 = s \,. \tag{3.2}$$

We introduce dual coordinates x_j, $1 \leq j \leq 3$ and y, such that

$$p_1 = x_2 - x_1, \qquad p_2 = x_3 - x_2, \qquad l = y - x_2 \,. \tag{3.3}$$

The external points x_j completely fix the external momenta, while y is an internal point corresponding to shifted loop momentum. Since p_j and l are expressed as differences between dual coordinates x_j and y in Eq. (3.3), we are free to apply the same translation to all the dual coordinates. We choose to fix the translation "gauge freedom" by taking,

$$x_2 = 0 \,, \tag{3.4}$$

so the explicit expressions for the dual coordinates are

$$x_1 = -p_1 \,, \qquad x_2 = 0 \,, \qquad x_3 = p_2 \,, \qquad y = l \,. \tag{3.5}$$

With this gauge choice, in terms of these dual coordinates, Eqs. (3.1) and (3.2) become

$$\rho_1 = (y - x_1)^2 \,, \qquad \rho_2 = y^2 \,, \qquad \rho_3 = (y - x_3)^2 \,, \tag{3.6}$$

and

$$x_1^2 = x_2^2 = x_3^2 = 0, \qquad (x_2 - x_1)^2 = (x_3 - x_2)^2 = 0, \qquad (x_3 - x_1)^2 = s \,. \tag{3.7}$$

Eqs. (3.6) and (3.7) imply that

$$y \cdot x_1 = \frac{1}{2}(\rho_2 - \rho_1), \qquad y \cdot x_2 = 0 \,, \qquad y \cdot x_3 = \frac{1}{2}(\rho_2 - \rho_3) \,. \tag{3.8}$$

As discussed in Section II, the key property of the dual conformal transformations (2.21) is that when acting on inverse propagators, they return results proportional to the inverse propagator itself, as shown in Eq. (2.22). In order to use dual conformal transformations to generate IBP relations, we restrict to the subset (2.24) where the transformations do not shift the external points.

The shift under the transformation of the loop momentum gives an IBP-generating vector

$$\Delta y^\mu \frac{\partial}{\partial y^\mu} = \Delta l^\mu \frac{\partial}{\partial l^\mu} = v^\mu \frac{\partial}{\partial l^\mu} \,, \tag{3.9}$$

that satisfies the key condition of Eq. (2.4) that it does not raise the power of propagators in the IBP identity. Applying Eq. (2.21) to x_j^μ, and using $x_j^2 = 0$ from Eq. (3.7), Eq. (2.24) becomes

$$0 = (\beta - b \cdot x_j) x_j, \quad j = 1, 2, 3 \,. \tag{3.10}$$

One solution to Eq. (3.10) is

$$\beta = s\,, \qquad\qquad b = -2(x_1 + x_3)\,, \tag{3.11}$$

where we used Eq. (2.3). This gives, using Eq. (2.21),

$$\Delta l = v = -l^2(x_1 + x_3) + [s + 2\,l\,\cdot(x_1 + x_3)]\,l\,. \tag{3.12}$$

The IBP-generating vector $v^\mu \partial_\mu$ satisfies

$$v^\mu \frac{\partial}{\partial l^\mu}\rho_i = \mathcal{W}_i \rho_i, \quad 1 \le i \le 3\,, \tag{3.13}$$

where $\mathcal{W}_i$ follows from Eq. (2.22),

$$\mathcal{W}_i = 2\beta - b\cdot(l + x_i)\,. \tag{3.14}$$

The divergence of the vector follows from Eq. (2.23),

$$\mathcal{W}_{\text{measure}} = \frac{\partial v^\mu}{\partial l^\mu} = (\beta - b\cdot l)d\,. \tag{3.15}$$

We obtain the IBP relation

$$\begin{aligned} 0 &= \int d^d l\, \frac{\partial}{\partial l^\mu} \frac{v^\mu}{\rho_1\rho_2\rho_3} \\ &= \int d^d l\, \left(\mathcal{W}_{\text{measure}} - \mathcal{W}_1 - \mathcal{W}_2 - \mathcal{W}_3\right) \frac{1}{\rho_1\rho_2\rho_3} \\ &= \int d^d l\, \big[(d-6)\beta + b\cdot(x_1 + x_2 + x_3) - (d-3) b\cdot l\big] \frac{1}{\rho_1\rho_2\rho_3} \\ &= \int d^d l\, \big[(d-4)s + 2(d-3)(x_1 + x_3)\cdot l\big] \frac{1}{\rho_1\rho_2\rho_3}\,. \end{aligned} \tag{3.16}$$

In the last line above, we have used the explicit solution for β and b^μ in Eq. (3.11). Simplifying the result using Eqs. (3.5) and (3.8), the final IBP relation is

$$\begin{aligned} 0 &= \int d^d l \big[(d-4)s + (d-3)(2\rho_2 - \rho_1 - \rho_3)\big] \frac{1}{\rho_1\rho_2\rho_3} \\ &= (d-4)s I_{\text{tri}} + 2(d-3) I^{(s)}_{\text{bub}}\,, \end{aligned} \tag{3.17}$$

where I_{tri} is the scalar triangle integral in Fig. 2 and $I^{(s)}_{\text{bub}}$ is the scalar bubble integral obtained from the term proportional to ρ_2 which cancels the propagator $1/\rho_2 = 1/l^2$, so that the mass of both external legs is s. The terms proportional to ρ_1 and ρ_3 in the second line of Eq. (3.17) give bubble integrals with massless external legs, which vanish in dimensional regularization and are discarded in the last line. Eq. (3.17) corresponds to a well known relation between the one-external-mass triangle and the bubble integral (see e.g. the fourth appendix of Ref. [42]). The coefficient of the triangle integral in Eq. (3.17) vanishes as $d \to 4$ while the coefficient of the bubble integral does not. This is due to infrared singularities of the triangle integral. In fact, the triangle integral allows a leading Landau singularity, whose connection with IBP-generating vectors will be explored in Section VIII. This simple example illustrates the basic principle behind using dual conformal symmetry to generate useful IBP relations.

B. Embedding-space treatment of one- and two-external-mass triangle

To streamline dual conformal transformations and the construction of IBP-generating vectors we use the embedding formalism [22, 27] summarized in Section II E. This reduces d-dimensional conformal transformations to simpler $(d+2)$-dimensional Lorentz transformations. The algorithm involves solving for all $(d+2)$-dimensional Lorentz transformations that leave the external momenta invariant. This is used to construct a matrix that encodes the action of the IBP vector, so that the IBP relations can be conveniently constructed. We will use the above one-external-mass triangle as an example, before explaining the generalization. Here we apply Lorentz rotations that act in the subspace of external points. One can also consider Lorentz rotations in the embedding space that only act in the space orthogonal to the external points, as we do in Section IV B.

Using Eqs. (2.34) and (2.35), the scalar triangle integral in Fig. 2 is written in the SO$(d, 2)$ embedding space,

$$I_{\rm tri} = \int \frac{d^{d+2}Y\, \delta(Y^2/2)}{(YI)^{d-3}\,{\rm Vol(GL(1))}} \frac{(-1)^3}{(YX_1)(YX_2)(YX_3)}\,, \tag{3.18}$$

where Y and X_j are as defined in Eqs. (2.31) and (2.32) and as in Eq. (2.35) Y^2 is a shorthand for $(YY) = Y^a Y_a$. The factor $(-1)^3$ comes from the minus sign on the right hand side of Eq. (2.34).

We define a subset of infinitesimal $d+2$ dimensional Lorentz transformations Δ_ω by an antisymmetric 4×4 matrix ω^{ij}, acting on a $(d+2)$ dimensional point Z^a as

$$\Delta_\omega Z^a = \sum_{1\le i,j\le 4} (ZX_i)\omega^{ij} X_j^a\,, \tag{3.19}$$

where X_1, X_2, X_3 are the three external points in Eq. (3.18) and $X_4 = I$, where I is defined in Eq. (2.30). We will choose the ω^{ij} such that the above Lorentz transformation leaves X_1, X_2, X_3 invariant up to a GL(1) gauge scaling. This means that under the transformations only the loop-momentum shifts by an infinitesimal amount, captured by the IBP vector,

$$\frac{1}{2}\omega^{ij} u(X_i, X_j) = \omega^{ij}(X_i Y) X_j^a \frac{\partial}{\partial Y^a}\,. \tag{3.20}$$

The summation over $1 \le i, j \le 4$ is implicit, and we have used the definition of $u(X_i, X_j)$ in Eq. (2.38).

The Lorentz transformations in Eq. (3.19) acts on X_k as

$$\begin{aligned}\Delta_\omega X_k^a &= g_{ki}\,\omega^{ij} X_j^a \\ &\equiv \bar\omega_k{}^j X_j^a\,,\end{aligned} \tag{3.21}$$

where we defined the "embedding space gram matrix" as,

$$g_{ij} = (X_i X_j) = \begin{pmatrix} 0 & 0 & -s & 1 \\ 0 & 0 & 0 & 1 \\ -s & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \tag{3.22}$$

where we identify X_4 with I and the last row and column contain entries of unity due to the gauge choice $(X_j I) = 1$ in Eq. (2.32). We then impose the condition that X_1, X_2, X_3 but not $X_4 = I$, are left invariant by the Lorentz transformation:

$$\Delta_\omega X_k^a = \alpha_k\, X_k^a\,, \qquad \text{if } k = 1, 2, 3, \tag{3.23}$$

where α_k can be absorbed into the $GL(1)$ invariance of the integrand (2.26) which takes $(d+2)$-dimensional vectors to be equivalent if they are scaled. The second line in Eq. (3.21) defines the "IBP matrix", and depends on the free parameters ω^{ij} which we determine below,

$$\bar\omega \equiv g\,\omega = \begin{pmatrix} s\omega_{13} - \omega_{14} & s\omega_{23} - \omega_{24} & -\omega_{34} & -s\omega_{34} \\ -\omega_{14} & -\omega_{24} & -\omega_{34} & 0 \\ -\omega_{14} & -s\omega_{12} - \omega_{24} & -s\omega_{13} - \omega_{34} & -s\omega_{14} \\ -\omega_{12} - \omega_{13} & \omega_{12} - \omega_{23} & \omega_{13} + \omega_{23} & \omega_{14} + \omega_{24} + \omega_{34} \end{pmatrix}. \tag{3.24}$$

Eq. (3.23) implies

$$\bar\omega_k{}^j = 0 \quad \text{if } k = 1, 2, 3,\ j \ne k, 1 \le j \le 4\,, \tag{3.25}$$

i.e., the non-diagonal entries have to vanish in all but the last rows. This gives four independent homogeneous linear constraints on the six possible components of the antisymmetric matrix ω,

$$\begin{aligned} \omega_{14} = \omega_{34} &= 0\,, \\ -s\omega_{12} - \omega_{24} &= 0\,, \\ s\omega_{23} - \omega_{24} &= 0\,. \end{aligned} \tag{3.26}$$

The two independent solutions are

$$\omega^{(1)} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & s \\ 0 & -1 & 0 & 0 \\ 0 & -s & 0 & 0 \end{pmatrix}, \qquad \omega^{(2)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.27}$$

under which the IBP matrix in Eq. (3.24) becomes

$$\bar{\omega}^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -s & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & s \end{pmatrix}, \qquad \bar{\omega}^{(2)} = \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -s & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}, \tag{3.28}$$

respectively. To compute IBP relations, the IBP vector Eq. (3.20) acts on (YX_k) as

$$\begin{aligned} \Delta_\omega(YX_k) &= \frac{1}{2}\omega^{ij}u(X_i,X_j)(YX_k) = \omega^{ij}(X_iY)(X_jX_k) = -g_{kj}\omega^{ji}(YX_i) \\ &= -\bar{\omega}_k{}^i(YX_i), \end{aligned} \tag{3.29}$$

where we used the antisymmetry of ω, and $u(X_i,X_j)$ is defined in Eq. (2.38). In terms of matrix components $\bar{\omega}_i{}^j$ that are nonvanishing for either solution, the resulting IBP relation is,

$$\begin{aligned} 0 &= \int \frac{d^{d+2}Y\,\delta(Y^2/2)}{\mathrm{Vol(GL(1))}}\,\Delta_\omega\left(\frac{(-1)^3}{(YI)^{d-3}(YX_1)(YX_2)(YX_3)}\right) \\ &= \int \frac{d^{d+2}Y\,\delta(Y^2/2)}{\mathrm{Vol(GL(1))}}\,\frac{(-1)^3}{(YI)^{d-3}(YX_1)(YX_2)(YX_3)}\left\{\left[\left(\sum_{i=1}^{3}\bar{\omega}_i{}^i\right)+(d-3)\bar{\omega}_4{}^4\right]\right. \\ &\qquad\qquad \left.+\frac{1}{(YI)}\left[(d-3)\sum_{i=1}^{3}\bar{\omega}_4{}^i(YX_i)\right]\right\} \\ &= \int d^dl\,\frac{1}{\rho_1\rho_2\rho_3}\left\{\left[\left(\sum_{i=1}^{3}\bar{\omega}_i{}^i\right)+(d-3)\bar{\omega}_4{}^4\right]-(d-3)(\bar{\omega}_4{}^1\rho_1+\bar{\omega}_4{}^2\rho_2+\bar{\omega}_4{}^3\rho_3)\right\} \\ &= \int d^dl\,\frac{1}{\rho_1\rho_2\rho_3}\left\{\left[\left(\sum_{i=1}^{3}\bar{\omega}_i{}^i\right)+(d-3)\bar{\omega}_4{}^4\right]-(d-3)\bar{\omega}_4{}^2\rho_2\right\}, \end{aligned} \tag{3.30}$$

where on the last line we dropped the contributions proportional to ρ_1 and ρ_3 because those generate scaleless bubble integrals that vanish in dimensional regularization.

Substituting the first solution for $\bar{\omega}$ in Eq. (3.28) ,

$$\bar{\omega}_1^{(1)1} = \bar{\omega}_3^{(1)3} = 0\,, \qquad \bar{\omega}_4^{(1)4} = -\bar{\omega}_2^{(1)2} = s, \qquad \bar{\omega}_4^{(1)2} = -2\,, \tag{3.31}$$

into Eq. (3.30) yields,

$$0 = s(d-4)I_{\text{tri}} + 2(d-3)I^{(s)}_{\text{bub}}\,, \tag{3.32}$$

reproducing Eq. (3.17).

For the second solution in Eq. (3.27), we have

$$\bar{\omega}_1^{(2)1} = -\bar{\omega}_3^{(2)3} = s\,, \qquad \bar{\omega}_2^{(1)2} = \bar{\omega}_4^{(1)4} = 0\,, \qquad \bar{\omega}_4^{(1)2} = 0\,, \tag{3.33}$$

so the IBP relation (3.30) is trivial because it involves only integrals that vanish in dimensional regularization.

As another example, consider the triangle with two external masses shown in Fig. 3. Following the same procedure as in the one-external-mass case, we introduce dual coordinates as usual

$$p_1 = x_2 - x_1\,, \qquad p_2 = x_3 - x_2\,, \qquad l = y - x_2\,. \tag{3.34}$$

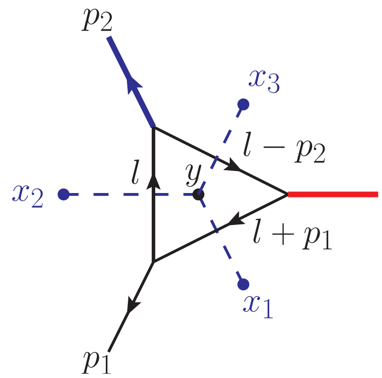

FIG. 3: The one-loop triangle with outgoing external momenta p_1, p_2, $-p_1 - p_2$. All internal propagators are massless, and the massive external legs, shown as thick lines, have masses $p_2^2 = t$ and $(-p_1 - p_2)^2 = s$. The dashed (blue) line indicates the dual diagram.

Following a similar analysis as for the single-external-mass case, we find only a single solution that leaves all the external momenta invariant. The associated IBP matrix is

$$\bar{\omega} = \begin{pmatrix} -(s-t) & 0 & 0 & 0 \\ 0 & -(s-t) & 0 & 0 \\ 0 & 0 & s-t & 0 \\ 2 & -2 & 0 & s-t \end{pmatrix}, \tag{3.35}$$

which encodes the action of the IBP-generating vector through Eq. (3.29). The resulting IBP relation, expressed in terms of the non-vanishing matrix components ${\bar{\omega}_i}^j$, is

$$\begin{aligned} 0 &= \int d^d l \frac{1}{\rho_1 \rho_2 \rho_3} \left\{ \left[\sum_{i=1}^{3} {\bar{\omega}_i}^i + (d-3){\bar{\omega}_4}^4 \right] - (d-3){\bar{\omega}_4}^1 \rho_1 - (d-3){\bar{\omega}_4}^2 \rho_2 - (d-3){\bar{\omega}_4}^3 \rho_3 \right\} \\ &= (d-4)(s-t) I_{\text{tri}}^{(s,t)} - 2(d-3) I_{\text{bub}}^{(t)} + 2(d-3) I_{\text{bub}}^{(s)} , \end{aligned} \tag{3.36}$$

where $I_{\text{bub}}^{(s)}$ is the bubble diagram obtained by canceling the propagator $l - q_2$, and $I_{\text{bub}}^{(s)}$ is the bubble diagram obtained by canceling the propagator $l - q_1$. When $t = 0$, $I_{\text{bub}}^{(t)}$ is a scaleless integral which vanishes in dimensional regularization, so the above IBP relation Eq. (3.36) becomes the same as the previous IBP relation Eq. (3.32) found for the triangle with only one massive external leg.

C. The Higgs to $b\bar{b}$ decay triangle

As a more sophisticated example to illustrate the use of dual conformal transformations in the presence of a mass, consider the one-loop triangle integral involved in the decay of the Higgs to a $b\bar{b}$ quark pair, with the bottom quark mass appearing in both internal and external lines, as depicted in Fig. 4. Internal masses are included in the embedding formalism, as described at the end of Section II E.

Introducing dual coordinates as usual, the three propagators are written as squared differences between dual coordinates,

$$\rho_1 = (y - x_1)^2, \qquad \rho_2 = (y - x_2)^2, \qquad \rho_3 = (y - x_3)^2 , \tag{3.37}$$

with gauge choice $x_2 = 0$, while the kinematic invariants are written as

$$x_2^2 = 0, \qquad x_1^2 = x_3^2 = m_b^2, \qquad (x_2 - x_1)^2 = (x_3 - x_2)^2 = m_b^2, \qquad (x_1 - x_3)^2 = m_H^2 , \tag{3.38}$$

involving three massive external legs and two massive internal propagators. Eqs. (3.37) and (3.38) imply that

$$y \cdot x_1 = \frac{1}{2}(\rho_2 - \rho_1 + m_b^2) , \qquad y \cdot x_2 = 0 , \qquad y \cdot x_3 = \frac{1}{2}(\rho_2 - \rho_3 + m_b^2) . \tag{3.39}$$

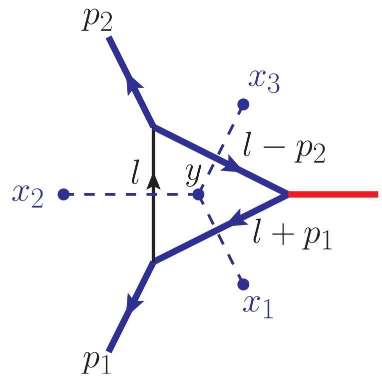

FIG. 4: The one-loop triangle that appears in the decay of a Higgs boson to a $b\bar{b}$ quark pair. The outgoing external momenta are p_1, p_2, $-p_1 - p_2$. The Higgs leg, shown as a thick (red) line on the rightmost part of the figure, has squared mass $(-p_1 - p_2)^2 = m_H^2$. The bottom-quark lines, appearing in both external legs and internal propagators, are shown as thick (blue) lines with squared mass m_b^2.

The embedding space Gram matrix is, using the mapping Eq. (2.45) for the massive case and identifying I with X_4,

$$g_{ij} = (X_i X_j) = \begin{pmatrix} 2m_b^2 & 0 & 2m_b^2 - m_H^2 & 1 \\ 0 & 0 & 0 & 1 \\ 2m_b^2 - m_H^2 & 0 & 2m_b^2 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} . \tag{3.40}$$

Using the general algorithm illustrated in Subsection III B, there is only one solution to the antisymmetric matrix ω^{ij} such that the IBP vector

$$\frac{1}{2}\omega^{ij} u(X_i, X_j) \,, \tag{3.41}$$

leaves all external momenta invariant. The solution is

$$\omega = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & m_H^2 - 4m_b^2 \\ 0 & -1 & 0 & 0 \\ 0 & -(m_H^2 - 4m_b^2) & 0 & 0 \end{pmatrix} , \tag{3.42}$$

which gives the IBP matrix,

$$\bar{\omega} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -(m_H^2 - 4m_b^2) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & (m_H^2 - 4m_b^2) \end{pmatrix} , \tag{3.43}$$

which encodes the action of the IBP-generating vector through Eq. (3.29). The resulting IBP relation, expressed in terms of the matrix components $\bar{\omega}_i{}^j$, is

$$\begin{aligned} 0 &= \int d^d l \, \frac{1}{\rho_1 \rho_2 \rho_3} \left\{ \left[\left(\sum_{i=1}^{3} \bar{\omega}_i{}^i \right) + (d-3)\bar{\omega}_4{}^4 \right] - (d-3)\bar{\omega}_4{}^1 \rho_1 - (d-3)\bar{\omega}_4{}^2 \rho_2 - (d-3)\bar{\omega}_4{}^3 \rho_3 \right\} \\ &= (d-4)(m_H^2 - 4m_b^2) I_{\text{tri}}^{Hb\bar{b}} + 2(d-3) I_{\text{bub}}^{(H)} - 2(d-3) I_{\text{bub}}^{(b)} \,, \end{aligned} \tag{3.44}$$

where $I_{\text{tri}}^{Hb\bar{b}}$ is the scalar triangle diagram, $I_{\text{bub}}^{(H)}$ is the bubble sub-diagram obtained by canceling the propagator with momentum l, and $I_{\text{bub}}^{(b)}$ is the bubble sub-diagram obtained by canceling either the propagator with momentum $l + p_1$ or the one with momentum $l - p_2$. The IBP relation for the one-external-mass triangle, Eq. (3.32) can be reproduced from the above IBP relation Eq. (3.44) by setting $m_H^2 = s$, $m_b^2 = 0$.

For higher-loop planar integrals with up to four external legs of any topology, the algorithm presented above can be adapted to find nontrivial dual conformal transformations that leaves all the external momenta invariant. We start with the embedding-space Gram matrix for the specific integral topology as in Eq. (3.22), with X_{N+1} defined to be equal to I. Then we repeat the subsequent calculations to produce the IBP matrix as in Eq. (3.24), leading to homogeneous linear constraints as in Eq. (3.25). Solving the linear constraints gives the IBP vectors and relations. As discussed in the previous subsection, for any solution of the antisymmetric matrix ω that has a vanishing last column, we will obtain IBP relations that only involve integrals with canceled propagators, therefore such solutions may be discarded if we are interested in the IBP reduction of top-level integrals. In the next section we will describe another class of useful dual conformal transformations orthogonal to all external momenta, which will be useful at higher loops.

IV. IBP FOR PLANAR TWO-LOOP INTEGRALS

In this section we discuss the more interesting case of higher-loop integrals. With generic mass configurations (e.g. with all external and internal masses being different from each other), a complete set of IBP-generating vectors is tabulated in Ref. [12]. Here we apply dual conformal symmetry to uncover extra IBP-generating vectors for planar two-loop integrals with massless lines. In Section VI we will extend this to the nonplanar case.

A. Conformal transformations in transverse dimensions

In the direct treatment of Section III A, the parameter of the conformal boost in Eq. (3.11), with x_i^μ given in Eq. (3.5), is a linear combination of external momenta. However, another interesting possibility is a conformal boost in a direction orthogonal to all external momenta, which gives unitarity-compatible IBP-generating vectors for *every* planar integral at any loop order.

Consider a general L-loop N-point diagram. For a planar N-point diagram at L loops, we take the dual coordinates to be $x_1, x_2, \ldots, x_N$. It is easy to fix the translation gauge freedom such that every x_i^μ is written as a linear combination of the external momenta p_i^μ. (For example, if we fix $x_1 = 0$, then $x_i = \sum_{j=1}^{i-1} p_j$.) In Eq. (2.22), we choose the conformal-boost parameter b^μ to be any vector that is orthogonal to all external momenta, and do not include a scaling transformation (i.e. setting $\beta = 0$). This gives $\Delta(x_i - x_j)^2 = 0$ for all pairs of i, j, which means all Mandelstam variables are left invariant. Therefore it is always possible to keep each individual external momentum invariant by adding a compensating Lorentz transformation.

In the planar case, the $\mathrm{SO}(d,2)$ embedding formalism gives a convenient way of proceeding. This eliminates the need to fix a gauge for the translation degrees of freedom of the dual coordinates. For illustration, we focus on $d = 4 - 2\epsilon$ dimensional loop integrals with N external momenta, where $N \le 5$. Generally, the embedding-space reference points $X_1, X_2, \ldots, X_N$ and the point at infinity I together span $N+1$ "physical" dimensions, leaving an orthogonal "transverse" space of dimension $(d+2) - (N+1) = d - (N-1)$. This directly corresponds to the subspace of ordinary $\mathrm{SO}(d-1,1)$ spacetime orthogonal to the $N-1$ dimensions spanned by the external momenta.[1] In addition, in the $(N+1)$-dimensional "physical" space spanned by $X_1, X_2, \ldots, X_N, I$, one can always find one vector $\tilde{I}$ that satisfies the N conditions,

$$(\tilde{I} X_i) = 0, \quad 1 \le i \le N\,. \tag{4.1}$$

In particular, if the top-left $N \times N$ sub-block $\tilde{g}_{ij} = (X_i X_j)$ of the embedding-space Gram matrix is non-singular, then the above $\tilde{I}$ can be found by projecting I onto the space orthogonal $X_1, X_2, \ldots X_N$,

$$\tilde{I}^a = I^a - (I X_i)(\tilde{g}^{-1})_{ij} X_j^a\,. \tag{4.2}$$

We can also define a set of vectors that span the transverse space. Let N_k, with $1 \le k \le d + 1 - N$, be an orthonormal basis of this orthogonal space. The $\mathrm{SO}(d,2)$ Lorentz transformations in Eq. (2.36), with $Z_i = \tilde{I}$ and $Z_j = N_k$ for any $1 \le k \le d+1-N$, leaves all X_j $(1 \le j \le N)$ invariant, since it only acts in the transverse space. So we obtain a valid unitarity-compatible IBP-generating vector

$$u(\tilde{I}, N_k) = \sum_A \left[(\tilde{I}\, Y_A) \left(N_k \frac{\partial}{\partial Y_A} \right) - (N_k\, Y_A) \left(\tilde{I} \frac{\partial}{\partial Y_A} \right) \right], \tag{4.3}$$

---

[1] For example, for a five-point diagram, with dimensional regularization the transverse space has dimension $4 - 2\epsilon - (5 - 1) = -2\epsilon$.

following the notation of the one-loop version in Eq. (2.38). However, the IBP relation from the multi-loop version of Eq. (2.37),

$$0 = \int \left(\prod_A \frac{d^{d+2}Y_A \, \delta(Y_A^2/2)}{\mathrm{Vol(GL(1))}} \right) u(\tilde{I}, N_k)\, \mathcal{I}\,, \tag{4.4}$$

breaks the Lorentz symmetry in the $d-(N+1)$ dimensional transverse space, since it introduces vectors N_k not present in the original problem, so it is not ideal. A remedy is to contract the Lorentz indices to give IBP-generating vectors that are invariant under the Lorentz symmetry of the transverse directions. We can write down the following L different vectors,

$$\frac{1}{(-Y_B I)}(N_k Y_B) u(\tilde{I}, N_k) = u\left(\tilde{I}, \frac{\tilde{Y}_{B\perp}}{(-Y_B I)}\right), \tag{4.5}$$

where the index k is summed and $1 \le B \le L$ specifies one of the independent loop momentum. The label B is not summed in Eq. (4.5). The contraction over the index k ensures Lorentz invariance in the transverse directions, while the normalization factor $1/(-Y_B I)$ ensures GL(1) gauge invariance. $Y_{B\perp}$ is the projection of Y_B onto the transverse space, using the inverse of the $(N+1)\times(N+1)$ Gram matrix $g_{ij} = (X_i X_j)$ with $X_{N+1} \equiv I$,

$$Y^a_{B\perp} = (N_k Y_B) N^a_k = Y^a_B - (Y_B X_i) g^{-1}_{ij} X^a_j \,. \tag{4.6}$$

This results in the IBP relations (see Eq. (2.44) for the one-loop analog),

$$0 = \int \left(\prod_A \frac{d^{d+2}Y_A \, \delta(Y_A^2/2)}{\mathrm{Vol(GL(1))}} \right) \sum_A \left[(\tilde{I}\, Y_A) \left(N_k \frac{\partial}{\partial Y_A} \right) - (N_k\, Y_A) \left(\tilde{I} \frac{\partial}{\partial Y_A} \right) \right] \left(\frac{(N_k Y_B)}{(-Y_B I)} \mathcal{I} \right), \tag{4.7}$$

where there is implicit summation over k, and B is a fixed loop label $1, 2, \ldots, L$.

The right hand side of Eq. (4.5) is an example of an IBP-generating vector defined using reference vectors with dependence on loop momenta. The IBP relation from such a vector is a superposition of familiar SO$(d,2)$ Lorentz symmetry identities, as in Eq. (4.7). IBP relations are obtained from the vector in explicit components,

$$u\left(\tilde{I}, \frac{\tilde{Y}_{B\perp}}{(-Y_B I)}\right) = \left(\frac{(\tilde{I} Y_A)}{(-Y_B I)} \tilde{Y}^a_{B\perp} - \frac{(\tilde{Y}_{B\perp} Y_A)}{(-Y_B I)} \tilde{I}^a \right) \frac{\partial}{\partial Y^a_A}\,, \tag{4.8}$$

then calculating the total divergence, as in Eq. (2.44). As before, in this expression A is summed over but B is not.

Since the IBP relations we derived earlier already suffice to reduce the triangle integrals to bubble integrals, we do not need the additional IBP relations coming from the transverse space.[2] But these relations are needed at the two-loop level.

B. Global and loop-by-loop conformal transformations

Now consider Lorentz transformations in the embedding space that affect the external momenta. To simplify the discussion we focus on two loops. We trivially extend the definition of the infinitesimal Lorentz transformation in Eq. (2.38) to simultaneously transform both Y_1 and Y_2,

$$u_{12}(Z_i, Z_j) = \sum_{A=1}^{2} u_A(Z_i, Z_j) = \sum_{A=1}^{2} (Z_{[i}\, Y_A) \left(Z_{j]} \frac{\partial}{\partial Y_A} \right). \tag{4.9}$$

Similarly, we will define loop-by-loop Lorentz transformations, namely

$$u_1(Z_i, Z_j), \quad u_2(Z_i, Z_j) \tag{4.10}$$

---

[2] These additional IBP relations in fact vanish on the maximal cut, for the three different triangle integrals considered in the previous section.

acting only on Y_1 and only on Y_2, respectively. For appropriate Z_i and Z_j, $u_{12}(Z_i, Z_j)$ can be considered a global $\mathrm{SO}(d,2)$ transformation (instead of acting only on the loop momentum points) that leaves all the external momenta invariant, so that Eq. (4.9) is a two-loop IBP-generating vector that does not lead to propagators raised to higher powers. The situation is entirely analogous to the one-loop case, and allows one-loop IBP-generating vectors to be reused at higher loops. A difference from the one-loop case is that we need the IBP-generating vectors arising from transverse directions, as explained in Subsection IV A, which may be considered as loop-momentum-dependent global conformal transformations.

For some of the more complicated two-loop integral topologies such as the penta-box discussed in Subsection IV E, IBP-generating vectors from global conformal transformations are *not* sufficient. To deal with this, we construct a class of loop-by-loop unitarity-compatible IBP-generating vectors. Consider the inverse propagators,

$$-\frac{(Y_1 X_i)}{(Y_1 I)}, \quad -\frac{(Y_2 X_j)}{(Y_2 I)}, \quad \frac{(Y_1 Y_2)}{(Y_1 I)(Y_2 I)}, \quad \text{with } i \in \sigma_1, \quad j \in \sigma_2\,, \tag{4.11}$$

where σ_1 and σ_2 are both subsets of $\{1, 2, \ldots, N\}$. If an $\mathrm{SO}(d,2)$ transformation parametrized by the antisymmetric matrix $\omega^{ij}_{(1)}$ leaves all the X_i points ($i \in \sigma_1$) invariant, the action of the transformation on Y_1 alone gives the IBP-generating vector

$$V_1^a \frac{\partial}{\partial Y_1^a} = \frac{1}{2} \omega^{ij}_{(1)} u_1(X_i, X_j)\,, \tag{4.12}$$

which does not raise the power of any propagator denominator of the form $-(Y_1 X_i)/(Y_1 I)$. The vector also does not raise the power of any propagator denominator of the form $-(Y_2 X_j)/(Y_2 I)$ because the vector does not involve derivatives w.r.t. the second loop momentum. However, the vector may double the power of the propagator denominator $\rho_c \equiv -(Y_1 Y_2)/(Y_1 I)(Y_2 I)$, so this is not yet a valid unitarity-compatible vector.

Similarly, if a conformal transformation parametrized by $\omega^{ij}_{(2)}$ leaves all the X_j points with $j \in \sigma_2$ invariant, we can write down an IBP-generating vector

$$V_2^a \frac{\partial}{\partial Y_2^a} = \frac{1}{2} \omega^{ij}_{(2)} u_2(X_i, X_j)\,, \tag{4.13}$$

which again does not increase the power of any propagator denominator except for ρ_c. Our final IBP-generating vector, to be denoted by $\mathrm{cross}(V_1, V_2)$, is

$$\begin{aligned} \mathrm{cross}(V_1^a \partial_{1\,a}, V_2^a \partial_{2\,a}) &= \frac{1}{(Y_1 I)(Y_2 I)} \left\{ \left[V_1^b \frac{\partial}{\partial Y_1^b} (Y_1 Y_2) \right] V_2^a \frac{\partial}{\partial Y_2^a} - \left[V_2^b \frac{\partial}{\partial Y_2^b} (Y_1 Y_2) \right] V_1^a \frac{\partial}{\partial Y_1^a} \right\} \\ &= \frac{1}{(Y_1 I)(Y_2 I)} \left\{ (V_1 Y_2) V_2^a \frac{\partial}{\partial Y_2^a} - (V_2 Y_1) V_1^a \frac{\partial}{\partial Y_1^a} \right\}, \end{aligned} \tag{4.14}$$

where the overall prefactor $1/((Y_1 I)(Y_2 I))$ is needed for GL(1) gauge invariance. This is designed to annihilate $(Y_1 Y_2)$. As a result, this IBP-generating vector does not raise the power of the propagator denominator ρ_c. To see this, in Eq. (2.4) we have

$$\mathcal{W}_c = \frac{\mathrm{cross}(V_1^a \partial_{1\,a}, V_2^a \partial_{2\,a}) \rho_c}{\rho_c} = -\frac{\mathrm{cross}(V_1^a \partial_{1\,a}, V_2^a \partial_{2\,a}) \big((Y_1 I)(Y_2 I) \big)}{(Y_1 I)(Y_2 I)}\,, \tag{4.15}$$

which evaluates to an expression with polynomial dependence on loop momenta, because the gauge $(Y_1 I) = (Y_2 I) = 1$ eliminates the denominators.

C. The triangle-box

As an explicit example, consider the two-loop triangle-box diagram shown in Fig. 5. The inverse propagators are

$$\begin{aligned} &\rho_1 = (l_1 + p_1)^2, && \rho_2 = l_1^2, && \rho_3 = (l_1 - p_2)^2, \\ &\rho_4 = (l_2 + p_1)^2, && \rho_5 = (l_2 - p_2)^2, && \rho_6 = (l_2 - l_1)^2\,, \end{aligned} \tag{4.16}$$

while an "irreducible numerator", which cannot be written as a linear combination of inverse propagators, is

$$\rho_7 = l_2^2\,. \tag{4.17}$$

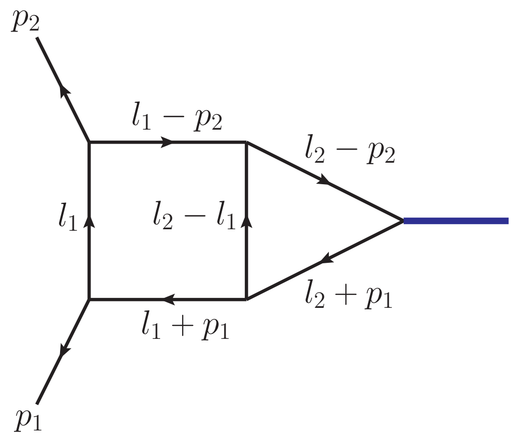

FIG. 5: The triangle-box diagram.

Notice that l_2 is not the momentum of any propagator, due to our choice of momentum routing. The external kinematic invariants are identical to those of the one-loop triangle with one external mass in Section III,

$$p_1^2 = p_2^2 = 0, \quad (p_1 + p_2)^2 = s\,. \tag{4.18}$$

Introducing dual coordinates as usual, the six inverse propagators and one irreducible numerator becomes

$$\begin{aligned} &\rho_1 = (y_1 - x_1)^2, \quad \rho_2 = (y_1 - x_2)^2, \quad \rho_3 = (y_1 - x_2)^2, \\ &\rho_4 = (y_2 - x_1)^2, \quad \rho_5 = (y_2 - x_3)^2, \quad \rho_6 = (y_2 - y_1)^2, \quad \rho_7 = (y_2 - x_2)^2\,, \end{aligned} \tag{4.19}$$

with the kinematic invariants written as

$$(x_2 - x_1)^2 = (x_3 - x_2)^2 = 0, \quad (x_3 - x_1)^2 = -s\,. \tag{4.20}$$

The triangle-box integral, with the irreducible numerator ρ_7 raised to the m-th power, is written as

$$\begin{aligned} I_m^{\text{tri - box}} &= \int d^d l_1 \int d^d l_2 \frac{\rho_7^m}{\rho_1\rho_2\rho_3\rho_4\rho_5\rho_6} \\ &= \int \frac{d^{d+2}Y_1\, \delta(Y_1^2/2)}{\text{Vol(GL(1))}} \int \frac{d^{d+2}Y_2\, \delta(Y_2^2/2)}{\text{Vol(GL(1))}} \\ &\times \frac{(-1)^{6+m}(Y_2X_2)^m}{(Y_1I)^{d-4}(Y_2I)^{d-3+m}(Y_1X_1)(Y_1X_2)(Y_1X_3)(Y_2X_1)(Y_2X_3)(Y_1Y_2)}\,. \end{aligned} \tag{4.21}$$

Since the external momenta are identical to those for the one-loop triangle, the same subgroup of conformal transformations in dual space leaves the external momenta invariant. Therefore, we can reuse the IBP-generating vectors for the one-loop triangle. The IBP generating vector is parametrized as

$$\frac{1}{2}\omega^{ij}u_{12}(X_i, X_j) = \sum_{A=1}^{2} \omega^{ij}(X_iY_A)X_j^a \frac{\partial}{\partial Y_A^a}\,, \tag{4.22}$$

which differs from the corresponding one-loop expression Eq. (3.20) only by an additional summation over the loop label A. We reuse the first solution $\omega_{(1)}$ for the antisymmetric matrix ω^{ij} in Eq. (3.27) found at one loop. The action of the IBP-generating vector is a straightforward generalization of the one-loop expression Eq. (3.29),

$$\begin{aligned} &\Delta_{\omega(1)}(Y_AX_1) = \Delta_{\omega(1)}(Y_AX_3) = 0, \\ &\Delta_{\omega(1)}(Y_AX_2) = s(Y_AX_2), \\ &\Delta_{\omega(1)}(Y_AI) = -\left[(Y_AX_1) - 2(Y_AX_2) + (Y_AX_3)\right] - s(Y_AI), \\ &\Delta_{\omega(1)}(Y_1Y_2) = 0\,. \end{aligned} \tag{4.23}$$

Other than the appearance of the loop label A which may be either 1 or 2, the only difference from the one-loop expression is the last line, namely the trivial statement that (Y_1Y_2) is invariant under simultaneous Lorentz transformations of Y_1 and Y_2.

IBP relations can be computed in a way similar to how it is done at one loop in Eq. (3.32), in terms of the non-vanishing components of the first solution for $\bar{\omega}$ in Eq. (3.28),

$$
\begin{aligned}
0 =& \int \frac{d^{d+2}Y_1\, \delta(Y_1^2/2)}{\mathrm{Vol(GL(1))}} \int \frac{d^{d+2}Y_2\, \delta(Y_2^2/2)}{\mathrm{Vol(GL(1))}} \\
&\times \Delta_{\omega(1)} \frac{(-1)^{6+m}(Y_2X_2)^m}{(Y_1I)^{d-4}(Y_2I)^{d-3+m}(Y_1X_1)(Y_1X_2)(Y_1X_3)(Y_2X_1)(Y_2X_3)(Y_1Y_2)} \\
=& \int \frac{d^{d+2}Y_1\, \delta(Y_1^2/2)}{\mathrm{Vol(GL(1))}} \int \frac{d^{d+2}Y_2\, \delta(Y_2^2/2)}{\mathrm{Vol(GL(1))}} \\
&\times \frac{(Y_2X_2)^m}{(Y_1I)^{d-4}(Y_2I)^{d-3+m}(Y_1X_1)(Y_1X_2)(Y_1X_3)(Y_2X_1)(Y_2X_3)(Y_1Y_2)} \\
&\times \Bigg\{ (-1)^{6+m}\left[-\bar{\omega}_2{}^2(m-1) + (d-4)\bar{\omega}_4{}^4 + (d-3+m)\bar{\omega}_4{}^4\right] \\
&+(-1)^{5+m}\left[-\frac{(d-4)}{(Y_1I)}\left(\sum_{i=1}^{3}\bar{\omega}_4{}^i(Y_1X_i)\right) - \frac{(d-3+m)}{(Y_2I)}\left(\sum_{i=1}^{3}\bar{\omega}_4{}^i(Y_2X_i)\right)\right]\Bigg\}\,.
\end{aligned}
\tag{4.24}
$$

It is illuminating to look at Eq. (4.24) on the maximal cut of the triangle-box, which sets

$$
(Y_1X_1) = (Y_1X_2) = (Y_1X_3) = (Y_2X_1) = (Y_2X_3) = (Y_1Y_2) = 0\,. \tag{4.25}
$$

After translating Eq. (4.24) back to $\mathrm{SO}(d-1,1)$ loop-momentum space, imposing the maximal cut, and substituting $\bar{\omega}_i{}^j$ for their explicit values, we obtain

$$
0 = 2(d-4+m)s\, I_m^{\text{tri-box}} + 2(d-3+m) I_{m+1}^{\text{tri-box}} + \text{ daughter integrals}, \tag{4.26}
$$

using the notation of Eq. (4.21) and "daughter integrals" refer to integrals where some of the triangle-box propagators are canceled. This is a recursion relation which reduces all the triangle-box integrals to the scalar integral $I_0^{\text{tri-box}}$ and integrals with canceled propagators.

We will further show that the scalar triangle-box integral is also reducible to integrals with canceled propagators, by constructing another IBP relation using transformations in the transverse directions as explained in Subsection IV A. We define

$$
\tilde{I} = X_2\,, \tag{4.27}
$$

which satisfies Eq. (4.1) and also define $Y_{1\perp}$ according to Eq. (4.6) with B set to 1,

$$
Y_{1\perp}^a = Y_1^a - (Y_1X_i)g_{ij}^{-1}X_j^a\,, \tag{4.28}
$$

which is the projection of Y_1 onto the transverse space orthogonal to X_1, X_2, X_3, I. Using the IBP-generating vector Eq. (4.5) with $B=2$,

$$
\frac{-1}{(Y_1I)} u_{12}(\tilde{I}, \tilde{Y}_{1\perp})\,, \tag{4.29}
$$

the IBP relations can be written down as a total divergence as in Eq. (2.44) (but generalized to more than one loop by trivially adding a summation over loop labels 1 and 2), with $\mathcal{I}$ set to

$$
\mathcal{I} = \frac{(-1)^6 s}{(Y_1I)^{d-4}(Y_2I)^{d-3}(Y_1X_1)(Y_1X_2)(Y_1X_3)(Y_2X_1)(Y_2X_3)(Y_1Y_2)}\,. \tag{4.30}
$$

Explicit calculation gives the IBP relation, again dropping terms that vanish on the maximal cut for the purpose of illustration,

$$
0 = -(d-3)s\, I_0^{\text{tri-box}} + \text{daughter integrals}\,. \tag{4.31}
$$

Combined with the recursion relation Eq. (4.26), this shows that all triangle-box integrals can be reduced to zero on the maximal cut. In other words, all these integrals can be reduced to integrals with canceled propagators, if we retain all terms proportional to inverse propagators in the calculation of the IBP relations.

D. The double box

Consider now the two-loop double-box integral in Fig. 1. The inverse propagators with the assigned momentum labels are

$$\rho_1 = l_1^2\,, \qquad \rho_2 = (l_1-p_1)^2\,, \qquad \rho_3 = (l_1-p_1-p_2)^2\,, \qquad \rho_4 = (l_2-p_1-p_2)^2\,,$$
$$\rho_5 = (l_2+p_4)^2, \qquad \rho_6 = l_2^2\,, \qquad \rho_7 = (l_2-l_1)^2\,, \tag{4.32}$$

while a choice of irreducible numerators is

$$\rho_8 = (l_1+p_4)^2\,, \qquad \rho_9 = (l_2-p_1)^2\,. \tag{4.33}$$

To write every inverse propagator in the dual-space form, as either $(y_1-y_2)^2$ or $(y_A-x_j)^2$, we define the $\mathrm{SO}(d-1,1)$ dual coordinates x_j and y_A such that

$$x_2^\mu - x_1^\mu = p_1^\mu\,, \qquad x_3^\mu - x_2^\mu = p_2^\mu\,, \qquad x_4^\mu - x_3^\mu = p_3^\mu\,, \qquad x_1^\mu - x_4^\mu = p_4^\mu\,,$$
$$y_1^\mu - x_1^\mu = l_1^\mu\,, \qquad y_2^\mu - x_1^\mu = l_2^\mu\,. \tag{4.34}$$

under which the seven inverse propagators become

$$\rho_1 = (y_1-x_1)^2\,, \qquad \rho_2 = (y_1-x_2)^2\,, \qquad \rho_3 = (y_1-x_3)^2\,, \qquad \rho_4 = (y_2-x_3)^2\,,$$
$$\rho_5 = (y_2-x_4)^2\,, \qquad \rho_6 = (y_2-x_1)^2\,, \qquad \rho_7 = (y_2-y_1)^2\,, \tag{4.35}$$

and the two irreducible numerators become

$$\rho_8 = (y_1-x_4)^2\,, \qquad \rho_9 = (y_2-x_2)^2\,. \tag{4.36}$$

A convenient visualization of the dual points is shown in Fig. 1. In terms of these quantities, we define the planar double-box integrand as

$$\Omega_1^{\mathrm{P}} = d^d l_1 d^d l_2 \frac{st}{\rho_1 \ldots \rho_7}\,. \tag{4.37}$$

where

$$s = (p_1+p_2)^2 = (x_1-x_3)^2\,, \qquad t = (p_2+p_3)^2 = (x_2-x_4)^2\,, \tag{4.38}$$

are Mandelstam invariants needed to cancel overall conformal weights.

As usual for planar integrals, we map the dual coordinates y_A^μ and x_j^μ to $\mathrm{SO}(d,2)$ embedding-space points Y_A and X_j, following Eqs. (2.31) and (2.32). If we use the algorithm presented in Section III B to find infinitesimal $\mathrm{SO}(d,2)$ Lorentz transformations that leave all X_j invariant, we find two such transformations in the notation of Eq. (2.38):

$$u_{12}(X_1,X_3), \qquad u_{12}(X_2,X_4)\,, \tag{4.39}$$

following the notation of Eq. (4.9). For the one-loop box diagram, both transformations vanish on the maximal cut because $(YX_i)=0$, $1\le i\le 4$. But for the two-loop double box topology, (Y_1X_4) and (Y_2X_2) are proportional to the irreducible numerators, so the IBP-generating vector,

$$u_{12}(X_2,X_4)\,, \tag{4.40}$$

still gives an IBP-generating vector that does not vanish on the maximal cut. Eq. (4.40) is essentially the same as the first IBP-generating vector for the double box in Ref. [14] obtained using computational algebraic geometry. There is another IBP-generating vector for the double box following the discussion of Subsection IV A. We define another $\mathrm{SO}(d,2)$ embedding-space point $Y_{1\perp}$,

$$Y_{1\perp}^a = Y_1^a - (Y_1X_i)g_{ij}^{-1}X_j^a\,, \tag{4.41}$$

where as usual, $g_{ij}=(X_iX_j)$ is the embedding-space Gram matrix, with X_5 identified with I. We also define the $\mathrm{SO}(d,2)$ embedding-space point $\tilde I$,

$$\tilde I^a = s(X_2^a + X_4^a) + t(X_1^a + X_3^a) + st\, I^a\,, \tag{4.42}$$

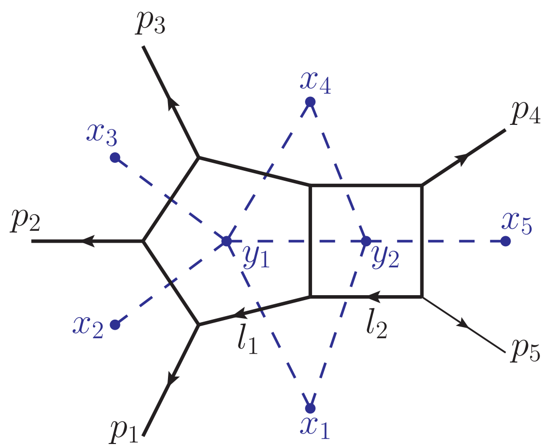

FIG. 6: The penta-box integral.

which is the same as Eq. (4.2) but with an extra overall factor st, and satisfies Eq. (4.1). Using the IBP-generating vector Eq. (4.5) with $B = 1$ we have,

$$u_{12}(\tilde{I}, Y_{1\perp}) \,. \tag{4.43}$$

We have checked using computer algebra that the two IBP-generating vectors, Eqs. (4.40) and (4.43), with all possible choices of numerators in $\mathcal{I}$ in the two-loop generalization of Eq. (2.44), generate a complete set of IBP relations that reduce all double box tensor integrals to two double box master integrals and daughter integrals (i.e. integrals with canceled propagators). It is worth noting that the two vectors we found are written down in a very compact form, whereas in Ref. [14] nearly one page is needed to display the vectors found from computational algebraic geometry.

E. The penta-box

As a more complex example consider the two-loop five-point penta-box shown in Fig. 6, along with dual coordinates x_i and y_A introduced as usual. There are five independent kinematic variables, which may be chosen as

$$s_{12}, s_{23}, s_{34}, s_{45}, s_{51} \,, \tag{4.44}$$

where $s_{ij} = (p_i + p_j)^2$. The embedding-space Gram matrix is, identifying I with X_6,

$$g_{ij} = (X_i X_j) = \begin{pmatrix} 0 & 0 & -s_{12} & -s_{45} & 0 & 1 \\ 0 & 0 & 0 & -s_{23} & -s_{51} & 1 \\ -s_{12} & 0 & 0 & 0 & -s_{34} & 1 \\ -s_{45} & -s_{23} & 0 & 0 & 0 & 1 \\ 0 & -s_{51} & -s_{34} & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} . \tag{4.45}$$

With $\tilde{I}$ defined as in Eq. (4.2), we obtain two IBP-generating vectors from conformal transformations in transverse directions, by setting $B = 1, 2$ in Eq. (4.5),

$$u\left(\tilde{I}, \frac{\tilde{Y}_{1\perp}}{(-Y_1 I)}\right), \tag{4.46}$$

$$u\left(\tilde{I}, \frac{\tilde{Y}_{2\perp}}{(-Y_2 I)}\right) \,. \tag{4.47}$$

Next, we examine conformal transformations which do not explicitly involve transverse directions. As in Eq. (3.19), we write down a conformal transformation parametrized as

$$\Delta_\omega Z^a = \sum_{1 \le i,j \le 6} (Z X_i) \omega^{ij} X_j^a \,, \tag{4.48}$$

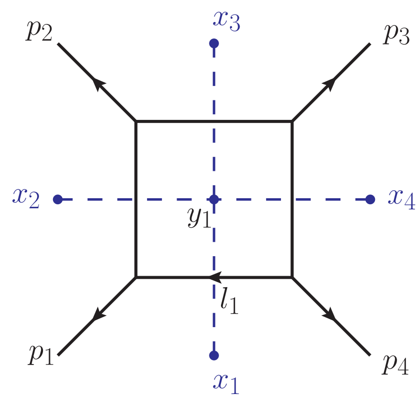

FIG. 7: The one-loop box diagram and its dual diagram.

where ω is a 6×6 antisymmetric matrix. Unlike the previous three-point and four-point examples in this paper, we are not able to find a solution for ω^{ij} which leaves all external momenta invariant. However, all is not lost. As discussed in the latter half of Subsection IV B, we can look for two different conformal transformations for the two sub-loops, and combine the two to give a unitarity-compatible IBP-generating vector.

We find one solution $\omega^{ij}_{(1)}$ which leaves x_1, x_2, x_3, x_4, or equivalently p_1, p_2, p_3, invariant, and three solutions $\omega^{ij}_{(2a)}, \omega^{ij}_{(2b)}, \omega^{ij}_{(2c)}$ which leaves x_1, x_4, x_5, or equivalently p_4, p_5 invariant. These solutions are tabulated in Appendix A.[3] Therefore, the following IBP-generating vectors do not increase the power of any propagator except the vertical central propagator in Fig. 6,

$$\begin{aligned} &\frac{1}{2}\omega^{ij}_{(1)} u_1(X_i, X_j), \quad \frac{1}{2}\omega^{ij}_{(2a)} u_2(X_i, X_j), \\ &\frac{1}{2}\omega^{ij}_{(2b)} u_2(X_i, X_j), \quad \frac{1}{2}\omega^{ij}_{(2c)} u_2(X_i, X_j)\,. \end{aligned} \tag{4.49}$$

These vectors can be combined to give IBP-generating vectors that do not increase the power of any propagator. Using the notation of Eqs. (4.12), (4.13) and (4.14), these IBP-generating vectors are

$$\begin{aligned} &\mathrm{cross}\left(\frac{1}{2}\omega^{ij}_{(1)} u_1(X_i, X_j), \frac{1}{2}\omega^{ij}_{(2a)} u_2(X_i, X_j)\right), \\ &\mathrm{cross}\left(\frac{1}{2}\omega^{ij}_{(1)} u_1(X_i, X_j), \frac{1}{2}\omega^{ij}_{(2b)} u_2(X_i, X_j)\right), \\ &\mathrm{cross}\left(\frac{1}{2}\omega^{ij}_{(1)} u_1(X_i, X_j), \frac{1}{2}\omega^{ij}_{(2c)} u_2(X_i, X_j)\right). \end{aligned} \tag{4.50}$$

We have checked, using computer algebra, that the five IBP-generating vectors in Eqs. (4.46), (4.47) and (4.50) are sufficient to reduce all penta-box integrals to three master integrals. Again, the five vectors are given by compact analytic expressions, in contrast to lengthy expressions one generally finds using computational algebraic geometry.

This formalism generalizes straightforwardly, e.g. to the six-point case, although one would need to check that the IBP relations are complete for each individual diagram topology, which is left to future work.

V. DIFFERENTIAL EQUATIONS FOR PLANAR INTEGRALS

In this section we briefly comment on applications of the ideas described in previous sections to constructing differential equations for integrals. An infinitesimal dual conformal transformation produces differential equations when we remove the restriction to the sub-algebra that keeps external legs invariant. We present a treatment in the embedding space, which simplifies the transformations and has the advantage that there is no need to fix the

---

[3] In quoting the number of solutions, we have ignored the solutions which ultimately do not lead to independent new IBP relations.

translation gauge for the dual coordinates. In the nonplanar case, covered in Section VI, where the transformations for some kinematic invariants become less obvious it will be simpler to use a "direct" treatment.

Consider the one-loop box, shown in Fig. 7 which has the same external momenta and the SO$(d,2)$ points X_i as the double box in Section IV D. Consider an infinitesimal SO$(d,2)$ Lorentz transformation ΔZ given by

$$\Delta Z^a = (Z_1 Z) Z_2^a - (Z_2 Z) Z_1^a \,, \tag{5.1}$$

with parameters

$$Z_1 = X_2, \quad Z_2 = s\, I + (X_1 + X_3)\,, \tag{5.2}$$

which satisfies

$$\begin{aligned} &(Z_1X_1) = (Z_1X_2) = (Z_1X_3) = 0, \quad (Z_1X_4) = -t, \quad (Z_1 I) = 1 \\ &(Z_2X_1) = (Z_2X_3) = 0, \quad (Z_2X_2) = (Z_2X_4) = s, \quad (Z_2 I) = 2\,. \end{aligned} \tag{5.3}$$

The transformation of the SO$(d,2)$ points are

$$\begin{aligned} &\Delta X_1 = \Delta X_3 = 0, \quad \Delta X_2 = -sX_2\,, \\ &\Delta X_4 = -tZ_2 - sZ_1 = -st\, I - t\, X_1 - t\, X_3 - sX_2\,, \\ &\Delta Y = (YX_2)(s\, I + X_1 + X_3) - s(YI)X_2 - (YX_1)X_2 - (YX_3)X_2\,, \end{aligned} \tag{5.4}$$

which shows X_1, X_2, and X_3 are invariant up to a GL(1) gauge scaling. In other words the d dimensional dual coordinates x_1^μ, x_2^μ and x_3^μ are left invariant. The factor (YI), which appears in the integration measure, transforms as

$$\Delta(YI) = (I\Delta Y) = 2(YX_2) - (YX_1) - (YX_3) - s(YI)\,. \tag{5.5}$$

As a result, $s = (x_1 - x_3)^2$ is invariant, while explicit calculation shows

$$\Delta t = 2(s+t)t\,. \tag{5.6}$$

So the transformation produces differential equations in the t variable,

$$\begin{aligned} 2(s+t)t\frac{\partial}{\partial t}\left(stI^{\text{box}}\right) &= \int \frac{d^{d+2}Y\,\delta(Y^2/2)}{\text{Vol(GL(1))}}\,\Delta\!\left(\frac{(X_1X_3)(X_2X_4)}{(YI)^{d-4}(YX_1)(YX_2)(YX_3)(YX_4)}\right) \\ &= \int \frac{d^{d+2}Y\,\delta(Y^2/2)}{\text{Vol(GL(1))}}\,\frac{(X_1X_3)(X_2X_4)}{(YX_1)(YX_2)(YX_3)(YX_4)}\,\Delta\!\left(\frac{1}{(YI)^{d-4}}\right) \\ &= \int \frac{d^{d+2}Y\,\delta(Y^2/2)}{\text{Vol(GL(1))}}\,\frac{(X_1X_3)(X_2X_4)}{(YX_1)(YX_2)(YX_3)(YX_4)}\,\frac{1}{(YI)^{d-4+1}} \\ &\qquad \times (-d+4)\left[2(YX_2) - (YX_1) - (YX_3) - s(YI)\right] \\ &= \epsilon\left[-2s\left(stI^{\text{box}}\right) + 4st\, I^{\text{tri},t} - 4st\, I^{\text{tri},s}\right], \end{aligned} \tag{5.7}$$

where the last line consists of the box, the t-channel triangle, and the s-channel triangle integrals. After summing s and t channel versions of this equation it immediately reproduces Eq. (4.11) of Ref. [43].

It is noteworthy that the right hand side of the differential equation so derived is proportional to ϵ [30]. It is perhaps not too surprising that this structure naturally arises in our approach. If we ignore the effect of the regulator, the combination stI^{box} is invariant under dual conformal transformations in four dimensions. However, the box integral is infrared singular so a regulator is required. Dimensional regularization breaks the invariance, so instead of finding zero on the right-hand-side we find terms proportional to ϵ. Besides leading to simpler differential equations, integrals with such symmetries are expected to have interesting properties, including uniform transcendentality [44] and $d\log$ forms [45]. It would be interesting to further explore these ideas at higher loops, not only for the planar case, but also for nonplanar integrals in the context of the approach of Section VI.

We end this section with some discussions about the applicability of this method to more complicated integral topologies. First, let us look at the number of legs allowed. In this simple example, a conformal boost changes the dimensionless ratio of Mandelstam variables, s/t, while a scaling transformation rescales both s and t. Together these two transformations allow the *whole phase space* of external kinematic invariants to be explored. For massless planar diagrams, this breaks down when there are six or more external legs, because nontrivial conformally invariant cross

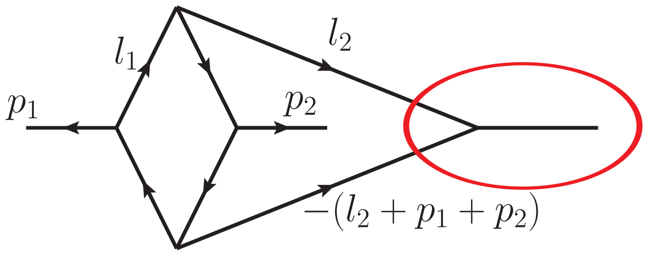

FIG. 8: The crossed triangle-box, with two massless legs p_1 and p_2, and one massive leg shown as a thick line. We remove the right-most part of the diagram enclosed in a (red) ellipse, in order to open up the diagram into a one-loop planar diagram.

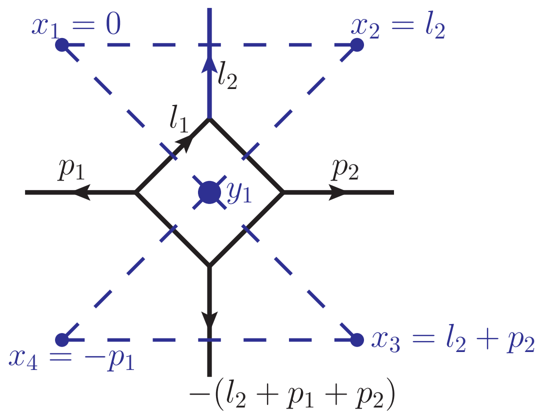

FIG. 9: This figure is obtained from Fig. 8 by removing the rightmost part enclosed in the (red) ellipse, including the massive leg. The result is a planar diagram, allowing dual coordinates x_i to be introduced. Each of the dashed (blue) lines corresponds to one of the six propagators in the integral.

ratios exist [21], and conformal transformations only allow us to explore a subspace of the phase space with the same cross ratios.

Second, consider the ϵ factorization properties of the differential equations for more general integrals. For any integrand that is dual conformal invariant, our method automatically leads to differential equations where there is an explicit factor of ϵ on the right hand side. For more complicated examples beyond the one-loop box, it is generally necessary to perform unitarity-compatible IBP reduction to bring the right hand side into a linear combination of master integrals. Assuming that IBP reduction of the right hand side does not introduce singularities, this gives a symmetry-based understanding of Henn's ϵ form of differential equations. For planar integrals that are not invariant, we still obtain differential equations without raised propagator powers. This allows unitarity-compatible IBP reduction to be used to simplify the differential equations, even though we no longer would have ϵ factorization prior to IBP reduction. Third, the applicability of our method to nonplanar topologies will be demonstrated in the next section, where differential equations are derived for the nonplanar double box by identifying a symmetry analogous to dual conformal symmetry.

VI. NONPLANAR ANALOG OF DUAL CONFORMAL SYMMETRY

In this section we find a nonplanar analog of dual conformal transformations at two loops. We do so by working out the symmetries of two-loop integrals with three or four external legs.

A. Hidden symmetry of a two-loop nonplanar three-point integral

We start by deriving IBP-generating vectors for a two-loop nonplanar integral topology, the crossed triangle-box shown in Fig. 8, with two massless outgoing external momenta p_1 and p_2, and one outgoing massive external momentum $-(p_1+p_2)$ on the right. The kinematics is given by

$$p_1^2 = p_2^2 = 0, \quad (p_1+p_2)^2 = s\,. \tag{6.1}$$

Our strategy is to open up the nonplanar diagram by removing vertices in the graph. This strategy has been pursued in Ref. [23] to find symmetries of leading singularities of nonplanar integrands. Here we will find symmetries of the *complete off-shell* integrand, up to anomalies introduced by dimensional regularization of infrared singularities, similar to the situation in the planar case. A key hint comes from the fact that in the planar case, dual conformal transformations generate polynomial tangent vectors of unitarity cut surfaces, under which each propagator transforms with a polynomial weight as in Eq. (2.8). Therefore, we will first find transformations of nonplanar integrands with this property, before verifying that such transformations are in fact a symmetry of the integrand with appropriate numerators.

To open up the nonplanar diagram in Fig. 8 into a planar one, we remove the massive external leg and the vertex attached to it enclosed in the (red) circle, producing a planar one-loop diagram in Fig. 9, in which two "external" legs depend on the second loop momentum. Dual coordinates may be introduced for this planar one-loop diagram, as illustrated by the dashed lines in Fig. 9. In this case, we find it more convenient to directly work with conformal transformations in d dimensions rather than using the embedding formalism. The inverse propagators are expressed as squared differences between pairs of points in dual spacetime,

$$\begin{aligned}
&\rho_1 = l_1^2 = (y_1-x_1)^2\,, && \rho_2 = (l_1-l_2)^2 = (y_1-x_2)^2\,,\\
&\rho_3 = (l_1-l_2-p_2)^2 = (y_1-x_3)^2\,, && \rho_4 = (l_1+p_1)^2 = (y_1-x_4)^2\,,\\
&\rho_5 = l_2^2 = (x_2-x_1)^2\,, && \rho_6 = [-(l_2+p_1+p_2)]^2 = (x_4-x_3)^2\,,
\end{aligned} \tag{6.2}$$

while the only irreducible numerator can be chosen as

$$\rho_7 = (l_2+p_1)^2 = (x_2-x_4)^2\,. \tag{6.3}$$

While the external momentum p_1 and p_2 each can be written as the difference between two dual coordinates, this is no longer true for the massive external momenta $-(p_1+p_2)$, in contrast to the planar case. Choosing a gauge $x_1=0$ to fix the translation degree of freedom, the dual coordinates are positioned at

$$x_1 = 0, \qquad x_2 = l_2, \qquad y_1 = l_1, \qquad x_3 = l_2+p_2, \qquad x_4 = -p_1\,. \tag{6.4}$$

Using these variables, the crossed triangle-box integral in Fig. 8, with m powers of the irreducible numerator, is

$$\begin{aligned}
I_m^{\text{ctb}} &= \int d^d y_1 \int d^d x_2 \frac{\rho_7^m}{\rho_1\rho_2\rho_3\rho_4\rho_5\rho_6}\\
&= \int d^d y_1 \int d^d x_2 \frac{(x_2-x_4)^{2m}}{(y_1-x_1)^2(y_1-x_2)^2(y_1-x_3(x_2))^2(y_1-x_4)^2(x_2-x_1)^2(x_4-x_3(x_2))^2}\,,
\end{aligned} \tag{6.5}$$

where x_3 is taken to be a function of $x_2 = l_2$.

The expression in Eq. (6.5) is in a form where we can conveniently apply conformal transformations. An infinitesimal transformation, consisting of a conformal boost with parameter b^μ, a scaling with parameter β, and a Lorentz transformation $\Omega^\mu{}_\nu$, is given by

$$\Delta z^\mu = \frac{1}{2} z^2\, b^\mu + (\beta - b\cdot z) z^\mu + \Omega^\mu{}_\nu z^\nu\,. \tag{6.6}$$

Under the transformation, each inverse propagator of the form $(z_1-z_2)^2$ has a weight given by Eq. (2.22),

$$[2\beta - b\cdot(z_1+z_2)]\,. \tag{6.7}$$

For a propagator given by $1/(z_1-z_2)^2$, the weight has an opposite sign. Meanwhile, the integration measures $d^d y_1$ and $d^d x_2$ have a weight given by Eq. (2.23),

$$\frac{\partial \Delta y_1^\mu}{\partial y_1^\mu} = (\beta - b\cdot y_1)\, d\,, \qquad \frac{\partial \Delta x_2^\mu}{\partial x_2^\mu} = (\beta - b\cdot x_2)\, d\,. \tag{6.8}$$

For the nonplanar integral in Eq. (6.5), the total weight, from the integration measures, irreducible numerators, and propagators, is

$$\begin{aligned}
\mathcal{W}(m,b,\beta) =& d\,(\beta - b\cdot y_1) + d\,(\beta - b\cdot x_2) + m\,[2\beta - b\cdot(x_2+x_4)]\\
&- \left(\sum_{i=1}^{4} [2\beta - b\cdot(y_1+x_i)]\right) - [2\beta - b\cdot(x_1+x_2)] - [2\beta - b\cdot(x_3+x_4)]\,,
\end{aligned} \tag{6.9}$$

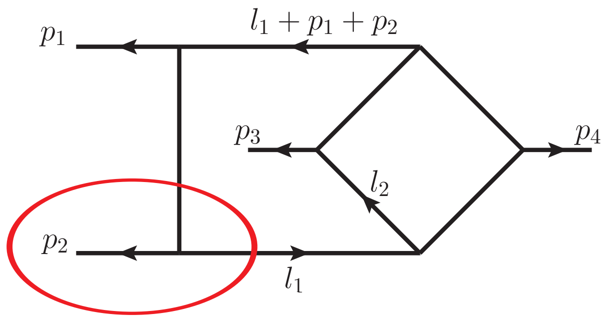

FIG. 10: The two-loop nonplanar crossed box. The part of the diagram enclosed in a red ellipse will later be removed, so that the diagram is broken up into a one-loop planar diagram.

which, using the explicit expression Eq. (6.4), becomes

$$\mathcal{W}(m,b,\beta) = 2\beta(d+m-6) + b\cdot[-(d-4)(l_1+l_2) + 2p_2 - 2p_1 - m(l_2-p_1)]\,. \tag{6.10}$$

We obtain IBP-generating vectors when the transformation Eq. (6.6) leaves both p_1 and p_2 invariant, i.e.

$$\Delta p_1 = \Delta x_1 - \Delta x_4 = 0\,, \qquad \Delta p_2 = \Delta x_3 - \Delta x_2 = 0\,. \tag{6.11}$$

A solution for such a transformation is

$$b = p_2\,, \qquad \beta = -\frac{p_1\cdot p_2}{2} = -\frac{s}{4}\,, \qquad \Omega^{\mu}{}_{\nu} = \frac{1}{2}p_{1\,\nu}p_2^{\mu} - \frac{1}{2}p_{2\,\nu}p_1^{\mu}\,. \tag{6.12}$$

The weight (6.10) is then (using $p_2^2=0$),

$$\begin{aligned}\mathcal{W}(m,p_2,-\frac{s}{4}) &= -\frac{s}{2}(d+m-6) + p_2\cdot[-(d-4)(l_1+l_2) - 2p_1 - m\,(l_2-p_1)]\\ &= \frac{1}{2}(d-4+m)s + \left(d-4+\frac{m}{2}\right)(\rho_7-\rho_6) + \frac{1}{2}(d-4)(\rho_3-\rho_2)\,.\end{aligned} \tag{6.13}$$

Remarkably, the above expression vanishes when $d=4$ and $m=0$. This shows the integrand of the scalar integral I_0^{ctb} is invariant under a nontrivial infinitesimal transformation.[4]

The IBP relation obtained from Eq. (6.13) is

$$\begin{aligned}0 = \int d^d l_1 \int d^d l_2 \frac{\rho_7^m \mathcal{W}(m,p_2,-\frac{s}{4})}{\rho_1\rho_2\rho_3\rho_4\rho_5\rho_6} &= \frac{1}{2}(d-4+m)s\, I_m^{\text{ctb}} + \left(d-4+\frac{m}{2}\right) I_{m+1}^{\text{ctb}}\\ &+ \text{daughter integrals}\,,\end{aligned} \tag{6.14}$$

which reduces all integrals of this topology to the scalar master integral I_0^{ctb} and daughter integrals with canceled propagators. We checked that Eq. (6.14) agrees with maximal-cut IBP relations obtained from computational algebraic geometry. Since this is a single scale integral, differential equations are not useful unless additional scales are introduced [31]; in any case its value is given in Ref. [46].

B. Hidden symmetry of two-loop four-point nonplanar integrals

Consider now the two-loop four-point nonplanar integral with massless external legs displayed in Fig. 10. In this case, if we follow the same procedure as for the nonplanar triangle-box, we find no solution for a generalized dual conformal transformation that leaves the external points invariant, so the construction does not generate IBP relations. However,

[4] By "nontrivial", we mean that the transformation is not a Lorentz transformation (of both external and loop momenta) which trivially leave the integral invariant.

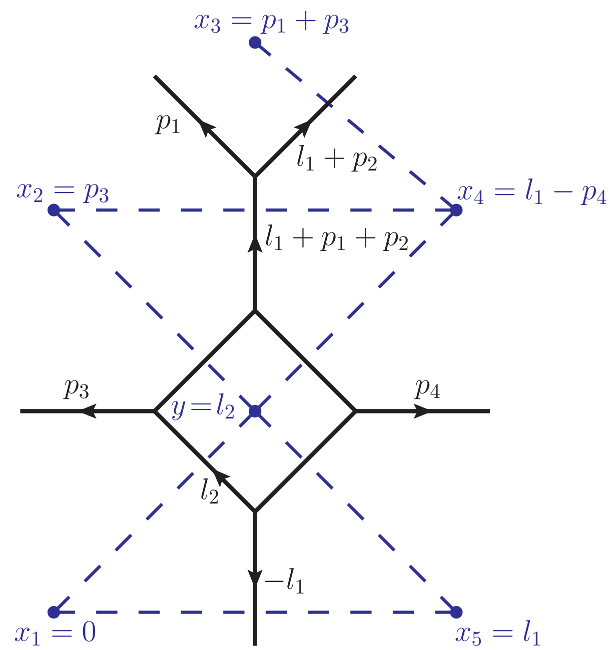

FIG. 11: The planar one-loop diagram obtained by removing the bottom left part of Fig. 10. This allows one to introduce dual coordinates x_i. Each of the dashed lines corresponds to one of the six propagators in the integral.

by relaxing this condition, we have no difficulty finding an invariance of the integrals with appropriate numerators. We use it to construct differential equations along the lines of Section V, implying that the symmetry determines the analytic structure. As we emphasize in the subsequent section, this implies that the nonplanar sector of the two-loop four-point $\mathcal{N} = 4$ super-Yang–Mills amplitude has a hidden symmetry analogous to dual conformal symmetry.

In order to define an analog of dual conformal symmetry we open the diagram by removing the part of the diagram in Fig. 10 enclosed by a red ellipse, including the leg with external momentum p_2. This opens up the two-loop diagram into a one-loop diagram with "fake", loop-momentum-dependent external legs as shown in Fig. 11. With this construction every propagator momentum is expressed as the difference between two dual-space points. Each of the external momenta p_1, p_3, and p_4 is also expressed as the difference between two dual coordinates, though the same is *not* true for p_2 (in contrast to the planar case). Although one might worry that this may cause problems with the construction, we shall see that it does not.

We take the inverse propagators as,

$$\begin{aligned}
&\rho_1 = l_1^2 = (x_1 - x_5)^2, \qquad \rho_2 = (l_1 + p_2)^2 = (x_4 - x_3)^2\,, \qquad \rho_3 = (l_1 + p_1 + p_2)^2 = (x_4 - x_2)^2\,,\\
&\rho_4 = (l_1 - l_2)^2 = (y - x_5)^2\,, \qquad \rho_5 = l_2^2 = (y - x_1)^2\,,\\
&\rho_6 = (l_2 - p_3)^2 = (y - x_2)^2\,, \qquad \rho_7 = (l_2 - l_1 + p_4)^2 = (y - x_4)^2\,,
\end{aligned} \tag{6.15}$$

and we have chosen the gauge

$$x_1 = 0\,, \qquad x_2 = p_3\,, \qquad x_3 = p_1 + p_3\,, \qquad x_4 = l_1 - p_4\,, \qquad x_5 = l_1\,, \qquad y = l_2\,. \tag{6.16}$$

We can define two numerators (which are not independent irreducible numerators, nevertheless are convenient for notational purposes),

$$\rho_8 = (l_1 - p_3)^2 = (x_5 - x_2)^2, \quad \rho_9 = (l_1 - p_4)^2 = (x_4 - x_1)^2\,. \tag{6.17}$$

We also note that

$$u = (p_1 + p_3)^2 = (x_3 - x_1)^2\,. \tag{6.18}$$

Refs. [32, 33] expressed the two-loop four-point amplitude in terms of integrals that have only logarithmic singularities, reflecting a property of the full amplitude. In this representation the two nonplanar integrands that appear

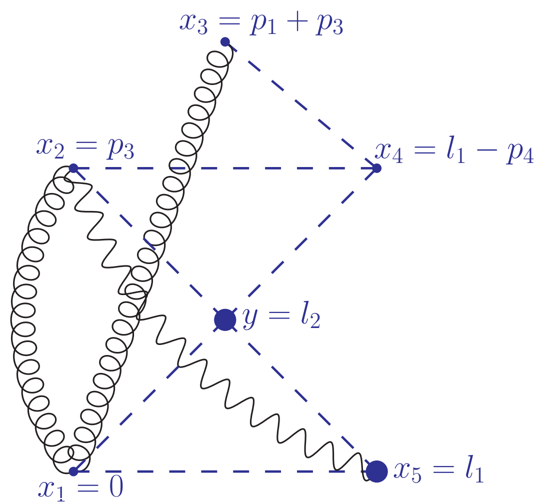

FIG. 12: Weight diagram for the integrand (6.19) under the conformal boost (6.23).

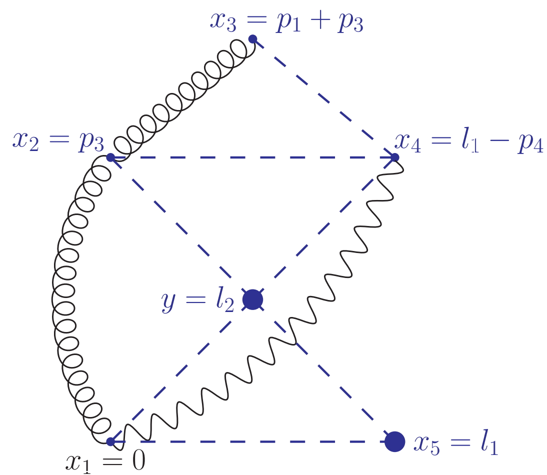

FIG. 13: Weight diagram for the integrand (6.20) under the conformal boost (6.23).

in the amplitude (up to relabelings) are,

$$\Omega_1^{\rm NP} = d^d l_1 d^d l_2 \frac{su\rho_8}{\rho_1 \ldots \rho_7}\,, \tag{6.19}$$

$$\Omega_2^{\rm NP} = d^d l_1 d^d l_2 \frac{st\rho_9}{\rho_1 \ldots \rho_7}\,. \tag{6.20}$$

The normalization of each is chosen so it has unit leading singularity [33]. Our task will be to find a hidden symmetry responsible for the simple analytic properties after integration.

A conformal boost on the x_i's and y with parameter b^μ gives

$$\begin{aligned}\Delta p_4 &\equiv \Delta x_5 - \Delta x_4 & (6.21)\\ &= (l_1 \cdot p_4) b - (p_4 \cdot b) l_1 - (l_1 \cdot b - p_4 \cdot b) p_4\,. & (6.22)\end{aligned}$$

The appearance of loop momentum in the transformation of external momentum is not surprising, given that once we cut a nonplanar diagram open internal momenta effectively become "external". This is, of course, not desirable if we wish to use the transformations to construct differential equations. To remove the loop-momentum dependence of this variation, we simply choose

$$b = p_4\,. \tag{6.23}$$

While this restricts the transformations to a subset of conformal transformations, we shall see that this is sufficient for constructing differential equations analogous to those of the planar case. Applying the conformal transformation gives

$$\begin{aligned}
\Delta p_1 &= \Delta x_3 - \Delta x_2 = \frac{1}{2}(u\,p_4 - t\,p_3 + u\,p_1)\,,\\
\Delta p_3 &= \Delta x_2 - \Delta x_1 = -\frac{1}{2}s\,p_3\,,\\
\Delta p_4 &= \Delta x_5 - \Delta x_4 = 0\,,
\end{aligned} \tag{6.24}$$

so that the masslessness of these three external legs is preserved. In fact, the masslessness of p_1, p_3, and p_4 are trivially preserved by the properties of conformal transformations, since each of these three momenta is the difference between two points in dual space. Remarkably, the same is nontrivially true of the second leg, as can be readily checked,

$$\Delta p_2^2 = 2p_2\cdot\Delta p_2 = -2p_2\cdot(\Delta p_1+\Delta p_3+\Delta p_4) = 0\,. \tag{6.25}$$

This ensures that the transformation preserves the masslessness of all external legs, which is essential for the construction to be useful. In addition we have,

$$\begin{aligned}
\Delta s &= \Delta(2p_3\cdot p_4) = 2(p_3\cdot\Delta p_4 + p_4\cdot\Delta p_3) = -\frac{s}{2}s\,,\\
\Delta t &= \Delta(2p_1\cdot p_4) = 2(p_1\cdot\Delta p_4 + p_4\cdot\Delta p_1) = -\frac{t+2s}{2}t\,.
\end{aligned} \tag{6.26}$$

Note that applying Eq. (2.22) directly gives,

$$\Delta u = -b\cdot(x_3+x_1)u = -p_4\cdot(p_1+p_3)u = -\frac{u}{2}(t+s) = -\Delta s - \Delta t\,, \tag{6.27}$$

which is consistent with momentum conservation. It will be convenient for later purposes to write down the weights of s, t, and u under the transformation, as dot products between $(-p_4)$ and other momenta,

$$\begin{aligned}
\mathcal{W}_s &\equiv \frac{\Delta s}{s} = -p_4\cdot p_3\,,\\
\mathcal{W}_t &\equiv \frac{\Delta t}{t} = -p_4\cdot(p_1+2p_3)\,,\\
\mathcal{W}_u &\equiv \frac{\Delta u}{u} = -p_4\cdot(p_1+p_3)\,.
\end{aligned} \tag{6.28}$$

Meanwhile, a numerator of the form $(z_i-z_j)^2$ has the weight $-p_4\cdot(z_i+z_j)$, while an extra minus sign is present in the weight for a propagator of the form $1/(z_i-z_j)^2$. The weight of the integration measure is given by $-d\,p_4\cdot(x_5+y) = -d\,p_4\cdot(l_1+l_2)$. We can now straightforwardly prove that the nonplanar contributions to the $\mathcal{N}=4$ super-Yang–Mills amplitudes are invariant under this transformation. Namely, in $d=4$, the two integrands Eqs. (6.19) and (6.20) in the amplitudes transform as,

$$\Delta\Omega_1^{\rm NP} = \Delta\Omega_2^{\rm NP} = 0\,. \tag{6.29}$$

A pictorial way to derive the above equation is as follows. We have shown that the weights of the Mandelstam variables, numerators, propagators, and integration measures are each written in the form $-p_4\cdot W$ for some "weight vector" W^μ. So it is convenient to represent the weight of the integrand diagrammatically as in Fig. 12 and 13. In the diagrams, the weight of a propagator of the form $1/(z_1-z_2)^2$ is represented by a dashed line connecting two points z_1 and z_2, contributing $-(z_1^\mu+z_2^\mu)$ to the weight vector W^μ. The weight of a numerator of the form $(z_1-z_2)^2$ is represented by a wiggly line connecting two points z_1 and z_2, contributing $z_1^\mu+z_2^\mu$ to the weight vector W^μ. The weight of the Mandelstam variables appearing in the numerator is represented by a coil-like line connecting two points z_1 and z_2, again contributing $z_1^\mu+z_2^\mu$ to the weight vector W^μ. To reproduce Eq. (6.28), for $\mathcal{W}_s$ we choose $z_1=0=x_1$ and $z_2=p_3=x_2$, for $\mathcal{W}_t$ we choose $z_1=p_3=x_2$ and $z_2=p_1+p_3=x_3$, and for $\mathcal{W}_u$ we choose $z_1=0=x_1$ and $z_2=p_1+p_3=x_3$. Finally, the weight of the integration measure is indicated by large black dots at the two points x_5 and y. In our notation, a large black dot at any point z contributes $d\,z^\mu$ to the weight vector W^μ, with d being the spacetime dimension. The total weight vector $\sum W^\mu$ can now be read off from the diagram in the following manner: at each vertex (i.e. a dual space point) z^μ, we count the number of wiggly lines and coil-like lines joining the vertex, subtract the number of dashed lines joining the vertex, and add the spacetime dimension d if a large black dot appears

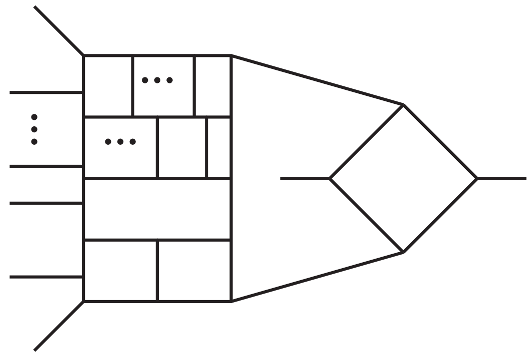

FIG. 14: An illustrative multi-loop diagram, where a similar analysis as for the nonplanar double box in Fig. 10 identifies a hidden symmetry.

at the vertex. The final number is multiplied by z^μ and included in $\sum W^\mu$. For the first integrand $\Omega_1^{\rm NP}$ in Eq. (6.19), the weight diagram Fig. 12 gives the weight vector

$$\begin{aligned}\sum W_1^\mu &= (2-2)x_1^\mu + (2-2)x_2^\mu + (1-1)x_3^\mu - 3x_4^\mu + (1-2+d)x_5^\mu + (-4+d)y^\mu \\ &= (d-4)(l_1^\mu + l_2^\mu) + 3p_4^\mu \,. \end{aligned} \tag{6.30}$$

Using this, we arrive at

$$\Delta\Omega_1^{\rm NP} = \Big(-p_4\cdot \sum W_1\Big)\,\Omega_1^{\rm NP} = -(d-4)p_4\cdot(l_1+l_2)\Omega_1^{\rm NP}\,. \tag{6.31}$$

Since the transformation changes the Mandelstam variables as in Eq. (6.26), we arrive at a differential equation for the Feynman integral,

$$\left(-\frac{s^2}{2}\frac{\partial}{\partial s} - \frac{t(t+2s)}{2}\frac{\partial}{\partial t}\right)\int \Omega_1^{\rm NP} = -(d-4)\int p_4\cdot(l_1+l_2)\,\Omega_1^{\rm NP}\,. \tag{6.32}$$

Since $\Omega_1^{\rm NP}$ has mass dimension $2(d-4)$, we trivially obtain another differential equation from the simultaneous scaling of all Mandelstam variables,

$$\frac{s}{2}\left(s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t}\right)\int \Omega_1^{\rm NP} = \frac{s}{2}(d-4)\int \Omega_1^{\rm NP}\,. \tag{6.33}$$

Adding Eqs. (6.32) and (6.33), we obtain the derivative of the integral against t only

$$\frac{tu}{2}\frac{\partial}{\partial t}\int \Omega_1^{\rm NP} = \epsilon\left[2\int p_4\cdot(l_1+l_2) - s\right]\Omega_1^{\rm NP}\,. \tag{6.34}$$

For the second integrand $\Omega_2^{\rm NP}$ in Eq. (6.20), again we read off the weight vector from Fig. 13. This leads to results similar to those for $\Omega_1^{\rm NP}$,

$$\Delta\Omega_2^{\rm NP} = \Big(-p_4\cdot \sum W_1\Big)\,\Omega_1^{\rm NP} = -(d-4)p_4\cdot(l_1+l_2)\Omega_2^{\rm NP}\,, \tag{6.35}$$

and

$$\frac{tu}{2}\frac{\partial}{\partial t}\int \Omega_2^{\rm NP} = \epsilon\left[2\int p_4\cdot(l_1+l_2) - s\right]\Omega_2^{\rm NP}\,. \tag{6.36}$$

If there were no infrared singularities, we would be able to set $\epsilon = 0$ and the symmetry would be exact.

It is interesting that a similar analysis extends to any nonplanar diagram with a single crossed box at any loop order, as illustrated in Fig. 14. In particular, if we consider this diagram with a numerator obtained from a corresponding planar dual conformal invariant one, except for the single crossed box which is given a similar factor as in the two-loop cases (6.19) and (6.20), then the resulting nonplanar integral possesses a higher-loop analog of dual conformal symmetry. A way to show the invariance is to remove a three vertex from the crossed box and perform a similar analysis to the one of the previous section for the two-loop crossed box. In this case, it is convenient to remove a vertex from the crossed double box, instead of the other parts of the diagram. We have checked that the analog of dual conformal symmetry is present for this class of nonplanar integrals, at any loop order. As for the two-loop case, we can use it to generate differential equations to constrain the integrals whose right hand side is proportional to the dimensional regularization parameter, ϵ.

VII. INVARIANCE OF THE NONPLANAR TWO-LOOP FOUR-POINT $\mathcal{N} = 4$ SUPER-YANG–MILLS AMPLITUDE

In the previous section we identified a new symmetry of the nonplanar integrands that appear in the two-loop four-point amplitude of $\mathcal{N} = 4$ super-Yang–Mills theory. In this section we comment on symmetries of the full amplitude.

From Eq. (3.15) of Ref. [33] we have the full two-loop four-point amplitude of $\mathcal{N} = 4$ super-Yang-Mills theory as

$$\begin{aligned}\mathcal{A}_4^{\text{2-loop}} = -\frac{g^6}{4(2\pi)^{2D}} \sum_{\mathcal{S}_4} \Big[& c^{\rm P}_{1234} A^{\rm tree}(1,2,3,4) \int \Omega^{\rm P} \\ & - c^{\rm NP}_{1234} \Big(A^{\rm tree}(1,2,4,3) \int \Omega^{\rm NP}_1 + A^{\rm tree}(1,2,3,4) \int \Omega^{\rm NP}_2 \Big) \Big] ,\end{aligned} \tag{7.1}$$

where $c^{\rm P}_{1234}$ and $c^{\rm NP}_{1234}$ are the planar and nonplanar color factors obtained by dressing the diagrams in Figs. 1 and 10 with $\tilde{f}^{abc}$ color factors at each diagram vertex. The planar integrands are given in Eq. (4.37) while the nonplanar integrands are given in Eqs. (6.19) and (6.19). The $A^{\rm tree}$'s are color-ordered tree amplitudes of $\mathcal{N} = 4$ super-Yang–Mills theory, with the indicated ordering of legs. The overall sum is over all 24 permutations of the external legs; the permutations act on the external color, polarization and momentum labels. The form in Eq. (7.1) differs from the one originally given in Ref. [34] by terms that vanish via the color Jacobi identity. In the original form, the individual nonplanar integrals do not reflect the analytic properties of the final amplitude, such as having only logarithmic singularities and no poles at infinity.

In order to understand the transformation properties we divide the amplitude into sectors determined by the tree amplitude prefactors. These tree amplitudes have differing overall weights under the transformations, which are easy to determine using the identities [47],

$$st A^{\rm tree}(1,2,3,4) = su A^{\rm tree}(1,2,4,3) = tu A^{\rm tree}(1,3,2,4) \,. \tag{7.2}$$

From here we can see that the tree amplitudes transform with different overall weights under Eq. (6.24),

$$\Delta\left(\frac{A^{\rm tree}(1,2,3,4)}{A^{\rm tree}(1,2,4,3)}\right) = \Delta\left(\frac{t}{u}\right) = -\frac{st}{2u} \,, \tag{7.3}$$

where we used Eqs. (6.26) and (6.27).

In Eq. (7.1), the coefficient of each tree amplitude factor is invariant under the four-dimensional symmetry. In Refs. [33], the ordering of the tree factors were chosen to adjust the factors of s, t, and u so that the remaining integrals have unit leading singularities. Not surprisingly, these factors are exactly what is needed to make the coefficient of each tree invariant under the four-dimensional symmetry.

As a side note, we can adjust the transformations in each sector so that a uniform transformation is applied to the external momenta in all sectors of the amplitude. In doing so, the transformations on internal momenta necessarily differ in the various sectors, as expected from the fact that there is no uniform sets of momenta or dual variables in the nonplanar sector. This may be accomplished by adjusting Lorentz and scaling transformations. However, since the different sectors transform with a different weight there is no need to do this.

The transformations described in the previous section can be taken as a direct analog of dual conformal symmetry of planar $\mathcal{N} = 4$ super-Yang–Mills theory, but applicable to the nonplanar sector as well. Like dual conformal symmetry in the planar case, the infinitesimal generators of the new symmetries can be identified as polynomial tangent vectors of unitarity cut surfaces.

This opens the possibility of finding numerators of higher-loop integrals with desired properties of having simple analytic properties and associated DEs, not by detailed studies of the singularity structure of the integrands [30, 32, 33], but by demanding that given integrands be invariant under symmetries analogous to dual conformal symmetry. For nonplanar diagrams that can be obtained from a planar one by a single replacement of a box subdiagram by a crossed box, as in Fig. 14, the obvious candidate transformations follow those described in the previous section. It would be very interesting to systematically study these cases, as well as ones with multiple twists. We expect such integrals to be direct building blocks for nonplanar $\mathcal{N}=4$ super-Yang–Mills amplitudes. More generally, it seems likely that a symmetry along the lines described here is responsible for the simple analytic properties [32, 33] of general nonplanar amplitudes at any loop order.

VIII. IBP-GENERATING VECTORS AND LANDAU EQUATIONS

In previous sections we obtained IBP-generating vectors from the point of view of conformal symmetries. In the following we will see that the symmetry generators occur with vanishing or degenerate mass configurations. We will show that the defining relations of the vectors connect directly to Landau equations, which determine the presence/absence of the exceptional vectors.

For simplicity we consider N-point one-loop integrals in the following. When IBP-generating vectors are expressed in dual coordinates, they fulfill a number of consistency requirements [22]: (i) They are tangent to a quadric $(YY)=0$, which defines the integration contour, (ii) being vector fields in projective space, they are homogeneous in the variable Y, (iii) and vectors are defined modulo vectors proportional to Y^a which induce a $GL(1)$ scaling and give trivial identities.

Maintaining propagator powers of the integrals, yields the further conditions,

$$V^a \frac{\partial}{\partial Y^a}(X_i Y) = \mathcal{W}_i (X_i Y)\,, \qquad 1 \le i \le N\,, \tag{8.1}$$

and in total we have the linear relations,

$$(X_i V) - (X_i Y)\mathcal{W}_i = 0\,, \qquad 2(YV) = 0\,, \tag{8.2}$$

for the data of the IBP-generating vector V^a and the weight functions $\mathcal{W}_k$. The above relations hold up to terms proportional to (YY).

It is convenient to write Eqs. (8.2) in matrix form in terms of the vectors in the problem. To this end we expand the IBP-generating vector in terms of the infinity point, X_j and transverse points N_j, the number of which varies in dimensional regularization, but do not include Y, i.e.

$$V = \left(\sum_{j=1}^{N} X_j v^j\right) + I v^{N+1} + \left(\sum_{k=N+2}^{d+2} N_k v^k\right). \tag{8.3}$$

We obtain the matrix,

$$C = \begin{pmatrix} (X_1X_1) & (X_1X_2) & \cdots & 1 & 0 & 0 & \cdots & -(X_1Y) & 0 & 0 \\ (X_2X_1) & (X_2X_2) & \cdots & 1 & 0 & 0 & \cdots & 0 & -(X_2Y) & 0 \\ \cdots & & & & & & & & & \\ (X_NX_1) & (X_NX_2) & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & -(X_NY) \\ 2(X_1Y) & 2(X_2Y) & \cdots & 2(YI) & 2(YN_1) & 2(YN_2) & \cdots & 0 & \cdots & 0 \end{pmatrix}. \tag{8.4}$$

which has been simplified using our gauge choice, $(X_iI)=1$ of the dual conformal embedding, and using the transversality $(N_iX_j)=(N_iI)=0$. This matrix acts on the column vector $(v^i,\mathcal{W}_k)$. The entries of the matrix C are monomials of either degree 0 or degree 1 in Y, which leads to, in an approach using computational algebraic geometry (not covered here), simple syzygy equations that could be solved efficiently. The top-left $(N+1)\times(N+1)$ sub-block of the above matrix is closely related to the "embedding space Gram matrix" used in Section III. The top-left $N\times N$ sub-block defines the Cayley determinant $\Delta_C = \det(\{(X_iX_j)\}_{i,j=1,N})$ [40].

From Eq. (8.2) we see that the IBP-generating vectors $(v^i,\mathcal{W}_k)$ correspond to the kernel of the matrix C. The generic vectors in the kernel of the matrix C are obtained by applying Cramer's rule, with vector components being appropriate minors of the linear system. Cramer's rule gives,

$$c_{a_1}(a_2,a_3,\cdots,a_{N+2}) - c_{a_2}(a_1,a_3,\cdots,a_{N+2}) + \cdots + (-1)^{N+1} c_{a_{N+2}}(a_1,a_2,\cdots,a_{N+1}) = 0\,. \tag{8.5}$$

The expressions c_{a_i} stand for columns of the original matrix C and $(a_1, a_2, \cdots)$ stand for the leading minors of the matrix C given by the determinants of the matrices composed of the columns $(a_1, a_2, \cdots) \equiv \det(c_{a_1}, c_{a_2}, \cdots)$.[5] Eq. (8.5) is a statement that $N+2$ vectors in $N+1$ dimensions are necessarily linearly dependent.

Interesting solutions are the ones which do not vanish on the cuts, i.e. on the surfaces defined by $(X_i Y) = 0$. The respective IBP-generating vectors are then obtained by choosing $(N+1)$ columns, where we distinguish the following subclasses using the entries of the last row:

1. Two (YN_i) factors and no (YI) factor,
$$V = \Delta_C \, (YN_{[i|})N_{|j]} \,. \tag{8.6}$$
We can factor out Δ_C from the above expression, so we still use the transverse-space rotation vector that is left, when $\Delta_C = 0$.

2. One (YN_i) factor and one (YI) factor,
$$\begin{aligned} V \;&=\; \Delta_C \, (YN_k)I - (1, 2, \cdots, N+1)N_k \\ &\quad + \sum_{j=1}^{N} (-1)^{N+j+1} X_j (1, 2, \cdots, \widehat{j}, \cdots, N+1, N+1+k) \,, \end{aligned} \tag{8.7}$$
which matches the vector (4.3) up to normalization by Δ_C. The hat-symbol, $\widehat{j}$ indicates that the j^{th} column is omitted.

The above are the generic IBP-generating vectors. We have omitted the vectors that vanish on the maximal cut.

In addition, exceptional vectors appear for particular kinematic configurations. For one-loop integrals we consider the case of a vanishing Cayley determinant, $\Delta_C = 0$. A maximal-cut analysis predicts a scaling vector in addition to rotation vectors [12, 48]. The following construction shows how the vector arises from Lie-algebra generators acting on the Gram-matrix $\{(X_i X_j)\}_{i,j=1,N}$. We consider a generator with the first $(N+1)$ components non-vanishing and allow for non-vanishing $\mathcal{W}$'s. The generator has to give zero when contracted with either of the rows. Starting with the last row we find the general form,

$$v^i = \sum_{b=1}^{N+1} \omega^{ij} C_{N+1,j} \,, \tag{8.8}$$

and an antisymmetric matrix ω^{ij}. We attempt to find a solution of minimal degree, thus we assume ω^{ij} to be constant. In the language of computational algebraic geometry, this is the general form to parametrize the principal syzygies of the first $(N+1)$ elements of the last row.

An interesting consistency condition appears when considering the (YI) terms. The argument uses that (YI) enters through the last column of ω^{ij}, with $\omega^{N+1,N+1} = 0$ from antisymmetry. Thus, in order for the (YI) to cancel for each contraction, we require that the column vector $\omega^{N+1,j}$ is in the kernel of the $\{(X_i X_j)\}_{i,j=1,N}$ block of the linear system, confirming that the condition $\Delta_C = 0$ is necessary for these types of vectors.

This brings us to a close connection between the C matrix in Eq. (8.4) and the Landau equations [24, 25, 40], whose embedding-space version is,

$$\sum_{i=1}^{N} \alpha_i X_i + Y = 0 \,, \qquad \alpha_j (X_j Y) = 0 \,. \tag{8.9}$$

(For simplicity we omitted the extension of the Landau equations which in addition capture the singularities at infinity.) The leading Landau singularity appears for all propagators vanishing, $(X_j Y) = 0$, and requiring that the points X_i and Y must be linearly dependent. The equations are easiest analyzed when contracted with an independent set of points $\{Y, X_j, N_k, I\}$. The non-trivial conditions then are, setting $(X_j Y) = 0$,

$$(N_i Y) = 0 \,, \qquad \sum_{j=1}^{N} \alpha_j (X_j X_k) = 0 \,, \qquad \left(\sum_{i=1}^{N} \alpha_i (X_i I) \right) + (YI) = 0 \,. \tag{8.10}$$

---

[5] This notation has been used in e.g. Ref.[45] in the literature.

Remarkably, this can be re-written, using the transpose of the C matrix, as

$$C^T \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \\ 1 \end{pmatrix} = 0\,. \tag{8.11}$$

The conditions Eq. (8.10) in turn imply $\Delta_C = 0$ while the overall normalization of the α's is fixed by the last equations in (8.10) using the gauge conditions $(X_i I) = (YI) = 1$. In this way the Landau conditions (8.9) determining a singular point of the on-shell surface relate to the solutions of the IBP-generator equations (8.2).

Thus, we have given the form of the generic IBP-generating vectors in dual coordinates. In addition, we showed that exceptional vectors appear for degenerate kinematic configurations. We pointed out the close relation between the defining equations for unitarity-compatible vectors fields and the Landau equations. In this way, Landau equations signal the appearance of exceptional IBP-generating vectors. This connection between Landau equations and IBP relations might be expected, given that Landau equations signal a singular kinematic configuration in which the set of master integrals is reduced and additional integral relations appear.

The minor construction generalizes naturally to higher-loop computations, so that minors of the linear system of the IBP-generator equations play a central role for constructing IBP-vectors [12]. Additional investigations are needed to systematize the description of exceptional vectors at higher loops. Here the on-shell surfaces have a more complicated topology, with the characteristic sizes determined by combinations of the kinematic invariants (like the Cayley determinant Δ_C).

IX. CONCLUSION

In this paper we studied hidden symmetries of $\mathcal{N} = 4$ super-Yang–Mills theory as a means for generating compact integration-by-parts (IBP) relations [13] and differential equations (DEs) [28] for loop integrals encountered in generic theories. For the planar case, the hidden symmetry is the well-studied dual conformal symmetry [2]. By exploiting the connection between dual conformal symmetry and polynomial tangent vectors of unitarity cut surfaces, we were able to find an analogous symmetry for the nonplanar sector of the two-loop four-point amplitude as well. Besides being useful for generating IBP relations and DEs, this points to the exciting possibility that dual conformal symmetry can be generalized to the nonplanar sector of $\mathcal{N} = 4$ super-Yang–Mills theory.

Dual conformal transformations and their nonplanar analogs have the important property that they do not increase propagator powers, resulting in IBP relations and DEs that are naturally compatible with unitarity [14]. Such IBP relations had been previously described using computational algebraic geometry [14, 17, 18]. Our approach, based on exploiting hidden symmetries, provides new analytic insights and on the practical side gives compact expressions for the IBP-generating vectors and DEs.

In describing the symmetries we found it useful to work with both "direct" dual conformal transformations in d dimensions and the embedding formalism [27], which linearizes the transformations by going to $(d + 2)$ dimensions. The Gram matrix defined in the embedding formalism also clarifies the connection to Landau equations.

To illustrate these ideas, we presented a variety of examples at one and two loops. With up to four massless legs and a small number of mass parameters, it is straightforward to find several dual conformal transformations which leave the external momenta invariant, and lead to a sufficient number of IBP relations to solve generic cases. For example, the dual conformal transformations generate a complete set of IBP relations for the planar two-loop double box integral. We also studied a five-point example, namely the planar penta-box integral. In this case, we need additional IBP-generating vectors from combining separate conformal transformations for the left loop and right loop, generalizing the strategy of Ref. [12]. These additional vectors still have a simple analytic form. For illustration, we also looked at a simpler three-point nonplanar integral, and obtained IBP relations that reduce all integrals to top-level master integral and daughter integrals.

We also described DEs, where the integrals do not have raised propagator powers, for both planar and nonplanar cases that arise when external momenta are allowed to change under the transformation. For one- and two-loop integrals with appropriately chosen numerators that make the transformation weights cancel in four dimensions, the method directly gives a DEs where the right hand side is proportional to the dimensional regularization parameter ϵ [30]. This holds before IBP reduction to a basis of master integrals, because the equations follow from a symmetry that is exact in four dimensions. For massless kinematics, the method is applicable with up to five external legs. At higher points, when nontrivial conformal cross ratios are present, the method generates a subset of the DEs.

Our results point to promising directions for future studies. In various one- and two-loop examples we showed the utility of dual conformal invariance for generating both IBP relations and DEs, as well as presented a nonplanar symmetry analogous to dual conformal symmetry. An obvious direction for future studies is to try to generalize this to arbitrary loop orders and for any number of external legs. The unitarity-compatible IBP-generating vectors and DEs constructed via dual conformal symmetry and its generalizations are particularly simple, making it desirable to extend these ideas as widely as possible. The ability to generate relatively simple DEs becomes especially attractive when existing methods suffer from computational bottlenecks that occur in more complicated cases. It is also worth studying whether the compact expressions generated from our symmetry considerations can improve computational efficiency in numerical unitarity approaches at two loops and beyond [8]. We also noted an important connection between IBP-generating vectors and Landau equations, which would be interesting to pursue.

On the more formal side, we know that dual conformal symmetry [2] strongly restricts the analytic properties of the planar sector of $\mathcal{N} = 4$ super-Yang–Mills theory. In particular, the integrands have no double poles or poles at infinity [45]. These analytic properties also appear to carry over to the nonplanar sector [32, 33]. Here we took initial steps to identify a symmetry that can explain this. We explicitly constructed a symmetry of the nonplanar two-loop four-point $\mathcal{N} = 4$ amplitude, and used it to construct a differential equation for determining its value. As in the planar case, the symmetry is intimately connected to polynomial tangent vectors of unitarity cut surfaces. As for dual conformal invariance the symmetry is anomalous due to infrared singularities. We noted that for the class of integrals with a single crossed box and the remaining part planar, the symmetry extends straightforwardly to all loop orders with an arbitrary number of external legs. An important next step would be to extend this to more general nonplanar cases.

We look forward to exploring these ideas for simplifying computations of multi-loop integrals needed for scattering cross sections at particle colliders, as well as for understanding hidden symmetries of the nonplanar sector of $\mathcal{N} = 4$ super-Yang–Mills theory. These two issues are intertwined, as we found here.

Acknowledgments

We thank Samuel Abreu, Fernando Febres Cordero, Julio Parra-Martinez, Ben Page, Chia-Hsien Shen, Jaroslav Trnka and Yang Zhang for many useful and interesting discussions. This research is supported by the Department of Energy under Award Numbers DE-SC0009937. H.I.'s work is supported by a Marie Skłodowska-Curie Action Career-Integration Grant PCIG12-GA-2012-334228 of the European Union. This research is supported by the Munich Institute for Astro- and Particle Physics (MIAPP) of the DFG cluster of excellence 'Origin and Structure of the Universe'.

Appendix A: Sub-loop IBP-generating vectors for the penta-box

In this appendix, we tabulate the antisymmetric matrices in Eq. (4.49) of Subsection IV E, which parametrize conformal transformations that leave a subset of external momenta invariant. The matrices are,

$$\omega_{(1)} = \begin{pmatrix} 0 & s_{34} & s_{51} & s_{34}-s_{51} & s_{23} & s_{23}s_{34} \\ -s_{34} & 0 & 0 & -s_{34} & s_{12}-s_{45} & -s_{34}s_{45} \\ -s_{51} & 0 & 0 & -s_{51} & s_{23}-s_{45} & -s_{45}s_{51} \\ -s_{34}+s_{51} & s_{34} & s_{51} & 0 & s_{12} & s_{12}s_{51} \\ -s_{23} & -s_{12}+s_{45} & -s_{23}+s_{45} & -s_{12} & 0 & -s_{12}s_{23} \\ -s_{23}s_{34} & s_{34}s_{45} & s_{45}s_{51} & -s_{12}s_{51} & s_{12}s_{23} & 0 \end{pmatrix},$$

$$\omega_{(2a)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -s_{45} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -s_{23} & 0 \\ 0 & s_{45} & 0 & s_{23} & 0 & s_{23}s_{45} \\ 0 & 0 & 0 & 0 & -s_{23}s_{45} & 0 \end{pmatrix},$$

$$\omega_{(2b)} = \begin{pmatrix} 0 & -s_{34} & -s_{23}+s_{51} & 0 & 0 & -s_{23}s_{34} \\ s_{34} & 0 & s_{45} & -s_{12}+s_{34} & 0 & s_{34}s_{45} \\ s_{23}-s_{51} & -s_{45} & 0 & -s_{51} & 0 & -s_{45}s_{51} \\ 0 & s_{12}-s_{34} & s_{51} & 0 & 0 & s_{12}s_{51} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ s_{23}s_{34} & -s_{34}s_{45} & s_{45}s_{51} & -s_{12}s_{51} & 0 & 0 \end{pmatrix},$$

$$\omega_{(2c)} = \begin{pmatrix} 0 & 0 & 0 & 0 & s_{23} & 0 \\ 0 & 0 & 0 & 0 & -s_{45} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -s_{23} & s_{45} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{A1}$$

[1] J. M. Drummond, J. Henn, V. A. Smirnov and E. Sokatchev, JHEP **0701**, 064 (2007) [hep-th/0607160];
J. M. Henn, J. Phys. A **44**, 454011 (2011) [arXiv:1103.1016 [hep-th]];
V. Del Duca, L. J. Dixon, J. M. Drummond, C. Duhr, J. M. Henn and V. A. Smirnov, Phys. Rev. D **84**, 045017 (2011) [arXiv:1105.2011 [hep-th]];
B. Basso and L. J. Dixon, Phys. Rev. Lett. **119**, no. 7, 071601 (2017) [arXiv:1705.03545 [hep-th]].

[2] J. M. Drummond, G. P. Korchemsky and E. Sokatchev, Nucl. Phys. B **795**, 385 (2008) [arXiv:0707.0243 [hep-th]].

[3] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Nucl. Phys. B **425**, 217 (1994) [hep-ph/9403226].
Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Nucl. Phys. B **435**, 59 (1995) [hep-ph/9409265].

[4] G. Ossola, C. G. Papadopoulos and R. Pittau, Nucl. Phys. B **763**, 147 (2007) [hep-ph/0609007].

[5] D. Forde, Phys. Rev. D **75**, 125019 (2007) [arXiv:0704.1835 [hep-ph]].

[6] C. F. Berger, Z. Bern, L. J. Dixon, F. Febres Cordero, D. Forde, H. Ita, D. A. Kosower and D. Maitre, "An Automated Implementation of On-Shell Methods for One-Loop Amplitudes," Phys. Rev. D **78** (2008) 036003 [arXiv:0803.4180 [hep-ph]].

[7] C. F. Berger *et al.*, Phys. Rev. Lett. **102**, 222001 (2009) [arXiv:0902.2760 [hep-ph]];
R. K. Ellis, K. Melnikov and G. Zanderighi, Phys. Rev. D **80**, 094002 (2009) [arXiv:0906.1445 [hep-ph]];
F. Cascioli, P. Maierhofer and S. Pozzorini, Phys. Rev. Lett. **108**, 111601 (2012) [arXiv:1111.5206 [hep-ph]];
G. Bevilacqua, M. Czakon, M. V. Garzelli, A. van Hameren, A. Kardos, C. G. Papadopoulos, R. Pittau and M. Worek, Comput. Phys. Commun. **184**, 986 (2013) [arXiv:1110.1499 [hep-ph]];
G. Cullen, N. Greiner, G. Heinrich, G. Luisoni, P. Mastrolia, G. Ossola, T. Reiter and F. Tramontano, Eur. Phys. J. C **72**, 1889 (2012) [arXiv:1111.2034 [hep-ph]];
S. Badger, B. Biedermann, P. Uwer and V. Yundin, Comput. Phys. Commun. **184**, 1981 (2013) [arXiv:1209.0100 [hep-ph]];;
Z. Bern, L. J. Dixon, F. Febres Cordero, S. Höche, H. Ita, D. A. Kosower, D. Matre and K. J. Ozeren, Phys. Rev. D **88**, no. 1, 014025 (2013) [arXiv:1304.1253 [hep-ph]];
J. Alwall *et al.*, JHEP **1407**, 079 (2014) [arXiv:1405.0301 [hep-ph]];
S. Actis, A. Denner, L. Hofer, J. N. Lang, A. Scharf and S. Uccirati, Comput. Phys. Commun. **214**, 140 (2017) [arXiv:1605.01090 [hep-ph]].

[8] S. Abreu, F. Febres Cordero, H. Ita, M. Jaquier, B. Page and M. Zeng, arXiv:1703.05273 [hep-ph];
S. Abreu, F. Febres Cordero, H. Ita, M. Jaquier and B. Page, Phys. Rev. D **95**, no. 9, 096011 (2017) [arXiv:1703.05255 [hep-ph]].
[9] S. Badger, H. Frellesvig and Y. Zhang, JHEP **1204** (2012) 055 [arXiv:1202.2019 [hep-ph]];
Y. Zhang, JHEP **1209** (2012) 042 [arXiv:1205.5707 [hep-ph]].
[10] P. Mastrolia and G. Ossola, "On the Integrand-Reduction Method for Two-Loop Scattering Amplitudes," JHEP **1111** (2011) 014 [arXiv:1107.6041 [hep-ph]];
P. Mastrolia, E. Mirabella, G. Ossola and T. Peraro, Phys. Lett. B **718** (2012) 173 [arXiv:1205.7087 [hep-ph]];
P. Mastrolia, T. Peraro and A. Primo, JHEP **1608**, 164 (2016) [arXiv:1605.03157 [hep-ph]].
[11] R. K. Ellis, W. T. Giele and Z. Kunszt, JHEP **0803**, 003 (2008) [arXiv:0708.2398 [hep-ph]];
W. T. Giele, Z. Kunszt and K. Melnikov, JHEP **0804** (2008) 049 [arXiv:0801.2237 [hep-ph]].
[12] H. Ita, Phys. Rev. D **94**, no. 11, 116015 (2016) [arXiv:1510.05626 [hep-th]].
[13] F.V. Tkachov, Phys. Lett. B **100**, 65 (1981);
K. G. Chetyrkin and F. V. Tkachov, Nucl. Phys. B **192**, 159 (1981).
[14] J. Gluza, K. Kajda and D. A. Kosower, Phys. Rev. D **83**, 045012 (2011) [arXiv:1009.0472 [hep-th]].
[15] H. Hauser and G. Müller, Manuscripta Mathematica, 80(1):309337, 1993.
[16] Y. Zhang, arXiv:1612.02249 [hep-th].
[17] R. M. Schabinger, JHEP **1201**, 077 (2012) [arXiv:1111.4220 [hep-ph]].
[18] K. J. Larsen and Y. Zhang, Phys. Rev. D **93**, no. 4, 041701 (2016) [arXiv:1511.01071 [hep-th]].
[19] S. J. Parke and T. R. Taylor, Phys. Lett. **157B**, 81 (1985) Erratum: [Phys. Lett. B **174**, 465 (1986)].
[20] Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Rev. Lett. **70**, 2677 (1993) [hep-ph/9302280].
[21] J. M. Drummond, J. Henn, G. P. Korchemsky and E. Sokatchev, Nucl. Phys. B **795**, 52 (2008) [arXiv:0709.2368 [hep-th]].
[22] S. Caron-Huot and J. M. Henn, JHEP **1406**, 114 (2014) [arXiv:1404.2922 [hep-th]].
[23] R. Frassek and D. Meidinger, JHEP **1605**, 110 (2016) [arXiv:1603.00088 [hep-th]].
[24] L. D. Landau, Nucl. Phys. **13**, 181 (1959).
[25] R. J. Eden, P. V. Landshoff, D. I. Olive and J. C. Polkinghorne, "The Analytic S-Matrix," Cambridge University Press, 1966;
G. F. Sterman, "An Introduction to quantum field theory," Cambridge University Press, 1993.
[26] G. Passarino and M. J. G. Veltman, Nucl. Phys. B **160**, 151 (1979).
[27] D. Simmons-Duffin, JHEP **1404**, 146 (2014) [arXiv:1204.3894 [hep-th]].
[28] A. V. Kotikov, Phys. Lett. B **259**, 314 (1991);
Z. Bern, L. J. Dixon and D. A. Kosower, Nucl. Phys. B **412**, 751 (1994) [hep-ph/9306240];
E. Remiddi, Nuovo Cim. A **110**, 1435 (1997) [hep-th/9711188].
[29] M. Zeng, JHEP **1706**, 121 (2017) [arXiv:1702.02355 [hep-th]].
[30] J. M. Henn, Phys. Rev. Lett. **110**, 251601 (2013) [arXiv:1304.1806 [hep-th]];
J. M. Henn, J. Phys. A **48**, 153001 (2015) [arXiv:1412.2296 [hep-ph]].
[31] J. M. Henn, A. V. Smirnov and V. A. Smirnov, JHEP **1403**, 088 (2014) [arXiv:1312.2588 [hep-th]].
[32] N. Arkani-Hamed, J. L. Bourjaily, F. Cachazo, A. Postnikov and J. Trnka, JHEP **1506**, 179 (2015) [arXiv:1412.8475 [hep-th]];
Z. Bern, E. Herrmann, S. Litsey, J. Stankowicz and J. Trnka, JHEP **1506**, 202 (2015) [arXiv:1412.8584 [hep-th]].
[33] Z. Bern, E. Herrmann, S. Litsey, J. Stankowicz and J. Trnka, JHEP **1606**, 098 (2016) [arXiv:1512.08591 [hep-th]].
[34] Z. Bern, J. S. Rozowsky and B. Yan, Phys. Lett. B **401**, 273 (1997) [hep-ph/9702424];
Z. Bern, L. J. Dixon, D. C. Dunbar, M. Perelstein and J. S. Rozowsky, Nucl. Phys. B **530**, 401 (1998) [hep-th/9802162].
[35] D. R. Grayson and M. E. Stillman, "Macaulay2, a software system for research in algebraic geometry", available at http://www.math.uiuc.edu/Macaulay2/;
W. Decker, G.-M. Greuel, G. Pfister, and H. Schönemann, Singular4-0-2 – A computer algebra system for polynomial computations http://www.singular.uni-kl.de, 2015.
[36] S. Laporta and E. Remiddi, Nucl. Phys. B **704**, 349 (2005) [hep-ph/0406160].
[37] S. Müller-Stach, S. Weinzierl and R. Zayadeh, Commun. Num. Theor. Phys. **6**, 203 (2012) [arXiv:1112.4360 [hep-ph]].
[38] R. N. Lee, JHEP **0807**, 031 (2008) [arXiv:0804.3008 [hep-ph]].
[39] D. A. Kosower and K. J. Larsen, Phys. Rev. D **85**, 045017 (2012) [arXiv:1108.1180 [hep-th]];
S. Caron-Huot and K. J. Larsen, JHEP **1210**, 026 (2012) [arXiv:1205.0801 [hep-ph]];
M. Søgaard, JHEP **1309**, 116 (2013) [arXiv:1306.1496 [hep-th]];
H. Johansson, D. A. Kosower and K. J. Larsen, Phys. Rev. D **89**, no. 12, 125010 (2014) [arXiv:1308.4632 [hep-th]];
M. Søgaard and Y. Zhang, JHEP **1312**, 008 (2013) [arXiv:1310.6006 [hep-th]];
S. Abreu, R. Britto, C. Duhr and E. Gardi, JHEP **1706**, 114 (2017) [arXiv:1702.03163 [hep-th]];
M. Søgaard and Y. Zhang, JHEP **1407**, 112 (2014) [arXiv:1403.2463 [hep-th]];
M. Harley, F. Moriello and R. M. Schabinger, JHEP **1706**, 049 (2017) [arXiv:1705.03478 [hep-ph]];
J. Bosma, M. Søgaard and Y. Zhang, arXiv:1704.04255 [hep-th].
[40] S. Abreu, R. Britto, C. Duhr and E. Gardi, JHEP **1706**, 114 (2017) [arXiv:1702.03163 [hep-th]].
[41] L. F. Alday, J. M. Henn, J. Plefka and T. Schuster, JHEP **1001**, 077 (2010) [arXiv:0908.0684 [hep-th]].
[42] Z. Bern and G. Chalmers, Nucl. Phys. B **447**, 465 (1995) [hep-ph/9503236].
[43] Z. Bern, L. J. Dixon and D. A. Kosower, Nucl. Phys. B **412**, 751 (1994) [hep-ph/9306240].
[44] A. V. Kotikov, L. N. Lipatov, A. I. Onishchenko and V. N. Velizhanin, Phys. Lett. B **595**, 521 (2004) Erratum: [Phys.

Lett. B **632**, 754 (2006)] [hep-th/0404092].
[45] N. Arkani-Hamed, J. L. Bourjaily, F. Cachazo, A. B. Goncharov, A. Postnikov and J. Trnka, arXiv:1212.5605 [hep-th].
[46] R. J. Gonsalves, Phys. Rev. D **28**, 1542 (1983);
G. Kramer and B. Lampe, J. Math. Phys. **28**, 945 (1987).
[47] Z. Bern, J. J. M. Carrasco and H. Johansson, Phys. Rev. D **78**, 085011 (2008) [arXiv:0805.3993 [hep-ph]].
[48] H. Ita, PoS LL **2016**, 080 (2016) [arXiv:1607.00705 [hep-ph]].