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Lattice Simulations of 10d Yang-Mills toroidally compactified to 1d, 2d and 4d

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Toroidally compactified Yang-Mills theory on the lattice is studied by using the Hybrid Monte Carlo algorithm. When the compact dimensions are small, the theory naturally reduces to Yang-Mills with scalars. We confirm previous analytical and numerical results for pure gauge theory with scalars in $(0+1)$ dimensions and at high temperatures to Super-Yang-Mills in $(1+1)$ dimensions. In $(1+1)$ dimensions, our simulations confirm the previously conjectured phase diagram. Furthermore, we find evidence for the sequential breaking of the center symmetry in $(1+1)$ dimensions as a function of the volume. In $(3+1)$ dimensions we present first simulation results for the eigenvalue distribution of the Polyakov and Wilson loops, finding localized, non-uniform and center-symmetric configurations as a function of the lattice coupling.

I. INTRODUCTION

Toroidally compactified Yang-Mills theories on a d -dimensional lattice are of interest for several reasons. The case of $d = 4$ has obvious applications to QCD; sometimes calculations simplify at small volume, and one can have some hope to learn lessons about the large-volume theory; see e.g. Refs. [1–4]. Another important application is the gauge-Higgs unification scenario; just as a four-dimensional vector field is obtained from the five-dimensional metric via the Kaluza-Klein mechanism [5, 6], scalars in four dimensions can be obtained from five-dimensional vectors via compactification; see e.g. Refs. [7, 8] for previous work on this subject. Certain phenomenological models with large extra dimensions, such as the universal extra dimensions [9], add further motivations. Lattice studies along these directions can be found e.g. in Refs. [10–14]. Yet another application comes from supersymmetric Yang-Mills theory and superstring theory; the compactification of the $d = 10$ $SU(N)$ theory to p dimensions leads to Yang-Mills theory with $d - p$ adjoint scalars, which is the bosonic part of the maximal Super-Yang-Mills theory describing N -coincident $D(p - 1)$ -branes. As demonstrated in Refs. [15, 16], such a theory is useful for understanding the phase diagram of $(p + 1)$ -dimensional Super-Yang-Mills, by interpreting the former to be the high-temperature limit of the latter. The main argument behind this connection is the fact that in this limit the fermions acquire a large thermal mass due to antiperiodic boundary conditions in the temporal direction, and therefore effectively do not contribute to some observables¹

In all these cases, the $(\mathbb{Z}_N)^d$ center symmetry plays a crucial role. This symmetry is characterized by $W_i \rightarrow e^{2\pi i n_i/N} W_i$, $n_i \in \mathbb{Z}$, where W_i is the Wilson line winding on the i -th dimension (see section II B for a lattice definition). The center symmetry along each direction can be broken when that direction is small. Dimensional reduction can make sense after the center symmetry has been broken.

From string theory point of view, this transition can be interpreted as the black hole/black string transition or its higher-dimensional analogue [15, 16, 18].

In this paper we use the Hybrid Monte Carlo method for studying this theory. Firstly, as a sanity check, we study the $d = 10$ theory on an $N_t \times 1^9$ lattice, which reduces to a $(0 + 1)$ -

¹ Clearly, approximating SYM by just its bosonic content is limited to observables that are not sensitive to the effective number of degrees of freedom. In particular, the approximation fails for the Stefan Boltzmann limit of the free energy of SYM at high temperature. However, lattice studies in full QCD have indicated that the pure bosonic theory is able to offer quantitatively accurate descriptions for the Wilson loop expectation value at high temperatures [17].

dimensional theory with 9 scalars in the continuum limit. We observe good agreement with previous simulation result [15, 19].

Next we consider the compactification to two dimensions, by taking the lattice size to be $N_t \times N_s \times 1^8$. The lattice spacing a is taken to be the same for all directions. At sufficiently small lattice spacing, the theory reduces to two-dimensional Yang-Mills with 8 scalars. Although the scalars can acquire mass through radiative corrections, we expect this mass to grow only logarithmically with the lattice spacing and hence it likely is not important for the small lattices used in this paper; we can expect our simulation results to be close to the massless theory, which is the bosonic part of maximally supersymmetric two-dimensional Yang-Mills theory. This begs the natural question: how close are the two theories, or in other words, how important are the effects from fermions? One clear difference is that the supersymmetric theory has only a deconfined phase, while the bosonic theory has both confined and deconfined phases. As we will see, deep in the deconfined phase, our simulation results for the bosonic theory are very close to the results for the supersymmetric theory obtained before by Catterall, Joseph and Wiseman [20] (see also [21] for other simulation results on the same theory and [22] for a review of this and related topics).

By using the same simulation code, it is possible to study classical real-time dynamics without fermions. The observation that bosonic and supersymmetric theories give close results suggests that the classical treatment could offer a potentially quantitatively good description in the deconfined phase. Previous studies in other dimensions, e.g. about the thermalization [23–25] and scrambling [26, 27], might be justified in this way. It would be interesting if various real-time aspects of string theory, such as the fast scrambling [28–30] and the black hole/black string topology change [15, 20, 31, 32], could be studied in a similar manner.

In the remainder of this work we study the dynamics of the theory on an $N_t \times N_s^3 \times 1^6$ lattice, which can be thought to approximate $\mathcal{N} = 4$ Super-Yang-Mills (SYM) in $(3 + 1)$ dimensions in the high-temperature limit. We map out the location of the deconfinement transition on various lattice sizes and calculate the corresponding thermodynamic pressure as a function of the lattice coupling. We then continue to demonstrate that simulating the full ten-dimensional theory is technically possible and then summarize and give our conclusions. Detailed treatment of two somewhat more technical aspects can be found in the appendices.

II. YANG-MILLS IN TOROIDAL COMPACTIFICATION

We consider the $SU(N)$ pure Yang-Mills theory on d -dimensional torus, whose action is defined by

$$S = \frac{1}{4g_{(d)}^2} \sum_{i,j=1}^d \int d^d x \text{Tr} F_{ij}^2, \quad (1)$$

where the field strength F_{ij} is defined by $F_{ij} = \partial_i A_j - \partial_j A_i - i[A_i, A_j]$. Note that the coupling constant $g_{(d)}^2$ has a dimension of $(\text{mass})^{4-d}$. Also, note that this action uses holographic normalization convention, whereas standard QCD convention would have a prefactor of $\frac{1}{2g_{(D)}^2}$ instead. Formally, the lattice regularization can be obtained by using the standard plaquette action,

$$S = \beta_{\text{eff}} \sum_{x \in T^d, \square} \left(1 - \frac{1}{N} \text{Re Tr } U_{\square} \right), \quad (2)$$

where $\beta_{\text{eff}} \equiv \frac{Na^{d-4}}{g_{(d)}^2}$ is the d -dimensional effective lattice coupling. Note that β_{eff} has mass dimension zero and that β_{eff} scales as $\propto N^2$ in the large N limit. At $d > 4$, this theory is power-counting non-renormalizable, and it is most likely not well-defined in the continuum limit. Still, it can be treated as a cut-off theory, and there are efforts motivated by the extensions of the standard model with extra dimensions (see e.g. in [10–13]).

If we take the lattice to be $T^p \times 1^{d-p}$, Yang-Mills theory coupled to $d - p$ adjoint scalars can be obtained. Again, when $p > 4$, the theory is not renormalizable. In this paper we concentrate on $p \leq 4$ where the continuum theory is well-defined. Note that, unless counter-terms are added, the scalar masses receive radiative corrections so that the masses may become large in the continuum limit. On the lattice, this has been studied in the context of torelon masses (see for instance [12, 33, 34]). One finds that heavy scalars effectively decouple from the theory, so that e.g. $d = 5$ Yang-Mills reduces to usual four-dimensional Yang-Mills in the continuum limit except if the coupling and extra dimension size are fine-tuned [12].

In this work, we do not fine-tune the scalar masses but rather check if our results are sensitive to the number of scalars $d - p$ by performing lattice simulations with different values for the parent dimension d . If the scalars become very heavy, then one expects results to be insensitive to the choice of d while conversely a sensitivity to d implies that the scalars have not (completely) decoupled from the theory.

A. Expected Phase Diagram and String Theory Interpretation

In this section we review the previous numerical and analytical results concerning the phase diagrams. For simplicity we consider only the square tori, i.e. all compactification periods are taken to be the same value L .

Let us start with pure Yang-Mills without scalars. Narayanan and Neuberger [4, 35] have studied T^3 and T^4 , taking the continuum and large- N limits. (Note that, at finite N , the phase transition cannot take place at finite volume.) It has been observed that the center symmetry breaks down sequentially as $(\mathbb{Z}_N)^d \rightarrow (\mathbb{Z}_N)^{d-1} \rightarrow \cdots \rightarrow \{1\}$. One observable sensitive to this sequential breaking pattern is $P_i \equiv \text{Tr} W_i / N$ since every time one of the $(\mathbb{Z}_N)$'s breaks, one of P_i 's gets a nonzero expectation value. When the center symmetry is broken to $(\mathbb{Z}_N)^p$, the system can be described approximately by Yang-Mills with $d - p$ scalars. Therefore, it is natural to expect the center breaking pattern [4, 35] also for theories with adjoint scalars; see the right panel of Fig. 2 for $(1 + 1)$ -d theory with scalars.

The case of $d = 10$, $p \leq 4$ theory can be regarded as the high temperature limit of maximally supersymmetric Yang-Mills theory in $(p + 1)$ dimensions, which has an interpretation in terms of dual superstring theory. As a concrete example, let us first consider $d = 10$, $p = 1$ [15, 16]. The corresponding $(1 + 1)$ -dimensional maximal SYM is dual to black 1-brane in type IIB superstring theory at low temperature and at large- N . The finite temperature and $1/N$ corrections correspond to stringy corrections. In the center-symmetric phase, the Wilson line phases are distributed uniformly (top-left of Fig. 1), while in the center-broken phase a non-uniform distribution without gap at $\pm\pi$ (top-middle of Fig. 1) and the localized distribution (top-right of Fig. 1) can exist. The Wilson line phases correspond to the positions of D0-branes in the T-dual picture on the gravity side; hence these three phases correspond to the uniform black string, the non-uniform black string² and the black hole (bottom of Fig. 1). The thermodynamic study on the gravity side [15, 16] shows that there exist two phases at low temperature, the uniform black string phase at large volume and the black hole phase at small volume, which are separated by a first-order phase transition. The $(0 + 1)$ -dimensional bosonic theory has a similar phase diagram at high temperature; the center symmetry is broken at small volume (high temperature, deconfined phase) and unbroken at large volume (low temperature, confined phase). However, unlike the low tem-

² Note that, in this context, ‘black string’ does not mean the black 1-brane; it means the smeared, string-shaped distribution of D0-branes.

perature region of the $(1+1)$ -dimensional maximal SYM, the center-symmetric phase further splits into the non-uniform distribution without gap and the localized distribution [19, 36]. By combining these arguments, the phase diagram of the $(1+1)$ -dimensional maximal SYM becomes should qualitatively look like the left panel of Fig. 2.

FIG. 1. Possible Wilson line phase distributions in $(1+1)$ -dimensional maximal SYM (top) and dual gravity interpretation (bottom). The Wilson line phases correspond to the positions of D0-branes in the T-dual picture on the gravity side. The uniform distribution corresponds to the uniform black string (left), non-uniform distribution without gap at $\pm\pi$ corresponds to the non-uniform string (middle), and the localized distribution corresponds to the black hole (right).

The same analysis applies to $(2+1)$ - and $(3+1)$ -dimensional maximal SYM theories, just by considering the higher-dimensional counterparts of the black string. The thermodynamic analysis on the gravity side [18] gives the center breaking pattern observed in the bosonic theory, namely $(\mathbb{Z}_N)^2 \rightarrow \mathbb{Z}_N \rightarrow \{1\}$ and $(\mathbb{Z}_N)^3 \rightarrow (\mathbb{Z}_N)^2 \rightarrow \mathbb{Z}_N \rightarrow \{1\}$. Therefore it is natural to expect p -dimensional Yang-Mills with adjoint scalars, which resembles the high temperature region of $(p+1)$ -dimensional SYM, has the center breaking pattern $(\mathbb{Z}_N)^p \rightarrow (\mathbb{Z}_N)^{p-1} \rightarrow \dots \rightarrow \{1\}$. In [16] a part of the phase diagram of the 2d Yang-Mills with heavy adjoint scalars has been studied analytically, and the same conclusion have been obtained.

B. Numerical setup

We will be employing a Hybrid Monte Carlo simulation of the *quantum* $SU(N)$ Yang-Mills system with arbitrary N for a system in d dimensional Euclidean space-time using standard

FIG. 2. Conjectured phase diagram of 2d SYM (left) [16] and 2d bosonic theory (right) [16, 18] at finite temperature, which is expected for massive scalars. The present work provides the first numerical check on the conjectured bosonic phase diagram. At the high-temperature region of SYM, two transitions (uniform to non-uniform, and non-uniform to localized) are expected [19, 36].

lattice methods [37]. Our numerical code is publicly available as a service to the community [41].

As observables we will consider Wilson/Polyakov loops which are defined as $W_i = \mathcal{P} e^{i \oint dx A_i(x)}$ for a loop in the i^{th} (Euclidean) direction, where $\mathcal{P}$ denotes path ordering. On the lattice, this is easily implemented through a product of link matrices

$$W_i = \prod_{m=1}^{m=n_i} U_i(m), \quad (3)$$

where n_i denotes the number of lattice sites in the direction i . We will distinguish the case where $i = t$ (the Euclidean “time” direction) and $i = (x, y, z) = s$ (one of the Euclidean “space” directions), corresponding to the case of Polyakov and Wilson loops, respectively. Of particular interest will be the expectation value of the absolute value of the trace (cf. Ref. [20]),

$$\langle P_i \rangle = \left\langle \left| \frac{1}{N} \text{Tr } W_i \right| \right\rangle, \quad (4)$$

averaged over directions transverse to i and over all configurations. Also, we will study the distribution of eigenvalues $\{e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_N}\}$ of the Wilson/Polyakov loops on the unit circle. The phases of the eigenvalues are required to sum to 2π times the integer winding number for the $SU(N)$ gauge group. We will ignore non-trivial topologies in this study and re-center the phases of the eigenvalues so they sum to zero if a non-trivial winding number

occurs. The distribution of eigenvalue phases on the interval $[-\pi, \pi]$ will then be of interest as an order parameter for the systems considered here.

Another observable will be the equation of state, e.g. the functional dependence of the energy density ε on the temperature T . Given a partition function $Z = \int dU e^{-S}$, the energy density and pressure P are calculated as derivatives w.r.t. temperature and volume V ,

$$\begin{aligned}\varepsilon &= -\frac{1}{V} \frac{\partial \ln Z}{\partial (1/T)} = \frac{T}{V} \left\langle a_t \frac{\partial S}{\partial a_t} \right\rangle \\ P &= T \frac{\partial \ln Z}{\partial V} = \frac{-T}{(d-1)V} \left\langle a_s \frac{\partial S}{\partial a_s} \right\rangle.\end{aligned}\quad (5)$$

Note that derivatives w.r.t temperature $T = \frac{1}{N_t a_t}$ and volume $V = (N_s a_s)^{d-1}$ for a lattice with a fixed number of sites N_t, N_s in the temporal and spatial directions have been recast as derivatives w.r.t. a_t, a_s , the (in principle different) lattice spacings in these directions. For an anisotropic lattice with different lattice spacings in the temporal and spatial directions the action is given by

$$S = \frac{N^2 a_s^{d-3}}{\lambda_{(d)} a_t} \sum_x R_{\text{temp}} + \frac{N^2 a_t a_s^{d-5}}{\lambda_{(d)}} \sum_x R_{\text{spat}}, \quad (6)$$

where the “temporal” and “spatial” plaquettes are given as (cf. [37]).

$$R_{\text{temp}} \equiv \sum_{i=1}^{d-1} \left(1 - \frac{1}{N} \text{Re Tr } U_{ti} \right), \quad R_{\text{spat}} \equiv \sum_{i=1}^{d-1} \sum_{j>i}^{d-1} \left(1 - \frac{1}{N} \text{Re Tr } U_{ij} \right). \quad (7)$$

Toroidal compactification simply amounts to the replacement $\frac{a_s^d}{\lambda_{(d)}} \rightarrow \frac{a_s^p}{\lambda_{(p)}}$. Away from $p = 4$, the coupling is dimensionful and can be scaled away in all physically relevant observables. Thus only the explicit dependence on the lattice spacings contributes to the derivatives in (5). Setting $a_t = a_s = a$ in the end (isotropic lattice) leads to

$$\begin{aligned}\varepsilon &= \frac{T}{V} \frac{N^2 a^{p-4}}{\lambda_{(p)}} \sum_x (-\langle R_{\text{temp}} \rangle + \langle R_{\text{spat}} \rangle), \\ P &= \frac{T}{(p-1)V} \frac{N^2 a^{p-4}}{\lambda_{(p)}} \sum_x (-\langle R_{\text{temp}} \rangle (p-3) - \langle R_{\text{spat}} \rangle (p-5)).\end{aligned}\quad (8)$$

As a special case, let us mention that for $p = 1$ (matrix model quantum mechanics), we have

$$\varepsilon V = E = \frac{N^2 a^3 T}{\lambda_{(1)}} \sum_x (-\langle R_{\text{temp}} \rangle + \langle R_{\text{spat}} \rangle). \quad (9)$$

For $p = 4$, in particular, it is advantageous to consider a different strategy to calculate thermodynamic properties. Namely, note that a different definition of the pressure is $P = \frac{T}{V} \ln Z$. Considering

$$\frac{\partial \ln Z}{\partial \beta_{\text{eff},p}} = - \left\langle \frac{\partial S}{\partial \beta_{\text{eff},p}} \right\rangle = - \sum_x (\langle R_{\text{temp}} \rangle + \langle R_{\text{spat}} \rangle) . \quad (10)$$

Integration of this result leads to the pressure. However, care must be taken to remove the zero-temperature contribution. In practice, this is done by subtracting $\langle R_{\text{temp}} \rangle + \langle R_{\text{spat}} \rangle$ calculated on a lattice with N_s^D sites at each value of β_{eff} [37],

$$\begin{aligned} \frac{P}{T^{D-p}} &= \int_{\beta_0}^{\beta} d\bar{\beta} \Delta S(\bar{\beta}) , \\ \Delta S(\bar{\beta}) &= N_t^{D-p} [S(N_s, N_s, \bar{\beta}) - S(N_t, N_s, \bar{\beta})] , \\ S(N_t, N_s, \bar{\beta}) &\equiv \frac{1}{N_s^{D-1} N_t} \sum_x (\langle R_{\text{temp}} \rangle + \langle R_{\text{spat}} \rangle)|_{\bar{\beta}} . \end{aligned} \quad (11)$$

III. RESULTS

A. $d = 10 \rightarrow 1d$: Matrix Model Quantum Mechanics

Starting with ten-dimensional $\text{SU}(N)$ Yang-Mills and compactifying down to $p = 1$ dimension leads to a Lagrangian corresponding to bosonic matrix model quantum mechanics. This theory has been well-studied both analytically as well as numerically [15, 19, 36]. We conduct simulations for fixed values of $\beta_{\text{eff},1} = \frac{N^2}{\lambda_{(1)} a^3}$ on an isotropic lattice, where the continuum limit $a \rightarrow 0$ corresponds to $\beta_{\text{eff},1} \rightarrow \infty$. Using the dimensionful 't Hooft coupling $\lambda_{(1)}$, we can consider dimensionless quantities such as $a\lambda_{(1)}^{1/3}$, $T\lambda_{(1)}^{-1/3}$, which we will denote by a bar, e.g. $\bar{a} \equiv a\lambda_{(1)}^{1/3}$, $\bar{T} \equiv T\lambda_{(1)}^{-1/3}$. The results for the Polyakov loop $\langle P_t \rangle$ and the energy E , obtained by evaluating (4,9) on the lattice configurations, are shown in Fig. 3. Also shown in Fig. 3 are numerical results from Ref. [19], as well as high-temperature-expansion (HTE) analytic results from Ref. [42]. As can be seen from Fig. 3, there is very good agreement between the present study and previous results. The difference between simulating $\text{U}(N)$ and $\text{SU}(N)$ gauge theories does not seem to be relevant.

We observe a rapid change in the expectation value of the Polyakov loop as a function of temperature. Defining the susceptibility $\chi_t \equiv \frac{d\langle P_{t,s} \rangle}{dT}$, we define $\bar{T}_t$ by the location of the peak in χ_t (see Fig. 4). For a resolution of $\beta_{\text{eff}} = \frac{N^2}{0.04^3}$ we find $\bar{T}_t \simeq 0.916(30)$; increasing the

FIG. 3. Results from simulating 10d $SU(N)$ compactified to 1d (matrix model quantum mechanics) for $\beta_{\text{eff}} = \frac{N^2}{0.04^3}$ (effective lattice spacing $\bar{a} = 0.04$) for various values of N , and $\beta_{\text{eff}} = \frac{N^2}{0.01^3}$ for $N = 4$. Shown are the expectation value of the Polyakov loop $\langle P_t \rangle$ and the expectation value of the energy E (minus the energy at zero temperature E_0 which is calculated from the lattice with the largest value of N_t) as a function of the dimensionless temperature $\bar{T}$ (full symbols). For comparison, analytic results from a high-temperature-expansion (HTE) [42], as well as previous lattice simulation results are shown [19]. The dash-dotted line indicates the location of T_t and is a guide to the eye.

resolution to $\beta_{\text{eff}} = \frac{N^2}{0.02^3}$ we find $\bar{T}_t \simeq 0.885(15)$, consistent with previous results [15, 19]. As can be seen from Fig. 4, the peak in the susceptibility and hence the change in the Polyakov loop expectation value becomes more pronounced as the number of colors is increased, and also corresponds to a change in the eigenvalue distribution of the Polyakov loop (see Fig. 4). As the temperature is decreased, the distribution of the Polyakov loop eigenvalues becomes more and more uniform, but because of the limited statistics for this study we cannot rule out the possibility of a second phase transition from non-uniform to a fully uniform distribution.

Comparing the value of $\bar{T}_t$ with the results for the energy in Fig. 3, we are led to identify $\bar{T}_t$ with the critical temperature for the confinement-deconfinement transition in the bosonic 1d Matrix Model. Since results shown in Fig. 4 indicate that the Polyakov loop eigenvalue distribution also changes qualitatively from localized ($\bar{T} > \bar{T}_t$) to non-uniform ($\bar{T} < \bar{T}_t$) at the same critical temperature $\bar{T}_t$, we are led to identify the confined phase with the non-uniform eigenvalue phase and the deconfined phase with the localized eigenvalue phase.

FIG. 4. Left: 1d Polyakov loop susceptibility as a function of temperature for various N and $\beta_{\text{eff}} = \frac{N^2}{0.04^3}$. For large N , and this resolution, we find the susceptibility peak to lie at $\bar{T}_t = 0.916(30)$. Right: The eigenvalue distribution for the 1d Polyakov loop is found to change qualitatively from non-uniform ($\bar{T} < \bar{T}_t$) to localized ($\bar{T} > \bar{T}_t$).

B. Arbitrary $d \rightarrow 1\text{d}$ and 2d : Sensitivity to Number of Scalars

To check for the sensitivity to the number of scalars, we perform simulations with varying parent dimension d , always compactifying down to $p = 1$. Similar studies have been performed by other groups before [43, 44]. We monitor the Polyakov loop expectation value and locate the peak in its susceptibility, defining the location of the critical (deconfinement) temperature T_t . We find indications that as d is increased above $d = 10$, the susceptibility peak and hence the strength of the transition at fixed N becomes more pronounced. Conversely, for $d < 10$, the transition seems to weaken at fixed N and we have to increase N to clearly identify a peak in the Polyakov loop susceptibility. The results for $\bar{T}_t$ versus parent dimension d are shown in Fig. 5. For comparison, also the analytic results for $\bar{T}_t$ obtained in a large d expansion from Ref. [36] are shown. We find remarkably good agreement with the next-to-leading order analytic result even for $d = 4$. This agreement (and the implied dependence of $\bar{T}_t$ on the effective number of scalars $d - 1$) suggests that in our simulations the scalars have not decoupled from the theory.

Similar findings hold true for the case of the reduction from $d = 10$ to $p = 2$ (2d). The results for $\bar{T}_t$ exhibit a clear dependence on the parent dimension d , which is qualitatively similar to the trend seen in the 1d case. We are not aware of any analytic calculations in

FIG. 5. Dependence on the critical temperature for the Polyakov loop (defined via the peak in the susceptibility) as a function of varying parent dimension d . Left: Reduction to 1d. For comparison, the analytic results obtained in a large d approximation from Ref. [36] are shown. Right: Reduction to 2d with $\bar{r}_s = 0.707$. The behavior of $\bar{T}_t$ is qualitatively similar to 1d.

the large d limit for this case.

C. $d = 10 \rightarrow 2d$: Yang-Mills on a circle

Compactification of the 10d theory onto $(1+1)$ dimensions corresponds to Yang-Mills theory at finite temperature compactified on a circle. Using the two-dimensional 't Hooft coupling $\lambda_{(2)} = \frac{N^2}{\beta_{\text{eff},2} a^2}$ we follow Ref. [20] to define a dimensionless circle radius $\bar{r}_s = N_s a \lambda_{(2)}^{1/2}$ and a “temporal” radius $\bar{r}_t = N_t a \lambda_{(2)}^{1/2} = 1/\bar{T}$. In terms of these radii, the supersymmetric (SYM) version of this system is conjectured to have three possible phases, corresponding to situations which the Wilson loop eigenvalue distribution is localized, non-uniform and uniform. Holography has been used to study this system in the strong coupling (low temperature) limit, predicting the existence of a Gregory-Laflamme transition occurring at $\bar{r}_s^2 \simeq 2.29 \bar{r}_t$, cf. Refs. [15, 16, 20] which is characterizing the change from the uniform to localized phase (the non-uniform phase is expected to be thermodynamically disfavored, cf. [15, 20]). At weak coupling (high temperatures), a deconfinement-confinement transition occurring at $\bar{r}_s^3 \simeq 1.35 \bar{r}_t$ has been found in previous studies [15, 36, 42]. The high temperature result can be gleaned from results in section III A through identifying $T \rightarrow \lambda_{(2)}^{1/2}/\bar{r}_s$ and $\lambda_{(1)} = \lambda_{(2)}/a \rightarrow \lambda_{(2)}^{3/2}/\bar{r}_t$ since the temporal circle becomes very short and temporal and spatial elements trade their respective meaning. In this manner it is obvious that our results

FIG. 6. Left: Polyakov and Wilson loop for pure $SU(N)$ as a function of temperature for $\bar{r}_s = 0.707$ (combined data with $\beta_{\text{eff}} = \frac{N^2}{0.0707^2}$ for $\bar{T} < 1.6$ and $\beta_{\text{eff}} = \frac{N^2}{0.0353^2}$, respectively) compared to results for SYM for $N=3,4$ from Ref. [20]. Right: Polyakov and Wilson loop susceptibilities in pure $SU(N)$ as a function of temperature for fixed circle radius $\bar{r}_s = 0.707$. The dash-dotted lines indicate the location of $\bar{T}_{t,s}$ where the corresponding susceptibilities peak and are a guide to the eye.

in the high temperature limit must match the previous finding of $\bar{r}_s^3 = \frac{\bar{r}_t}{(0.885(15))^3} \simeq 1.44(7)\bar{r}_t$.

The phase diagram of the 2d theory differs between the supersymmetric theory and the purely bosonic theory (see Fig. 2). The supersymmetric theory should always be in the deconfined phase characterized by a nonzero Polyakov loop expectation value. The bosonic theory studied in this work is rather different: first of all, the phase diagram must be invariant under the exchange of the temporal and spatial circles, $\bar{r}_t \leftrightarrow \bar{r}_s$. By regarding this theory as the high-temperature limit of 3d SYM, the dual gravity calculation valid at the low-temperature region suggests a form of the phase diagram shown in the right panel of Fig. 2 [18]. The conjectured phase diagram for the purely bosonic theory has not been confirmed previously by numerical simulations, except for the case of toroidal compactification of the toroidally compactified $d = 4$ Yang-Mills theory for $\bar{r}_t = \bar{r}_s = \bar{r}_y = \bar{r}_z$ where a symmetry breaking pattern of $(\mathbb{Z}_N)^4 \rightarrow (\mathbb{Z}_N)^3 \rightarrow (\mathbb{Z}_N)^2 \rightarrow \mathbb{Z}_N \rightarrow \{1\}$ has been observed [35].

We first study the Wilson and Polyakov loop expectation values for fixed spatial circle radius as a function of temperature in Fig. 6. We find that both the Wilson and Polyakov loops exhibits a rapid change as a function of temperature as N is increased. Defining again the susceptibilities $\chi_{t,s} \equiv \frac{d\langle P_{t,s} \rangle}{dT}$, at any given spatial radius $\bar{r}_s$ for large enough N we observe a peak at specific (different) temperatures $\bar{T}_{t,s}(\bar{r}_s)$ (see figure 6). The peak for χ_t is visible

FIG. 7. The eigenvalue distribution for the Polyakov loop (left) and Wilson loop (right). Both are for the 2d theory for $N = 8$ and a circle radius of $\bar{r}_s = 0.707$, calculated at $\beta_{\text{eff}} \simeq \frac{N^2}{0.035^2}$ and $\beta_{\text{eff}} \simeq \frac{N^2}{0.07^2}$, respectively. Note the change in the Polyakov loop behavior above/below $\bar{T}_t \simeq 1.232(10)$ and the Wilson loop behavior above/below $\bar{T}_s \simeq 2.7(3)$.

for any $N \geq 4$ and becomes more pronounced as N is increased, while the peak for χ_s only emerges for $N \geq 16$. Using the peak position from χ_t for $N \geq 8$ to define $\bar{T}_t$ and the peak for χ_s for $N = 16$ to defined $\bar{T}_s$ we find $\bar{T}_t(\bar{r}_s \simeq 0.7) \simeq 1.232(10)$ and $\bar{T}_s(\bar{r}_s \simeq 0.7) \simeq 2.7(3)$.

Also shown in Fig. 6 are the results for SYM theory (gauge theory plus fermions) from Ref. [20]. At high temperatures the anti-periodic boundary conditions for the Fermions in SYM implies that they acquire a large mass and effectively decouple from the theory. Thus one expects good agreement between the full SYM theory and the pure bosonic $SU(N)$ theory simulations for high temperatures, while deviations are expected at low temperatures³. Fig. 6 seems to corroborate this expectation. We have performed a more extensive comparison between the SYM and pure $SU(N)$ gauge theory for a same-size lattice in appendix A, from which we expect quantitative agreement between the SYM and pure gauge theory for $T \geq 5\lambda_{(2)}^{1/2}$.

The different temperatures $\bar{T}_{t,s}$ are accompanied by corresponding changes in the distribution of eigenvalues of the Polyakov and Wilson loops (see Fig. 7). Starting from low temperatures $\bar{T} < \bar{T}_t$ we find the distribution of the Polyakov loop eigenvalues to be non-

³ The SYM simulations from Ref. [20] are based on a different discretization scheme, and require stabilization of flat directions in the scalar potential. Without supersymmetry, these flat directions are lifted at the quantum level, which is why stabilization is not needed in simulations of the purely bosonic theory. The fact that quantum effects lift the flat directions can be seen by calculating the interaction between eigenvalues. Essentially the same calculation has been performed in Ref. [45] for a different motivation.

FIG. 8. Left: 2d Energy density for $\bar{r}_s = 0.707$ for various values of N , calculated for $\beta_{\text{eff}} = \frac{N^2}{0.07^2}$. The vertical line labeled $\bar{T}_t$ indicates the peak position of the Polyakov loop and is a guide to the eye. Right: critical temperatures $\bar{T}_t, \bar{T}_s$ for the Polyakov loop and Wilson loop for $N=8$ and $N=16$, respectively, as a function of spatial circle radius $\bar{r}_s$. The line labeled 'inferred' are results for $\bar{T}_t$ from SU(8) and switching $\bar{r}_s \leftrightarrow \bar{r}_t$.

uniform, but non-gapped, and the Wilson loop eigenvalues to be localized. For $\bar{T}_t < \bar{T} < \bar{T}_s$ the Polyakov loop and Wilson loop eigenvalues are both localized, and for $\bar{T} > \bar{T}_s$ the Polyakov loop eigenvalues are localized and the Wilson loop eigenvalues are non-uniform.

Comparing the temperatures $\bar{T}_t$ with the energy density as a function of temperature shown in Fig. 8, we are led to identify this temperature with the deconfinement-confinement transition as was the case for the 1d compactification. As can be seen from Fig 6, the deconfinement-confinement transition temperature $\bar{T}_t$ is clearly separated from the temperature $\bar{T}_s$. Because of the mapping of the high-temperature behavior and the behavior of the one-dimensional theory discussed in section III A, we identify $\bar{T}_s$ with a critical temperature signaling a Gregory-Laflamme (GL) type instability. This feature of having distinct confinement and GL critical temperatures is unique to the pure gauge SU(N) theory considered here and is not expected to occur for the full SYM theory. In 3d and 4d cases, the confined phases can exist when the theories are compactified to two- and three-spheres [46].

Also shown in Fig. 8 are the locations of the critical temperatures for the Polyakov loop ($\bar{T}_t$) and Wilson loop ($\bar{T}_s$) as a function of the length of the spatial circle $\bar{r}_s$. The different transition temperatures correspond to boundaries of the phases with two unbroken center symmetries, one unbroken and one broken center symmetry, respectively, and both center

FIG. 9. Time history of P_t, P_s for $N = 16$ and $r_t = r_s$. Note that the time length of a single Monte Carlo trajectory is 25 in lattice units.

symmetries broken. For large values of the spatial circle radius $\bar{r}_s \gg 1$ we find that the value of $\bar{T}_t(\bar{r}_s)$ becomes approximately independent of $\bar{r}_s$, consistent with the expectation from the infinite volume limit. However, note that $\bar{r}_s \gg 5$ would be necessary to unambiguously demonstrate this volume independence. We find qualitative agreement with the conjectured bosonic phase diagram in Fig. 2. Close to $\bar{r}_t, \bar{r}_s \simeq 1$, Fig. 2 suggests the presence of a single transition line (“neck”) connecting the deconfined, center symmetry broken phase to the deconfined, center symmetry restored phase. On this line, one expects the center symmetry to be partially broken, and we address this question in the following subsection.

D. Sequential Breaking of $\mathbb{Z}_N$ symmetries in 2d and 4d

When the circumferences of the temporal and spatial circles are taken to be the same, $\bar{r}_t = \bar{r}_s$ (for $p = 2$), or $\bar{r}_t = \bar{r}_x = \bar{r}_y = \bar{r}_z$ (for $p = 4$), and if $\bar{r}_t$ is varied from ∞ to 0, then the $(\mathbb{Z}_N)^p$ center symmetry should break down sequentially as $(\mathbb{Z}_N)^p \rightarrow (\mathbb{Z}_N)^{p-1} \rightarrow \dots \rightarrow \{1\}$. Thus in a single configuration, the center symmetry along the temporal direction may be broken (and as a consequence $P_t \simeq 1$), while the spatial direction may still center symmetry and hence $P_s \simeq 0$. Unfortunately, since $\bar{r}_t = \bar{r}_s$, P_t and P_s will flip roles from configuration to configuration in the lattice ensemble⁴, so that on average $\langle P_t \rangle = \langle P_s \rangle$ even if one of the center symmetries (but not the other) is broken. An example of this ‘two-state’ signal is shown in Fig. 9. Therefore, the expectation value of the Polakov/Wilson loops are not

⁴ Note that for large N , such ‘tunneling’ will be suppressed.

FIG. 10. Expectation value of the ordered Polyakov and Wilson loops for the 2d theory (left) and the 4d theory (right) for equal size lattices.

sensitive to this partially broken center symmetric phase. In order to construct an operator that is sensitive, we can (for each configuration) order the expectation values of P_i by size and average the ordered values over ensembles, which ensures that flipped roles of P_t, P_s do not average out. In Fig. 10 we study this observable as a function of the temperature (for the 2d theory) or the effective lattice coupling β_{eff} (for the 4d theory).

As can be seen from Fig. 10, there is a region ($\bar{T} \in [0.9, 1.1]$ for 2d and — to a lesser extent — for 4d and $\beta_{\text{eff}} \in [1, 5]$) where the average of the maximum Polyakov/Wilson loops does not match the minimum Polyakov/Wilson loops. For instance for 2d and $\bar{T} \simeq 1$, the maximal Polyakov/Wilson loop value of about 0.6 indicates that the corresponding eigenvalue distribution is localized (see e.g. Figures 6,7), corresponding to a breaking of the associated $\mathbb{Z}_N$ symmetry. Yet the expectation value of the minimum P_i at this temperature indicates that the corresponding center symmetry remains unbroken. Therefore, the results shown for the 2d theory in Fig. 10 suggest unbroken $(\mathbb{Z}_N)^2$ symmetry for temperatures $\bar{T} \lesssim 0.9$, a remaining $(\mathbb{Z}_N)^1$ symmetry for $\bar{T} \in [0.9, 1.1]$ and fully broken center symmetry for $\bar{T} \gtrsim 1.1$. For the 4d theory, the $N = 6, 8$ results available do not allow to clearly identify a sequential center symmetry breaking pattern. We intend to return to this issue in a future study for larger N .

N	N_t	N_s	β_t/N^2
3	2	8	0.546(2)
3	3	12	0.612(1)
3	4	16	0.63(1)
4	2	8	0.591(3)
6	2	8	0.619(3)

TABLE I. Table of critical 4d lattice coupling values for various fixed-size lattices and N .

E. $d = 10 \rightarrow 4\mathbf{d}$

If six of the original ten dimensions are compactified we are studying the bosonic part of $\mathcal{N} = 4$ SYM in $(3 + 1)$ dimensions. This case is special because for $p = 4$ the coupling is dimensionless and we would have to perform scale setting in order to express results in terms of physical quantities. We leave this to future work and report results in terms of the effective four-dimensional lattice coupling β_{eff} . In this work we only consider bare (unrenormalized) results for the Polyakov/Wilson loop expectation values (see e.g. Ref. [47] for a discussion on how to extract the renormalized quantities from lattice simulations). At low values of β_{eff} , both the bare Polyakov and Wilson loop expectation values saturates at the same constant value. For a fixed-size lattice, the expectation value of the Polyakov loop changes discontinuously as β_{eff} is changed (see Fig. 11). In analogy to QCD studies in $(3+1)$ dimensions [48], we define this value of the lattice coupling as the critical coupling β_t . In analogy to our results for 2d above, the expectation value for the Wilson loop starts to change at a generically different value of $\beta_{\text{eff}} = \beta_s$ (see Fig. 11). As can be seen from Fig 11, for the values of N and lattice volumes simulated here, the change in the expectation value of the Wilson loop is gradual in β , unlike the dependence seen in the Polyakov loop. For this reason it is difficult to unambiguously identify a critical value β_s given the present statistics. For $N = 6$ we find $\beta_s \simeq 2.4(1)$ on a 2×8^3 lattice. The values of β_t depend on the chosen lattice size as well as on N and are shown in Tab. I. Also shown in Fig. 11 are the results for the pressure in the 4d theory. From this figure it can be seen that again the critical Polyakov coupling β_t corresponds to the location of deconfinement transition. Note that the pressure results were calculated on a coarse (2×8^3) lattice, which suffers from significant lattice artifacts at high temperature (cf. the discussion in Ref. [48]), which explains why the

FIG. 11. 4d simulations on a $N_t = 2$, $N_s = 8$ lattice as a function of effective lattice coupling β_{eff} . Left: bare Polyakov and Wilson loops. Right: Pressure as a function of lattice coupling. β_t denotes the critical lattice coupling at which the Polyakov loop susceptibility has a peak (see results in Tab. I). The line labeled 'Stefan-Boltzmann' is the Stefan-Boltzmann result for 8 adjoint bosonic degrees of freedom.

numerical results do not seem to converge to the continuum Stefan-Boltzmann behavior.

We proceed to study the eigenvalue distribution for the Wilson and Polyakov loops respectively. In Fig. 12 the eigenvalue distributions are shown for the case of $N = 4$ on a fixed-size lattice of $N_t = 2$, $N_s = 8$. In the left panel, localization of the Polyakov loop eigenvalues can be seen for $\beta_{\text{eff}} = 1.25$ and 2.5 , which indicates the system is in the deconfined phase. At $\beta_{\text{eff}} = 0.5$, $\mathbb{Z}_4$ -symmetric distribution with four peaks can be seen. It is rather different from the $p = 1$ and $p = 2$ cases, in which rather uniform, continuous distributions have been observed. This can be attributed to the larger fluctuations of eigenvalues in lower dimensions due to larger infrared effects. A similar situation can be found in the treatment of the eigenvalues of scalar fields in supersymmetric Yang-Mills theories; in 3d and 4d one can choose the values by hand because of super-selection, while in 1d and 2d super-selection cannot work and the eigenvalues should be determined dynamically [49]. As for the Wilson loop eigenvalues, four peaks can be seen at $\beta_{\text{eff}} = 0.5$ and 1.25 , while the distribution is uniform at $\beta_{\text{eff}} = 2.5$. This is natural because super-selection requires large volume, which corresponds to smaller β_{eff} for a fixed lattice size.

To test for the sensitivity to the presence of scalars, we can again change the value of the parent dimension d . The $N = 3$ results for β_t on a 4×16^3 lattice can be readily

FIG. 12. The eigenvalue distribution for the Polyakov loop (left) and Wilson loop (right). Both are for the 4d theory for $N = 4$ on a fixed-size lattice of 2×8^3 points. Note the change in the Polyakov loop behavior above/below $\beta_{\text{eff}} \simeq 0.56N^2$ and the Wilson loop behavior above/below $\beta_{\text{eff}} \simeq 2.4N^2$.

compared to the case of $d = 4$ (pure glue QCD) where $\beta_t(d = 4) = 5.6908(2)$ [48] or $\beta_t(d = 4) = 0.63231(2)$ (see Ref. [50] for a study of pure glue thermodynamics in $d = 4$ for different N). With our present statistics, this result is numerically indistinguishable from the case of $d = 10$, $p = 4$ case given in Tab. I. This suggests that the scalars have acquired sizable mass comparable to the temperature scale of the confinement deconfinement transition. In order to simulate smaller scalar masses, we are led to larger lattice spacings (or coarser lattices), e.g. $N_t = 2$, $N_s = 8$. For this case the we find $\beta_t = 0.546(2)N^2$ for $N = 3$ for $d = 10$, while $\beta_t(d = 4) = 5.55(11)N^2$, suggesting a mild dependence on the number of scalars. However, the difference between the number of parent theory dimension (which translates to the number of scalars simulated in the 4d theory) is more apparent when considering the action difference ΔS , cf. Eq. (11). In the left panel of Fig. 13, the results for the ΔS are shown for the theory with 6 scalars and no scalars on a 2×8^3 lattice. Recalling that this quantity effectively is the derivative of the free energy with respect to the lattice coupling (cf. Eq. 11), the pronounced difference between the two theories translates into a difference in the free energy at high temperature. Thus, at least on coarse lattices such as 2×8^3 , the scalars have not decoupled from the theory.

FIG. 13. Left: Simulation results for the action difference ΔS (cf. Eq. (11)) for the 4d theory and $N=3$ on a 2×8^3 lattice with parent dimensions $d = 10$ (“6 scalars”) and $d = 4$ (“no scalars”), respectively. Right: $d = 10$ simulation results for the Polyakov and Wilson loop expectation values on a fixed size 1×3^9 lattice.

F. $d = 10$

It is also technically possible to simulate the original 10d gauge theory on the lattice. This theory is not expected to exist in the sense that it does not have a well defined continuum limit. Yet one may potentially be interested in the 10d theory as a cutoff theory. The lattice spacing a provides such a cutoff and we have performed lattice simulations on a $N_t = 1$, $N_s = 3$ lattice. The results for the Polyakov and Wilson loop expectation values are shown in the right panel of Fig. 13.

IV. SUMMARY AND CONCLUSIONS

Simulating supersymmetric gauge theories on the lattice has interesting connections to gravitational theories via holographic dualities. This work is meant as a step in this direction via simulating the high-temperature phase of these supersymmetric theories, which can be approximated as pure gauge theories in this limit. We have performed lattice simulations of pure gauge 10 dimensional Yang-Mills $SU(N)$ theory toroidally compactified to $p = 1, 2, 4$ dimensions. The compactification naturally turns the gauge field components in the compactified directions into scalars without any extra programming effort. A possible downside of this approach is the generation of potentially sizable scalar masses.

We find excellent agreement with simulation results by other group for the 1d case and for the high-temperature limit of the supersymmetric 2d case. We have confirmed the conjectured phase diagram for the bosonic theory for the 2d case and have presented first results for the bosonic 4d case. More work is clearly needed, for instance to determine the scalar masses in the bosonic lattice simulations, so we intend to revisit this (and other) issues in the future. Another future application of this work concerns the real time simulation of the (semi-classical) pure gauge theory dynamics with our code package. The simulation code is flexible concerning the number of parent dimensions d as well as the number of colors d , easy to use, and is publicly available as a service to the community [41].

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Appendix A: Comparison between SYM and pure $SU(N)$ lattice simulations for 2d

In this appendix we compare the results from our pure $SU(N)$ simulations in 2d to those for SYM (gauge theory plus fermions) from Ref. [20]. We compare lattice data for the Wilson loop expectation value on a lattices with $N_s = 8$ and $N_t = 2, 4$ for various values of β_{eff} . Results are shown in Fig. 14. Note that results from Ref. [20], which were calculated using QCD normalization convention [51], to holographic normalization convention (cf. Eq. (1).

The comparison shown in Fig. 14 indicates that results for the SYM simulation indeed coincide with the purely bosonic $SU(N)$ simulations for high temperatures (low temporal radius). In practice, we find good agreement between the full SYM and bosonic $SU(N)$ simulations for $\bar{r}_t \leq 0.2$ (corresponding to $T \geq 5\lambda_{(2)}^{1/2}$).

FIG. 14. Comparison between pure $SU(N)$ simulations for the 2d case (this work) to lattice simulations of SYM for same-size lattices as a function of temporal radius $\bar{r}_t$. All pure $SU(N)$ results shown in this figure are for the deconfined phase.

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