Tensions and correlations in |V_{cb}| determinations

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Tensions and correlations in $|V_{cb}|$ determinations

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Recently several papers extracted $|V_{cb}|$ using the Belle measurement [1] of the exclusive $\bar{B} \to D^*\ell\bar{\nu}$ unfolded differential decay rates, available for the first time. Depending on the theoretical inputs, some of the fits yield higher $|V_{cb}|$ values, compatible with those from inclusive semileptonic B decays. Since these four fits use mostly the same data, if their correlations were close to 100%, the tension between them would be over 5σ. We determine the correlations, find that the tension between the results is less than 3σ, and explore what might lead to improving the consistency of the fits. We find that fits that yield the higher values of $|V_{cb}|$, also suggest large violations of heavy quark symmetry. These fits are also in tension with preliminary lattice QCD data on the form factors. Without additional experimental data or lattice QCD input, there are no set of assumptions under which the tension between exclusive and inclusive determinations of $|V_{cb}|$ can be considered resolved.

I. INTRODUCTION

Using the unfolded $\bar{B} \to D^*\ell\,\bar{\nu}$ spectra from Belle [1], several theory papers [2–4] could perform fits to the data for the first time, using different theoretical approaches. Using the BGL parametrization [5, 6] for the $\bar{B} \to D^*\ell\,\bar{\nu}$ form factors, a substantial shift in the extracted value of $|V_{cb}|$ was found [3, 4], compared to the Belle [1] analysis using the CLN [7] parametrization,

$$|V_{cb}|_{\rm CLN} = (38.2 \pm 1.5) \times 10^{-3}\,, \qquad [1]\,, \tag{1a}$$

$$|V_{cb}|_{\rm BGL} = (41.7^{+2.0}_{-2.1}) \times 10^{-3}\,, \qquad [3]\,, \tag{1b}$$

$$|V_{cb}|_{\rm BGL} = (41.9^{+2.0}_{-1.9}) \times 10^{-3}\,, \qquad [4]\,. \tag{1c}$$

The main result in Ref. [1] was $|V_{cb}|_{\rm CLN} = (37.4 \pm 1.3) \times 10^{-3}$, obtained from a fit inside the Belle framework, before unfolding. Only Eq. (1a) quoted in the Appendix of [1] can be directly compared with Eqs. (1b) and (1c). These papers, as well as this work, use the same fixed value of $\mathcal{F}(1)$ [8] (see Eq. (4) below), so the differences in the extracted values of $|V_{cb}|$ are due to the extrapolations to zero recoil, where heavy quark symmetry gives the strongest constraint on the rate [9–13]. Intriguingly, the BGL fit results for $|V_{cb}|$ are compatible with those from inclusive $B \to X_c\ell\bar{\nu}$ measurements [14]. If one assumed, naively, a 100% correlation between the fits yielding Eqs. (1a), (1b), and (1c), then the tension between Eqs. (1a) and (1b) or between Eqs. (1a) and (1c) would be above 5σ.

The BGL [5, 6] fit implements constraints on the $B \to D^*\ell\bar{\nu}$ form factors based on analyticity and unitarity [15–17]. The CLN [7] fit imposes, in addition, constraints on the form factors from heavy quark symmetry, and relies on QCD sum rule calculations [18–20] of the subleading Isgur-Wise functions [13, 21], without accounting for their uncertainties. Ref. [2] performed combined fits to $\bar{B} \to D^*\ell\,\bar{\nu}$ and $\bar{B} \to D\ell\,\bar{\nu}$, using predictions of the heavy quark effective theory (HQET) [22, 23], including all $\mathcal{O}(\Lambda_{\rm QCD}/m_{c,b})$ uncertainties and their correlations for the first time. The effect of relaxing the QCD sum rule inputs in the CLN fit was found to be small compared to the difference of the CLN and BGL results.

The recent papers using the BGL parametrization [3, 4] assert that the higher values obtained for $|V_{cb}|$ are due to the too restrictive functional forms used in the CLN fits. It was previously also noticed that the CLN gives a poorer fit to the $B \to D\ell\bar{\nu}$ data than BGL [24]. The effects on $|V_{cb}|$ due to additional theoretical inputs were also explored in Refs. [25, 26].

Based on our work in Ref. [2], we explore which differences between the BGL and CLN fits are responsible for the different extracted $|V_{cb}|$ values, study the consistency and compatibility of the fits, and the significance of the shift in the extracted value of $|V_{cb}|$.

II. DEFINITIONS

The $B \to D^*\ell\bar{\nu}$ form factors which occur in the standard model are defined as

$$\begin{aligned}
\langle D^* |\, \bar{c}\gamma^\mu b \,| \overline{B}\rangle &= i\sqrt{m_B m_{D^*}}\, h_V\, \varepsilon^{\mu\nu\alpha\beta}\, \epsilon^*_\nu v'_\alpha v_\beta\,, \\
\langle D^* |\, \bar{c}\gamma^\mu \gamma^5 b \,| \overline{B}\rangle &= \sqrt{m_B m_{D^*}}\, \big[h_{A_1}(w+1)\epsilon^{*\mu} \\
&\qquad - h_{A_2}(\epsilon^*\cdot v)v^\mu - h_{A_3}(\epsilon^*\cdot v)v'^\mu\big],
\end{aligned} \tag{2}$$

where v is the four-velocity of the B and v' is that of the D^*. The form factors $h_{V,A_{1,2,3}}$ depend on $w = v\cdot v' = (m_B^2 + m_{D^*}^2 - q^2)/(2m_B m_{D^*})$. Neglecting lepton masses, only one linear combination of h_{A_2} and h_{A_3} is measurable. In the heavy quark limit, $h_{A_1} = h_{A_3} = h_V = \xi$ and $h_{A_2} = 0$, where ξ is the Isgur-Wise function [9, 10]. Each of these form factors can be expanded in powers of $\Lambda_{\rm QCD}/m_{c,b}$ and α_s. It is convenient to parametrize deviations from the heavy quark limit via the form factor ratios

$$R_1(w) = \frac{h_V}{h_{A_1}}\,, \qquad R_2(w) = \frac{h_{A_3} + r_{D^*} h_{A_2}}{h_{A_1}}\,, \tag{3}$$

form factors	BGL	CLN	CLNnoR	noHQS
axial $\propto \epsilon_\mu^*$	$b_0,\, b_1$	$h_{A_1}(1),\ \rho^2_{D^*}$	$h_{A_1}(1),\ \rho^2_{D^*}$	$h_{A_1}(1),\ \rho^2_{D^*},\ c_{D^*}$
vector	$a_0,\, a_1$	$\left\{ R_1(1),\ R_2(1) \right.$	$\left\{ R_1(1),\ R_1'(1) \right.$	$\left\{ R_1(1),\ R_1'(1) \right.$
$\mathcal{F}$	$c_1,\, c_2$		$\left\{ R_2(1),\ R_2'(1) \right.$	$\left\{ R_2(1),\ R_2'(1) \right.$

TABLE I. The fit parameters in the BGL, CLN, CLNnoR, and noHQS fits, and their relationships with the form factors.

which satisfy $R_{1,2}(w) = 1 + \mathcal{O}(\Lambda_{\rm QCD}/m_{c,b},\, \alpha_s)$ in the $m_{c,b} \gg \Lambda_{\rm QCD}$ limit, and $r_{D^*} = m_{D^*}/m_B$.

The $B \to D^*\ell\bar\nu$ decay rate is given by

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}w} = \frac{G_F^2 |V_{cb}|^2\, m_B^5}{48\pi^3}\, (w^2-1)^{1/2}\, (w+1)^2\, r_{D^*}^3 (1-r_{D^*})^2 \times \left[1 + \frac{4w}{w+1}\, \frac{1-2wr_{D^*}+r_{D^*}^2}{(1-r_{D^*})^2}\right] \mathcal{F}(w)^2\,, \qquad (4)$$

and the expression of $\mathcal{F}(w)$ in terms of the form factors defined in Eq. (2) is standard in the literature [27]. In the heavy quark limit, $\mathcal{F}(w) = \xi(w)$. We further denote

$$\rho^2_{D^*} = -\frac{1}{h_{A_1}(1)}\, \frac{\mathrm{d}h_{A_1}(w)}{\mathrm{d}w}\bigg|_{w=1}\,, \qquad (5)$$

which is a physical fit parameter in the CLN approach, and is a derived quantity in the other fits.

III. NEW FITS, LATTICE QCD, AND THEIR TENSIONS

The constraints built into the CLN fit can be relaxed by ignoring the QCD sum rule inputs and the condition $R_{1,2}(w) = 1 + \mathcal{O}(\Lambda_{\rm QCD}/m_{c,b},\, \alpha_s)$ following from heavy quark symmetry. (Ref. [2] showed that only ignoring the QCD sum rule inputs, and using only $w=1$ lattice QCD data, leaves $|V_{cb}| = (38.8 \pm 1.2) \times 10^{-3}$.) Thus, we write

$$\begin{aligned} R_1(w) &= R_1(1) + (w-1) R_1'(1)\,, \\ R_2(w) &= R_2(1) + (w-1) R_2'(1)\,, \end{aligned} \qquad (6)$$

and treat $R_{1,2}(1)$ and $R'_{1,2}(1)$ as fit parameters. We refer to this fit as "CLNnoR". It has the same number of fit parameters as BGL, and allows $\mathcal{O}(1)$ heavy quark symmetry violation, but the constraints on the form factors are nevertheless somewhat different than in BGL.

While this CLNnoR fit is a simple modification of the CLN fit widely used by BaBar and Belle, it still relies on heavy quark symmetry and model-dependent input on subleading Isgur-Wise functions. The reason is that both CLN and CLNnoR use a cubic polynomial in $z = (\sqrt{w+1}-\sqrt{2})/(\sqrt{w+1}+\sqrt{2})$ to parametrize the form factor h_{A_1}, with its four coefficients determined by two parameters, $h_{A_1}(1)$ and $\rho^2_{D^*}$, derived from unitarity constraints on the $B \to D$ form factor. Therefore, we also consider a "noHQS" scenario, parametrizing h_{A_1} by a quadratic polynomial in z, with unconstrained coefficients,

$$h_{A_1}(w) = h_{A_1}(1) \big[1 - 8\rho^2_{D^*} z + (53.\, c_{D^*} - 15.) z^2\big], \qquad (7)$$

	CLN	CLNnoR	noHQS	BGL
$\lvert V_{cb}\rvert \times 10^3$	38.2 ± 1.5	41.5 ± 1.9	41.8 ± 1.9	41.5 ± 1.8
$\rho^2_{D^*}$	1.17 ± 0.15	1.6 ± 0.2	1.8 ± 0.4	1.54 ± 0.06
c_{D^*}	$\rho^2_{D^*}$	$\rho^2_{D^*}$	2.4 ± 1.6	fixed: 15./53.
$R_1(1)$	1.39 ± 0.09	0.36 ± 0.35	0.48 ± 0.48	0.45 ± 0.28
$R_2(1)$	0.91 ± 0.08	1.10 ± 0.19	0.79 ± 0.36	1.00 ± 0.18
$R_1'(1)$	fixed: -0.12	5.1 ± 1.8	4.3 ± 2.6	4.2 ± 1.2
$R_2'(1)$	fixed: 0.11	-0.89 ± 0.61	0.25 ± 1.3	-0.53 ± 0.42
χ^2 / ndf	35.2 / 36	27.9 / 34	27.6 / 33	27.7 / 34

TABLE II. Summary of CLN, CLNnoR, noHQS, and BGL fit results.

	$\lvert V_{cb}\rvert_{\rm CLN}$	$\lvert V_{cb}\rvert_{\rm CLNnoR}$	$\lvert V_{cb}\rvert_{\rm noHQS}$	$\lvert V_{cb}\rvert_{\rm BGL}$
$\lvert V_{cb}\rvert_{\rm CLN}$	1.	0.75	0.69	0.76
$\lvert V_{cb}\rvert_{\rm CLNnoR}$		1.	0.95	0.97
$\lvert V_{cb}\rvert_{\rm noHQS}$			1.	0.97
$\lvert V_{cb}\rvert_{\rm BGL}$				1.

TABLE III. Correlation matrix of the four extracted $|V_{cb}|$ values. For BGL the outer functions of Ref. [4] were used. All results are derived by bootstrapping [28] the unfolded distributions of Ref. [1] using the published covariance.

keeping the same prefactors as in CLN, to permit comparison between $\rho^2_{D^*}$ and c_{D^*} (in the CLN fit $c_{D^*} = \rho^2_{D^*}$).

The fit parameters in the BGL, CLN, CLNnoR, and noHQS fits are summarized in Table I. The results of these fits for $|V_{cb}|$, $\rho^2_{D^*}$, c_{D^*}, $R_{1,2}(1)$, and $R'_{1,2}(1)$ are shown in Table II. The BGL, CLNnoR, and noHQS results are consistent with each other, including the uncertainties, and the fit quality. The correlations of these four fit results for $|V_{cb}|$ are shown in Table III and have been derived by creating a bootstrapped [28] ensemble of the unfolded distributions of Ref. [1], using the published covariance. Each set of generated decay distributions in the ensemble is fitted with the BGL, CLN, CLNnoR, and noHQS parametrizations, and the produced ensemble of $|V_{cb}|$ values is used to estimate the covariance between them. The correlation of the CLN fit with either BGL, CLNnoR, or noHQS is substantially below 100%. This reduces the tension between these fits to below 3σ.

As soon as $R'_{1,2}(1)$ are not constrained to their values imposed in the CLN framework, large deviations from those constraints are observed. The BGL, CLNnoR, and noHQS results favor a large value for $R_1'(1)$, in tension with the heavy quark symmetry prediction, $R_1'(1) =$

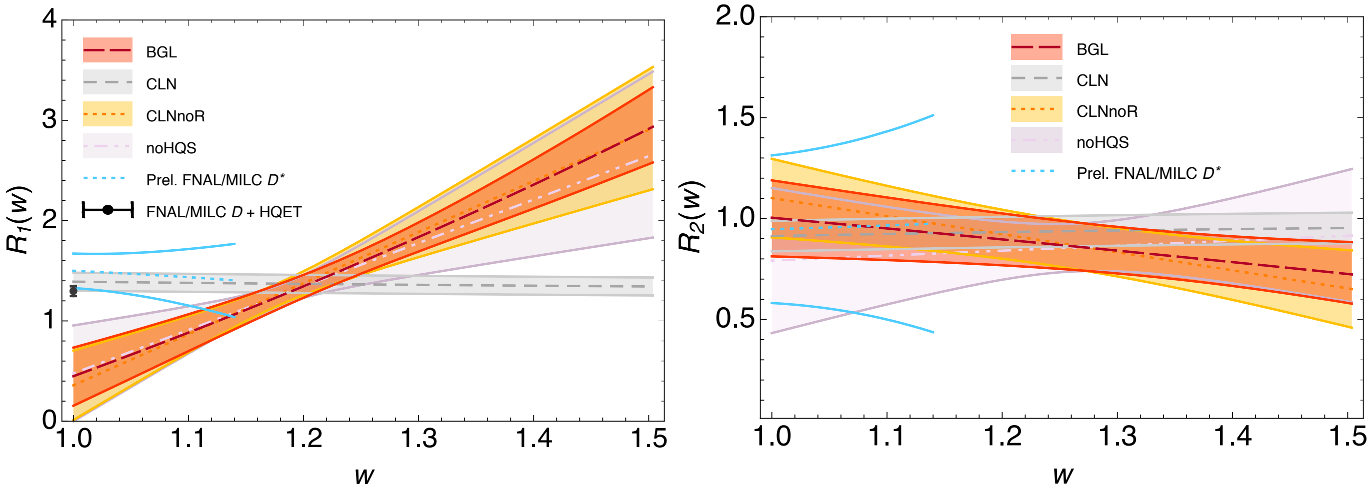

FIG. 1. The form factor ratios $R_1(w)$ (left) and $R_2(w)$ (right) for the BGL (red long dashed), CLN (gray dashed), CLNnoR (orange dotted) fits, and noHQS (purple dot-dot-dashed). The BGL, CLNnoR, and noHQS fits for R_1 suggest a possibly large violation of heavy quark symmetry, in conflict with lattice QCD predictions. The blue lines show our estimated bounds, based on preliminary FNAL/MILC lattice results [29]. The black data point for $R_1(1)$ follows from the FNAL/MILC $B \to D\,\ell\bar\nu$ result and heavy quark symmetry (see details in the text).

$\mathcal{O}(\Lambda_{\rm QCD}/m_{c,b},\, \alpha_s)$.

These aspects of the BGL, CLNnoR, and noHQS fits are also in tension with lattice QCD results. Recently the first preliminary lattice results were made public on the $B \to D^*\ell\bar\nu$ form factors away from zero recoil, at finite lattice spacing [29]. The results are fairly stable over a range of lattice spacings. Assuming that the continuum extrapolation will not introduce a sizable shift (the chiral logs are not large [30, 31]) we can estimate the projections for the $R_{1,2}(w)$ form factor ratios. We approximate the predicted form factors in a narrow range of w using a linear form, with a normalization and slope chosen such that they encompass all reported lattice points and uncertainties in Ref. [29]. At zero recoil, we estimate the ranges $1.3 \lesssim R_1(1) \lesssim 1.7$ and $0.6 \lesssim R_2(1) \lesssim 1.3$ from looking at the (preliminary) plots in Ref. [29], which should be viewed as bounds on these values, as the final lattice QCD results will likely have smaller uncertainties. Figure 1 shows $R_{1,2}(w)$ derived from the results of our fit scenarios, as well as these lattice QCD constraints.

We can obtain another independent prediction for $R_1(1)$ based on lattice QCD and heavy quark symmetry, using the result for the $B \to D\,\ell\bar\nu$ form factor [32]. Using the $\mathcal{O}(\Lambda_{\rm QCD}/m_{c,b},\, \alpha_s)$ expressions [2], the f_+ form factor (see Eq. (2.1) in Ref. [32]) and the subleading Isgur-Wise function η are related at zero recoil via

$$\frac{2\sqrt{r_D}}{1+r_D} f_+(1) = 1 + \hat\alpha_s \left(C_{V_1} + C_{V_2}\,\frac{2r_D}{1+r_D} + C_{V_3}\,\frac{2}{1+r_D} \right) - (\varepsilon_c - \varepsilon_b)\,\frac{1-r_D}{1+r_D}\,[2\,\eta(1) - 1] + \dots\,, \qquad (8)$$

since other subleading Isgur-Wise functions enter suppressed by $w-1$. Here $r_D = m_D/m_B$, $\varepsilon_{c,b} = \bar\Lambda/m_{c,b}$ is treated as in Ref. [2], and hereafter the ellipsis denotes $\mathcal{O}(\varepsilon_{c,b}^2,\, \alpha_s\,\varepsilon_{c,b},\, \alpha_s^2)$ higher order corrections. Using $f_+(w=1) = 1.199 \pm 0.010$ [32] one finds $\eta(1) = 0.35 \pm 0.10$. The uncertainty in this relation and the extracted value of $\eta(1)$ is dominated by $\mathcal{O}(\Lambda_{\rm QCD}^2/m_c^2)$ corrections parametrized by several unknown matrix elements [33], which we estimate with $\varepsilon_c^2 \sim 0.05$. Thus,

$$R_1(1) = 1.34 - 0.12\,\eta(1) + \ldots = 1.30 \pm 0.05\,. \qquad (9)$$

(Recall that both the α_s terms and a $\bar\Lambda/(2m_c)$ correction enhance $R_1(1)$.) This estimate is shown with the black dot and error bar in the left plot in Fig. 1. It shows good consistency with our estimate from the preliminary direct calculation of the $B \to D^*\ell\bar\nu$ form factors, as shown in the region bounded by the blue curves.

Another clear way to see that the central values of the BGL, CLNnoR, and noHQS fit results cannot be accommodated in HQET, without a breakdown of the expansion, is by recalling [2] that besides Eq. (9), also

$$\begin{aligned} R_2(1) &= 0.98 - 0.42\,\eta(1) - 0.54\,\hat\chi_2(1) + \dots\,,\\ R_1'(1) &= -0.15 + 0.06\,\eta(1) - 0.12\,\eta'(1) + \dots\,, \qquad (10)\\ R_2'(1) &= 0.01 - 0.54\,\hat\chi_2'(1) + 0.21\,\eta(1) - 0.42\,\eta'(1) + \dots\,. \end{aligned}$$

Here η and $\hat\chi_2$ are subleading Isgur-Wise functions. Eqs. (9) and (10) have no solutions close to the BGL, CLNnoR, or noHQS fit results in Table II with $\mathcal{O}(1)$ values for $\eta(1)$, $\eta'(1)$, $\hat\chi_2(1)$, and $\hat\chi_2'(1)$.

Figure 2 shows $\mathrm{d}\Gamma/\mathrm{d}w$ in the four fit scenarios, as well as the Belle data [1]. The shaded bands show the uncertainties of the CLN and noHQS fits, which are comparable to the uncertainties of the other two fits. The BGL, CLNnoR, and noHQS fits show larger rates near zero and maximal recoil, in comparison to CLN. The CLN fit shows a larger rate at intermediate values of w.

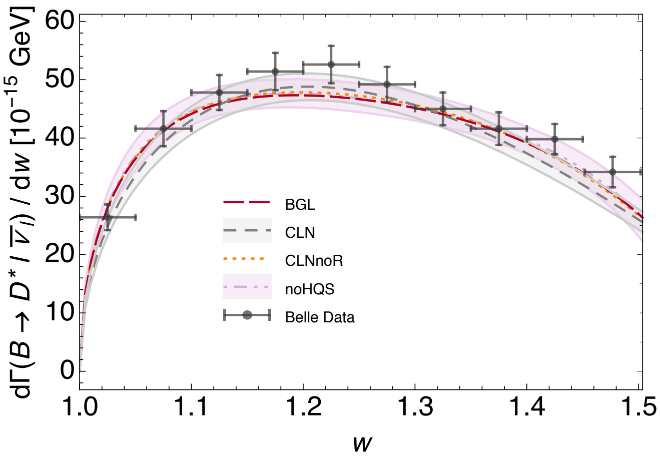

FIG. 2. $\mathrm{d}\Gamma/\mathrm{d}w$ for the fit scenarios shown in Fig. 1.

IV. CONCLUSIONS

Our results show that the tensions concerning the exclusive and inclusive determinations of $|V_{cb}|$ cannot be considered resolved. The central values of the BGL, CLNnoR, and noHQS fits, which all give good descriptions of the data, suggest possibly large deviations from heavy quark symmetry. These results are also in tension with preliminary lattice QCD predictions for the form factor ratio R_1, which use the same techniques as for the determination of $\mathcal{F}(1)$ used to extract $|V_{cb}|$ from $B \to D^*\ell\bar{\nu}$. If the resolution of the tension between lattice QCD and the fits for R_1 is a fluctuation in the data, then we would expect the extracted value of $|V_{cb}|$ to change in the future. If the resolution of the tension is on the lattice QCD side, then it may also affect the calculation of $\mathcal{F}(1)$ used to extract $|V_{cb}|$. We look forward to higher statistics measurements in the future, and a better understanding of the composition of the inclusive semileptonic rate as a sum of exclusive channels [34, 35], which should ultimately allow unambiguous resolution of these questions.

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