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Color Characters for White Hot String Bits

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Abstract

The state space of a generic string bit model is spanned by $N \times N$ matrix creation operators acting on a vacuum state. Such creation operators transform in the adjoint representation of the color group $U(N)$ (or $SU(N)$ if the matrices are traceless). We consider a system of b species of bosonic bits and f species of fermionic bits. The string, emerging in the $N \to \infty$ limit, identifies $P^+ = mM\sqrt{2}$ with M the bit number operator and $P^- = H\sqrt{2}$ with H the system Hamiltonian. We study the thermal properties of this string bit system in the case $H = 0$, which can be considered the tensionless string limit: the only dynamics is restricting physical states to color singlets. Then the thermal partition function $\mathrm{Tr}e^{-\beta mM}$ can be identified, putting $x = e^{-\beta m}$, with a generating function $\chi_0^{bf}(x)$, for which the coefficient of x^n in its expansion about $x = 0$ is the number of color singlets with bit number $M = n$. This function is a purely group theoretic object, which is well-studied in the literature. We show that at $N = \infty$ this system displays a Hagedorn divergence at $x = 1/(b + f)$ with ultimate temperature $T_H = m/\ln(b + f)$. The corresponding function for finite N is perfectly finite for $0 < x < 1$, so the $N = \infty$ system exhibits a phase transition at temperature T_H which is absent for any finite N. We demonstrate that the low temperature phase is unstable above T_H. The lowest-order $1/N$ asymptotic correction, for $x \to 1$ in the high temperature phase, is computed for large N. Remarkably, this is related to the number of labeled Eulerian digraphs with N nodes. Systematic methods to extend our results to higher orders in $1/N$ are described.

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1 Introduction

Recent work on string bits at finite temperature [1, 2] explored the nature of a transition from a stringy low temperature phase to a high temperature bit phase. Here we give a follow up to [2], in which we supplement the numerical analysis of that paper. and extend consideration to more general string bit models. There is considerable overlap of many of our mathematical results with earlier work on gauge theories at zero coupling [3, 4].

The state space of string bit models [5, 6, 7, 8, 9] is spanned by monomials of matrix creation operators $(\bar{A}_K)_\alpha^{\ \beta} = ((A_K)^\alpha_\beta)^\dagger$ acting on the vacuum $|0\rangle$ annihilated by the A's. In this notation $\alpha, \beta = 1, \ldots, N$ label the fundamental and antifundamental representations, respectively, of a color $U(N)$ group. In other words each string bit lies in the adjoint representation of $U(N)$. The index K denotes collectively all of the physical properties of a single string bit: spin, flavor, momentum, etc. A string interpretation can be given to the color singlet subspace of string bit states, which can be spanned by states of the form

$$\mathrm{tr}\ (\bar{A}_{K_1} \cdots \bar{A}_{K_k}) \cdots \mathrm{tr}\ (\bar{A}_{L_1} \cdots \bar{A}_{L_l})|0\rangle \ , \tag{1}$$

where each trace creates a closed chain from the vacuum $|0\rangle$.

For a closed chain to behave like a continuous string (necessary for Lorentz invariance) the number of bits in the corresponding trace must be infinite. But the trace operator description of a multi-string state is unambiguous only if $N = \infty$ [10]. Otherwise by the Cayley-Hamilton theorem, as soon as the number of bits in a trace is larger than N, that trace can be expressed in terms of products of smaller traces, such that the total number of bits remains the same. Of course Nature only requires approximate Lorentz invariance, in which case N might be finite, albeit ridiculously large.

The thermal partition function of the string bit system is taken to be

$$\mathrm{Tr} e^{-\beta(mM+H)} \ ,$$

where M is the bit number operator and H is the Hamiltonian of the system. This hypothesis is motivated by the connection to the lightcone description [11] of the emergent string for which mM is identified as $P^+/\sqrt{2}$ and H is identified as $P^-/\sqrt{2}$, so that $mM + H$ is identified as $P^0 = (P^+ + P^-)/\sqrt{2}$. Simple examples of H that produce stringy low energy eigenstates (see [8, 9]) have the property that non-singlet eigenstates are separated from singlet eigenstates by a gap that is infinite in energy units of the low lying states. Such a low energy phase exhibits the Hagedorn transition [12, 13] at infinite N [14, 15, 16]: the partition function diverges above some finite temperature $T_H = \beta_H^{-1}$. The central result revealed by the numerical studies in [2] is that this phenomenon is present in a much simpler system with $H = 0$ but with allowed states restricted to color singlets [3]. That is the quantity

$$Z \ = \ \mathrm{Tr} e^{-\beta m M} \mathcal{P}_0 \ , \tag{2}$$

where $\mathcal{P}_0$ projects onto the color singlet subspace, generically exhibits the Hagedorn phenomenon.

In [2] the focus was on the simplest string bit model with one bosonic bit a_α^β and one fermionic bit b_α^β. Then the bit number operator is $M = \mathrm{tr}\,[\bar{a}a + \bar{b}b]$. Then Z can be constructed from the character generating function

$$\chi(x,\theta) = \mathrm{Tr}e^{-\beta m M + iG_k\theta_k} \quad = \quad \prod_{k,l} \frac{1 + xe^{i(\theta_k-\theta_l)}}{1 - xe^{i(\theta_k-\theta_l)}} \;, \tag{3}$$

where $x = e^{-\beta m}$ and G_k is the Cartan subalgebra of $U(N)$. To impose singlet restrictions one simply integrates this character over the group[4]

$$Z \quad = \quad \int [d\theta]\chi(x,\theta) \equiv \chi_0(x) \;, \tag{4}$$

so the coefficient of x^n is precisely the number of color singlet states with total bit number $M = n$. In this simplified model the Z is a purely group theoretic object: the coefficient of x^n counts the total number of color singlets that reside in the symmetrized tensor products of adjoints times the antisymmetrized tensor products of adjoints such that the total number of adjoints is $M = n$.

In this article we consider a general system of string bits containing f species of fermionic bits and b species of bosonic bits:

$$\chi^{bf}(x,\theta) = \mathrm{Tr}e^{-\beta m M + iG_k\theta_k} \quad = \quad \prod_{k,l} \frac{(1 + xe^{i(\theta_k-\theta_l)})^f}{(1 - xe^{i(\theta_k-\theta_l)})^b} \;. \tag{5}$$

If the bit creation operators are traceless, and hence are adjoints under $SU(N)$ rather than $U(N)$, the right side must be multiplied by $(1-x)^b/(1+x)^f$ which removes the contributions of the traces.

In [2] exact results for $\chi_0(x)$ for the case $f = b = 1$ were obtained for $N = 2,3,4,5,6$. But for large N, a steepest descent analysis was applied numerically to the $SU(N)$ model in [2]. Fixing x, the values of θ_k which maximized the integrand were obtained using MATHEMATICA for many values of N ranging up to 100. Then the logarithm of these maxima was fit to a form $f_0(x)N^2 + f_1(x)N + f_2(x) + f_3(x)\ln N + f_4(x)N\ln N$. The fit determined $f_0(x) \approx 0$ for $x < 1/2$ but $f_0(x) > 0$ and monotonically increasing for $x > 1/2$. The fit also determined $f_1(x)$ consistent with zero for $x < 1/2$, and positive and increasing for $x > 1/2$. On the other hand the fitted $f_2(x)$ was non zero and negative for most of the interval $0 < x < 1$, crossing zero at x just above 0.9 and becoming positive. Finally the fitted $f_4(x)$ was consistent with 0 over the whole range of x, and the fitted $f_3(x)$ was consistent with 0 for $x < 1/2$, but nonzero for $x > 1/2$ taking both positive and negative values.

The numerical plot of $f_0(x)$ given in [2], which determines the physics at infinite N, indicated that $f_0(x)$ approaches zero at finite slope (of roughly 0.7) as $x \to 1/2$ from $x > 1/2$. This is consistent with, but does not prove, the hypothesis that the infinite N phase transition

[4]More generally the projection onto a representation whose character is $\chi_R(\theta)$ is achieved by $Z_R = \int [d\theta]\chi_R^*(\theta)\chi(x,\theta)$, a formula exploited to good effect in [17]

is first order. The sharp onset of $f_0(x)$ at $x = 1/2$ is the signal of a Hagedorn transition, and the appearance of an N^2 term for $x > 1/2$ is a sign of the active role of the adjoint string bits in the high temperature phase. In this sense the transition can be viewed as a deconfinement transition.

In the present paper we obtain $\chi_0^{bf}(x)$ in the low temperature regime for $N = \infty$ by exploiting the fact that the uniform distribution $\theta_k = 2\pi k/N$, $k = 0, \ldots, N-1$, is a stationary point for all x. For this distribution we can exactly calculate the Hessian double derivative matrix which determines the Gaussian fluctuations about the stationary point, and, because it is a circulant matrix, one can write down all of it's eigenvalues, and take their large N limit. This shows that all eigenvalues are negative when $x < 1/(f + b)$ and N is sufficiently large. This information allows us to write down the $N \to \infty$ limit of χ_o^{bf} for $x < 1/(f + b)$:

$$\chi_0^{bf}(x) = \prod_{k=1}^{\infty} \frac{1}{(1 - (b + f)x^{2k-1})(1 - (b - f)x^{2k})}, \qquad x < \frac{1}{b + f}, \quad N = \infty, \tag{6}$$

which is a special case of formulas derived long ago in [3, 4] in the context of enumerating color singlets in gauge theories at zero coupling. Although the motivation for these studies was not the study of string bit models, these authors anticipated most of our results.

For $x > 1/(b+f)$ at least one eigenvalue is positive showing that the uniform distribution is then not a maximum. We have not succeeded in finding the non-uniform distribution of θ's necessary to complete our large N analysis of the high temperature phase. However, Sundborg's analysis [3] has led to an effective method [4] for obtaining the density of eigenvalues just above the critical point. In addition, The numerical studies in [2] give a convincing determination of the same distributions.

Our paper is organized as follows. In Section 2 we define the group characters for the string bit models studied in this paper and sketch the large N methods applied to character integrals. In Section 3 we analyze our string bit models at low temperature. While the main results of our analysis have been anticipated in [3, 4], our methods and motivation are somewhat different, and we think they offer new insights and clarifications. In Section 4 we discuss some ideas for solving the high temperature phase. In Section 5 we develop an effective field theory approach to generating the $1/N$ expansion in the low temperature phase. This perturbation expansion breaks down near the critical point, but we offer a partial resummation procedure generated by the Dyson equation, which at least makes the propagator finite at the critical point. In Section 6, we analyze the thermal partition function near infinite temperature $x \approx 1$ and discuss its relation to the problem of counting Eulerian digraphs. We close with a concluding Section 7.

2 $SU(N)$ Characters and Large N

In a general string bit model the bits are adjoint creation operators, b bosonic and f fermionic. Rearranging the products on the right of (5), the $U(N)$ character of this model can be written

$$\chi^{bf}(x,\theta_1,\ldots,\theta_N) = \frac{(1+x)^{Nf}}{(1-x)^{Nb}}\prod_{k<l}\frac{(1+xe^{i\theta_{kl}})^f(1+xe^{-i\theta_{kl}})^f}{(1-xe^{i\theta_{kl}})^b(1-xe^{-i\theta_{kl}})^b}\ , \tag{7}$$

where $\theta_{kl}\equiv\theta_k-\theta_l$. Here the coefficient of x^n is the character of all states with bit number $M=n$. The θ's parameterize the transformations due to the Cartan sub-algebra. String bit models rely on color confinement to yield string models. This confinement is a salient feature of the stable string bit models [8], in which it arises from the dynamics. For example, in the case $b=f=1$, this is accomplished by the Hamiltonian

$$H = \frac{T_0}{2mN}\mathrm{tr}\ \left[(\bar{a}^2-i\bar{b}^2)a^2-(\bar{b}^2-i\bar{a}^2)b^2+(\bar{a}\bar{b}+\bar{b}\bar{a})ba+(\bar{a}\bar{b}-\bar{b}\bar{a})ab\right]\ , \tag{8}$$

where T_0 is the rest tension of the emergent string. In this paper we replace this dynamics with the restriction of allowed states to color singlets, setting $H=0$. Since H is proportional to T_0, we can say that $H=0$ describes the tensionless limit of the emergent string theory. The projection onto color singlet states is achieved by averaging the character (7) over the group.

$$\chi_0^{bf} = D^{-1}\int[d\theta]\frac{(1+x)^{Nf}}{(1-x)^{Nb}}\prod_{k<l}\frac{(1+xe^{i\theta_{kl}})^f(1+xe^{-i\theta_{kl}})^f}{(1-xe^{i\theta_{kl}})^b(1-xe^{-i\theta_{kl}})^b} \tag{9}$$

$$D = \int[d\theta]\equiv\int d\theta_1\cdots d\theta_N\prod_{k<l}(1-e^{i\theta_{kl}})(1-e^{-i\theta_{kl}})\ . \tag{10}$$

2.1 Large N Approximation

Notice that $\ln\chi$, the logarithm of the integrand for χ_0, can be expressed as a double sum, each ranging from 1 to N: it generically should be of order N^2 as $N\to\infty$. This justifies approximating the integral by a steepest descent evaluation valid for large N. Define

$$L(\theta)\equiv\sum_{k>l}\ln\frac{(1-e^{i\theta_{kl}})(1-e^{-i\theta_{kl}})(1+xe^{i\theta_{kl}})^f(1+xe^{-i\theta_{kl}})^f}{(1-xe^{i\theta_{kl}})^b(1-xe^{-i\theta_{kl}})^b}\ , \tag{11}$$

where f is the number of fermion bits, b the number of bosonic bits, and $\theta_{kl}\equiv\theta_k-\theta_l$. We seek a maximum for L which requires it to be stationary:

$$0 = \frac{\partial L}{\partial\theta_k}=\sum_{l\neq k}\left[\cot\frac{\theta_{kl}}{2}-\frac{2(b+f)x(1+x^2)\sin\theta_{kl}+2(b-f)x^2\sin 2\theta_{kl}}{1+x^4-2x^2\cos 2\theta_{kl}}\right]\ . \tag{12}$$

For large N one should be able to treat the sum over l as an integral (see for example [18, 19, 20]), by first ordering the θ's so that $\theta_1 < \theta_2 < \cdots < \theta_N$ and then introducing a density function ρ via

$$\sum_l S_l \to N \int_{-\pi}^{\pi} d\theta \rho(\theta) S(\theta) \ , \qquad \int d\theta \rho(\theta) = 1 \ , \tag{13}$$

after which the stationarity condition becomes an integral equation

$$0 = \fint d\theta' \rho(\theta') \left[\cot \frac{\theta - \theta'}{2} - \frac{2(f+b)x(1+x^2)\sin(\theta-\theta') + 2(b-f)x^2 \sin 2(\theta-\theta')}{1+x^4-2x^2\cos 2(\theta-\theta')} \right] \ , \tag{14}$$

where $\fint$ denotes the principal value prescription for $\theta' = \theta$. Correspondingly

$$L \quad \to \quad L_0 \equiv N^2 \int_{\theta > \theta'} d\theta \rho(\theta) d\theta' \rho(\theta') \ln \frac{|1 - e^{i(\theta-\theta')}|^2 |1 + x e^{i(\theta-\theta')}|^{2f}}{|1 - x e^{(\theta-\theta')i}|^{2b}} \ . \tag{15}$$

The uniform distribution $\rho = 1/(2\pi)$, for $-\pi < \theta < \pi$ solves (14) for all x. However this stationary point may not be the global maximum. Indeed for this solution, it is easy to show that $L = 0$, and any solution for which $L_0 > 0$ would dominate it.

In fact $\theta_k = 2\pi k/N$, $k = 1, \ldots, N$ is a solution of (12) before passing to the continuum limit. To see this, call the summand in the equation evaluated on these θ's $S(k-l)$:

$$\begin{aligned} S(k-l) \quad &= \quad \cot \frac{2\pi(k-l)}{2N} \\ &\quad - \frac{2(f+b)x(1+x^2)\sin(2\pi(k-l)/N) + 2(b-f)x^2 \sin(4\pi(k-l)/N)}{1+x^4-2x^2\cos(4\pi(k-l)/N)} \ . \end{aligned} \tag{16}$$

Then by inspection we have $S(n+N) = S(n)$ and $S(n) = -S(-n)$. Then

$$\begin{aligned} \sum_{l \neq k} S(k-l) \quad &= \quad \sum_{l=1}^{k-1} S(k-l) + \sum_{l=k+1}^{N} S(k-l) = \sum_{l=1-k}^{-1} S(-l) + \sum_{l=1}^{N-k} S(-l) \\ &= \quad \sum_{l=1}^{N-1} S(-l) = \sum_{l=1}^{N-1} S(-(N-l)) = \sum_{l=1}^{N-1} S(l) = -\sum_{l=1}^{N-1} S(-l) \ . \end{aligned} \tag{17}$$

From the second line we see that $\sum S(-l) = -\sum S(-l)$ which implies $\sum S(-l) = 0$. This establishes that $\theta_k = 2\pi k/N$ solves the stationarity equations.

3 Low temperature solution at $N = \infty$

Actually, one can straightforwardly evaluate the integrand for $\theta_k = 2\pi k/N$, with $k = 1, 2, \ldots, N$, i.e. for the uniform distribution of the θ's, without passing to the continuum approximation of the sums.

$$L(\theta) \rightarrow \frac{1}{2}\sum_{k\neq l}\left[\ln|1-e^{2\pi i(k-l)/N}|^2+\ln|1+xe^{2\pi i(k-l)/N}|^{2f}-\ln|1-xe^{2\pi i(k-l)/N}|^{2b}\right]$$

$$= \frac{N}{2}\sum_{k=1}^{N-1}\left[\ln|1-e^{2\pi ik/N}|^2+\ln|1+xe^{2\pi ik/N}|^{2f}-\ln|1-xe^{2\pi ik/N}|^{2b}\right] . \tag{18}$$

Each of the sums can be evaluated as follows:

$$\sum_{k=1}^{N-1}\ln(1-ue^{2\pi ik/N}) = -\sum_{n=1}^{\infty}\frac{u^n}{n}\sum_{k=1}^{N-1}e^{2\pi ink/N} . \tag{19}$$

The inner sum is $N-1$ for n an integer multiple of N and if not it is -1. Then we have

$$\sum_{k=1}^{N-1}\ln(1-ue^{2\pi ik/N}) = -N\sum_{l=1}^{\infty}\frac{u^{lN}}{lN}+\sum_{n=1}^{\infty}\frac{u^n}{n} = \ln\frac{1-u^N}{1-u} . \tag{20}$$

And we need this formula for $u=1$ when it gives $\ln N$, and for $u=-x$ and $u=x$. Putting these things together gives

$$L \rightarrow N\ln N + Nf\ln\frac{1-(-)^N x^N}{1+x} - Nb\ln\frac{1-x^N}{1-x}$$

$$= N\ln N + N\ln\frac{(1-x)^b}{(1+x)^f} + O(x^N) . \tag{21}$$

The $N\ln N$ is just the $x=0$ limit representing the maximum of the group measure factor which is dropped upon normalizing the character. The integral we are approximating is multiplied by a factor $(1+x)^{(N-1)f}/(1-x)^{(N-1)b}$ for $SU(N)$ or with the -1 removed for $U(N)$. In the first case the N in the second term on the right will be replaced by 1, and in the second case it will be completely cancelled. Thus the net contribution of the maximum of the integrand to the character will be a factor $(1-x)^b/(1+x)^f$ for $SU(N)$ and 1 for $U(N)$. The rest of the large N approximation to the character comes from the Gaussian integral of the fluctuations about the uniform distribution, to which we now turn.

3.1 Gaussian fluctuations

One can form the second derivative matrix of L (exhibited here only for $f = b = 1$):

$$\begin{aligned}
\frac{\partial^2 L}{\partial\theta_m\partial\theta_k} &= \sum_{l\neq k}(\delta_{km}-\delta_{lm})\Bigg[-\frac{1}{2}\csc^2\frac{\theta_k-\theta_l}{2} - \frac{4x(1+x^2)\cos(\theta_k-\theta_l)}{1+x^4-2x^2\cos 2(\theta_k-\theta_l)} \\
&\quad + \frac{16x^3(1+x^2)\sin(\theta_k-\theta_l)\sin 2(\theta_k-\theta_l)}{(1+x^4-2x^2\cos 2(\theta_k-\theta_l))^2}\Bigg] \\
\frac{\partial^2 L}{\partial\theta_k\partial\theta_k} &= -\sum_{l\neq k}\Bigg[\frac{1}{2}\csc^2\frac{\theta_k-\theta_l}{2} + \frac{4x(1+x^2)\cos(\theta_k-\theta_l)}{1+x^4-2x^2\cos 2(\theta_k-\theta_l)} \\
&\quad - \frac{16x^3(1+x^2)\sin(\theta_k-\theta_l)\sin 2(\theta_k-\theta_l)}{(1+x^4-2x^2\cos 2(\theta_k-\theta_l))^2}\Bigg], \qquad m=k \qquad (22) \\
\frac{\partial^2 L}{\partial\theta_m\partial\theta_k} &= \frac{1}{2}\csc^2\frac{\theta_k-\theta_m}{2} + \frac{4x(1+x^2)\cos(\theta_k-\theta_m)}{1+x^4-2x^2\cos 2(\theta_k-\theta_m)} \\
&\quad - \frac{16x^3(1+x^2)\sin(\theta_k-\theta_m)\sin 2(\theta_k-\theta_m)}{(1+x^4-2x^2\cos 2(\theta_k-\theta_m))^2}, \qquad m\neq k\ . \qquad (23)
\end{aligned}$$

To test the stability of the uniform θ distribution, one can simply substitute $\theta_n \to 2\pi n/N$ in these formulas which makes the second derivative matrix a circulant matrix for which the eigenvalues can be written down:

$$\begin{aligned}
K_{kk} \equiv c_0 &= -\sum_{l=1}^{N-1}\Bigg[\frac{1}{2}\csc^2\frac{\pi l}{N} + \frac{4x(1+x^2)\cos(2\pi l/N)}{1+x^4-2x^2\cos(4\pi l/N)} \\
&\quad - \frac{16x^3(1+x^2)\sin(2\pi l/N)\sin(4\pi l/N)}{(1+x^4-2x^2\cos(4\pi l/N))^2}\Bigg], \qquad m=k \\
K_{km} \equiv c_{k-m} &= \frac{1}{2}\csc^2\frac{\pi(k-m)}{N} + \frac{4x(1+x^2)\cos(2\pi(k-m)/N)}{1+x^4-2x^2\cos(4\pi(k-m)/N)} \\
&\quad - \frac{16x^3(1+x^2)\sin(2\pi(k-m)/N)\sin(4\pi(k-m)/N)}{(1+x^4-2x^2\cos(4\pi(k-m)/N)))^2}\ , \qquad m\neq k\ . \qquad (24)
\end{aligned}$$

Then the eigenvalues μ_n of the matrix K are given by

$$\mu_n = \sum_{l=0}^{N-1} c_l e^{-2\pi i n l/N} = \sum_{l=0}^{N-1} c_l \cos\frac{2\pi n l}{N} = -\sum_{l=1}^{N-1} c_l\left(1-\cos\frac{2\pi n l}{N}\right), \qquad (25)$$

for which the eigenvectors are

$$V_k^n = \frac{e^{2\pi i n k/N}}{\sqrt{N}}, \qquad V^{m\dagger}V^n = \frac{1}{N}\sum_{k=1}^{N} e^{2\pi i(n-m)k/N} = \delta_{mn}\ . \qquad (26)$$

In the large N limit the sum in (25) goes over into an integral. Put $\theta = 2\pi l/N$ and regard c_l as a function $c(\theta)$. Then

$$\mu_n \to -\frac{N}{2\pi}\int_0^{2\pi} d\theta c(\theta)(1-\cos n\theta)\ . \qquad (27)$$

This integral can be done by changing variables to $z = e^{i\theta}$ whence the integral becomes a closed contour integral which can be evaluated by residues. But first we can simplify the job by recognizing that by construction (returning to general b, f)

$$c(\theta) = \frac{d}{d\theta}\left[\cot\frac{\theta}{2} - \frac{2(f+b)x(1+x^2)\sin(\theta) + 2(b-f)x^2\sin 2\theta}{1+x^4-2x^2\cos 2\theta}\right] . \tag{28}$$

Inserting this into the formula for μ_n and integrating by parts gives

$$\begin{aligned} \frac{\mu_n}{N} &= -\frac{n}{2\pi}\int d\theta \sin n\theta \Bigg[\cot\frac{\theta}{2} - \frac{(f+b)+(b-f)\cos(\theta - i\ln x)}{2\sin(\theta - i\ln x)} \\ &\quad - \frac{f+b+(b-f)\cos(\theta + i\ln x)}{2\sin(\theta + i\ln x)}\Bigg] . \end{aligned} \tag{29}$$

Now changing variables to $z = e^{i\theta}$ leads to the contour integral

$$\begin{aligned} -\frac{\mu_n}{N} &\rightarrow \frac{n}{2\pi i}\oint \frac{dz}{z^{n+1}}\frac{z^{2n}-1}{2}\Bigg[\frac{z+1}{z-1} - \frac{(f+b)xz}{x^2z^2-1} + \frac{(f-b)x^2z^2}{x^2z^2-1} \\ &\quad - \frac{(f+b)xz}{z^2-x^2} + \frac{(f-b)x^2}{z^2-x^2}\Bigg] . \end{aligned} \tag{30}$$

The contour starts out on the unit circle, so the only contributing poles are the $(n+1)$th order one at $z = 0$ and the last two terms in square brackets with the simple poles at $z = \pm x$. The former contributes an amount $n(1-(x^n + x^{-n})(f+b)/2)$ when n is odd, and $n(1-(b-f)(x^n+x^{-n})/2)$ when n is even. The latter produces $-n(x^n - x^{-n})(f+b)/2$ for n odd and $-n(x^n - x^{-n})(b-f)/2$ for n even. Thus

$$-\frac{\mu_n}{N} \rightarrow \begin{cases} n(1-(b+f)x^n) & n \quad \text{odd} \\ n(1-(b-f)x^n) & n \quad \text{even} \end{cases} , \qquad N \to \infty . \tag{31}$$

These eigenvalues are actually doubled because the limit $N \to \infty$ can be taken either with n fixed or $N - n$ fixed with identical results. The saddle point evaluation of the character integral about the uniform θ distribution includes a factor of $(\det' K)^{-1/2}$, where the prime indicates the deletion of the zero eigenvalue, multiplying the maximum value of the integrand. Of course $\det' K \propto \prod_{n=1}^{\infty}[\mu_n^2/(nN)]$ where we drop multiplicative constants since the character is normalized to unity at $x = 0$. Then the $U(N)$ superstring bit character in the limit $N \to \infty$ is

$$Z_{N=\infty} = \prod_{k=1}^{\infty}\frac{1}{(1-(b+f)x^{2k-1})(1-(b-f)x^{2k})} , \tag{32}$$

valid whenever the uniform distribution maximizes the character integrand, that is for $x < 1/(f+b)$. The $SU(N)$ character is obtained by multiplying this by $(1-x)^b/(1+x)^f$.

Interesting special cases of this formula are the simplest superstring bit model $f = b = 1$ where the answer is $\prod(1 - 2x^{2k-1})^{-1}$; the single bosonic bit $b = 1, f = 0$ with answer $\prod(1 - x^k)^{-1}$; and the single fermionic bit $f = 1, b = 0$:

$$\prod_{k=1}^{\infty} \frac{1}{(1-x^{2k-1})(1+x^{2k})} = \prod_{k=1}^{\infty} \frac{1-x^{2k}}{(1-x^{2k-1})(1-x^{4k})} = \prod_{k=1}^{\infty} \frac{1-x^{2(2k-1)}}{1-x^{2k-1}}$$
$$= \prod_{k=1}^{\infty}(1+x^{2k-1}), \qquad f=1, \quad b=0 \ . \tag{33}$$

For the case $b = f = 1$ we used our formula to generate the singlet multiplicities of levels up to $M = 60$. The results agree exactly with the selected multiplicities obtained in [21], by counting states in the multitrace basis. At $M = 60$ this agreement is with all 18 digits of the integer 837224873334502342. To get the asymptotic multiplicity for very large bit number, our formula can be approximated by its behavior near $x = 1/(b + f)$:

$$Z_{N=\infty} \approx \frac{1}{1-(b+f)x} \prod_{k=1}^{\infty} \frac{1}{(1-(b+f)^{-2k})(1-(b-f)(b+f)^{-2k})} \ . \tag{34}$$

The asymptotic multiplicity is the coefficient of x^M in this approximate formula:

$$g(M) \sim (b+f)^M \prod_{k=1}^{\infty} \frac{1}{(1-(b+f)^{-2k})(1-(b-f)(b+f)^{-2k})} \tag{35}$$
$$\to 2^M \prod_{k=1}^{\infty} \frac{1}{1-2^{-2k}} \approx 1.452353642449597 \cdot 2^M \ , \qquad b=f=1. \tag{36}$$

also in agreement with [21].

In order to assess the situation at finite N, we employed MATLAB to numerically compute the eigenvalues. Such a study, in the case $b = f = 1$ shows that, for $x < 1/2$ and large enough N, except for a single zero eigenvalue due to translational invariance, all the eigenvalues are negative. For lower values of N, positive eigenvalues develop even for $x < 1/2$. Indeed when x is close to $1/2$, N must be taken quite large to assure only negative eigenvalues: for instance, for $x = 0.499$, N must be greater than 501 to ensure that all eigenvalues are negative. The negativity of all eigenvalues shows that the uniform solution is then a MAXIMUM (at least locally). For $x > 1/2$ the same numerical study shows that at least one eigenvalue becomes positive, indicating that the uniform distribution is a saddle point and hence not a true maximum.

The analysis in [2] numerically searched for the *global* maximum of $L(\theta)$ and fit the results to the model N dependence described in the introduction. This analysis supports the conclusion that the uniform distribution is a global maximum ($f_0(x) = 0$) for $x < 1/2$. For $x > 1/2$ positive values for $f_0(x)$ were obtained, with a nonuniform θ distribution $\rho(\theta)$, which populates an interval $-a(x) < \theta < a(x)$ with no apparent gaps. The maximum $a(x)$ decreases monotonically from π to 0 as x increases from 1/2 to 1. Our efforts to find this nonuniform distribution analytically have so far come up short, but see the discussion of this issue in [3, 4].

4 Toward a high temperature solution as $N \to \infty$

The method that successfully solved the unitary one matrix problem [19, 20] was couched in terms of complex analysis. Following this idea for our problem, we define a function $F(z)$ of the complex variable z by (we assume supersymmetry ($f = b$) for simplicity)

$$F(z) \;=\; \int_{-\alpha}^{\alpha} d\theta' \rho(\theta') \left[\cot\frac{z-\theta'}{2} - \frac{4bx(1+x^2)\sin(z-\theta')}{1+x^4-2x^2\cos 2(z-\theta')}\right] . \tag{37}$$

Clearly this function is periodic $F(z+2\pi) = F(z)$ and has branch points, joined by a finite cut on the real axis, at $z = \pm\alpha \bmod 2\pi$ due to the first term in square brackets. In addition the second term causes branch points at $z = \pm\alpha + i\ln x \bmod \pi$ and at $z = \pm\alpha - i\ln x \bmod \pi$. These additional branch points are joined by finite cuts which overlap when $\alpha > \pi/2$ but do not overlap when $\alpha < \pi/2$. The normalization $\int d\theta\rho(\theta) = 1$ implies that $F(z) \to \mp i$ when $z \to \pm i\infty$.

If such an analytic function can be found, with the property that $F(\eta \pm i\epsilon) = \mp 2\pi i \rho(\eta)$ for real η in the range $-\alpha < \eta < \alpha$, then $\rho(\eta)$ will solve the integral equation. In the case of the unitary matrix problem a fairly simple guess involving the square root function led to the unique answer. Here the guesswork is more challenging.

For a start, notice that the second term in square brackets may be expressed in partial fractions as

$$\begin{aligned}
&\frac{4x(1+x^2)\sin(z-\beta)}{1+x^4-2x^2\cos 2(z-\beta)} \\
&\quad = \frac{1}{2}\Big[\cot\frac{z-\beta-i\ln x}{2} - \cot\frac{z-\beta-\pi-i\ln x}{2} \\
&\qquad\qquad + \cot\frac{z-\beta+i\ln x}{2} - \cot\frac{z-\beta-\pi+i\ln x}{2}\Big] \\
&\quad = \frac{1}{\sin(z-\beta-i\ln x)} + \frac{1}{\sin(z-\beta+i\ln x)} .
\end{aligned} \tag{38}$$

The next to last line shows that we can express the function $F(z)$ in terms of a simpler function

$$g(z) \;\equiv\; \int_{-\alpha}^{\alpha} d\theta' \rho(\theta') \cot\frac{z-\theta'}{2} \tag{39}$$

via

$$\begin{aligned}
F(z) \;=\; & g(z) - \frac{b}{2}\left[g(z-i\ln x) + g(z+i\ln x)\right. \\
& \left. - g(z-\pi-i\ln x) - g(z-\pi+i\ln x)\right], \qquad f = b.
\end{aligned} \tag{40}$$

It remains to be seen whether this aids the problem of guessing the answer!

5 Effective Field Theory: $1/N$ Expansion

As we have seen, for N large enough, the uniform distribution $\theta_k^0 = 2\pi k/N$ is a maximum of $L(\theta)$ for $x < 1/(b+f)$. We can then write $\theta_k = \theta_k^0 + \delta\theta_k$ and expand L in a series of powers of $\delta\theta$:

$$L(\theta) \;\; = \;\; L(\theta^0) + \frac{1}{2}\delta\theta_k\delta\theta_l K_{kl} + \frac{1}{3!}\delta\theta_k\delta\theta_l\delta\theta_m \mathcal{V}^3_{klm} + \cdots \, , \tag{41}$$

where $K_{kl} = c_{k-l}$, the second derivative matrix has already been introduced in Eq.(25). In the discussion following that equation we noted that the eigenvalues of K in the large N limit are proportional to N and nonpositive for $x < 1/(b+f)$. Truncating the expansion at order $\delta\theta^2$ and integrating over the fluctuations is the approximation used to obtain the large N limit of the partition function.

The cubic and higher terms involve higher derivatives of L

$$\mathcal{V}^n_{k_a k_2 \cdots k_n} \;\; = \;\; \left.\frac{\partial^n L}{\partial\theta_{k_1}\cdots\partial\theta_{k_n}}\right|_{\theta=\theta^0} \, , \tag{42}$$

again evaluated at the uniform θ distribution. The $\delta\theta$'s can be expanded in eigenvectors of K_{kl},

$$\delta\theta_k \;\; = \;\; \sum_{n=0}^{N-1} \lambda_n V^n_k = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \lambda_n e^{2\pi i n k/N}, \tag{43}$$

and the expansion (41) recast as a series in powers of λ. Then the coefficient of $\lambda^m/m!$ is

$$\tilde{\mathcal{V}}^m_{n_1\cdots n_m} \;\; = \;\; \frac{1}{N^{m/2}} \sum_{k_1\cdots k_m} e^{2\pi i (n_1 k_1 + \cdots + n_m k_m)/N} \mathcal{V}^m_{k_1\cdots k_m} \, . \tag{44}$$

For $N \to \infty$, one can show that $\tilde{\mathcal{V}}^m$ scales as $N^{2-m/2}$. We have already seen that the eigenvalues of K scale as N, which is this scaling law for $m = 2$. To illustrate how this scaling law works at higher order, we work it out explicitly for $m = 3$. Start by noticing that $L = (1/2)\sum_{k\neq l} \mathcal{L}(\theta_k - \theta_l)$. Then

$$\frac{\partial^3 L}{\partial\theta_{k_1}\partial\theta_{k_2}\partial\theta_{k_3}} \;\; = \;\; \frac{1}{2}\sum_{k\neq l}\prod_{j=1}^{3}(\delta_{k_j k} - \delta_{k_j l})\mathcal{L}'''(\theta_k - \theta_l) \, , \tag{45}$$

where the primes denote derivative with respect to the argument. Next substitute $\theta_k \to$

$2\pi k/N$ and calculate

$$\begin{aligned}
\tilde{\mathcal{V}}^3_{n_1n_2n_3} &= \frac{1}{2N^{3/2}}\sum_{k\neq l}\prod_{j=1}^{3}(e^{2\pi i n_j k/N}-e^{2\pi i n_j l/N})\mathcal{L}'''\left(\frac{2\pi(k-l)}{N}\right)\\
&= \frac{1}{2N^{3/2}}\sum_{l=1}^{N}\sum_{k=1}^{N-1}\prod_{j=1}^{3}(e^{2\pi i n_j (k+l)/N}-e^{2\pi i n_j l/N})\mathcal{L}'''\left(\frac{2\pi k}{N}\right)\\
&= \frac{1}{2N^{3/2}}\sum_{l=1}^{N}e^{2\pi i l(n_1+n_2+n_3)/N}\sum_{k=1}^{N-1}\prod_{j=1}^{3}(e^{2\pi i n_j k/N}-1)\mathcal{L}'''\left(\frac{2\pi k}{N}\right)\\
&= \frac{1}{2N^{1/2}}\delta_{n_1+n_2+n_3}\sum_{k=1}^{N-1}\prod_{j=1}^{3}(e^{2\pi i n_j k/N}-1)\mathcal{L}'''\left(\frac{2\pi k}{N}\right)\\
&\to \frac{\sqrt{N}}{4\pi}\delta_{n_1+n_2+n_3,0}\int_0^{2\pi}d\theta(e^{in_1\theta}-1)(e^{in_2\theta}-1)(e^{in_3\theta}-1)\mathcal{L}'''(\theta)\ , \qquad (46)
\end{aligned}$$

showing that $\tilde{\mathcal{V}}^3 = O(\sqrt{N})$ as desired. A clarification: Before passing to the continuum approximation the $\delta_{n_1+n_2+n_3}$ is 1 for $n_1+n_2+n_3 = 0$ mod N, with the n_j ranging from 0 to $N-1$. We take $N\to\infty$ keeping either n_j fixed or $N-n_j$ fixed. In the latter case, we replace n_j with the negative integer $n_j - N$, before passing to the continuum approximation. In the last line the $\delta_{n_1+n_2+n_3,0}$ is 1 for $n_1+n_2+n_3=0$ and mod N is suspended.

The integral on the right side of (46) may be integrated twice by parts with vanishing surface terms, using

$$\left[(e^{in_1\theta}-1)(e^{in_2\theta}-1)(e^{in_3\theta}-1)\right]'' = -2i\left[n_1^2\sin n_1\theta + n_2^2\sin n_2\theta + n_3^2\sin n_3\theta\right]\ , \qquad (47)$$

which requires the constraint $n_1+n_2+n_3=0$. This manipulation shows that the vertex involves the same integral that occurred in the evaluation of (31):

$$I_n \equiv \frac{1}{2\pi}\int_0^{2\pi}\sin n\theta\mathcal{L}'(\theta) = \begin{cases}1-(b+f)x^n & n \quad \text{odd}\\ 1-(b-f)x^n & n \quad \text{even}\end{cases}\ , \qquad n>0. \qquad (48)$$

Otherwise $I_{-n} = -I_n$ and $I_0 = 0$.

Then the cubic vertex can at $N=\infty$ can be written

$$\tilde{\mathcal{V}}^3_{n_1n_2n_3} = -i\sqrt{N}\delta_{n_1+n_2+n_3,0}\left[n_1^2 I_{n_1} + n_2^2 I_{n_2} + n_3^2 I_{n_3}\right]\ . \qquad (49)$$

For x near $1/(b+f)$, n_k's equal to ±1 will be important, because their propagator blows up when $x = 1/(b+f)$. For the cubic vertex, at most two of the n_k can have these values, for example $n_1 = n_2 = \pm 1$ and $n_3 = \mp 2$. Since $n_3 = 0$ is not available, n_1 and n_2 must have the same sign.

The generalization to higher m is obvious:

$$\tilde{\mathcal{V}}^m_{n_1\cdots n_m} \to \frac{N^{2-m/2}}{4\pi}\delta_{\sum_j n_j,0}\int_0^{2\pi}d\theta\prod_{j=1}^{m}(e^{in_j\theta}-1)\mathcal{L}^{(m)}(\theta)\ . \qquad (50)$$

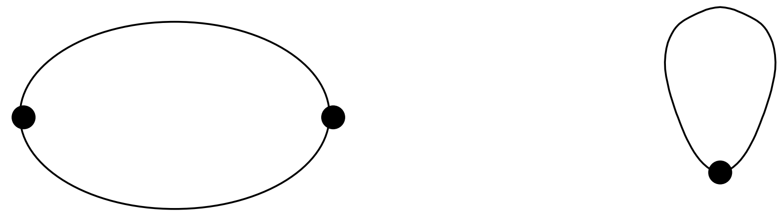

Figure 1: Lowest order self energy diagrams.The diagram on the left represents $\hat{\Pi}_1^0$ and that on the right Π_1^0.

Again $m-1$ integrations by parts can be performed with vanishing surface terms, for which we need

$$\frac{d^{m-1}}{d\theta^{m-1}}\prod_{j=1}^{m}(e^{in_j\theta}-1)=(-i)^m\sum_{l=1}^{m-1}(-)^l\sum_{k_1<k_2<\cdots<k_l}(n_{k_1}+\cdots+n_{k_l})^{m-1}\sin(n_{k_1}+\cdots+n_{k_l})\theta\ .$$

Then

$$\tilde{\mathcal{V}}^m_{n_1\cdots n_m}\ \ \rightarrow\ \ -i^m\frac{N^{2-m/2}}{2}\delta_{\sum_j n_j,0}\sum_{l=1}^{m-1}(-)^l\sum_{k_1<k_2<\cdots k_l}(n_{k_1}+\cdots+n_{k_l})^{m-1}I_{n_{k_1}+\cdots+n_{k_l}}\ . \quad (51)$$

5.1 $1/N$ Perturbations near the Critical Point

We have defined the $1/N$ expansion in the low temperature phase. But since there is no actual phase transition at finite N, we should be able to reach the high temperature phase if we keep N finite. The $1/N$ expansion breaks down near $x=1/(b+f)$ because the propagator of the nth mode is $(N(1-(b+f)x^n))^{-1}$, so the $n=\pm1$ mode propagator blows up at the critical point. Thus a partial summation of all orders in the expansion is required.

First lets examine the lowest order correction to the "self-energy" for $n=\pm1$ described by the diagrams in Fig. 1.

$$\hat{\Pi}_1^0 \ = \ -\frac{8}{N^2}\frac{(2I_2-I_1)^2}{I_1I_2} \quad (52)$$

$$\Pi_1^0 \ = \ \frac{8}{N^2}\frac{2I_2-I_1}{I_1}=\frac{8}{N^2}\frac{1+(b+f)x-2(b-f)x^2}{1-(b+f)x}\ , \quad (53)$$

where the normalization is such that the corrected inverse propagator for the first mode is $N(I_1+\hat{\Pi}_1^0+\Pi_1^0)/2$. We see that both contributions blow up like $(1-(b+f)x)^{-1}$ at $x=1/(b+f)$. This singularity in the propagator is responsible for the Hagedorn phenomenon.

At finite N a singularity on the real interval $0<x<1$ must be absent, which means the singularity must move off the real axis in the exact propagator. There is a simple partial summation which accomplishes this which is specified by a truncated Dyson equation depicted in Fig. 2:

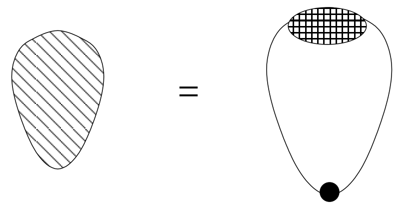

Figure 2: Diagrammatic representation of the Dyson equation. The left side signifies Π^D, and the propagator with the shaded bubble represents the corresponding propagator $\Delta_D = (2/N)(I_1 + \Pi^D)^{-1}$.

$$\Pi^D = \frac{8}{N^2}\frac{2I_2 - I_1}{I_1 + \Pi^D} . \tag{54}$$

This is a quadratic algebraic equation with solutions

$$\Pi^D_\pm = \frac{1}{2}\left[-I_1 \pm \sqrt{I_1^2 + \frac{32}{N^2}(2I_2 - I_1)}\right] \tag{55}$$

We should choose the plus sign so that $\Pi^D \to 0$ as $N \to \infty$, so the propagator becomes

$$\Delta_D = \frac{2}{N}\frac{1}{I_1 + \Pi^D} = \frac{4}{N}\left[I_1 + \sqrt{I_1^2 + \frac{32}{N^2}(2I_2 - I_1)}\right]^{-1} . \tag{56}$$

It is seen that there are no singularities of Δ_D on the real axis. The original Hagedorn singularity has been replaced by two complex branch points of order $1/N$ away from the real axis. when $N \to \infty$, these two branch points coalesce and form the Hagedorn pole.

After this partial summation, the Feynman rules are modified as follows: (1) The propagator for the first mode is replaced by Δ_D, and (2) Corrections to the first mode propagator obtained by attaching any number of the second diagram of Fig. 1 to any propagator, including the propagator in that diagram are deleted.

At finite N the modified first mode propagator Δ_D has no singularities on the real axis in the interval $0 < x < 1$ corresponding to real temperatures $0 < T < \infty$. The Hagedorn singularity at $T = T_H$ has been replaced by two branch points off the real axis. These branch points are responsible for different behavior of the $N \to \infty$ limit depending on whether I_1 is positive or negative. If $I_1 > 0$ the two terms inside the square brackets are both positive so the limit gives

$$\Delta_D \sim \frac{2}{NI_1}\left[1 - O\left(\frac{1}{N^2 I_1^2}\right)\right], \qquad I_1 > 0, \quad N \to \infty . \tag{57}$$

In contrast if $I_1 < 0$, the first term in square brackets is negative and we obtain

$$\Delta_D \sim \frac{N|I_1|}{4(2I_2 - I_1)} + O\left(\frac{1}{N|I_1|}\right), \qquad I_1 < 0, \quad N \to \infty. \tag{58}$$

We stress that Δ_D does not include the full second order self energy since it excludes the first diagram of Fig.1. If that diagram had been included in the Dyson equation the two branch points would be on the real x axis.

5.2 High Temperature

Just above the phase transition the $n_j = \pm 1$ modes are the only unstable ones. Then a nonzero vertex, involving only these unstable modes, requires that m is even and therefore half the modes are $+1$ and the other half are -1..These nonzero vertices reduce to

$$\tilde{\mathcal{V}}^m = \frac{N^{2-m/2}}{4\pi} \int_0^{2\pi} d\theta (2 - 2\cos\theta)^{m/2} \mathcal{L}^{(m)}(\theta) \ . \tag{59}$$

As long as $x < 1/(b+f)$ the eigenvalues of K are all negative and of order $O(N)$. In this case the integral over fluctuations limits the λ's to be of order $N^{-1/2}$, so the term λ^m is of order N^{2-m}. Thus successive terms in the expansion are smaller by a factor of $1/N$. Using this effective field theory, one can develop a Feynman graph expansion in which the propagator is determined by the λ^2 term and the vertices are $\tilde{\mathcal{V}}^n$ with $n > 2$.

The structure of the perturbation expansion in powers of $1/N$ can also be used to motivate the large N behavior of the high temperature phase. When $x = 1/(b+f)$, the coefficient of the $\lambda_1^* \lambda_1$ term vanishes and the strength of this mode is not limited by the quadratic term, so the higher order terms must be relied on to damp the λ_1 integral. When $x > 1/(b+f)$ the coefficient of the $\lambda_1^* \lambda_1$ term becomes positive driving toward a higher maximum. This term becomes comparable to the higher order terms only for λ_1 of order $\sqrt{N}$ so the new maximum of L must be of order N^2. Unfortunately, for λ this large, all of the higher order terms are comparable and a perturbative description is lost. However nonperturbative numerical methods as described in [2] have succeeded in getting a convincing determination of the high temperature phase.

6 The Limit $x \to 1$: Asymptotics of Eulerian Digraphs

As noted in [2] in the case $b = f = 1$ the $x \to 1$ limit of the partition function is given by

$$Z(x \to 1) = \left(\frac{2}{1-x}\right)^{N-1} \frac{\int_{-\pi}^{\pi} \exp\{L(\theta)|_{x=1}\} \prod_{1 \leqslant k \leqslant N} d\theta_k}{N!(2\pi)^N} = \left(\frac{2}{1-x}\right)^{N-1} \frac{R_N}{N!} \ , \tag{60}$$

where

$$R_N = \operatorname*{Res}_{\mathbb{T}^N} \left(\frac{\prod_{1 \leqslant i < j \leqslant N} (z_i + z_j)^2}{\prod_{1 \leqslant k \leqslant N} (z_k)^N} \right) \ . \tag{61}$$

Eq. 60 is obtained by substituting $e^{i\theta_k} \to z_k$ in the integral. On the other hand, it is well-known [22] that the number of labeled Eulerian digraphs with N nodes is given by $ED(N)$, the coefficient of $\prod_{j=1}^N z_j^{N-1}$ in

$$\prod_{j=1}^N \prod_{k=1+j}^N (z_j + z_k)^2 \ .$$

In other words, in addition to deciding the leading order N-dependence at high temperature, $R_N \equiv ED(N)$ also counts the number of Eulerian digraphs.

McKay showed [23] that $ED(N)$ is given asymptotically for large N by

$$ED\left(N\right) \underset{N\to\infty}{\sim} \left(\frac{2^N}{\sqrt{\pi N}}\right)^{N-1} e^{-1/4}\sqrt{N}\left(1+O\left(\frac{1}{\sqrt{N}}\right)\right) \ . \tag{62}$$

We find that the lowest-order correction to this result is

$$ED\left(N\right) \underset{N\to\infty}{\sim} \left(\frac{2^N}{\sqrt{\pi N}}\right)^{N-1} e^{-1/4}\sqrt{N}\left(1+\frac{3}{16N}+O\left(\frac{1}{N^2}\right)\right) \ . \tag{63}$$

We emphasize that the correction is $O\left(1/N\right)$ and not $O\left(1/\sqrt{N}\right)$. In any case, the correction in (63) significantly reduces the relative error, as shown for $N \leq 16$ in the Appendix.

A systematic procedure to obtain the correction in (63) and higher order corrections, at least in principle, is described in the following.

6.1 Methodology

The basic plan follows that of McKay [23] but pursues the structure of the ensuing gaussian integrals in more detail. Applying Cauchy residue theorem to $ED(N)$, we obtain an exact integral expression, as given by

$$ED\left(N\right) = \left(\frac{2^N}{\pi}\right)^{N-1} \int_{-\pi/2}^{\pi/2} \cdots \int_{-\pi/2}^{\pi/2} \prod_{j=1}^{N-1}\left(\cos^2\left(\alpha_j\right)\ \prod_{k=1+j}^{N-1}\cos^2\left(\alpha_j-\alpha_k\right)\right)\, d\alpha_1 \ \cdots \ d\alpha_{N-1} \ . \tag{64}$$

Here the α variables are related to the θ's introduced earlier by $\alpha_j = \theta_j/2$. For example, $ED\left(1\right)=1$, $ED\left(2\right)=2$, $ED\left(3\right)=10$, $ED\left(4\right)=152$, etc. We then write

$$\cos^2\left(\alpha\right) = \exp\left(2\ln\left(\cos\alpha\right)\right) = \exp\left(-\alpha^2-\frac{1}{6}\alpha^4-\frac{2}{45}\alpha^6+O\left(\alpha^8\right)\right) \tag{65}$$

and extend the integrations as $\int_{-\pi/2}^{\pi/2} \to \int_{-\infty}^{\infty}$ for any truncated series expansion of $\ln\left(\cos\alpha\right)$ in the exponential.

A straightforward adaptation of the analysis given by McKay shows, for any truncation of the series in the exponential (65), the additional contributions coming from $\int_{-\infty}^{-\pi/2}$ and $\int_{\pi/2}^{\infty}$ will be exponentially suppressed, for large N, when compared to the dominant asymptotic behavior or to any $\left(1/N\right)^n$ corrections to that dominant behavior.

Therefore, to obtain the requisite asymptotic behavior, we need to evaluate for large N

$$\left\langle \exp\left(-\frac{1}{6}\sum_{j=1}^{N-1}\alpha_j^4-\frac{1}{6}\sum_{j<k}\left(\alpha_j-\alpha_k\right)^4-\frac{2}{45}\sum_{j=1}^{N-1}\alpha_j^6-\frac{2}{45}\sum_{j<k}\left(\alpha_j-\alpha_k\right)^6+O\left(\alpha^8\right)\right)\right\rangle \tag{66}$$

where the averaging $\langle \cdots \rangle$ is to be done with a "skewed gaussian" measure on $\mathbb{E}_{N-1}$, namely,

$$\langle f \rangle \equiv \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f\left(\alpha_1, \cdots, \alpha_{N-1}\right)\ d\mu\ , \tag{67}$$

$$d\mu = \frac{1}{\sqrt{\pi^{N-1}\left(\frac{1}{N}\right)^{N-2}}}\ \exp\left(-\sum_{j=1}^{N-1}\alpha_j^2 - \sum_{j=1}^{N-2}\sum_{k=1+j}^{N-1}\left(\alpha_j - \alpha_k\right)^2\right)\ d\alpha_1\ d\alpha_2\ \cdots\ d\alpha_{N-1}\ . \tag{68}$$

This measure is normalized so that the $(N-1)$-fold integration $\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} d\mu = 1$ for all N.

Now the exponentiated bilinear form in the measure may be diagonalized by changing variables, as noted by McKay. Explicitly, let

$$\alpha_j = \frac{1}{\sqrt{N}}\ \left(\beta_j + \frac{1}{\left(1+\sqrt{N}\right)}\sum_{k=1}^{N-1}\beta_k\right)\ , \qquad \beta_j = \sqrt{N}\ \alpha_j - \frac{1}{\left(1+\sqrt{N}\right)}\sum_{k=1}^{N-1}\alpha_k\ ,$$

$$\alpha_j - \alpha_k \ = \ \frac{1}{\sqrt{N}}\ \left(\beta_j - \beta_k\right)\ . \tag{69}$$

This change of variables gives

$$d\mu = \left(\frac{1}{\sqrt{\pi}}\right)^{N-1} \exp\left(-\sum_{j=1}^{N-1}\beta_j^2\right)\ d\beta_1\ \cdots\ d\beta_{N-1} \tag{70}$$

in addition to

$$\sum_{j=1}^{N-2}\sum_{k=1+j}^{N-1}\left(\alpha_j - \alpha_k\right)^{2n} = \frac{1}{N^n}\sum_{j=1}^{N-2}\sum_{k=1+j}^{N-1}\left(\beta_j - \beta_k\right)^{2n} \tag{71}$$

for any n. Indeed, the β-variables are useful to evaluate the integrals of the remaining non-diagonal terms in the exponentials, especially the nested double-summation terms. But even without this change of variables it is readily apparent from considering a few examples that *all positive powers of* $\sum_{j=1}^{N-2}\sum_{k=1+j}^{N-1}\left(\alpha_j - \alpha_k\right)^4$ *must be included in the averages* to obtain correct asymptotic results to $O\left(1\right)$ and to $O\left(1/N\right)$, as well as to higher orders in inverse powers of N. In particular, as discussed in more detail in the next subsection,

$$\left\langle \exp\left(-\frac{1}{6}\sum_{j=1}^{N-2}\sum_{k=1+j}^{N-1}\left(\alpha_j - \alpha_k\right)^4\right)\right\rangle \underset{N\to\infty}{\sim} \exp\left(-\frac{1}{6}\sum_{j=1}^{N-2}\sum_{k=1+j}^{N-1}\left\langle\left(\alpha_j - \alpha_k\right)^4\right\rangle\right) + \ O\left(\frac{1}{N}\right)$$

$$\underset{N\to\infty}{\sim} e^{-1/4} + \ O\left(\frac{1}{N}\right)\ . \tag{72}$$

This result, along with the normalization factor exhibited in the measure (68) and the prefactor in (64), immediately gives the dominant asymptotic behavior (62).

6.2 First-order Corrections

Define

$$\Sigma_{2n} = \sum_{j=1}^{N-1} \alpha_j^{2n} \ , \quad \Sigma\Sigma_{2n} = \sum_{j=1}^{N-2} \sum_{k=1+j}^{N-1} (\alpha_j - \alpha_k)^{2n} \tag{73}$$

and find by direct calculation

$$\langle \Sigma_2 \rangle = \frac{(N-1)}{N} \ , \quad \langle \Sigma_4 \rangle = \frac{3\,(N-1)}{N^2} \ , \quad \langle \Sigma_6 \rangle = \frac{15\,(N-1)}{N^3} \ , \quad \langle \Sigma_8 \rangle = O\left(\frac{1}{N^3}\right) \ , \tag{74}$$
$$\langle \Sigma\Sigma_2 \rangle = \frac{(N-1)\,(N-2)}{2N} \ , \quad \langle \Sigma\Sigma_4 \rangle = \frac{3\,(N-1)\,(N-2)}{2N^2} \ ,$$
$$\langle \Sigma\Sigma_6 \rangle = \frac{15\,(N-1)\,(N-2)}{2N^3} \ , \quad \langle \Sigma\Sigma_8 \rangle = O\left(\frac{1}{N^2}\right) \ .$$

Moreover, again by direct calculation,

$$\left\langle (\Sigma\Sigma_4)^2 \right\rangle = \frac{(N-1)\,(N-2)}{N^4} \left(\frac{9}{4}N^2 + \frac{51}{4}N - 6\right) \ , \tag{75}$$
$$\left\langle (\Sigma\Sigma_4)^3 \right\rangle = \frac{(N-1)\,(N-2)}{N^6} \left(\frac{27}{8}N^4 + \frac{135}{2}N^3 + \frac{3267}{8}N^2 - \frac{837}{4}N + 54\right) \ ,$$
$$\left\langle (\Sigma\Sigma_4)^4 \right\rangle = \frac{(N-1)\,(N-2)}{N^8} \left(\frac{81}{16}N^6 + \frac{3483}{16}N^5 + \frac{58\,185}{16}N^4 + \frac{413\,505}{16}N^3 - \frac{53\,703}{8}N^2 \right.$$
$$\left. + \frac{18\,819}{2}N - 3240\right) \ ,$$

etc. Also, while it is not required for the problem at hand, it is not too difficult to show for any integer n that

$$\langle (\Sigma\Sigma_2)^n \rangle = \frac{\Gamma\left(\frac{1}{2}N - 1 + n\right)}{\Gamma\left(\frac{1}{2}N - 1\right)} \ \frac{(N-1)^n}{N^n} \ . \tag{76}$$

The asymptotic behavior of this last result nicely illustrates some features that are relevant to the problem at hand. An averaged exponential of $\Sigma\Sigma_2$ gives

$$\langle \exp\left(-\lambda\ \Sigma\Sigma_2\right) \rangle = \left(1 + \frac{\lambda\,(N-1)}{N}\right)^{1-\frac{1}{2}N} \tag{77}$$

while an averaged exponential of $(\Sigma\Sigma_2)^2$ gives

$$\left\langle \exp\left(-\lambda \left(\frac{\Sigma\Sigma_2}{N}\right)^2\right) \right\rangle = \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}N - 1 + 2n\right)}{\Gamma\left(\frac{1}{2}N - 1\right)} \ \frac{(1-N)^{2n}}{N^{2n}} \frac{(-\lambda/N^2)^n}{n!} \ . \tag{78}$$

The latter series is *divergent*. This is not unexpected [5] Nevertheless, asymptotically for large N,

$$\left\langle \left(\frac{\Sigma\Sigma_2}{N} \right)^{2n} \right\rangle = \frac{1}{4^n} \left(1 + \frac{4n\,(n-2)}{N} + \frac{n\,(24n^3 - 112n^2 + 132n - 29)}{3N^2} + O\left(\frac{1}{N^3} \right) \right) \quad (79)$$

and each of the individual terms in the asymptotic behavior can be exponentially summed as convergent series. For example,

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-\lambda}{4} \right)^n = e^{-\frac{1}{4}\lambda} \ , \quad (80)$$

$$\frac{4}{N} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-\lambda}{4} \right)^n n\,(n-2) = e^{-\frac{1}{4}\lambda} (4+\lambda) \left(\frac{\lambda}{4N} \right) \ , \quad (81)$$

$$\frac{1}{3N^2} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-\lambda}{4} \right)^n n \left(24n^3 - 112n^2 + 132n - 29 \right) =$$
$$e^{-\frac{1}{4}\lambda} \left(-15 - 9\lambda - 2\lambda^2 + \frac{3}{8}\lambda^3 \right) \left(\frac{\lambda}{12N^2} \right) \ . \quad (82)$$

Thus we obtain as a warm-up exercise

$$\left\langle \exp\left(-\lambda \left(\frac{\Sigma\Sigma_2}{N} \right)^2 \right) \right\rangle \underset{N\to\infty}{\sim}$$
$$e^{-\frac{1}{4}\lambda} \left(1 + \frac{\lambda}{4N} (4+\lambda) + \frac{\lambda}{12N^2} \left(-15 - 9\lambda - 2\lambda^2 + \frac{3}{8}\lambda^3 \right) + O\left(\frac{1}{N^3} \right) \right) . (83)$$

Similar techniques can now be used to obtain (63), but for that result we need to evaluate $\langle (\Sigma\Sigma_4)^n \rangle$. This is not as simple as (76).

In general, as is evident from the examples in (75),

$$\langle (\Sigma\Sigma_4)^n \rangle = \frac{(N-1)\,(N-2)}{N^{2n}} \sum_{k=0}^{2(n-1)} A_k\,(n)\ \ N^k \ . \quad (84)$$

---

[5] For example, consider just a single integral as follows.

$$I\,(a,b) = \int_0^\infty e^{-ax^2 - bx^4} dx = \frac{1}{2\sqrt{a}} \sum_{n=0}^{\infty} \frac{\left(-\frac{b}{a^2} \right)^n}{n!}\, \Gamma\left(2n + \frac{1}{2} \right) = \frac{1}{2}\sqrt{\frac{\pi}{a}}\, \text{hypergeom}\left(\left[\frac{1}{4}, \frac{3}{4} \right], [\,], -4\frac{b}{a^2} \right) \ .$$

This is a divergent series in powers of b. However, as an expansion in a, a convergent series is obtained.

$$I\,(a,b) = \frac{1}{4b^{1/4}} \sum_{n=0}^{\infty} \frac{\left(-\frac{a}{\sqrt{b}} \right)^n}{n!}\, \Gamma\left(\frac{n+1/2}{2} \right) = \frac{\pi}{8}\sqrt{\frac{2a}{b}}\; e^{\frac{a^2}{8b}} \left(I_{-1/4}\left(\frac{a^2}{8b} \right) - I_{1/4}\left(\frac{a^2}{8b} \right) \right) \ .$$

The first of these series correctly gives the asymptotic expansion of $I\,(a,b)$ for large a with b fixed, while the second nicely exhibits the large b behavior for any fixed a.

All the $A_k(n)$ coefficients in this case are not yet available. Fortunately, however, to obtain (63) only the two leading coefficients are required, namely, A_{2n-2} and A_{2n-3}. These are *not* so difficult to obtain. By direct calculation the leading large N coefficient is $A_{2n-2} = \left(\frac{3}{2}\right)^n$, followed by $A_{2n-3} = \left(\frac{3}{2}\right)^n \frac{1}{3}(n-1)(13n-9)$. Therefore

$$\langle (\Sigma\Sigma_4)^n \rangle \underset{N\to\infty}{\sim} \left(\frac{3}{2}\right)^n \left(1 + \frac{13n^2 - 22n}{3N} + O\left(\frac{1}{N^2}\right)\right) , \tag{85}$$

and these leading terms may be exponentially summed to obtain

$$\langle \exp(-\lambda\, \Sigma\Sigma_4) \rangle \underset{N\to\infty}{\sim} e^{-\frac{3}{2}\lambda} \left(1 + \frac{18\lambda + 39\lambda^2}{4N} + O\left(\frac{1}{N^2}\right)\right) . \tag{86}$$

In particular, for $\lambda = 1/6$,

$$\left\langle \exp\left(-\frac{1}{6}\, \Sigma\Sigma_4\right) \right\rangle \underset{N\to\infty}{\sim} e^{-\frac{1}{4}} \left(1 + \frac{49}{48N} + O\left(\frac{1}{N^2}\right)\right) . \tag{87}$$

In view of the above results, the asymptotic behavior of interest is

$$\left\langle \exp\left(-\frac{1}{6}\, \Sigma_4 - \frac{1}{6}\, \Sigma\Sigma_4 - \frac{2}{45}\, \Sigma_6 - \frac{2}{45}\, \Sigma\Sigma_6 + O\left(\alpha^8\right)\right) \right\rangle \tag{88}$$

$$\underset{N\to\infty}{\sim} \left\langle \left(1 - \frac{1}{6}\, \Sigma_4 - \frac{2}{45}\, \Sigma\Sigma_6 + O\left(\frac{1}{N^2}\right)\right) \exp\left(-\frac{1}{6}\, \Sigma\Sigma_4\right) \right\rangle . \tag{89}$$

The "1" term is given by (87) up to but not including $O(1/N^2)$. But then two more averages involving $\exp(-\,\Sigma\Sigma_4/6)$ are needed. They are

$$\left\langle (\Sigma_4) \exp\left(-\frac{1}{6}\, \Sigma\Sigma_4\right) \right\rangle \underset{N\to\infty}{\sim} 3\, e^{-\frac{1}{4}} \left(\frac{1}{N} + O\left(\frac{1}{N^2}\right)\right) , \tag{90}$$

$$\left\langle (\Sigma\Sigma_6) \exp\left(-\frac{1}{6}\, \Sigma\Sigma_4\right) \right\rangle \underset{N\to\infty}{\sim} \frac{15}{2}\, e^{-\frac{1}{4}} \left(\frac{1}{N} + O\left(\frac{1}{N^2}\right)\right) . \tag{91}$$

Finally, $49/48 - 1/2 - 1/3 = 3/16$ and we obtain

$$\left\langle \exp\left(-\frac{1}{6}\, \Sigma_4 - \frac{1}{6}\, \Sigma\Sigma_4 - \frac{2}{45}\, \Sigma_6 - \frac{2}{45}\, \Sigma\Sigma_6 + O\left(\alpha^8\right)\right) \right\rangle \underset{N\to\infty}{\sim} e^{-\frac{1}{4}} \left(1 + \frac{3}{16N} + O\left(\frac{1}{N^2}\right)\right) \tag{92}$$

These results complete the derivation of the first-order correction. Putting everything together — the normalization factor exhibited in the measure (68), the prefactor in (64), and (92) — we obtain (63).

6.3 Central Moments

In our opinion, it would not be surprising if the central limit theorem underlies some of the results given above. Perhaps the following comments shed some light on this issue.

Define the nth central moment of f as

$$\sigma_n [f] \equiv \langle (f - \langle f \rangle)^n \rangle = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \langle f^{n-k} \rangle \langle f \rangle^k \tag{93}$$

where $\langle f^0 \rangle \equiv 1$, and consider central moments of $\Sigma\Sigma_4$. For example, from (74) and (75),

$$\sigma_2 [\Sigma\Sigma_4] = \frac{3}{2N^4} (N-1)(N-2)(13N-7) = \frac{39}{2N} + O\left(\frac{1}{N^2}\right) \tag{94}$$

which is $O(1/N)$ compared to $O(1)$ for $\left\langle (\Sigma\Sigma_4)^2 \right\rangle$. On the other hand, for all $n \geq 3$, We find

$$\sigma_n [\Sigma\Sigma_4] = O\left(\frac{1}{N^2}\right) \tag{95}$$

as opposed to $\langle (\Sigma\Sigma_4)^n \rangle = O(1)$. This fact alone permits a recursive calculation of both the $O(1)$ and the $O(1/N)$ terms in $\langle (\Sigma\Sigma_4)^n \rangle$ for $n \geq 3$.

It could be interesting to exploit additional relations of this sort to determine higher-order corrections to the asymptotic behavior of $ED(N)$. While research on the number of digraphs has continued up to the present, as evident in [24] and references cited therein, higher-order asymptotic corrections are as yet unexplored.

7 Conclusion

We have gone some distance toward understanding the physics of string bits at finite temperature. In particular we have elucidated the concept of a Hagedorn transition in a system whose only dynamics is a restriction to the singlet sector. One can think of this simplified dynamics as underlying a tensionless string, since the P^- of the emergent string, proportional to T_0 is zero. Extending the analysis to a string bit model with $T_0 > 0$ is a clear direction for future research.

The partition function of the system studied here also has the purely group theoretic interpretation as a generating function for the number of color singlet states in a system of bosonic and fermionic creation operators each in the adjoint representation. For finite N the partition function is finite and smooth in the whole range $0 < x < 1$. But only in what we have called the low temperature phase $x < 1/(b+f)$ does the finite N partition function have a finite $N \to \infty$ limit. The corresponding limit for $x > 1/(b+f)$ does not exist, but its logarithm grows quadratically with N. In particular in the case $b = f = 1$ at $x = 1$ the partition function is related to known counting functions for digraphs, which have been successfully analyzed for large N [22, 23, 2], and for which we have calculated $1/N$ corrections in this article..

We have briefly touched on setting up a systematic $1/N$ expansion for our system. For the low temperature phase it is well defined and analytically tractable and deserves further study. It would also be of interest to do a better job on the high temperature phase for $x_H < x < 1$. In the context of effective field theory about the uniform θ distribution characterizing the low temperature phase, some qualitative information about the system near the critical point can be gleaned, but we have not been able to draw rigorous quantitative conclusions. However the numerical studies of [2] and Section 6 of the present article give a rather detailed account of the physics of the high temperature phase.

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Appendix: Relative Errors

The relative error is given by

$$\Delta(N) = \left(ED(N) - ED_{\text{asymp}}(N)\right)/ED(N)$$

where $ED(N)$ is exact and $ED_{\text{asymp}}(N)$ is either McKay's result (62) or the corrected result (63). Numerical results for $1 \leq N \leq 16$ are given in the following Table. To compute these relative errors, exact results for $ED(N)$ for $N \leq 16$ were taken from [22].

N	$\Delta(N)$ using ED_{asymp} in (62)	$\Delta(N)$ using ED_{asymp} in (63)
1	22.120×10^{-2}	7.517×10^{-2}
2	12.122×10^{-2}	3.883×10^{-2}
3	8.400×10^{-2}	2.675×10^{-2}
4	5.777×10^{-2}	1.360×10^{-2}
5	4.335×10^{-2}	7.471×10^{-3}
6	3.496×10^{-2}	4.801×10^{-3}
7	2.939×10^{-2}	3.390×10^{-3}
8	2.537×10^{-2}	2.531×10^{-3}
9	2.233×10^{-2}	1.965×10^{-3}
10	1.995×10^{-2}	1.571×10^{-3}
11	1.802×10^{-2}	1.285×10^{-3}
12	1.644×10^{-2}	1.071×10^{-3}
13	1.511×10^{-2}	9.064×10^{-4}
14	1.398×10^{-2}	7.771×10^{-4}
15	1.301×10^{-2}	6.737×10^{-4}
16	1.217×10^{-2}	5.896×10^{-4}