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Gluon Amplitudes as $2d$ Conformal Correlators

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Abstract

Recently, spin-one wavefunctions in four dimensions that are conformal primaries of the Lorentz group $SL(2, \mathbb{C})$ were constructed. We compute low-point, tree-level gluon scattering amplitudes in the space of these conformal primary wavefunctions. The answers have the same conformal covariance as correlators of spin-one primaries in a $2d$ CFT. The BCFW recursion relation between three- and four-point gluon amplitudes is recast into this conformal basis.

1 Introduction

The $4d$ Lorentz group acts as the global $SL(2, \mathbb{C})$ conformal group on the celestial sphere at null infinity. This implies that the $4d$ massless quantum field theory (QFT) scattering amplitudes, recast as correlators on the celestial sphere, share some properties with those of $2d$ CFTs. When gravity is included, the plot thickens: the global conformal group is enhanced to the infinite-dimensional local group [1–4],[1] suggesting an even tighter connection with $2d$ CFT. This $4d$-$2d$ connection is of interest both for the ambitious goal of a holographic reformulation of flat space quantum gravity, as well as for its potentially strong yet unexploited mathematical implications for the rich subject of QFT scattering amplitudes.

A perhaps-not-too-distant goal is to find out whether or to what extent there is any set of QFT scattering amplitudes that can be approximately[2] generated by some kind of $2d$ CFT. It has been clear from the outset that such CFTs would not be of the garden variety with operators in highest weight representations. Recently it has emerged [10] from the study of two-point functions, that the unitary principal continuous series (which has appeared in a variety of CFT studies [11–14]) of the Lorentz group plays a central role. Beyond that it is not clear what to expect, and concrete computations are in order. In this paper we compute and explore some basic properties of amplitudes - in particular the three- and four-point tree-level gluon amplitudes - presented as $2d$ conformal correlators on the celestial sphere. This case is of special interest both because it has been shown that soft gluons generate a $2d$ current algebra [15, 16] and because of the plethora of beautiful results about these amplitudes in the momentum-space representation. In principle, one may hope to find a $2d$ CFT of some kind that generates the $4d$ tree amplitudes. Indeed several constructions including the scattering equations [17] and twistor space [18–20] seem close to achieving this goal.

The $SL(2, \mathbb{C})$ conformal presentation of $4d$ QFT scattering amplitudes was discussed long ago by Dirac [21]. Recently, soft photon and gluon theorems were recast as $2d$ Kac-Moody-Ward identities [15, 16, 22–24]. Massless [3] and massive [25] scalar wavefunctions in four-dimensional Minkowski spacetime that are primaries of the Lorentz group $SL(2, \mathbb{C})$ were constructed. Such solutions, called *conformal primary wavefunctions*, are labeled by a point $z, \bar{z}$ in $\mathbb{R}^2$ and a conformal dimension Δ, rather than the three independent components

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[1]Up to IR divergences at one loop [5–9].

[2]Exact CFTs are expected only with the inclusion of gravity. What we have in mind here is something like a large N approximation.

of an on-shell four-momentum. The massless spin-one conformal primary wavefunctions in four dimensions were also constructed in [3]. A comprehensive survey of conformal primary wavefunctions with or without spin in arbitrary spacetime dimensions was performed in [10]. In particular, it was shown there that conformal primary wavefunctions in $\mathbb{R}^{1,d+1}$ with conformal dimensions on the principal continuous series $\Delta \in \frac{d}{2} + i\mathbb{R}$ of $SO(1, d+1)$ form a complete set of delta-function-normalizable solutions to the wave equation. The factorization singularity of amplitudes in the conformal basis is studied in [26, 27].

In this paper we study low-point $4d$ tree-level gluon scattering amplitudes in the space of conformal primary wavefunctions. We find that the tree-level color-ordered MHV four-point amplitude $1^- 2^- \to 3^+ 4^+$ takes the form

$$\tilde{\mathcal{A}}_{--++}(z_i, \bar{z}_i) = I(z_{ij}, \bar{z}_{ij}) \, \delta(\sum_{i=1}^{4} \lambda_i) \delta(|z - \bar{z}|) \frac{z^{\frac{5}{3}}}{(z-1)^{\frac{1}{3}}}, \qquad 1 < z, \qquad (1.1)$$

where $z, \bar{z}$ are the cross ratios, $I(z_{ij}, \bar{z}_{ij})$ is a product of powers of $z_{ij}, \bar{z}_{ij}$ that is fixed by conformal covariance, and $1 + i\lambda_i$ are the conformal dimensions of the four primaries. The delta-function for the imaginary part of the cross ratio is shown to be implied by $4d$ translation invariance (which is generally obscured in the conformal basis). As an important check on this formula, we show that is has a BCFW representation in a factorization channel involving the product of three-point functions.

Going forward, more might be learned from a $2d$ conformal block expansion of the four-point function (1.1). Interestingly, the BCFW relation resembles the OPE expansion of a $2d$ CFT four-point correlator. We leave these directions to future work.

The paper is organized as follows. In Section 2 we review the massless vector conformal primary wavefunction in four spacetime dimensions. The change of basis from plane waves to the conformal primary wavefunctions is implemented by a Mellin transform. In Sections 3 and 4, we compute Mellin transforms of the tree-level three- and four-point amplitudes, and show that the answers transform as spin-one $2d$ conformal correlators. In Section 5, we write the BCFW recursion relation for the MHV four-point amplitude in the conformal primary wavefunction basis. In Appendix A we set up our conventions for the spinor helicity variables. In Appendix B, we review Mellin transforms on inner products of two gauge boson one-particle states.

2 A Conformal Basis for Gauge Bosons

In this section we review the massless vector conformal primary wavefunctions in $\mathbb{R}^{1,3}$ [3,10], i.e. solutions to the four-dimensional Maxwell equation that transform as two-dimensional spin-one conformal primaries. In particular, the transition from momentum space to conformal primary wavefunctions is implemented by a Mellin transform. Later sections will derive and study low-point amplitudes in the new conformal basis.

Let us begin by setting up the notations. We will restrict ourselves to four-dimensional Minkowski space $\mathbb{R}^{1,3}$ with spacetime coordinates X^μ ($\mu = 0, 1, 2, 3$). We use $z, \bar{z}$ to denote a point in $\mathbb{R}^2$. We will sometimes use ∂_+ (∂_-) to denote ∂_z ($\partial_{\bar{z}}$).

2.1 Massless Vector Conformal Primary Wavefunctions

Scattering problems of gauge bosons are usually studied in the plane wave basis which, in Lorenz gauge $\partial^\mu A_\mu = 0$, consists of

$$\epsilon_{\mu\ell}(p)\, e^{\mp i|\vec{p}|X^0 \pm i\vec{p}\cdot\vec{X}}\,, \qquad \ell = \pm 1\,. \tag{2.1}$$

Here $\epsilon_{\mu\ell}(p)$ are the polarization vectors for the helicity ℓ one-particle states. They satisfy $\epsilon_\ell(p) \cdot p = 0$, $\epsilon_\pm(p)^* = \epsilon_\mp(p)$, and $\epsilon_\ell(p) \cdot \epsilon_{\ell'}(p)^* = \delta_{\ell\ell'}$. Plane wave solutions are labeled by three continuous variables and two signs:

- a spatial momentum $\vec{p}$,

- a $4d$ helicity $\ell = \pm 1$,

- a sign distinguishing an incoming solution from an outgoing one.

To make the two-dimensional conformal symmetry manifest, an alternative set of solutions for the Maxwell equation, called *massless vector conformal primary wavefunctions*

$$V_{\mu J}^{\Delta\pm}(X^\mu; z, \bar{z})\,, \tag{2.2}$$

were constructed in [3,10]. These solutions are again labeled by three continuous variables and two signs:

- a point $z, \bar{z}$ in $\mathbb{R}^2$ and a conformal dimension Δ,

- a $2d$ spin $J = \pm 1$,

- a sign distinguishing an incoming solution from an outgoing one.

The defining properties of the massless vector conformal primary wavefunction $V_{\mu J}^{\Delta\pm}(X^\mu; z, \bar{z})$ are:

1. It satisfies the Maxwell equation,

$$\left(\frac{\partial}{\partial X^\rho} \frac{\partial}{\partial X_\rho} \delta_\nu^\mu - \frac{\partial}{\partial X^\nu} \frac{\partial}{\partial X_\mu} \right) V_{\mu J}^{\Delta\pm}(X^\mu; z, \bar{z}) = 0 \, . \tag{2.3}$$

2. It transforms as a four-dimensional vector and a two-dimensional (quasi-)conformal primary with spin $J = \pm 1$ and has dimension Δ under an $SL(2,\mathbb{C})$ Lorentz transformation:

$$V_{\mu J}^{\Delta\pm}\left(\Lambda^\mu{}_\nu X^\nu; \frac{az + b}{cz + d}, \frac{\bar{a}\bar{z} + \bar{b}}{\bar{c}\bar{z} + \bar{d}} \right) = (cz + d)^{\Delta+J}(\bar{c}\bar{z} + \bar{d})^{\Delta-J} \Lambda_\mu{}^\rho V_{\rho J}^{\Delta\pm}(X^\mu; z, \bar{z}) \, , \tag{2.4}$$

where $a, b, c, d \in \mathbb{C}$ with $ad - bc = 1$ and $\Lambda^\mu{}_\nu$ is the associated $SL(2,\mathbb{C})$ group element in the four-dimensional representation.[3]

By construction, scattering amplitudes of conformal primary wavefunctions transform covariantly as a two-dimensional conformal correlators of spin-one primaries with conformal dimensions Δ_i.

Let us write down the explicit expression for the massless vector conformal primary wavefunctions. We define a "unit" null vector q^μ associated to $z, \bar{z}$ as

$$q^\mu(z, \bar{z}) = (1 + |z|^2, z + \bar{z}, -i(z - \bar{z}), 1 - |z|^2) \, . \tag{2.5}$$

Under an $SL(2,\mathbb{C})$ transformation $z \to z' = (az + b)/(cz + d), \bar{z} \to \bar{z}' = (\bar{a}\bar{z} + \bar{b})/(\bar{c}\bar{z} + \bar{d})$, the null vector q^μ transforms as a vector up to a conformal weight,

$$q^\mu \to q^{\mu\prime} = (cz + d)^{-1}(\bar{c}\bar{z} + \bar{d})^{-1} \Lambda^\mu{}_\nu q^\nu \, . \tag{2.6}$$

The derivative of q^μ with respect to z and $\bar{z}$ are respectively the polarization vectors of helicity $+1$ and -1 one-particle states propagating in the q^μ direction:

$$\partial_z q^\mu = \sqrt{2} \, \epsilon_+^\mu(q) = (\bar{z}, 1, -i, -\bar{z}) \, , \qquad \partial_{\bar{z}} q^\mu = \sqrt{2} \, \epsilon_-^\mu(q) = (z, 1, i, -z) \, . \tag{2.7}$$

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[3]For an explicit expression for $\Lambda^\mu{}_\nu$ in terms of a, b, c, d, see [25].

Bases	Plane Waves	Conformal Primary Wavefunctions
Notations	$\epsilon_{\mu\ell}(p)\exp\left[\mp i\lvert\vec{p}\rvert X^0 \pm i\vec{p}\cdot\vec{X}\right]$	$V_{\mu J}^{\Delta,\pm}(X;z,\bar{z})$
Continuous Labels	$\vec{p}$	$\Delta, z, \bar{z}$
Discrete Labels	4d Helicity ℓ Incoming vs. Outgoing	2d Spin J Incoming vs. Outgoing

Table 1: A comparison between the plane waves and the massless vector conformal primary wavefunctions. The two set of solutions are labeled by some continuous labels and discrete labels. The continuous labels for the former are a spatial momentum $\vec{p}$, while those for the latter are $\Delta, z, \bar{z}$. The discrete labels consist of a sign distinguishing between incoming and outgoing wavefunctions, and a sign for the 4d helicity or the 2d spin.

They satisfy $q \cdot \partial_z q = q \cdot \partial_{\bar{z}} q = 0$, $\partial_z q \cdot \partial_z q = \partial_{\bar{z}} q \cdot \partial_{\bar{z}} q = 0$ and $\partial_z q \cdot \partial_{\bar{z}} q = 2$.

The explicit expression for the conformal primary wavefunctions was given in terms of the spin-one bulk-to-boundary propagator in the three-dimensional hyperbolic space H_3 in [3, 10]. To compute gauge invariant physical observables such as scattering amplitudes, we can choose a convenient gauge representative for the conformal primary wavefunction. In [3, 10] it was shown that, for $\Delta \neq 1$, the vector conformal primary wavefunction is gauge equivalent to

$$V_{\mu J}^{\Delta,\pm}(X^\mu; z, \bar{z}) = \mathcal{N} \frac{\partial_J q_\mu}{(-q \cdot X \mp i\epsilon)^\Delta}, \quad \Delta \neq 1, \tag{2.8}$$

where $\mathcal{N} = (\mp i)^\Delta \Gamma(\Delta)/\sqrt{2}$ is a normalization constant chosen for later convenience. From now on we will assume $\Delta \neq 1$. (When $\Delta = 1$, the conformal primary wavefunction itself is a total derivative in X^μ, so is a pure gauge. We have put in an $i\epsilon$-prescription to circumvent the singularity at the light sheet where $q \cdot X = 0$.) There is another set of conformal primary wavefunctions that are shadow to (2.8) [10]. We leave the study of scattering amplitudes in the shadow basis for future investigation.

Finally, we need to determine the range of the conformal dimension Δ. It was shown in [10] that the conformal primary wavefunctions are a complete and delta-function-normalizable basis if Δ ranges over the one-dimensional locus

$$\mathcal{C} = \{\Delta \in 1 + i\mathbb{R}\}, \tag{2.9}$$

on the complex plane. See also [3, 28] for an alternative argument via the hyperbolic slicing of Minkowski space. For a given spin, this range of Δ is known as the principal continuous series of unitary representations of $SL(2, \mathbb{C})$.

2.2 Mellin Transform

The gauge representative (2.8) for the conformal primary wavefunction has the advantage that it is simply related to the plane wave (2.1) by a Mellin transform[4]

$$\mathcal{N} \frac{\partial_J q_\mu}{(-q \cdot X \mp i\epsilon)^\Delta} = \frac{\partial_J q_\mu}{\sqrt{2}} \int_0^\infty d\omega\, \omega^{\Delta-1}\, e^{\pm i\omega q \cdot X - \epsilon\omega}\,, \qquad \Delta = 1 + i\lambda\,, \qquad (2.12)$$

with the plus (minus) sign for an outgoing (incoming) wavefunction. Δ is the scaling dimension under the Lorentz boosts which preserve the particle trajectory while rescaling $q \cdot X$.

Notice that the spin $+1(-1)$, outgoing conformal primary wavefunction $A_{\mu+}^{\Delta,+}$ ($A_{\mu-}^{\Delta,+}$) has nontrivial projections to the $4d$ helicity $+1(-1)$ sector. It follows that a $4d$ helicity $+1$ (-1) *outgoing* one-particle state is mapped to a $2d$ spin $+1$ (-1) conformal operator under Mellin transform. By CPT, a $4d$ helicity $+1$ (-1) *incoming* one-particle state is mapped to a $2d$ spin -1 $(+1)$ conformal operator under Mellin transform.

When considering scattering amplitudes in the plane wave basis, it is often convenient to take all particles to be outgoing but some of them carrying negative energy. In the basis of conformal primary wavefunctions, however, we need to keep track of which particles are incoming and outgoing, and different crossing channels have to be treated separately. Nonetheless, we will label the helicity of an external gauge boson as if it were an outgoing particle.

This change of basis can be immediately extended to any gauge boson scattering amplitude. Consider a general n-point gauge boson scattering amplitude, which is a function of the $\omega_i, z_i, \bar{z}_i$ and helicities ℓ_i,

$$\mathcal{A}_{\ell_1 \cdots \ell_n}(\omega_i, z_i, \bar{z}_i)\,, \qquad (2.13)$$

with the momentum conservation delta function $\delta^{(4)}(\sum_{i=1}^n p_i^\mu)$ included. We can perform a Mellin transform on each of the external particles to go to the basis of conformal primary

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[4]The Mellin transform of a function $f(\omega)$ is defined by

$$\widetilde{f}(\Delta) = \int_0^\infty d\omega\, \omega^{\Delta-1} f(\omega)\,, \qquad (2.10)$$

while the inverse Mellin transform of $\tilde{f}(\Delta)$ is

$$f(\omega) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\Delta\, \omega^{-\Delta} \widetilde{f}(\Delta)\,, \qquad c \in \mathbb{R}\,. \qquad (2.11)$$

wavefunctions,

$$\tilde{\mathcal{A}}_{J_1 \cdots J_n}(\lambda_j, z_j, \bar{z}_j) = \prod_{i=1}^{n} \int_0^{\infty} d\omega_i \, \omega_i^{\lambda_i} \, \mathcal{A}_{\ell_1 \cdots \ell_n}(\omega_j, z_j, \bar{z}_j) \,, \tag{2.14}$$

where the $2d$ spin J_i is identified as the $4d$ helicity ℓ_i, i.e. $J_i = \ell_i$. From the defining properties of the conformal primary wavefunction (2.4), the resulting function $\tilde{\mathcal{A}}_{J_1 \cdots J_n}(\lambda_i, z_i, \bar{z}_i)$ is guaranteed to transform covariantly as a two-dimensional conformal correlators of spin-one primaries with dimensions $\Delta_i = 1 + i\lambda_i$, i.e.

$$\tilde{\mathcal{A}}_{J_1 \cdots J_n}\left(\lambda_j, \frac{az_j + b}{cz_j + d}, \frac{\bar{a}\bar{z}_j + \bar{b}}{\bar{c}\bar{z}_j + \bar{d}}\right) = \prod_{i=1}^{n} \left[(cz_i + d)^{\Delta_i + J_i}(\bar{c}\bar{z}_i + \bar{d})^{\Delta_i - J_i}\right] \tilde{\mathcal{A}}_{J_1 \cdots J_n}(\lambda_j, z_j, \bar{z}_j) \,. \tag{2.15}$$

We will sometimes use the inverse Mellin transform to convert an amplitude in the conformal primary wavefunction basis back to the plane wave basis,

$$\mathcal{A}_{\ell_1 \cdots \ell_n}(\omega_j, z_j, \bar{z}_j) = \prod_{i=1}^{n} \int_{-\infty}^{\infty} \frac{d\lambda_i}{2\pi} \, \omega^{-1 - i\lambda_i} \, \tilde{\mathcal{A}}_{J_1 \cdots J_n}(\lambda_j, z_j, \bar{z}_j) \,. \tag{2.16}$$

To sum up the above discussion, the change of basis from plane waves to the massless vector conformal primary wavefunction is implemented by a Mellin transform (2.11) with conformal dimension $\Delta = 1 + i\lambda$ lying on the principal continuous series. The $4d$ helicity of an one-particle state is identified as the $2d$ spin for the primary operator.

We now discuss some general properties of Mellin transforms of low-point gluon amplitudes. They scale homogeneously under uniform rescaling of the frequencies as

$$\mathcal{A}_{\ell_1 \cdots \ell_n}(\Lambda \omega_i, z_i, \bar{z}_i) = \Lambda^{-n} \mathcal{A}_{\ell_1 \cdots \ell_n}(\omega_i, z_i, \bar{z}_i) \,. \tag{2.17}$$

The momentum conservation delta function is included in $\mathcal{A}$ and contributes a factor of Λ^{-4} in (2.17). (Note this scaling also holds true for tree-level ϕ^4 theory, so that the analysis in the rest of this subsection carries over to that case as well.) It is convenient to change the integration variables to an overall frequency $s \equiv \sum_i \omega_i$ and a set of "simplex variables" $\sigma_i \equiv s^{-1}\omega_i \in [0, 1]$ with $\sum_{i=1}^{n} \sigma_n = 1$,

$$\prod_{i=1}^{n} \int_0^{\infty} d\omega_i \, \omega_i^{i\lambda_i}[...] = \int_0^{\infty} ds \, s^{n-1+i\sum_i \lambda_i} \prod_{i=1}^{n} \int_0^1 d\sigma_i \, \sigma_i^{\lambda_i} \delta(\sum_i \sigma_i - 1)[...] \,. \tag{2.18}$$

Let $A_{\ell_1 \cdots \ell_n}$ denote the stripped amplitude with the delta function and an overall power of s

factored out:

$$\mathcal{A}_{\ell_1 \cdots \ell_n}(\omega_j, z_j, \bar{z}_j) = s^{-n} A_{\ell_1 \cdots \ell_n}(\sigma_j, z_j, \bar{z}_j) \delta^{(4)}\left(\sum_i \varepsilon_i \sigma_i q_i\right), \qquad (2.19)$$

where $q_i(z_i, \bar{z}_i)$ is given in (2.5) (or its $(-+-+)$ signature analog which will be discussed in Section 3) and $\varepsilon_i = +1 (-1)$ for an outgoing (incoming) external particle. Then, using (2.17) and

$$\int_0^\infty d\omega \, \omega^{i\lambda - 1} = 2\pi \, \delta(\lambda), \qquad (2.20)$$

we can rewrite the amplitude in the conformal basis (2.14) as:

$$\tilde{\mathcal{A}}_{J_1 \cdots J_n}(\lambda_j, z_j, \bar{z}_j) = 2\pi \delta\left(\sum_i \lambda_i\right) \prod_{i=1}^n \int_0^1 d\sigma_i \, \sigma_i^{i\lambda_i} \, A_{\ell_1 \cdots \ell_n}(\sigma_j, z_j, \bar{z}_j) \delta^{(4)}\left(\sum_i \varepsilon_i \sigma_i q_i\right) \delta\left(\sum_i \sigma_i - 1\right).$$
$$(2.21)$$

We thus have a total of 5 delta functions inside the integral of (2.21), which localize the σ_i integrals for up to five-point scattering amplitudes.

We can then compute $\tilde{\mathcal{A}}_{J_1 \cdots J_n}(\lambda_j, z_j, \bar{z}_j)$ for $n \le 5$ easily in two steps. In the first step, we rewrite the delta functions as

$$\delta^{(4)}\left(\sum_i \varepsilon_i \sigma_i q_i\right) \delta\left(\sum_i \sigma_i - 1\right) = C(z_i, \bar{z}_i) \prod_{i=1}^{n \le 5} \delta(\sigma_i - \sigma_{*i}), \qquad (2.22)$$

for some function $C(z_i, \bar{z}_i)$. Here $\sigma_{*i}(z_j, \bar{z}_j)$'s are the solutions of σ_i's fixed by the momentum conservation delta functions. For $n = 3, 4$, the momentum conservation equations are over-constraining, and the function $C(z_i, \bar{z}_i)$ will contain delta functions in $z_i, \bar{z}_i$, restricting the angles of the external particle trajectories. For example, as we will see in Section 4, for $n = 4$, $C(z_i, \bar{z}_i) \sim \delta(|z - \bar{z}|)$ where $z, \bar{z}$ are the cross ratios.

In the second step, all the simplex integrals in σ_i can be done by simply evaluating the integrand at $\sigma_i = \sigma_{*i}(z_j, \bar{z}_j)$. The final result for the Mellin transform of the amplitude (2.21) is then:

$$\tilde{\mathcal{A}}_{J_1 \cdots J_n}(\lambda_j, z_j, \bar{z}_j) = 2\pi \delta\left(\sum_i \lambda_i\right) \left(\prod_{i=1}^{n \le 5} \sigma_{*i}^{i\lambda_i}\right) A_{\ell_1 \cdots \ell_n}(\sigma_{*j}, z_j, \bar{z}_j) C(z_i, \bar{z}_i) \prod_i \mathbf{1}_{[0,1]}(\sigma_{*i}), \quad (2.23)$$

where the indicator function $\mathbf{1}_{[0,1]}(x)$ defined as

$$\mathbf{1}_{[0,1]}(x) = \begin{cases} 1\,, & \text{if } x \in [0,1]\,, \\ 0\,, & \text{otherwise}\,, \end{cases} \tag{2.24}$$

comes from the restricted range of σ_i between 0 and 1 in their definitions. In Sections 3 and 4, we will give explicit expressions for the tree-level color-ordered three- and four-point amplitudes in the conformal basis.

3 Gluon Three-Point Amplitudes

In this section we derive the Mellin transforms of the tree-level MHV and anti-MHV three-point amplitudes. In the $(-+++)$ signature, the kinematics of massless scatterings forces the gluon three-point amplitude to vanish. To circumvent this issue, we will instead be working in the $(-+-+)$ signature in this section, in which $(z,\bar{z})$ as well as the spinor helicity variables $(|p],|p\rangle)$ are not related by complex conjugation, and the three-point function need not vanish. In particular (2.5) becomes:

$$q^\mu(z,\bar{z}) = (1 + z\bar{z}, z + \bar{z}, z - \bar{z}, 1 - z\bar{z})\,. \tag{3.1}$$

The Lorentz group in the $(-+-+)$ signature is $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$, which acts on z and $\bar{z}$ separately.

Let us start with the tree-level color-ordered MHV three-point amplitude. Letting the first and the second particles have negative helicities and the third particle have positive helicity, the momentum space amplitude is (see Appendix A for our conventions on the spinor helicity variables),

$$\begin{aligned} \mathcal{A}_{--+}(\omega_i, z_i, \bar{z}_i) &= \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle} \delta^{(4)}(p_1^\mu + p_2^\mu + p_3^\mu) \\ &= -2\frac{\omega_1\omega_2}{\omega_3}\frac{z_{12}^3}{z_{23}z_{31}}\delta^{(4)}\Big(\sum_i \varepsilon_i\omega_i q_i^\mu\Big)\,. \end{aligned} \tag{3.2}$$

The $2d$ spins of the corresponding conformal primaries are $J_1 = J_2 = -1$ and $J_3 = +1$. In writing down the above expression, we have assumed $z_{ij} \neq 0$, while $\bar{z}_{ij}$, which are independent real variables in the $(-+-+)$ spacetime signature, are allowed to vanish. The sign ε_i is $+1$ (-1) for an outgoing (incoming) particle when we rotate back to the $(-+++)$ signature. However, in the $(-+-+)$ signature, there is no invariant distinction between an incoming

and an outgoing wavefunction. Indeed, under $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$, the sign ε_i is not invariant and transforms as $\varepsilon_i \to \varepsilon_i \mathrm{sgn}((cz_i + d)(\bar{c}\bar{z}_i + \bar{d}))$, where $a, b, c, d, \bar{a}, \bar{b}, \bar{c}, \bar{d} \in \mathbb{R}$ with $ad - bc = \bar{a}\bar{d} - \bar{b}\bar{c} = 1$.

We will follow the route of Section 2.2. The delta function in (2.21) on the support of $z_{ij} \neq 0$ can be written

$$\delta^{(4)}(\sum_i \varepsilon_i \sigma_i q_i)\delta(\sum_i \sigma_i - 1)\bigg|_{z_{ij} \neq 0} = \frac{\delta(\bar{z}_{12})\delta(\bar{z}_{13})}{4\sigma_1\sigma_2\sigma_3 D_3^2} \delta(\sigma_1 - \frac{z_{23}}{D_3})\delta(\sigma_2 + \varepsilon_1\varepsilon_2\frac{z_{13}}{D_3})\delta(\sigma_3 - \varepsilon_1\varepsilon_3\frac{z_{12}}{D_3}),$$

$$\equiv \frac{\delta(\bar{z}_{12})\delta(\bar{z}_{13})}{4\sigma_1\sigma_2\sigma_3 D_3^2} \prod_{i=1}^{3} \delta(\sigma_i - \sigma_{*i}), \tag{3.3}$$

where the denominator is

$$D_3 = (1 - \varepsilon_1\varepsilon_2)z_{13} + (\varepsilon_1\varepsilon_3 - 1)z_{12}. \tag{3.4}$$

There is a similar term with support at $\bar{z}_{ij} \neq 0$ relevant for the anti-MHV three-point amplitude, which by symmetry of the left hand side is just the above expression with the substitution $z_{ij} \leftrightarrow \bar{z}_{ij}$. Thanks to the delta functions in (3.3), all the Mellin integrals collapse to evaluating the integrand on the solutions of σ_i:

$$\boxed{\tilde{\mathcal{A}}_{--+}(\lambda_i; z_i, \bar{z}_i) = -\pi\, \delta(\sum_i \lambda_i) \frac{\mathrm{sgn}(z_{12}z_{23}z_{31})\delta(\bar{z}_{13})\delta(\bar{z}_{12})}{|z_{12}|^{-1-i\lambda_3}|z_{23}|^{1-i\lambda_1}|z_{13}|^{1-i\lambda_2}} \prod_{i=1}^{3} \mathbf{1}_{[0,1]}(\sigma_{*i}),\ z_i, \bar{z}_i \in \mathbb{R}} \tag{3.5}$$

where σ_{*i} are given in (3.3). The indicator function $\mathbf{1}_{[0,1]}(x)$ is defined in (2.24). Importantly, in the $(-+-+)$ signature, $z_i, \bar{z}_i$ are independent real variables and the notation $|z_{ij}|$ stands for the absolute value of a real variable, rather than $\sqrt{z_{ij}\bar{z}_{ij}}$. Note that the sign function $\mathrm{sgn}(z_{12}z_{23}z_{31})$ is $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ invariant.

Note that the three-point MHV amplitude has a factor of $\frac{\sigma_i\sigma_j}{\sigma_k}$ for i, j, k distinct. This means that the denominator D_3 of σ_{*i} drop out except for the indicator function constraints coming from the domain of integration of the simplex variables σ_i. In particular, the Mellin transform depends on the choice of the crossing channel (i.e. dependence on ε_i) *only* through the ranges of support for z_i's constrained by the indicator functions $\prod_{i=1}^{3} \mathbf{1}_{[0,1]}(\sigma_{*i})$.

Let us decode the indicator functions $\prod_{i=1}^{3} \mathbf{1}_{[0,1]}(\sigma_{*i})$. Their physical origin is that, given a crossing channel, not every possible direction, parametrized by $z_i, \bar{z}_i$, is allowed by the four-dimensional massless kinematics. For example, the three-point function obviously vanishes if the three particles are all incoming or all outgoing. We therefore only need to consider the

two-to-one or one-to-two decay amplitudes, which will be denoted by $ij \rightleftarrows k$, correspond-ing to $\varepsilon_i = \varepsilon_j = -\varepsilon_k$. The two arrows of opposite directions are related by time reversal. The indicator functions $\prod_{i=1}^{3} \mathbf{1}_{[0,1]}(\sigma_{*i})$ constrain the three real z_i's to be in the following orderings for different crossing channels:

$$\prod_{i=1}^{3} \mathbf{1}_{[0,1]}(\sigma_{*i}): \quad \begin{array}{llll} a) & 12 \rightleftarrows 3 & \Rightarrow z_1 < z_3 < z_2 \text{ or } z_2 < z_3 < z_1 \\ b) & 13 \rightleftarrows 2 & \Rightarrow z_1 < z_2 < z_3 \text{ or } z_3 < z_2 < z_1 \\ c) & 23 \rightleftarrows 1 & \Rightarrow z_3 < z_1 < z_2 \text{ or } z_2 < z_1 < z_3 \,. \end{array} \qquad (3.6)$$

For each crossing channel, there are two possible orderings of the z_i's. Note that the ordering of two points z_1 and z_2 is not $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ invariant but depends on the sign of $(cz_1 + d)(cz_2 + d)$. On the other hand, the crossing channel is also not invariant in the $(-+-+)$ signature. The indicator functions are nonetheless $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ invariant if we take into account of the sign flip of ε_i mentioned above, i.e. $\varepsilon_i \to \varepsilon_i \text{sgn}((cz_i+d)(\bar{c}\bar{z}_i+\bar{d}))$.

Coming back to the full three-point function, under an $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ action, the Mellin transform $\tilde{\mathcal{A}}_{--+}(\lambda_i; z_i, \bar{z}_i)$ of the color-ordered MHV amplitude indeed transforms as a conformal three-point function of spin-one primaries with weights,

$$\begin{aligned} h_1 &= \frac{i}{2}\lambda_1\,, & \bar{h}_1 &= 1 + \frac{i}{2}\lambda_1\,, \\ h_2 &= \frac{i}{2}\lambda_2\,, & \bar{h}_2 &= 1 + \frac{i}{2}\lambda_2\,, \\ h_3 &= 1 + \frac{i}{2}\lambda_3\,, & \bar{h}_3 &= \frac{i}{2}\lambda_3\,. \end{aligned} \qquad (3.7)$$

Note that it is important to use the conformal covariance of the delta function as in (B.4).

Next, consider the color-ordered anti-MHV amplitude where the first and second particles have positive helicities and the third particle has negative helicity,

$$\begin{aligned} \mathcal{A}_{++-}(\omega_i, z_i, \bar{z}_i) &= \frac{[12]^3}{[23][31]} \delta^{(4)}(p_1^\mu + p_2^\mu + p_3^\mu) \\ &= 2\frac{\omega_1\omega_2}{\omega_3} \frac{\bar{z}_{12}^3}{\bar{z}_{23}\bar{z}_{31}} \delta^{(4)}\Big(\sum_i \varepsilon_i \omega_i q_i^\mu\Big)\,. \end{aligned} \qquad (3.8)$$

The $2d$ spins of the corresponding conformal primaries are $J_1 = J_2 = +1$ and $J_3 = -1$. In writing down the above expression, we have assumed $\bar{z}_{ij} \neq 0$. Its Mellin transform is given

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by

$$\tilde{\mathcal{A}}_{++-}(\lambda_i; z_i, \bar{z}_i) = \pi \, \delta(\sum_i \lambda_i) \frac{\mathrm{sgn}(\bar{z}_{12}\bar{z}_{23}\bar{z}_{31})\delta(z_{13})\delta(z_{12})}{|\bar{z}_{12}|^{-1-i\lambda_3}|\bar{z}_{23}|^{1-i\lambda_1}|\bar{z}_{13}|^{1-i\lambda_2}} \prod_{i=1}^{3} \mathbf{1}_{[0,1]}(\sigma'_{*i}), \quad z_i, \bar{z}_i \in \mathbb{R} \quad (3.9)$$

where σ'_{*i} are related to σ_{*i} in (3.3) by $z_{ij} \leftrightarrow \bar{z}_{ij}$.

Under an $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ action, $\tilde{\mathcal{A}}_{++-}(\lambda_i; z_i, \bar{z}_i)$ transforms as a conformal three-point function of spin-one primaries with weights,

$$
\begin{aligned}
h_1 &= 1 + \frac{i}{2}\lambda_1\,, & \bar{h}_1 &= \frac{i}{2}\lambda_1\,, \\
h_2 &= 1 + \frac{i}{2}\lambda_2\,, & \bar{h}_2 &= \frac{i}{2}\lambda_2\,, \\
h_3 &= \frac{i}{2}\lambda_3\,, & \bar{h}_3 &= 1 + \frac{i}{2}\lambda_3\,.
\end{aligned}
\quad (3.10)
$$

4   Gluon Four-Point Amplitudes

Let us move on to the tree-level color-ordered four-point MHV amplitude. It is convenient to work in the $(-+++)$ spacetime signature in this section. We take particles 1 and 2 to have negative helicities and particles 3 and 4 to have positive helicities.[5] We focus on the amplitude with color order (1234) [29],

$$\mathcal{A}_{--++}(\omega_i, z_i, \bar{z}_i) = \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \delta^{(4)}(\sum_{i=1}^{4} \varepsilon_i \omega_i q_i)\,, \quad (4.1)$$

where ε_i is $+1$ (-1) if the particle is outgoing (incoming). The $2d$ spins of the corresponding conformal primaries are $J_1 = J_2 = -1$ and $J_3 = J_4 = +1$.

Again following the route of Section 2.2, the four-point delta function in (2.21) can be

---

[5]Recall that we label the helicity of an external gluon as if it were an outgoing particle.

written as

$$\delta^{(4)}(\sum_{i=1}^{4} \varepsilon_i \sigma_i q_i) \delta(\sum_{i=1}^{4} \sigma_i - 1) = \frac{1}{4} \delta(|z_{12} z_{34} \bar{z}_{13} \bar{z}_{24} - z_{13} z_{24} \bar{z}_{12} \bar{z}_{34}|)$$

$$\times \delta\left(\sigma_1 + \frac{\varepsilon_1 \varepsilon_4}{D_4} \frac{z_{24} \bar{z}_{34}}{z_{12} \bar{z}_{13}}\right) \delta\left(\sigma_2 - \frac{\varepsilon_2 \varepsilon_4}{D_4} \frac{z_{34} \bar{z}_{14}}{z_{23} \bar{z}_{12}}\right) \delta\left(\sigma_3 + \frac{\varepsilon_3 \varepsilon_4}{D_4} \frac{z_{24} \bar{z}_{14}}{z_{23} \bar{z}_{13}}\right) \delta\left(\sigma_4 - \frac{1}{D_4}\right)$$

$$\equiv \frac{1}{4} \delta(|z_{12} z_{34} \bar{z}_{13} \bar{z}_{24} - z_{13} z_{24} \bar{z}_{12} \bar{z}_{34}|) \sum_{i=1}^{4} \delta(\sigma_i - \sigma_{*i}), \tag{4.2}$$

where the denominator D_4 is defined as

$$D_4 = (1 - \varepsilon_1 \varepsilon_4) \frac{z_{24} \bar{z}_{34}}{z_{12} \bar{z}_{13}} + (\varepsilon_2 \varepsilon_4 - 1) \frac{z_{34} \bar{z}_{14}}{z_{23} \bar{z}_{12}} + (1 - \varepsilon_3 \varepsilon_4) \frac{z_{24} \bar{z}_{14}}{z_{23} \bar{z}_{13}}. \tag{4.3}$$

Note that due to the first delta function constraint, all of the σ_{*i}'s are real. The Mellin integrals in σ_i all collapse to evaluating the integrand at $\sigma_i = \sigma_{*i}$. By repeatedly using the first delta function constraint in (4.2), we arrive at the following answer for the Mellin transform of the tree-level, color-ordered MHV four-point amplitude,

$$\tilde{\mathcal{A}}_{--++}(\lambda_i, z_i, \bar{z}_i) = -\frac{\pi}{4} \delta(\sum_k \lambda_k) \delta\left(\frac{|z - \bar{z}|}{2}\right) \left(\prod_{i<j}^{4} z_{ij}^{\frac{h}{3} - h_i - h_j} \bar{z}_{ij}^{\frac{\bar{h}}{3} - \bar{h}_i - \bar{h}_j}\right) z^{\frac{5}{3}} (1 - z)^{-\frac{1}{3}}$$

$$\times \prod_{i=1}^{4} \mathbf{1}_{[0,1]}(\sigma_{*i}), \tag{4.4}$$

where σ_{*i} are given in (4.2) and the indicator function $\mathbf{1}_{[0,1]}(x)$ is defined in (2.24). z and $\bar{z}$ are the conformal cross ratios,

$$z \equiv \frac{z_{12} z_{34}}{z_{13} z_{24}}, \qquad \bar{z} \equiv \frac{\bar{z}_{12} \bar{z}_{34}}{\bar{z}_{13} \bar{z}_{24}}, \tag{4.5}$$

and we have written our answer in terms of the weights

$$h_1 = \frac{i\lambda_1}{2}, \qquad h_2 = \frac{i\lambda_2}{2}, \qquad h_3 = 1 + \frac{i\lambda_3}{2}, \qquad h_4 = 1 + \frac{i\lambda_4}{2},$$

$$\bar{h}_1 = 1 + \frac{i\lambda_1}{2}, \qquad \bar{h}_2 = 1 + \frac{i\lambda_2}{2}, \qquad \bar{h}_3 = \frac{i\lambda_3}{2}, \qquad \bar{h}_4 = \frac{i\lambda_4}{2}, \tag{4.6}$$

with $h \equiv \sum_{i=1}^{4} h_i$ and $\bar{h} \equiv \sum_{i=1}^{4} \bar{h}_i$. As conventional in any CFT four-point function, we write the answer as a product of a (non-unique) prefactor that accounts for the appropriate

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conformal covariance, and a function of the cross ratios which is conformal invariant. Here, we chose this prefactor to be $\prod_{i<j}^{4} z_{ij}^{\frac{h}{3}-h_i-h_j} \bar{z}_{ij}^{\frac{\bar{h}}{3}-\bar{h}_i-\bar{h}_j}$. From (4.4), we find that the Mellin transform of the color-ordered tree-level four-point amplitude does transform as a CFT four-point function with the above weights.

From (4.4), we see that the delta function constrains the angular coordinates $z_i, \bar{z}_i$ on the celestial sphere such that the cross ratio z is real,

$$z - \bar{z} = 0 \,. \tag{4.7}$$

This constraint has a simple explanation. Using $SL(2,\mathbb{C})$ Lorentz invariance, we can arrange for the asymptotic positions of the first three particles (z_1, z_2, z_3) to all lie on the equator of the celestial sphere. Momentum conservation then clearly implies the fourth must also lie on the equator. Interestingly, this is exactly the locus where a Lorentzian CFT correlator in $(1+1)$ dimensions is singular as discussed in [30].

Let us decode the indicator functions $\prod_{i=1}^{4} \mathbf{1}_{[0,1]}(\sigma_{*i})$. The indicator functions is only non-vanishing for two-to-two scattering amplitudes, as the original amplitudes in momentum space vanish otherwise. We will denote a two-to-two crossing channel as $ij \rightleftharpoons k\ell$, corresponding to $\varepsilon_i = \varepsilon_j = -\varepsilon_k = -\varepsilon_\ell$. From (4.4), we see that the four-point function depends on the crossing channel *only* through the indicator functions $\prod_{i=1}^{4} \mathbf{1}_{[0,1]}(\sigma_{*i})$, which constrain the cross ratio to be in the following ranges for different crossing channels:

$$\prod_{i=1}^{4} \mathbf{1}_{[0,1]}(\sigma_{*i}): \quad \begin{array}{llll} a) & 12 \rightleftharpoons 34 & \Rightarrow & 1 < z \\ b) & 13 \rightleftharpoons 24 & \Rightarrow & 0 < z < 1 \\ c) & 14 \rightleftharpoons 23 & \Rightarrow & z < 0 \,, \end{array} \tag{4.8}$$

in the $(-+++)$ signature. Recall that the cross ratio is already constrained to be real $z = \bar{z}$ by the four-dimensional massless kinematics. The indicator functions are manifestly $SL(2,\mathbb{C})$ invariant since it depends on the positions of the four points only through the cross ratios $z, \bar{z}$.

The MHV tree-level four-point amplitudes in other color orders can be obtained immediately by multiplying the answer (4.4) by a function of z_{ij}. For example, the MHV amplitude in the color order (1324) is given by

$$\frac{\langle 12 \rangle^4}{\langle 13 \rangle \langle 32 \rangle \langle 24 \rangle \langle 41 \rangle} \delta^{(4)} \Big(\sum_{i=1}^{4} \varepsilon_i \omega_i q_i \Big) \,. \tag{4.9}$$

Its Mellin transform is simply (4.4) multiplied by $-\frac{z_{12}z_{34}}{z_{13}z_{24}} = -z$, so that the Mellin transform

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of the (1324) color-ordered amplitude again transforms as a conformal four-point function with weights (4.6), as expected.

5   BCFW Recursion Relation of Conformal Correlators

In this section we transform the BCFW recursion relation [31,32] from momentum space to the space of conformal primary wavefunctions. More explicitly, we perform Mellin transforms on both sides of the BCFW relation for the MHV four-point amplitude in terms of the three-point amplitudes. We will be working in the $(-+-+)$ signature in this section where the three-point amplitude is finite and well-defined. Throughout this section, $|z_i|$ denotes the absolute value of a real variable, rather than $\sqrt{z_i \bar{z}_i}$.

5.1   BCFW in Momentum Space

Let us review the BCFW recursion relation for the four-point MHV amplitude. We denote a stripped amplitude without the momentum conservation delta function as A_n, while the physical unstripped amplitude is denoted as $\mathcal{A}_n$. The stripped color-ordered MHV and anti-MHV three-point amplitudes are

$$A_3(1^-, 2^-, 3^+) = \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle}, \qquad A_3(1^+, 2^+, 3^-) = \frac{[12]^3}{[23][31]}. \tag{5.1}$$

On the other hand, the stripped color-ordered MHV four-point amplitude is

$$A_4(1^-, 2^-, 3^+, 4^+) = \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}. \tag{5.2}$$

Let p_i^μ be the four-momenta of the four external particles satisfying the momentum conservation $\sum_{i=1}^4 p_i^\mu = 0$. Their spinor helicity variables will be denoted by $|i\rangle$ and $|i]$. We define $P_{i,j}^\mu \equiv p_i^\mu + p_j^\mu$. To apply the BCFW recursion relation, we choose 1 and 4 to be the reference gluons and shift their spinor helicity variables by

$$
\begin{aligned}
|\widehat{1}\rangle &= |1\rangle, & |\widehat{1}] &= |1] + u|4], \\
|\widehat{4}\rangle &= |4\rangle - u|1\rangle, & |\widehat{4}] &= |4],
\end{aligned}
\tag{5.3}
$$

where

$$u \equiv -\frac{P_{3,4}^2}{\langle 1|P_{3,4}|4]}, \tag{5.4}$$

with $\langle i|p_j|k] = \langle ij\rangle[jk]$.

The BCFW recursion relation equates the four-point amplitude to the product of two three-point amplitudes with shifted momenta,

$$A_4(1^-, 2^-, 3^+, 4^+) = A_3(\widehat{1}^-, 2^-, -\widehat{P}_{1,2}^+)\frac{1}{P_{1,2}^2}A_3(\widehat{P}_{1,2}^-, 3^+, \widehat{4}^+). \tag{5.5}$$

In the following section we will rewrite the above BCFW recursion relation for the four-point amplitude in the space of conformal primary wavefunctions and verify that it is obeyed by our expressions.

5.2 BCFW of Conformal Correlators

We first want to determine the change in $\omega, z, \bar{z}$ under the BCFW shift. We can choose a reference frame so that

$$|p\rangle = \pm\sqrt{2\omega}\begin{pmatrix} -1 \\ -z \end{pmatrix}, \qquad |p] = \sqrt{2\omega}\begin{pmatrix} -\bar{z} \\ 1 \end{pmatrix}, \tag{5.6}$$

where p is a null vector that is parametrized by $\omega, z, \bar{z}$ as $p^\mu = \pm\omega(1+z\bar{z}, z+\bar{z}, z-\bar{z}, 1-z\bar{z})$. We choose a plus (minus) sign for an outgoing (incoming) momentum. Let $\hat{p}$ be the BCFW-shifted momentum parametrized by $\widehat{\omega}, \widehat{z}, \widehat{\bar{z}}$. Its spinor helicity variables are

$$|\hat{p}\rangle = \pm t\sqrt{2\widehat{\omega}}\begin{pmatrix} -1 \\ -\widehat{z} \end{pmatrix}, \qquad |\hat{p}] = t^{-1}\sqrt{2\widehat{\omega}}\begin{pmatrix} -\widehat{\bar{z}} \\ 1 \end{pmatrix}. \tag{5.7}$$

Recall that given a null momentum, the spinor variables are only well-defined up to a little group rescaling $|p\rangle \to t|p\rangle$, $|p] \to t^{-1}|p]$ which will be important momentarily. The BCFW shift (5.3) for incoming particle number 1 is

$$\begin{aligned}
\sqrt{2\widehat{\omega}_1}\begin{pmatrix} 1 \\ \widehat{z}_1 \end{pmatrix} &= t_1^{-1}\sqrt{2\omega_1}\begin{pmatrix} 1 \\ z_1 \end{pmatrix}, \\
\sqrt{2\widehat{\omega}_1}\begin{pmatrix} -\widehat{\bar{z}}_1 \\ 1 \end{pmatrix} &= t_1\left[\sqrt{2\omega_1}\begin{pmatrix} -\bar{z}_1 \\ 1 \end{pmatrix} + u\sqrt{2\omega_4}\begin{pmatrix} -\bar{z}_4 \\ 1 \end{pmatrix}\right].
\end{aligned} \tag{5.8}$$

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From above, we can express the hatted variables in terms of the unhatted ones,

$$\widehat{\omega}_1 = \omega_1 + u\sqrt{\omega_1\omega_4} = \frac{\bar{z}_{14}}{\bar{z}_{24}}\omega_1\,,$$

$$\widehat{z}_1 = z_1\,, \tag{5.9}$$

$$\widehat{\bar{z}}_1 = \frac{\sqrt{\omega_1}\bar{z}_1 + u\sqrt{\omega_4}\bar{z}_4}{\sqrt{\omega_1} + u\sqrt{\omega_4}} = \bar{z}_2\,.$$

The little group factor t_1 is given by $t_1 = \sqrt{\frac{\omega_1}{\widehat{\omega}_1}} = \sqrt{\frac{\bar{z}_{24}}{\bar{z}_{14}}}$.

Similarly, for outgoing particle number 4, the BCFW-shifted variables are

$$\widehat{\omega}_4 = \omega_4 + u\sqrt{\omega_4\omega_1} = \frac{z_{14}}{z_{13}}\omega_4\,,$$

$$\widehat{z}_4 = \frac{\sqrt{\omega_4}z_4 + u\sqrt{\omega_1}z_1}{\sqrt{\omega_4} + u\sqrt{\omega_1}} = z_3\,, \tag{5.10}$$

$$\widehat{\bar{z}}_4 = \bar{z}_4\,.$$

The little group factor is given by $t_4 = \sqrt{\frac{\widehat{\omega}_4}{\omega_4}} = \sqrt{\frac{z_{14}}{z_{13}}}$. Particles 2 and 3 are unaffected by the BCFW shift so that $\widehat{\omega}_i = \omega_i$, $\widehat{z}_i = z_i$, $\widehat{\bar{z}}_i = \bar{z}_i$ for $i = 2, 3$.

The unshifted (stripped) three-point amplitude $A_3(1^-, 2^-, -P^+)$ can be written in terms of the $\omega, z, \bar{z}$ coordinates as follows

$$A_3(1^-, 2^-, -P^+) = -2\frac{\omega_1\omega_2}{\omega_P}\frac{z_{12}^3}{z_{2P}z_{P1}} \equiv A_{--+}(\omega_i; z_i, \bar{z}_i)\,. \tag{5.11}$$

For the BCFW-shifted three-point amplitude, we need to keep track of the little group factors in (5.7),

$$A_3(\widehat{1}^-, 2^-, -\widehat{P}^+) = -2t_1^2\,\frac{\widehat{\omega}_1\widehat{\omega}_2}{\widehat{\omega}_P}\,\frac{\widehat{z}_{12}^3}{\widehat{z}_{2P}\widehat{z}_{P1}} = t_1^2\,A_{--+}(\widehat{\omega}_i; \widehat{z}_i, \widehat{\bar{z}}_i)\,. \tag{5.12}$$

Similarly, if

$$A_3(P^-, 3^+, \widehat{4}^+) = 2\frac{\omega_3\omega_4}{\omega_P}\frac{\bar{z}_{34}^3}{\bar{z}_{3P}\bar{z}_{P4}} \equiv A_{-++}(\omega_i; z_i, \bar{z}_i)\,, \tag{5.13}$$

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then

$$A_3(\widehat{P}^-, 3^+, \hat{4}^+) = 2t_4^{-2} \frac{\widehat{\omega}_3 \widehat{\omega}_4}{\widehat{\omega}_P} \frac{\widehat{\bar{z}}_{34}^3}{\widehat{\bar{z}}_{3P} \widehat{\bar{z}}_{P4}} = t_4^{-2} A_{-++}(\widehat{\omega}_i; \widehat{z}_i, \widehat{\bar{z}}_i). \tag{5.14}$$

The BCFW relation is conventionally written in terms of the stripped amplitudes. We would like to write it in terms of the physical unstripped amplitudes, as only these have well-defined Mellin transforms. We do not want to work directly with the standard BCFW-shifted variables $\widehat{\omega}_i, \widehat{z}_i, \widehat{\bar{z}}_i$ since the three-point function $\tilde{\mathcal{A}}_3$ is singular at these values. We will circumvent this difficulty by considering a modified shift of the variables $\omega_i, z_i, \bar{z}_i$,

$$\begin{aligned}
(\widetilde{\omega}_1, \widetilde{z}_1, \widetilde{\bar{z}}_1) &= \left(|1 + U\zeta| \omega_1, z_1, \frac{\bar{z}_1 + U\zeta \bar{z}_4}{1 + U\zeta} \right), \\
(\widetilde{\omega}_2, \widetilde{z}_2, \widetilde{\bar{z}}_2) &= (\omega_2, z_2, \bar{z}_2), \\
(\widetilde{\omega}_3, \widetilde{z}_3, \widetilde{\bar{z}}_3) &= (\omega_3, z_3, \bar{z}_3), \\
(\widetilde{\omega}_4, \widetilde{z}_4, \widetilde{\bar{z}}_4) &= \left(|1 + U\zeta^{-1}| \omega_4, \frac{z_4 + U\zeta^{-1} z_1}{1 + U\zeta^{-1}}, \bar{z}_4 \right),
\end{aligned} \tag{5.15}$$

where U is a free variable, the absolute values are such that $\widetilde{\omega}_i > 0$, and

$$\zeta \equiv \sqrt{\frac{z_{12} \bar{z}_{13}}{z_{24} \bar{z}_{34}}}. \tag{5.16}$$

Using (4.2), one can show that the quantity in the square root in (5.16) is positive. The tilde variables $\widetilde{\omega}_i, \widetilde{z}_i$ coincide with the standard BCFW-shifted variables $|\widehat{\omega}_i|, \widehat{z}_i$ if both the ratio ω_4/ω_1 and U equal to their on-shell values, i.e. if $\frac{\omega_4}{\omega_1} = \zeta^2$ and $U = u$. Indeed, the on-shell values of ω_1 and ω_4 can be computed using (4.2) to be

$$\frac{\omega_4}{\omega_1} = \frac{z_{12} \bar{z}_{13}}{z_{24} \bar{z}_{34}} = \zeta^2. \tag{5.17}$$

Even though the two shifts agree on-shell, there is an important Jacobian factor relating $\delta^{(4)}(p_1 + p_2 + p_3 + p_4)$ and $\delta^{(4)}(\widetilde{p}_1 + p_2 + p_3 + \widetilde{p}_4)$:

$$\begin{aligned}
\delta^{(4)}(p_1 + p_2 + p_3 + p_4) &= \left| \frac{\det\{\widetilde{p}_1, p_2, p_3, \widetilde{p}_4\}}{\det\{p_1, p_2, p_3, p_4\}} \right| \delta^{(4)}(\widetilde{p}_1 + p_2 + p_3 + \widetilde{p}_4) \\
&\to_{U=u} |1 - z| \, \delta^{(4)}(\widetilde{p}_1 + p_2 + p_3 + \widetilde{p}_4).
\end{aligned} \tag{5.18}$$

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Let us now rewrite the BCFW relation as

$$
\begin{aligned}
\mathcal{A}_4(1^-, 2^-, 3^+, 4^+) &= A_4(1^-, 2^-, 3^+, 4^+)\delta^{(4)}(\widehat{p}_1 + p_2 + p_3 + \widehat{p}_4) \\
&= t_1^2 t_4^{-2} \frac{1}{P_{1,2}^2} A_{--+}\left(\widehat{\omega}_1, \omega_2, \widehat{\omega}_{P_{1,2}}; \widehat{z}_i, \widehat{\bar{z}}_i\right) A_{-++}\left(\widehat{\omega}_{P_{1,2}}, \omega_3, \widehat{\omega}_4; \widehat{z}_i, \widehat{\bar{z}}_i\right) \\
&\times \int d^4 P\, \delta^{(4)}(\widehat{p}_1 + p_2 - P)\delta^{(4)}(P + p_3 + \widehat{p}_4)\,.
\end{aligned}
\tag{5.19}
$$

Next, we trivially insert $\int_{-\infty}^{\infty} dU \delta(U - u)$ into the integral and use

$$
\frac{1}{P_{1,2}^2}\delta(U - u) = -\mathrm{sgn}(z_{12}\bar{z}_{12})\frac{1}{|u|}\delta(\langle 1|P_{1,2}|4]U - P_{1,2}^2) = -\mathrm{sgn}(z_{12}\bar{z}_{12})\frac{1}{|U|}\delta(\widehat{P}_{1,2}^2(U))\,, \tag{5.20}
$$

where we have defined $\widehat{P}_{1,2}(U) = P_{1,2} + U|4]\langle 1|$. At the on-shell value of U, i.e. $U = u$, $\widehat{P}_{1,2}(u) = \hat{p}_1 + p_2$. Since the integrand only has support on $U = u$, we can replace $\widehat{P}_{1,2}(U)$ by P and $\widehat{\omega}_i, \widehat{z}_i$ by $\widetilde{\omega}_i, \widetilde{z}_i$ (which depend on U). The BCFW relation then takes the unstripped form

$$
\begin{aligned}
\mathcal{A}_4&(1^-, 2^-, 3^+, 4^+) \\
&= -\mathrm{sgn}(z_{12}\bar{z}_{12})|1 - z|t_1^2 t_4^{-2} \int_{-\infty}^{\infty} \frac{dU}{|U|} \int d^4 P\, \delta(P^2)\, A_{--+}\left(\widetilde{\omega}_1, \omega_2, \omega_P; \widetilde{z}_i, \widetilde{\bar{z}}_i\right) \delta^{(4)}(\widetilde{p}_1 + p_2 - P) \\
&\times A_{-++}\left(\omega_P, \omega_3, \widetilde{\omega}_4; \widetilde{z}_i, \widetilde{\bar{z}}_i\right) \delta^{(4)}(P + p_3 + \widetilde{p}_4) \\
&= -\mathrm{sgn}(z_{12}\bar{z}_{12})|1 - z|t_1^2 t_4^{-2} \int_{-\infty}^{\infty} \frac{dU}{|U|} \int d^4 P\, \delta(P^2)\, \mathcal{A}_{--+}\left(\widetilde{\omega}_1, \omega_2, \omega_P; \widetilde{z}_i, \widetilde{\bar{z}}_i\right) \mathcal{A}_{-++}\left(\omega_P, \omega_3, \widetilde{\omega}_4; \widetilde{z}_i, \widetilde{\bar{z}}_i\right)\,,
\end{aligned}
\tag{5.21}
$$

where $\mathcal{A}$ denotes the unstripped amplitude with the momentum conservation delta function included. The Jacobian factor $|1 - z|$ comes from the momentum conservation delta functions as explained in (5.18). Because of the delta function $\delta(P^2)$, P^μ is null and we can define ω_P, z_P, $\bar{z}_P$ as $P^\mu = \omega_P(1 + z_P\bar{z}_P, z_P + \bar{z}_P, z_P - \bar{z}_P, 1 - z_P\bar{z}_P)$. In $\mathcal{A}_{--+}$ above and from now on, the notation $\widetilde{z}_i$ collectively denotes $\widetilde{z}_3$, $\widetilde{z}_4$, and z_P. We use a similar collective notation in $\mathcal{A}_{-++}$.

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We now perform the Mellin transform on both sides of the BCFW relation,

$$
\begin{aligned}
\tilde{\mathcal{A}}_{--++}(\lambda_i, z_i, \bar{z}_i) &\equiv \prod_{i=1}^{4} \int_0^\infty d\omega_i\, \omega_i^{i\lambda_i}\, \mathcal{A}_4(1^-, 2^-, 3^+, 4^+) \\
&= -\mathrm{sgn}(z_{12}\bar{z}_{12})|1-z| \left|\frac{\bar{z}_{24}}{\bar{z}_{14}}\right|^{2+i\lambda_1} \left|\frac{z_{13}}{z_{14}}\right|^{2+i\lambda_4} \int_{-\infty}^\infty \frac{dU}{|U|} \int d^4 P\,\delta(P^2) \prod_{i=1}^{4} \int_0^\infty d\widetilde{\omega}_i\, \widetilde{\omega}_i^{i\lambda_i} \\
&\quad\times \mathcal{A}_{--+}\left(\widetilde{\omega}_1, \omega_2, \omega_P; \widetilde{z}_i, \widetilde{\bar{z}}_i\right) \mathcal{A}_{-++}\left(\omega_P, \omega_3, \widetilde{\omega}_4; \widetilde{z}_i, \widetilde{\bar{z}}_i\right) \\
&= -\mathrm{sgn}(z_{12}\bar{z}_{12})|1-z| \left|\frac{\bar{z}_{24}}{\bar{z}_{14}}\right|^{2+i\lambda_1} \left|\frac{z_{13}}{z_{14}}\right|^{2+i\lambda_4} \int_{-\infty}^\infty \frac{dU}{|U|} \int d^4 P\,\delta(P^2) \prod_{i=1}^{4} \int_0^\infty d\widetilde{\omega}_i\, \widetilde{\omega}_i^{i\lambda_i} \\
&\quad\times \prod_{i=1}^{4} \int_{-\infty}^\infty \frac{d\widetilde{\lambda}_i}{2\pi} \int_{-\infty}^\infty \frac{d\lambda_P}{2\pi} \int_{-\infty}^\infty \frac{d\lambda_{P'}}{2\pi} \left(\prod_{i=1}^{4} \widetilde{\omega}_i^{-1-i\widetilde{\lambda}_i}\right) \omega_P^{-2-i\lambda_P-i\lambda_{P'}} \\
&\quad\times \tilde{\mathcal{A}}_{--+}(\widetilde{\lambda}_1, \widetilde{\lambda}_2, \lambda_P; \widetilde{z}_j, \widetilde{\bar{z}}_j) \tilde{\mathcal{A}}_{-++}(\lambda_{P'}, \widetilde{\lambda}_3, \widetilde{\lambda}_4; \widetilde{z}_j, \widetilde{\bar{z}}_j)\,.
\end{aligned}
\tag{5.22}
$$

In the second line we have changed the integration variables to the shifted energy $\widetilde{\omega}_i$. In the third line we express the three-point amplitudes in terms of their Mellin transforms.

Finally, using

$$
\int d^4 P\,\delta(P^2) = \int_0^\infty \omega_P d\omega_P \int dz_P d\bar{z}_P\,,
\tag{5.23}
$$

we can perform the ω_P and the $\lambda_{P'}$ integrals to obtain,

$$
\begin{aligned}
\tilde{\mathcal{A}}_{--++}(\lambda_i, z_i, \bar{z}_i) &= -\mathrm{sgn}(z_{12}\bar{z}_{12})|1-z| \left|\frac{\bar{z}_{24}}{\bar{z}_{14}}\right|^{2+i\lambda_1} \left|\frac{z_{13}}{z_{14}}\right|^{2+i\lambda_4} \\
&\quad\times \int_{-\infty}^\infty \frac{dU}{|U|} \int_{-\infty}^\infty \frac{d\lambda_P}{2\pi} \int dz_P d\bar{z}_P \tilde{\mathcal{A}}_{--+}(\lambda_1, \lambda_2, \lambda_P; \widetilde{z}_j, \widetilde{\bar{z}}_j) \tilde{\mathcal{A}}_{-++}(-\lambda_P, \lambda_3, \lambda_4; \widetilde{z}_j, \widetilde{\bar{z}}_j)\,.
\end{aligned}
\tag{5.24}
$$

The above equation is our final result for the BCFW recursion relation in the space of conformal primary wavefunctions. The tilde variables are defined in (5.15) and $\widetilde{z}_P = z_P$, $\widetilde{\bar{z}}_P = \bar{z}_P$.

Let us check (5.24) by explicitly plugging in the three-point functions obtained in (3.5)

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and (3.9):

$$\tilde{\mathcal{A}}_{--+}(\lambda_1, \lambda_2, \lambda_P; \widetilde{z}_j, \widetilde{\bar{z}}_j) = -\pi\mathrm{sgn}(z_{12}z_{2P}z_{P1})\,\delta(\lambda_1 + \lambda_2 + \lambda_P)\frac{\delta(\widetilde{\bar{z}}_1 - \widetilde{\bar{z}}_2)\delta(\widetilde{\bar{z}}_{2P})}{|z_{12}|^{-1-i\lambda_P}|z_{2P}|^{1-i\lambda_1}|z_{1P}|^{1-i\lambda_2}}\,,$$

$$\tilde{\mathcal{A}}_{-++}(-\lambda_P, \lambda_3, \lambda_4,; \widetilde{z}_j, \widetilde{\bar{z}}_j) = \pi\mathrm{sgn}(\bar{z}_{34}\bar{z}_{4P}\bar{z}_{P3})\,\delta(\lambda_3 + \lambda_4 - \lambda_P)\frac{\delta(\widetilde{z}_4 - \widetilde{z}_3)\delta(\widetilde{z}_{3P})}{|\bar{z}_{43}|^{-1+i\lambda_P}|\bar{z}_{3P}|^{1-i\lambda_4}|\bar{z}_{4P}|^{1-i\lambda_3}}\,.$$

$$(5.25)$$

All the integrals can be performed by solving the delta functions. In particular we have

$$U = \frac{z_{34}}{z_{13}}\zeta\,. \tag{5.26}$$

Plugging the above value of U into another delta function we obtain

$$\delta\left(\frac{\bar{z}_{12} - U\zeta\bar{z}_{24}}{1 + U\zeta}\right) = \delta(|z - \bar{z}|)\left|\frac{\bar{z}_{34}\bar{z}_{14}}{\bar{z}_{24}^2\bar{z}_{13}}\right|\,, \tag{5.27}$$

where z and $\bar{z}$ are the cross-ratios (4.5). One can then verify that the righthand side of (5.24) correctly reproduces the $(-+-+)$ signature analog of the four-point function (4.4).[6]

Acknowledgments

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[6]There is a $\mathrm{sgn}(z_{ij}\bar{z}_{ij})$ in the $(-+-+)$ analog of (4.4) which is matched by the signs in the BCFW recursion relation.

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A Conventions

In this appendix we will review our conventions in the $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$ signature. The Levi-Civita symbols are normalized as $\epsilon^{12} = -\epsilon^{21} = +1$ and $\epsilon_{12} = -\epsilon_{21} = -1$. We denote the chiral and anti-chiral spinor indices of the Lorentz group $SL(2, \mathbb{C})$ as α and $\dot\alpha$, respectively. The index of a spinor λ_α is lowered and raised as $\lambda^\alpha = \epsilon^{\alpha\beta}\lambda_\beta$ and $\lambda_\alpha = \epsilon_{\alpha\beta}\lambda^\beta$.

A four-momentum p^μ can be represented by a two-by-two matrix as

$$p_{\alpha\dot\alpha} = p_\mu \sigma^\mu_{\alpha\dot\alpha}, \tag{A.1}$$

where $\sigma^\mu_{\alpha\dot\alpha} = (I, \vec\sigma)$. We also define

$$p^{\dot\alpha\alpha} = \epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}p_{\beta\dot\beta} = p_\mu\bar\sigma^{\mu\dot\alpha\alpha}, \tag{A.2}$$

where $\bar\sigma^{\mu\dot\alpha\alpha} = (I, -\vec\sigma)$. Using the identity $\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta} = -2\eta^{\mu\nu}$, the inner product between two four-momenta can be written as $\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}p_{\alpha\dot\alpha}q_{\beta\dot\beta} = -2p^\mu q_\mu$.

A null four-momentum $p_{\alpha\dot\alpha}$ has vanishing determinant so can always be written in terms of their spinor helicity variables,

$$p_{\alpha\dot\alpha} = |p]_\alpha \langle p|_{\dot\alpha}. \tag{A.3}$$

The spinor helicity variables are defined up to a little group rescaling, $|p\rangle \to t|p\rangle$ and $|p] \to t^{-1}|p]$. Similarly,

$$p^{\dot\alpha\alpha} = |p\rangle^{\dot\alpha}[p|^\alpha, \tag{A.4}$$

where the spinor helicity variables with upper indices are defined as

$$[p|^\alpha = \epsilon^{\alpha\beta}|p]_\beta, \qquad \langle p|_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}|p\rangle^{\dot\beta}. \tag{A.5}$$

We define the brackets of spinor helicity variables as

$$[pq] = [p|^\alpha|q]_\alpha = -\epsilon^{\alpha\beta}|p]_\alpha|q]_\beta = -[qp], \tag{A.6}$$

$$\langle pq\rangle = \langle p|_{\dot\alpha}|q\rangle^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\langle p|_{\dot\alpha}\langle q|_{\dot\beta} = -\langle qp\rangle. \tag{A.7}$$

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The inner product between two null momenta can be written as products of the brackets,

$$2p \cdot q = -p_{\alpha\dot{\alpha}} q_{\beta\dot{\beta}} \epsilon^{\alpha\beta} \epsilon^{\dot{\alpha}\dot{\beta}} = \langle pq \rangle [pq] \,. \tag{A.8}$$

We also define:

$$[p|k|q\rangle = [p|^\alpha k_{\alpha\dot{\alpha}} |q\rangle^{\dot{\alpha}} = [pk]\langle kq \rangle \,, \tag{A.9}$$

$$\langle q|k|p] = \langle q|_{\dot{\alpha}} k^{\dot{\alpha}\alpha} |p]_\alpha = \langle qk \rangle [kp] \,. \tag{A.10}$$

We can choose a frame and parametrize a null momentum p^μ by $\omega, z, \bar{z}$ as,

$$p^\mu = \pm\omega(1 + |z|^2, z + \bar{z}, -i(z - \bar{z}), 1 - |z|^2) \,, \tag{A.11}$$

with a plus (minus) sign for an outgoing (incoming) momentum. In terms of a two-by-two matrix, we have

$$p_{\alpha\dot{\alpha}} = \sigma^\mu_{\alpha\dot{\alpha}} p_\mu = |p]_\alpha \langle p|_{\dot{\alpha}} = \pm 2\omega \begin{pmatrix} -|z|^2 & \bar{z} \\ z & -1 \end{pmatrix} \,. \tag{A.12}$$

Next we want to express the spinor helicity variables in terms of $\omega, z, \bar{z}$. A priori, any such identification suffers from the ambiguity of little group rescaling $|p\rangle \to t|p\rangle, |p] \to t^{-1}|p]$, which in turn rescales the polarization vectors as $\epsilon^\mu_\pm(p) \to t^{\mp 2}\epsilon^\mu_\pm(p)$. For our purpose, however, the choice of conformal primary wavefunction in (2.8) fixes a particular normalization for the polarization vectors as in (2.7), $\epsilon^\mu_+(p = \pm\omega q) = \frac{1}{\sqrt{2}}\partial_z q^\mu$ and $\epsilon^\mu_-(p = \pm\omega q) = \frac{1}{\sqrt{2}}\partial_{\bar{z}} q^\mu$. In this normalization the spinor helicity variables can be written in terms of $\omega, z, \bar{z}$ as

$$|p]_\alpha = \sqrt{2\omega} \begin{pmatrix} -\bar{z} \\ 1 \end{pmatrix} \,, \qquad \langle p|_{\dot{\alpha}} = \pm\sqrt{2\omega} \begin{pmatrix} z \\ -1 \end{pmatrix} \,,$$
$$[p|^\alpha = \sqrt{2\omega} \begin{pmatrix} 1 \\ \bar{z} \end{pmatrix} \,, \qquad |p\rangle^{\dot{\alpha}} = \pm\sqrt{2\omega} \begin{pmatrix} -1 \\ -z \end{pmatrix} \,. \tag{A.13}$$

For two outgoing or two incoming particles, the brackets can be written as

$$[ij] = 2\sqrt{\omega_i \omega_j}(\bar{z}_i - \bar{z}_j) \,, \qquad \langle ij \rangle = -2\sqrt{\omega_i \omega_j}(z_i - z_j) \,. \tag{A.14}$$

On the other hand, the brackets between one incoming and one outgoing particle are

$$[ij] = 2\sqrt{\omega_i \omega_j}(\bar{z}_i - \bar{z}_j) \,, \qquad \langle ij \rangle = 2\sqrt{\omega_i \omega_j}(z_i - z_j) \,. \tag{A.15}$$

B   Inner Products of One-Particle States

In [10] (see also [25, 28] for the scalar case) the inner product between four-dimensional one-particle states of spin one was computed in the space of conformal primary wavefunctions. Here we review this calculation for completeness. Let us denote a massless one-particle state with helicity $\ell = \pm 1$ and three-momentum $\vec{p}$ by $|\vec{p}, \ell\rangle$, with the energy p^0 given by $p^0 = |\vec{p}|$. The inner product between such one-particle states is[7]

$$\langle p_2, \ell_2 | p_1, \ell_1 \rangle = 2p_1^0 (2\pi)^3 \, \delta_{\ell_1, -\ell_2} \, \delta^{(3)}(\vec{p}_1 + \vec{p}_2) \,. \tag{B.1}$$

The Mellin transform of this inner product is

$$
\begin{aligned}
\tilde{\mathcal{A}}_{J_1 J_2}(\lambda_i, z_i, \bar{z}_i) &= (2\pi)^3 \, \delta_{\ell_1, -\ell_2} \\
&\times \int_0^\infty d\omega_1 \, \omega_1^{i\lambda_1} \int_0^\infty d\omega_2 \, \omega_2^{i\lambda_2} \, \omega_1 (1 + |z_1|^2) \, \delta^{(2)}(\omega_1 z_1 - \omega_2 z_2) \delta(\omega_1(1 - |z_1|^2) - \omega_2(1 - |z_2|^2)) \\
&= (2\pi)^4 \, \delta_{\ell_1, -\ell_2} \, \delta(\lambda_1 + \lambda_2) \delta^{(2)}(z_1 - z_2) \,,
\end{aligned}
\tag{B.2}
$$

where we have used (2.20). The $2d$ spins are given by $J_1 = \ell_1$ and $J_2 = \ell_2$.

Let us consider the case with helicities $-\ell_1 = \ell_2 = +1$, while the other case follows similarly. The answer $\tilde{\mathcal{A}}_{-+}(\lambda_i, z_i, \bar{z}_i)$ is a contact term, but it has the same $SL(2, \mathbb{C})$ covariance as a two-point function of conformal primaries with weights (with $\lambda_2 = -\lambda_1$ fixed by the delta function above)

$$
\begin{aligned}
h_1 &= i\frac{\lambda_1}{2} \,, & \bar{h}_1 &= 1 + i\frac{\lambda_1}{2} \,, \\
h_2 &= 1 + i\frac{\lambda_2}{2} \,, & \bar{h}_2 &= i\frac{\lambda_2}{2} \,.
\end{aligned}
\tag{B.3}
$$

In particular, the $2d$ spins are $J_1 = h_1 - \bar{h}_1 = -1$ and $J_2 = h_2 - \bar{h}_2 = +1$. Indeed, under an $SL(2, \mathbb{C})$ transformation $z_i \to z_i' = \frac{az_i + b}{cz_i + d}$, the contact term $\delta(\lambda_1 + \lambda_2)\delta^{(2)}(z_1 - z_2)$ transforms as

$$
\begin{aligned}
\delta(\lambda_1 + \lambda_2)\delta^{(2)}(z_1' - z_2') &= |cz_1 + d|^4 \delta(\lambda_1 + \lambda_2)\delta^{(2)}(z_1 - z_2) \\
&= \left[\prod_{i=1}^2 (cz_i + d)^{\Delta_i + J_i} (\bar{c}\bar{z}_i + \bar{d})^{\Delta_i - J_i} \right] \delta(\lambda_1 + \lambda_2)\delta^{(2)}(z_1 - z_2) \,. \tag{B.4}
\end{aligned}
$$

---

[7]As before, we label a helicity of a gauge boson as it were an outgoing particle. That is why the inner product is only non-vanishing if $\ell_1 = -\ell_2$.

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