Aretakis charges and asymptotic null infinity

Hadi Godazgar, Mahdi Godazgar, and C. N. Pope

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Hadi Godazgar[1]**, Mahdi Godazgar**[2] **and C.N. Pope**[3,4]

[1] *Max-Planck-Institut für Gravitationsphysik (Albert-Einstein-Institut),*
Mühlenberg 1, D-14476 Potsdam, Germany.
hadi.godazgar@aei.mpg.de

[2] *Institut für Theoretische Physik,*
Eidgenössische Technische Hochschule Zürich,
Wolfgang-Pauli-Strasse 27, 8093 Zürich, Switzerland.
godazgar@phys.ethz.ch

[3] *George P. & Cynthia Woods Mitchell Institute for Fundamental Physics and Astronomy,*
Texas A&M University, College Station, TX 77843, USA.
pope@physics.tamu.edu

[4]*DAMTP, Centre for Mathematical Sciences,*
Cambridge University, Wilberforce Road, Cambridge CB3 0WA, UK

ABSTRACT

We construct a relation between the Aretakis charge of any extreme black hole and the Newman-Penrose charge. This is achieved by constructing a conformal correspondence between extreme black holes and what we call weakly asymptotically flat space-times. Under this correspondence the Newman-Penrose charge of the weakly asymptotically flat space-time maps on to the Aretakis charge of the extreme black hole. Furthermore, we generalise the conformal isometry displayed by the extreme Reissner-Nordström solution to a novel conformal symmetry that acts within a class of static STU supergravity black holes.

1 Introduction

The horizon of an extreme black hole is unstable under scalar perturbations [1–4]. More generally, at least for the case of extreme Reissner-Nordström (ERN) and extreme Kerr black holes, the horizon is also unstable under massive scalar, electromagnetic and linearised gravitational perturbations [5–7]. This instability is due to the existence of conserved charge(s) on the extremal horizon—the *Aretakis charge(s)*.

Beyond the calculational intricacies, our aim in this paper is to understand why such conserved charges exist on the extremal horizon; say as opposed to bifurcate horizons, where Aretakis charges do not exist. If we consider, for example, the ERN or extreme Kerr black holes, apart from the associated Aretakis charges on the horizon there exist another set of conserved charges, namely the *Newman-Penrose* (NP) charges [8] that are defined at future null infinity, which is strictly not a part of these space-times.[1] In fact, for any asymptotically flat space-time (see e.g. [9]), there exist NP charge(s) at its future null infinity.

Unlike the Aretakis charges, the existence of NP charges is not so surprising. In fact we would expect such charges to exist from a physical point of view, given that asymptotically flat space-times are defined precisely such that there is a notion of approaching "flat space-time" at large radial distance away from the gravitational source of interest. Since conserved charges exist in flat space-time, it is reasonable to expect that they will exist at future null infinity [10] (see also [11]).

For scalar perturbations on the ERN background, it has been shown [6, 12] that there is a simple bijective map relating the Aretakis and NP charges. The existence of such a map is due to the fact that the ERN metric displays a conformal isometry, generated by a spatial inversion that interchanges the extremal horizon and future null infinity [13]. The massless scalar wave equation is conformally covariant on a Ricci scalar-flat background. Therefore, as future null infinity maps on to the horizon under the spatial inversion, so do its NP charges map on to the Aretakis charges of the horizon.

The above argument provides an attractive explanation for the existence of Aretakis charges. However, the conformal isometry at the heart of this explanation is a rather special feature of the ERN black hole. Such an isometry does not exist in more general situations. For example, the extreme Kerr black hole does not have such a conformal isometry [13]. [2]

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[1]This is fortunate, because otherwise the existence of NP charges would lead to an instability of asymptotically flat space-times.

[2]Although, an axially symmetric scalar field propagating on the extreme Kerr(-Newman) background does display such a symmetry [12]. This is closely related to the fact that the radial equation displays such a property [13].

Furthermore, the argument for the existence of Aretakis charges associated with a massless scalar field for general extreme black holes [3,5] does not preclude non-asymptotically flat extreme black holes. Such space-times have no NP charges. Hence, an argument of the form above, relating different charges in the space-time to one another, is impossible here.

In this paper, we aim to study the relation between Aretakis and NP charges in a more general context. Throughout the paper, for lucidity, we confine ourselves to four dimensions and massless scalar perturbations. These restrictions do not affect our main results, which will follow through, almost trivially, for higher-dimensional space-times, as well as to other conformally-covariant fields.

We find that a more general class of solutions, the 4-charge static extreme STU black holes, also exhibit a kind of conformal symmetry of the type exhibited by the ERN black hole, albeit of a more general type. Under a spatial inversion, analogous to that performed in [13] for the ERN black hole, we find that a general 4-charge STU black hole maps on to a space-time that is conformally related to another such STU black hole but with different values for the charge parameters. Thus, while the transformation does not amount to an isometry, there is an intriguing symmetry within the class of static 4-charge STU black holes which means that the Aretakis charges of one black hole are related to the NP charges of another, and *vice-versa*. For a subclass of the 4-charge black holes, where the charges are set pairwise-equal (which includes the ERN black hole as a special case), we find that there is indeed a symmetry enhancement to a conformal isometry.

Motivated by the above observation for STU black holes we relate, via a spatial inversion, any extreme black hole space-time and its Aretakis charge(s), to a *weakly asymptotically flat* space-time and its associated NP charge(s). [3] If one makes the fall-off conditions for certain metric components slightly stronger, weakly asymptotically flat space-times reduce to asymptotically flat space-times in Bondi coordinates.

In section 2, we review the construction of the NP and Aretakis charges. In section 2.1, we derive the NP charge for *weakly asymptotically flat space-times*. This result emphasises the fact that NP charges do not only exist for asymptotically flat space-times, but also for more general space-times. In section 2.1.1, we show that for stationary spherically-symmetric space-times one has a hierarchy of NP charges and complements a similar result by Aretakis regarding stationary spherically symmetric extreme black holes [3]. In section 2.2, we review the construction in [5] of the Aretakis charge using Gaussian null coordinates introduced in the vicinity of the horizon [16]. We rederive the first Aretakis charge of the

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[3]Note that our definition of weakly asymptotic has no relation to the usage in [14,15].

ERN black hole as an example.

In section 3, we consider a class of static (extreme) black hole solutions of STU super-gravity, namely a four-charge family that includes as a sub-class the one-parameter ERN family. We find a novel conformal symmetry under spatial inversion, whereby the horizon of a particular solution with given parameters maps on to the future null infinity of another solution with related, but different, parameters, and *vice-versa*. If we specialise to a two-parameter subclass, by setting the four charges pairwise equal, we find that the conformal symmetry enhances to a conformal isometry. The ERN black holes are contained as a special case within the pairwise-equal class, in which all the charges are set equal. In contrast to the four-charge solutions, we find that no such generalised conformal symmetry exists for the most general eight-charge dyonic static STU black holes, the details of which are given in appendix A.

In section 4, we derive the main result of the paper: we find a conformal correspondence between weakly asymptotically flat space-times and extreme black holes. First, we write the extreme black hole metric in Gaussian null coordinates, which can always be chosen in the neighbourhood of the horizon. In these coordinates, the horizon corresponds to the null surface $r = 0$. We then perform a coordinate transformation, which includes the spatial inversion $r \to 1/r$. We identify the resulting metric as being conformally related to a space-time whose metric is naturally written in Bondi coordinates—a weakly asymptotically flat space-time. Since the original metric was valid in the neighbourhood of $\{r = 0\}$, we expect the weakly asymptotically flat metric to describe a space-time for which we can choose asymptotic coordinates (i.e. coordinates valid for large r), such that the metric coincides with that of the weakly asymptotically flat metric for large r. Moreover, note that the weakly asymptotically flat space-time we obtain via this map will not be Ricci-flat. However, it will be asymptotically Ricci-flat.

The results of section 2.1 guarantee the existence of NP charges for weakly asymptotically flat space-times. Thus, this correspondence maps the NP charge(s) of the associated weakly asymptotically flat space-time to the Aretakis charge(s) of the original extreme black hole under consideration. In section 4.1, we make this map between the Aretakis and NP charges precise for the massless scalar, and we consider the ERN black hole as an example. It should be stressed that the associated weakly asymptotically flat space-time has no direct physical significance. The construction merely highlights the correspondence between Aretakis and NP charges.

We end with some discussions in section 5.

2 Conserved charges at null infinity and the extremal horizon

2.1 Newman-Penrose charge

In this section, we derive the well-known Newman-Penrose (NP) charge [8] for a massless scalar field at future null infinity. The existence of a boundary that can be identified with future null infinity necessitates that the solution be asymptotically flat. Thus, NP charges are generally defined for such space-times. However here we slightly weaken the definition of asymptotic flatness, and show that NP charges can nevertheless still be defined. The main reason for doing this is that later we will need a notion of an NP charge for such slightly more general space-times. Another motivation for this exercise is to highlight the fact that NP charges arise in this more general context, and they are not merely tied to the notion of asymptotic flatness.

We define a *weakly asymptotically flat* space-time as one for which Bondi coordinates [17, 18] (u, r, x^I) can be introduced, such that the metric takes the Bondi form

$$ds^2 = -Fe^{2\beta}du^2 - 2e^{2\beta}dudr + r^2 h_{IJ}(dx^I - C^I du)(dx^J - C^J du), \qquad (2.1)$$

with the metric functions satisfying the fall-off conditions

$$\lim_{r\to\infty} r(F-1) = \tilde{F}(u, x^I), \qquad \lim_{r\to\infty} r^2\beta = \tilde{\beta}(u, x^I),$$

$$\lim_{r\to\infty} rC^I = \tilde{C}^I(u, x^I), \qquad \lim_{r\to\infty} r(h_{IJ} - \omega_{IJ}) = \tilde{h}_{IJ}(u, x^I), \qquad (2.2)$$

at large r, where the 2-dimensional space has the metric ω_{IJ} with coordinates x^I, and is assumed to be compact.

The differences between weakly asymptotically flat and asymptotically flat space-times are two-fold. First, for asymptotically flat space-times, ω_{IJ} is the metric on the unit round 2-sphere. Here, we assume only that ω_{IJ} is a positive-definite metric on a complete compact 2-manifold. Moreover, our assumption about the fall-off for C^I at large r is weaker than that for asymptotically flat space-times, for which it is required that

$$\lim_{r\to\infty} r^2 C^I = \tilde{C}^I(u, x^I). \qquad (2.3)$$

We can always choose a gauge in which

$$h \equiv \det(h_{IJ}) = \omega\, \zeta(r)^2, \qquad (2.4)$$

where $\omega = \det(\omega_{IJ})$ and $\zeta(r)$ is some function of r such that

$$\lim_{r\to\infty} r(\zeta - 1) = \text{constant}. \qquad (2.5)$$

Note that, we could of course, eliminate ζ by redefining r. However, for later convenience we choose to fix our gauge as in equation (2.4).

We now show that this more general class of space-times, which includes asymptotically flat space-times as a strict subset, contains an NP charge for a massless scalar field at future null infinity, which we define to be the surface $r = c$, $c \to \infty$.

Consider a massless scalar on the above background, satisfying

$$\Box_g \psi = \frac{1}{\sqrt{-g}} \partial_a \left(\sqrt{-g} \, g^{ab} \, \partial_b \psi \right) = 0. \tag{2.6}$$

We assume the fall-off condition

$$\lim_{r \to \infty} (r\psi) = \psi^{(1)}(u, x^I), \tag{2.7}$$

at large r. Using the fact that, with respect to coordiantes (u, r, x^I),

$$(e^{2\beta} g^{ab}) = \begin{pmatrix} 0 & -1 & 0 \\ -1 & F & -C^J \\ 0 & -C^I & e^{2\beta} h^{IJ}/r^2 \end{pmatrix} \tag{2.8}$$

and that

$$\sqrt{-g} = r^2 \, \zeta(r) \, e^{2\beta} \sqrt{\omega}, \tag{2.9}$$

equation (2.6) reduces, at leading order as $r \to \infty$, to

$$-\partial_u \left[\partial_r (r^2 \zeta) \, \psi + 2r^2 \zeta \, \partial_r \psi \right] - \frac{r^2 \zeta}{\sqrt{\omega}} \partial_I \left[\sqrt{\omega} \, C^I \partial_r \psi \right] + \frac{\zeta}{\sqrt{\omega}} \partial_I \left[e^{2\beta} \sqrt{\omega} h^{IJ} \partial_J \psi \right] = 0. \tag{2.10}$$

Note that the second term, which involves C^I, would drop out for asymptotically flat space-times. Multiplying equation (2.10) by r and taking the limit $r \to \infty$ gives

$$\lim_{r \to \infty} \left\{ -r^2 \partial_u \left[2 \, \partial_r (r\psi) + r \partial_r \zeta \, \psi \right] + \frac{r}{\sqrt{\omega}} \partial_I \left[\sqrt{\omega} \left(\omega^{IJ} \partial_J \psi - r^2 C^I \partial_r \psi \right) \right] \right\} = 0. \tag{2.11}$$

Now, integrating this equation over the compact space at infinity, whose metric is ω_{IJ}, we find that the second set of terms, which do not involve a u-derivative, reduce to a boundary contribution, which can be discarded. Thus, we have a conserved charge at future null infinity that is given by

$$H_{NP} = -\lim_{r \to \infty} \int d\omega \, r^2 \left[2\partial_r (r\psi) + r\partial_r \zeta \, \psi \right], \tag{2.12}$$

where $d\omega = \sqrt{\omega} \, dx^I$.

As mentioned earlier, we can always choose a gauge in which $\zeta = 1$. In this gauge, the NP charge reduces to

$$H_{NP}|_{\zeta=1} = -\lim_{r \to \infty} \int d\omega \, 2r^2 \partial_r (r\psi). \tag{2.13}$$

Note that this charge does not depend on the specific features of the space-time and it is therefore universal.

If we assume that the scalar field is analytic in $1/r$, and so it admits an expansion

$$\psi = \sum_{n=1}^{\infty} \frac{\psi^{(n)}(u, x^I)}{r^n}, \tag{2.14}$$

the Newman-Penrose charge reduces to

$$H_{NP} = 2 \int_{S^2} d\omega \, \psi^{(2)}. \tag{2.15}$$

2.1.1 Example: Spherically symmetric case

For the case of spherically symmetric backgrounds, one has a hierarchy of NP charges. A similar result also holds for Aretakis charges [3]. Given a stationary spherically symmetric space-time, one can choose coordinates $\{u, r, \theta, \phi\}$ such that the metric takes the form

$$ds^2 = -F(r) \, du^2 - 2dudr + r^2 \zeta(r) \, d\Omega^2, \tag{2.16}$$

where

$$\lim_{r \to \infty} r(F - 1) = \tilde{F}^{(1)}, \qquad \lim_{r \to \infty} r(\zeta - 1) = \tilde{\zeta}^{(1)}. \tag{2.17}$$

The scalar wave equation (2.6) on this background reduces to

$$-\partial_u \Big[2r^2 \zeta \, \partial_r \psi + \partial_r (r^2 \zeta) \, \psi \Big] + \partial_r \big(r^2 \zeta \, F \, \partial_r \psi \big) + \Delta \psi = 0, \tag{2.18}$$

where Δ is the scalar Laplacian on the unit 2-sphere.

In order to proceed further, we assume an analytic expansion for the scalar field,

$$\psi = \sum_{n=1}^{\infty} \frac{\psi^{(n)}(u, \theta, \phi)}{r^n}, \tag{2.19}$$

as well as for the metric functions F and ζ. We emphasise that this assumption is not essential (see [3]). However, we do so to make the appearance of the hierarchy of NP charges clearer.

Substituting the expansion (2.19) into the wave equation (2.18) gives

$$\sum_{n=1}^{\infty} \frac{1}{r^n} \Big\{ \partial_u \big[2n\zeta \, \psi^{(n+1)} - (r^2 \partial_r \zeta) \, \psi^{(n)} \big] + n(n-1) \, \zeta F \, \psi^{(n)} + \Delta \psi^{(n)}$$

$$- (n-1) \big[r^2 \partial_r (\zeta F) \big] \psi^{(n-1)} \Big\} = 0. \tag{2.20}$$

From this, we can extract a hierarchy of equations labelled by a positive integer n:

$$T_n = 0, \tag{2.21}$$

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where

$$T_n = \partial_u \left(\sum_{i=1}^{n+1} \alpha_n^{(i)} \, \psi^{(i)} \right) + \left[n(n-1) + \Delta \right] \psi^{(n)} + \sum_{i=1}^{n-1} \beta_n^{(i)} \, \psi^{(i)}, \tag{2.22}$$

with $\alpha_n^{(i)}$ and $\beta_n^{(i)}$ being certain functions of the metric components, which we leave implicit.

For $n = 1$, we have

$$2\partial_u \left(\psi^{(2)} + \lim_{r \to \infty} (r^2 \partial_r \zeta) \, \psi^{(1)} \right) + \Delta \psi^{(1)} = 0. \tag{2.23}$$

Integrating this equation over the sphere, we obtain the NP charge (2.12) that we derived more generally in the previous analysis.

For general $n > 1$, multiplying by the spherical harmonics $Y_{\ell,m}$ with $\ell = n - 1$ and integrating over the sphere gives

$$\int d\omega \, Y_{n-1,m} \left\{ \partial_u \left(\sum_{i=1}^{n+1} \alpha_n^{(i)} \, \psi^{(i)} \right) + \sum_{i=1}^{n-1} \beta_n^{(i)} \, \psi^{(i)} \right\} = 0, \tag{2.24}$$

where we have used the fact that

$$\int d\omega \, Y_{\ell,m} \, \Delta f = -\ell(\ell+1) \int d\omega \, Y_{\ell,m} \, f \tag{2.25}$$

for any function f. Hence, the only obstacle to obtaining an extra $(2n - 1)$ NP charges at each level n is the second set of terms. However, these terms can be iteratively rewritten to be of the same form as the first set of terms in (2.24), by making use of the lower-order equations $T_p = 0$, $p < n$. For example, we show how one can deal with the $\beta_n^{(n-1)} \psi^{(n-1)}$ term in equation (2.24). Multiplying equation $T_{n-1} = 0$ by $Y_{n-1,m}$ and integrating over the 2-sphere gives

$$\int d\omega \, Y_{n-1,m} \left\{ \partial_u \left(\sum_{i=1}^{n} \alpha_{n-1}^{(i)} \, \psi^{(i)} \right) + \sum_{i=1}^{n-1} \beta_{n-1}^{(i)} \, \psi^{(i)} \right\} = 0, \tag{2.26}$$

where $\beta_{n-1}^{(n-1)} = -2(n-1) \neq 0$. Therefore, this equation allows us to rewrite equation (2.24) as

$$\int d\omega \, Y_{n-1,m} \left\{ \partial_u \left(\sum_{i=1}^{n+1} \tilde{\alpha}_n^{(i)} \, \psi^{(i)} \right) + \sum_{i=1}^{n-2} \tilde{\beta}_n^{(i)} \, \psi^{(i)} \right\} = 0 \tag{2.27}$$

with $\tilde{\alpha}_n^{(i)}$, $\tilde{\beta}_n^{(i)}$ some functions of the metric components. Moreover, using equation $T_{n-2} = 0$, we can rewrite the $i = (n-2)$ term in the second set of terms above in terms of lower-order terms and other terms of the form $\partial_u(\ldots)$. Following this prescription down to equation $T_1 = 0$ gives a set of $(2n - 1)$ NP charges for each n:

$$\partial_u \, c_{n,m} = 0 \tag{2.28}$$

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where

$$c_{n,m} = \int d\omega \, Y_{n-1,m} \sum_{i=1}^{n+1} a_n^{(i)} \, \psi^{(i)} \tag{2.29}$$

for certain constants $a_n^{(i)}$ that are determined by implementing the above iterative procedure. Note that

$$a_n^{(n+1)} = 2n. \tag{2.30}$$

Therefore, unless $\psi^{(n+1)} = 0$ and $a_n^{(i)} \, \psi^{(i)} = 0$ for all $i < (n+1)$, which would not occur in a generic case, we have

$$c_{n,m} \neq 0. \tag{2.31}$$

2.2 Aretakis charge

In this section, we rederive the result [3, 5] that all extreme black holes admit a conserved quantity, associated with a massless scalar test field, on the horizon. Assuming that the horizon is Killing, one can introduce Gaussian null coordinates (v, r, x^I) in the vicinity of the horizon (the surface $r = 0$), such that the metric takes the form [16]

$$ds^2 = L(x)^2 \left[-r^2 F(r, x) dv^2 + 2dv dr \right] + \gamma_{IJ}(r, x) \left(dx^I - r \, h^I(r, x) dv \right) \left(dx^J - r \, h^J(r, x) dv \right), \tag{2.32}$$

with $F(r = 0, x^I) = 1$. The timelike Killing vector field is

$$k = \frac{\partial}{\partial v} \tag{2.33}$$

in these coordinates. Moreover, we are guaranteed the existence of a rotational Killing vector field [19] (see also [20]), which we take to be

$$m = \frac{\partial}{\partial \phi} \tag{2.34}$$

with $\phi \equiv x^1$. Furthermore, we assume that [4]

$$\lim_{r \to 0} r^{-1} h^2 = \tilde{h}^2(x^2). \tag{2.35}$$

Now consider a massless scalar field ψ on this background, well-behaved on the horizon, obeying

$$\Box_g \psi = \frac{1}{L^2 \sqrt{\gamma}} \partial_a \left(L^2 \sqrt{\gamma} \, g^{ab} \, \partial_b \psi \right) = 0, \tag{2.36}$$

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[4]In fact, we can assume the slightly weaker condition that $\lim_{r \to 0} h^2 = 0$.

where γ is the determinant of the 2-dimensional metric γ_{IJ}. Using the fact that

$$(L^2 g^{ab}) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & r^2 F & rh^J \\ 0 & rh^I & L^2 \gamma^{IJ} \end{pmatrix}, \tag{2.37}$$

where γ^{IJ} is the inverse of γ_{IJ}, equation (2.36) reduces *on the horizon $r = 0$* to

$$\frac{1}{\sqrt{\gamma}} \left\{ \partial_v \left[2\sqrt{\gamma}\partial_r \psi + \partial_r \sqrt{\gamma}\psi \right] + \partial_\phi (\sqrt{\gamma}\, h^\phi\, \psi) + \partial_I \left[\sqrt{\gamma} L^2 \gamma^{IJ} \partial_J \psi \right] \right\} = 0, \tag{2.38}$$

where we have used the fact that k and m are Killing vectors, and so the metric components do not depend on v and ϕ, as well as the fall-off condition (2.35). Integrating this equation over coordinates x^I with measure $\sqrt{\gamma}$, we identify the second and third set of terms, which do not involve v-derivatives, as total derivative terms. Since the horizon is compact we can disregard these terms, which leaves us with a conserved quantity, the Aretakis charge, on the horizon:

$$H_{Aretakis} = \lim_{r \to 0} \int dx^2 \sqrt{\gamma} \left[2\,\partial_r \psi + \tfrac{1}{2}\partial_r \log \gamma\, \psi \right]. \tag{2.39}$$

2.2.1   Example: Extreme Reissner-Nordström solution

The ERN metric can be put into the form (2.32) by redefining r in the standard metric

$$ds^2 = -\frac{(r-M)^2}{r^2}dv^2 + 2dvdr + r^2 d\Omega^2, \tag{2.40}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$, as follows

$$r \to M + M^2 r, \tag{2.41}$$

so that the horizon is now at $r = 0$.[5] This gives

$$ds^2 = M^2 \left[-\frac{r^2}{(1+Mr)^2}dv^2 + 2dvdr \right] + M^2(1+Mr)^2 d\Omega^2. \tag{2.42}$$

Comparing with metric (2.32), we have

$$L = M, \quad F = (1+Mr)^{-2}, \quad h^I = 0, \quad \gamma = M^4(1+Mr)^4 \sin^2\theta. \tag{2.43}$$

It is now simple to show, using equation (2.39), that the Aretakis charge for the ERN black hole [1,2] is [6]

$$H_{Aretakis} = 2M^2 \lim_{r \to 0} \int_{S^2} d\omega\, (\partial_r \psi + M\psi). \tag{2.44}$$

---

[5]Note that the new radial coordinate has dimensions of inverse length. This is done in order to be consistent with the assumed form of the metric in (2.32), where the function $F(r, x)$ is dimensionless.

[6]The form of the Aretakis charge in Refs. [1–3] is slightly different from that here. This is simply due to the different coordinate systems used. Here, we have shifted the horizon to $r = 0$.

3 Extreme black holes in STU supergravity

The existence of conserved charges was an unexpected property of extremal horizons. For the case of the ERN solution, at least, the existence of Aretakis charges has been successfully understand [6,12], in terms of NP charges, using the Couch-Torrence conformal isometry [13] that interchanges the horizon, on which Aretakis charges are defined, and future null infinity, on which NP charges are defined. Unfortunately, such a conformal isometry does not exist for general extreme black holes, such as the extreme Kerr black hole. We shall deal with the general case in the next section. For now, in this section, we present a slightly more involved Couch-Torrence-like conformal relation that takes a particular solution within a family of extreme solutions to a different solution within the family. For a sub-family of solutions among these examples, the conformal relation in fact enhances to an isometry.

3.1 The static 4-charge black holes

Here, we shall consider 4-charge static black holes in ungauged four-dimensional STU supergravity.[7] The relevant part of the bosonic Lagrangian is

$$\mathcal{L} = \sqrt{-g}\left[R - \tfrac{1}{2}(\partial\vec{\varphi})^2 - \tfrac{1}{4}\sum_{i=1}^{4} e^{\vec{a}_i \cdot \vec{\varphi}}\, F^{i\,\mu\nu}F^i_{\mu\nu}\right], \tag{3.1}$$

where the vectors $\vec{a}_i$ characterising the coupling of the three dilatons $\vec{\varphi}$ to the four gauge fields obey the relation

$$\vec{a}_i \cdot \vec{a}_j = 4\delta_{ij} - 1. \tag{3.2}$$

In a convenient choice of basis they are given by

$$\vec{a}_1 = (1,1,1), \quad \vec{a}_2 = (1,-1,-1), \quad \vec{a}_3 = (-1,1,-1), \quad \vec{a}_4 = (-1,-1,1). \tag{3.3}$$

The extreme static black-hole solutions are given by

$$ds^2 = -H^{-1/2}\,dt^2 + H^{1/2}\,(dr^2 + r^2 d\Omega^2), \qquad e^{-\frac{1}{2}\vec{a}_i \cdot \vec{\varphi}} = H_i\,H^{-1/4}, \tag{3.4}$$

$$A^i = (1 - H_i^{-1})\,dt,$$

where

$$H = \prod_{i=1}^{4} H_i, \qquad H_1 = 1 + \frac{q_i}{r}. \tag{3.5}$$

The charge parameters q_i should all be non-negative in order to have a black-hole solution with no naked singularity, and an event horizon at $r = 0$.

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[7]See also [21, 22].

We can define Eddington-Finkelstein coordinates u and v in the standard way:

$$u = t - r_* \,, \qquad v = t + r_* \,, \qquad r_* \equiv \int^r \sqrt{H(r')} \, dr' \,, \tag{3.6}$$

in terms of which the static black-hole metrics become

$$
\begin{aligned}
ds^2 &= -H^{-1/2} \, du^2 - 2dudr + H^{1/2} \, r^2 d\Omega^2 \,, \\
&= -H^{-1/2} \, dv^2 + 2dvdr + H^{1/2} \, r^2 d\Omega^2 \,.
\end{aligned}
\tag{3.7}
$$

The explicit expression for r_* is complicated in general.

The scalar wave equation $\Box \psi = 0$ becomes

$$
\begin{aligned}
0 &= -2\partial_r \partial_u \psi - \frac{1}{r^2 \, H^{1/2}} \, \partial_r \left(r^2 \, H^{1/2} \right) \partial_u \psi + \frac{1}{r^2 \, H^{1/2}} \partial_r (r^2 \partial_r \psi) + \frac{1}{r^2 \, H^{1/2}} \Delta \psi \,, \\
0 &= 2\partial_r \partial_v \psi + \frac{1}{r^2 \, H^{1/2}} \, \partial_r \left(r^2 \, H^{1/2} \right) \partial_v \psi + \frac{1}{r^2 \, H^{1/2}} \partial_r (r^2 \partial_r \psi) + \frac{1}{r^2 \, H^{1/2}} \Delta \psi \,,
\end{aligned}
\tag{3.8}
$$

where Δ is the scalar Laplacian on the unit 2-sphere.

3.2   Inversion and conformal symmetry

If we define a new radial coordinate

$$\tilde{r} = \frac{Q^2}{r} \,, \qquad \text{where } \; Q^4 = \prod_i q_i \,, \tag{3.9}$$

it is easy to see that the metric becomes

$$ds^2 = \frac{Q^2}{\tilde{r}^2} \, d\tilde{s}^2 \,, \tag{3.10}$$

where

$$d\tilde{s}^2 = -\widetilde{H}^{-1/2} \, dt^2 + \widetilde{H}^{1/2} \left(d\tilde{r}^2 + \tilde{r}^2 \, d\Omega^2 \right) , \tag{3.11}$$

$$\widetilde{H} = \prod_i \widetilde{H}_i \,, \qquad \text{where } \; \widetilde{H}_i = 1 + \frac{\tilde{q}_i}{\tilde{r}} \,, \qquad \tilde{q}_i = \frac{Q^2}{q_i} \,.$$

Thus the metric written in terms of the $\tilde{r}$ radial coordinate is conformally related to a metric of the original form, but with redefined charge parameters.

3.3   The pairwise-equal charge specialisation

Because the charge parameters in the metric $d\tilde{s}^2$ are different from those of the original metric ds^2, the inversion transformation (3.9) is not a conformal symmetry. However, it does become a conformal symmetry in the special case where the four electric charges are set pairwise equal, without loss of generality,

$$q_3 = q_1 \,, \qquad q_4 = q_2 \,, \tag{3.12}$$

since then we will have

$$Q^2 = q_1 \, q_2 \,, \qquad \tilde{q}_1 = q_2 \,, \qquad \tilde{q}_2 = q_1 \,, \tag{3.13}$$

and so the metric function $\widetilde{H}(\tilde{r})$ is equal to the original metric function $H(r)$. Note also that in the pairwise equal case the metric takes the very simple form, with no irrational functions,

$$ds^2_{\text{pwe}} = -\left[\left(1 + \frac{q_1}{r}\right)\left(1 + \frac{q_2}{r}\right)\right]^{-1} dt^2 + \left(1 + \frac{q_1}{r}\right)\left(1 + \frac{q_2}{r}\right)(dr^2 + r^2 d\Omega^2) \,. \tag{3.14}$$

Note that the ERN solution arises as the further specialisation where we set $q_1 = q_2 = q$ in (3.14) (after sending $r \to r - q$ to express the metric in the usual form).

The explicit expression for the r_* coordinate defined in (3.6), which is rather complicated in the general 4-charge case, becomes very simple in the case of pairwise-equal charges:

$$r_* = r + (q_1 + q_2) \log r - \frac{q_1 q_2}{r} \,. \tag{3.15}$$

The scalar wave equation, written in terms of advanced or retarded null coordinates as in (3.8), becomes quite simple also. In particular, if we expand in a power series in $1/r^n$ around infinity, or in r^n around the horizon,

$$\psi(r, u, \theta, \varphi) = \sum_{n \geq 1} \psi_n(u, \theta, \varphi) \, r^{-n} \,, \qquad \widetilde{\psi}(r, v, \theta, \varphi) = \sum_{n \geq 0} \widetilde{\psi}_n(v, \theta, \varphi) \, r^n \,, \tag{3.16}$$

then there will be just a fixed finite number of terms in the various NP or Aretakis charges. By contrast, in the general 4-charge solutions the number of terms will grow with n.

3.4 General static extreme STU black holes

We saw above that the static extreme STU black holes with four independent electric charges admit an inversion relation under which a given black hole maps into a conformally-transformed member of the same family with different values for the four charges. The charges of the original and the transformed black holes become the same in the special case where the four charges are taken to be pairwise equal.

The most general static black holes in STU supergravity have eight independent charges, with each of the four field strengths carrying both an electric and a magnetic charge. It is of interest to investigate whether an inversion relation of the kind we found for the extreme four-charge black holes extends to the most general eight-charge extreme black holes. We find that the answer is no. The eight-charge solutions are actually quite complicated (see, for example, [23]), but for our purposes it will suffice to consider the simpler case where only

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one of the four field strengths is turned on, and it carries independent electric and magnetic charges. For the purposes of describing this dyonic solution one can consistently truncate the STU theory to the so-called "Kaluza-Klein theory," that results from dimensionally reducing pure five-dimensional gravity to four dimensions.

The details of the extreme static Kaluza-Klein dyonic black hole are presented in appendix A, where we show that it does not map under inversion into a conformally related extreme dyonic black hole, except for special choices for the two charge parameters which then reduce to cases encompassed by our previous discussion. Of course, having established this result for the truncated system of the Kaluza-Klein solutions, it follows that there is also no inversion transformation relation generically for the eight-charge black holes.

Although our findings for the dyonic black holes are negative, the result does serve to highlight the fact that there is something rather special about the case where each field strength in the STU theory carries only a purely electric or a purely magnetic charge, as in the 4-charge solutions we discussed previously.

4   Correspondence between Aretakis and NP charges

In order to understand the existence of Aretakis charges on the horizon of general extreme black holes [3], we relate, via a conformal map, their geometry close to the horizon to the class of solutions that we introduced in section 2.1, namely *weakly asymptotically flat* space-times. Such space-times include asymptotically flat space-times as a subset.

The existence of conserved charges near null infinity is not so surprising given that we have asymptotic Killing isometries that give rise to conserved charges [10], such as those identified by Newman and Penrose [8]. If these NP charges are generated by an (asymptotically) conformal field, then they map on to the Aretakis charges on the extremal horizon via the conformal map that relates the two asymptotic space-times. We shall demonstrate this explicitly at the end of this section. However, before that we derive the conformal map.

As was explained in section 2.2, given an extreme black hole solution, we can always choose Gaussian null coordinates in an open patch in the neighbourhood of the horizon such that the metric takes the form given in (2.32), which we repeat here for convenience

$$ds^2 = L(x)^2\Big[-r^2 F(r,x)dv^2 + 2dvdr \Big] + \gamma_{IJ}(r,x)\Big(dx^I - r\,h^I(r,x)dv\Big)\Big(dx^J - r\,h^J(r,x)dv\Big).$$

(4.1)

We have Killing vectors

$$k = \partial/\partial v, \qquad m = \partial/\partial\phi \equiv \partial/\partial x^1. \tag{4.2}$$

Moreover, as before we assume that[8]

$$F(r,x) = 1 + \mathcal{O}(r), \quad h^\phi = \mathcal{O}(1), \quad h^2 = \mathcal{O}(r), \quad \gamma_{IJ} = \tilde{\gamma}_{IJ}(x) + \mathcal{O}(r), \tag{4.3}$$

where $\tilde{\gamma}_{IJ}$ is the metric on a compact space that is topologically spherical [19]. Now, we consider the coordinate change

$$v \to u, \qquad r \to \frac{1}{r}, \tag{4.4}$$

which gives

$$ds^2 = \frac{L^2}{r^2}\Big\{ - Fdu^2 - 2dudr + r^2 h_{IJ}(dx^I - C^I du)(dx^J - C^J du)\Big\}, \tag{4.5}$$

where

$$C^I = h^I/r, \qquad h_{IJ} = L^{-2}\gamma_{IJ}. \tag{4.6}$$

Using the fall-off conditions in (4.3)

$$F(r,x) = 1 + \mathcal{O}(1/r), \quad C^\phi = \mathcal{O}(1/r), \quad C^2 = \mathcal{O}(1/r^2), \quad h_{IJ} = \omega_{IJ} + \mathcal{O}(1/r) \tag{4.7}$$

with ω_{IJ} the metric on a topological sphere. We identify the conformally related metric in (4.5) as that of a weakly asymptotically flat space-time, with $\beta = 0$, as defined in section (2.1).

In summary, we have found that the metric in the neighbourhood of an extremal horizon is conformally related, see equation (4.5), to that of a weakly asymptotically flat metric. Thus, the Aretakis charge on the horizon generated by a conformal field maps on to the NP charge generated by that field at the future null infinity of the weakly asymptotically flat space-time associated with the extreme black hole via this conformal correspondence.

4.1 Massless scalars

More concretely, we take as an example the massless scalar field that was considered in section 2,

$$\Box_g\psi = 0. \tag{4.8}$$

---

[8]Apart from the leading order terms, we do not need to assume any analyticity properties for the metric functions. This seems like a reasonable assumption. For example, at null infinity, it is known that non-analytic terms may contribute at lower orders [24–26].

Assuming that the background extreme black hole, with metric g, is Ricci scalar-flat [9], the wave operator $\Box_g$ coincides with the conformally covariant wave operator

$$\mathcal{O}_g = \Box_g - \tfrac{1}{6}R(g). \tag{4.9}$$

In the previous discussion, we have identified a self-inverse diffeomorphism

$$\iota : (v, r, x^I) \longrightarrow \left(u = v, \ r' = \frac{1}{r}, \ x^I \right), \qquad \iota^{-1} : (u, r, x^I) \longrightarrow \left(v = u, \ r' = \frac{1}{r}, \ x^I \right), \tag{4.10}$$

such that given the metric g in Gaussian null coordinates obeys

$$(\iota^{-1})_*(g) = \Omega^2 \, \tilde{g}, \tag{4.11}$$

where

$$\Omega = L/r \tag{4.12}$$

and $\tilde{g}$ is the associated weakly asymptotically flat metric.

Now, suppose we have a solution of the wave equation on the associated background:

$$\Box_{\tilde{g}} \tilde{\psi} = 0. \tag{4.13}$$

Although the Ricci scalar $R(\tilde{g})$ will not generally be zero, the solution is nevertheless asymptotically Ricci-flat. Hence for large r, which is the region in which we are interested, we have

$$\mathcal{O}_{\tilde{g}} \tilde{\psi} \approx 0, \tag{4.14}$$

i.e. $\mathcal{O}_{\tilde{g}} \tilde{\psi}$ is asymptotically zero. Given such a solution $\tilde{\psi}$, the conformal covariance of the operator $\mathcal{O}$ implies that

$$\mathcal{O}_{\Omega^2 \tilde{g}} \psi = \Box_{\Omega^2 \tilde{g}} \psi = 0, \tag{4.15}$$

where

$$\psi = \Omega^{-1} \, \tilde{\psi} \tag{4.16}$$

and we have used the fact that $\Omega^2 \tilde{g}$ is Ricci scalar-flat. In conclusion, we have generated, using this map, a solution of the wave equation on the horizon of the original solution of interest,

$$\psi(v, r, x^I) = \frac{L}{r} \, \tilde{\psi} \left(v, \frac{1}{r}, x^I \right). \tag{4.17}$$

This relation can then be used to map the NP charge at null infinity of the dual spacetime to the Aretakis charge at the extremal horizon.

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[9] Any solution of the Einstein equation with $\Lambda = 0$ and traceless energy-momentum tensor $T_a{}^a = 0$ will have vanishing Ricci scalar. For backgrounds with non-vanishing Ricci scalar, one would naturally consider the conformally covariant wave operator.

4.2 Example: Extreme Reissner-Nordström solution

We demonstrate this correspondence explicitly for the ERN solution, without recourse to the special conformal isometry [13] that it exhibits. Starting with coordinates adapted to the extremal horizon, the metric takes the form (2.42),

$$ds^2 = M^2 \left[-\frac{r^2}{(1+Mr)^2} dv^2 + 2dvdr \right] + M^2(1+Mr)^2 d\Omega^2. \tag{4.18}$$

Now we let

$$v \to u, \qquad r \to \frac{1}{r}, \tag{4.19}$$

which gives

$$ds^2 = \frac{M^2}{r^2} \left\{ -\left(1 + \frac{M}{r}\right)^{-2} du^2 - 2dudr + r^2 \left(1 + \frac{M}{r}\right)^2 d\Omega^2 \right\}. \tag{4.20}$$

Hence, we identify the weakly asymptotically flat space-time dual to the ERN metric to be

$$ds^2 = -\left(1 + \frac{M}{r}\right)^{-2} du^2 - 2dudr + r^2 \left(1 + \frac{M}{r}\right)^2 d\Omega^2. \tag{4.21}$$

The determinant of this metric is given by

$$\sqrt{-g} = r^2 \left(1 + \frac{M}{r}\right)^2 \sin\theta, \tag{4.22}$$

from which we read off (compare with (2.9))

$$\zeta = \left(1 + \frac{M}{r}\right)^2. \tag{4.23}$$

Substituting this expression into equation (2.12) gives

$$H_{NP} = -2 \lim_{r \to \infty} \int d\omega \left[r^2 \partial_r(r\tilde{\psi}) - Mr\,\tilde{\psi} \right], \tag{4.24}$$

where $\tilde{\psi}$ solves the wave equation on the associated weakly asymptotically flat background given by metric (4.21). Assuming for convenience that for large r

$$\tilde{\psi} = \frac{\tilde{\psi}^{(1)}}{r} + \frac{\tilde{\psi}^{(2)}}{r^2} + \dots, \tag{4.25}$$

we find that

$$H_{NP} = 2 \int d\omega \left(\tilde{\psi}^{(2)} + M\tilde{\psi}^{(1)} \right). \tag{4.26}$$

Consider a solution ψ on the original ERN background, given by the metric (4.18), which we assume to take the form

$$\psi = \psi^{(0)} + r\psi^{(1)} + \dots \tag{4.27}$$

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near the horizon at $r = 0$. Equation (4.17) gives that

$$\psi(v, r, x^I) = \frac{M}{r} \tilde{\psi}(v, \frac{1}{r}, x^I).$$ (4.28)

Substituting the expansions of the scalar fields (4.25) and (4.27) gives

$$\psi^{(0)} = M\tilde{\psi}^{(1)}, \quad \psi^{(1)} = M\tilde{\psi}^{(2)}.$$ (4.29)

Hence we find

$$H_{NP} = \frac{2}{M} \int d\omega \left(\psi^{(1)} + M\psi^{(0)} \right).$$ (4.30)

Comparing this expression with the Aretakis charge (2.44), written in terms of the expansion of ψ (4.27), we find that

$$H_{Aretakis} = M^3 H_{NP}.$$ (4.31)

Thus, up to an irrelevant factor, we have identified the NP charge at null infinity of the associated weakly asymptotically flat solution with the Aretakis charge on the horizon of the ERN black hole.

5   Discussions

In this paper, we have found a correspondence between extreme black holes and weakly asymptotically flat space-times. These are space-times for which Bondi coordinates can be introduced. However, the metric functions fall off at a slower rate than that which asymptotic flatness would demand. Furthermore, the spatial sections of "future null infinity" are only required to be compact, rather than specifically S^2. In section 2.1, we found that weakly asymptotically flat space-times admit an NP charge at null infinity. What distinguishes asymptotically flat space-times in four dimensions is the existence of an enhanced symmetry group at null infinity—the BMS group. The existence of NP charges for the aformentioned more general space-times suggests that there cannot be a direct relation between NP charges and the BMS group. Another indication of this is that NP charges arise in higher-dimensional weakly asymptotically flat space-times, even though there is no BMS group in these cases.

Nevertheless, one might expect the BMS group to play a rôle in the existence of charges and the representations that they belong to. A question that remains unanswered, as far as we are aware, is whether NP charges have special additional features in space-times where the asymptotic symmetry group is enhanced, and if so how? We hope to return to this issue in the future.

Going back to the correspondence that we introduced in section 4, we demonstrated it explicitly for the ERN black hole in section 4.2. Of course, the existence of the Aretakis charges in terms of NP charges for the ERN black hole had already been explained in the literature. Surely, a more interesting example to consider would be the extreme Kerr(-Newman) black hole. The first step in this construction would be to construct Gaussian null coordinates in the neighbourhood of its horizon, such that the metric takes the form (2.32). Given how non-trivial it is to construct Bondi coordinates for the Kerr metric [27–29], we do not expect this to be an easy task. Nevertheless, it may be an illuminating one.

Acknowledgments

A   Inversion and the extreme dyonic Kaluza-Klein black holes

Here we present some details of the extreme static dyonic black hole in the truncation of STU supergravity to the Kaluza-Klein theory that results from the dimensional reduction of five-dimensional pure gravity on a circle. The four-dimensional theory is described by the Lagrangian

$$\mathcal{L} = \sqrt{-g}\left[R - \tfrac{1}{2}(\partial\varphi)^2 - \tfrac{1}{4}e^{-\sqrt{3}\varphi}\,F^2\right]. \tag{A.1}$$

The extreme static dyonic black hole is given by (see, for example, [30] for the details in the conventions we are using here)

$$ds^2 = -(H_1 H_2)^{-1/2}\,dt^2 + (H_1 H_2)^{1/2}\left(dr^2 + r^2\,d\Omega^2\right),$$

$$\varphi = \frac{\sqrt{3}}{2}\,\log\frac{H_2}{H_1},$$

$$H_1 = 1 + \frac{4Q^{2/3}\sqrt{P^{2/3}+Q^{2/3}}}{r} + \frac{8P^{2/3}\,Q^{4/3}}{r^2},$$

$$H_2 = 1 + \frac{4P^{2/3}\sqrt{P^{2/3}+Q^{2/3}}}{r} + \frac{8Q^{2/3}\,P^{4/3}}{r^2}, \tag{A.2}$$

where P and Q are the magnetic and electric charges. The gauge potential can be found also in [30], but we shall not need it here.

We now consider the inversion transformation under which we define a new radial coordinate

$$\tilde{r} = \frac{8PQ}{r}\,. \tag{A.3}$$

We then find that the metric (A.2) can be written as

$$ds^2 = \frac{8PQ}{\tilde{r}^2}\, d\tilde{s}^2\,, \tag{A.4}$$

where

$$d\tilde{s}^2 = -(\widetilde{H}_1\widetilde{H}_2)^{-1/2}\, dt^2 + (\widetilde{H}_1\widetilde{H}_2)^{1/2}\left(d\tilde{r}^2 + \tilde{r}^2\, d\Omega^2\right)\,, \tag{A.5}$$

with

$$\begin{aligned}
\widetilde{H}_1 &= 1 + \frac{4(PQ)^{1/3}\sqrt{P^{2/3}+Q^{2/3}}}{\tilde{r}} + \frac{8P^{4/3}\,Q^{2/3}}{\tilde{r}^2}\,,\\
\widetilde{H}_2 &= 1 + \frac{4(PQ)^{1/3}\sqrt{P^{2/3}+Q^{2/3}}}{\tilde{r}} + \frac{8Q^{4/3}\,P^{2/3}}{\tilde{r}^2}\,.
\end{aligned} \tag{A.6}$$

We see that for general P and Q there is no way to map the H_i functions into the $\widetilde{H}_i$ functions by choosing different parameter values. It can be done only if $P = 0$ or $Q = 0$ or $P = Q$. The first of these is just a single electric charge specialisation of the previous 4-charge case. The second is an equivalent single magnetic charge case. The third, although dyonic, is merely a duality rotation of the ERN black hole.

References

[1] S. Aretakis, "Stability and instability of extreme Reissner-Nordström black hole spacetimes for linear scalar perturbations I," *Comm. Math. Phys.* **307** (2011) 17–63, arXiv:1110.2007 [gr-qc].

[2] S. Aretakis, "Stability and instability of extreme Reissner-Nordström black hole spacetimes for linear scalar perturbations II," *Ann. Henri Poincaré* **12** (2011) 1491–1538, arXiv:1110.2009 [gr-qc].

[3] S. Aretakis, "Horizon instability of extremal black holes," *Adv. Theor. Math. Phys.* **19** (2015) 507–530, arXiv:1206.6598 [gr-qc].

[4] S. Aretakis, "A note on instabilities of extremal black holes under scalar perturbations from afar," *Class. Quant. Grav.* **30** (2013) 095010, arXiv:1212.1103 [gr-qc].

[5] J. Lucietti and H. S. Reall, "Gravitational instability of an extreme Kerr black hole," *Phys. Rev.* **D86** (2012) 104030, `arXiv:1208.1437 [gr-qc]`.

[6] J. Lucietti, K. Murata, H. S. Reall, and N. Tanahashi, "On the horizon instability of an extreme Reissner-Nordström black hole," *JHEP* **03** (2013) 035, `arXiv:1212.2557 [gr-qc]`.

[7] K. Murata, "Instability of higher dimensional extreme black holes," *Class. Quant. Grav.* **30** (2013) 075002, `arXiv:1211.6903 [gr-qc]`.

[8] E. T. Newman and R. Penrose, "New conservation laws for zero rest-mass fields in asymptotically flat space-time," *Proc. Roy. Soc. Lond.* **A305** (1968) 175–204.

[9] R. M. Wald, *General Relativity*. UCP, 1984.

[10] R. M. Wald and A. Zoupas, "A general definition of 'conserved quantities' in general relativity and other theories of gravity," *Phys. Rev.* **D61** (2000) 084027, `arXiv:gr-qc/9911095 [gr-qc]`.

[11] J. Goldberg, "Conservation laws, constants of the motion, and Hamiltonians," in *Topics in Mathematical Physics, General Relativity and Cosmology*, H. García-Compeán, B. Mielnik, M. Montesinos, and M. Przanowski, eds., p. 233. Aug., 2006.

[12] P. Bizon and H. Friedrich, "A remark about wave equations on the extreme Reissner-Nordström black hole exterior," *Class. Quant. Grav.* **30** (2013) 065001, `arXiv:1212.0729 [gr-qc]`.

[13] W. E. Couch and R. J. Torrence, "Conformal invariance under spatial inversion of extreme Reissner-Nordström black holes," *Gen. Rel. & Grav.* **16** (1984) no. 8, 789–792.

[14] P. T. Chrusciel, "Boundary conditions at spatial infinity from a Hamiltonian point of view," in *International School of Cosmology and Gravitation, 9th Course: Topological Properties and Global Structure of Space-Time Erice, Italy, May 12-22, 1985*, pp. 49–59. 1985. `arXiv:1312.0254 [gr-qc]`.

[15] P. Bizon and E. Malec, "On Witten's positive energy proof for weakly asymptotically flat space-times," *Class. Quant. Grav.* **3** (1986) L123.

[16] H. K. Kunduri, J. Lucietti, and H. S. Reall, "Near-horizon symmetries of extremal black holes," *Class. Quant. Grav.* **24** (2007) 4169–4190, `arXiv:0705.4214 [hep-th]`.

[17] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, "Gravitational waves in general relativity: 7. Waves from axisymmetric isolated systems," *Proc. Roy. Soc. Lond.* **A269** (1962) 21–52.

[18] R. K. Sachs, "Gravitational waves in general relativity: 8. Waves in asymptotically flat space-times," *Proc. Roy. Soc. Lond.* **A270** (1962) 103–126.

[19] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-time*. Cambridge University Press, 1975.

[20] S. Alexakis, A. D. Ionescu, and S. Klainerman, "Rigidity of stationary black holes with small angular momentum on the horizon," *Duke Math. J.* **163** (2014) no. 14, 2603–2615, `arXiv:1304.0487 [gr-qc]`.

[21] M. Cvetic and D. Youm, "Dyonic BPS saturated black holes of heterotic string on a six torus," *Phys. Rev.* **D53** (1996) 584–588, `arXiv:hep-th/9507090 [hep-th]`.

[22] M. Cvetic and A. A. Tseytlin, "Solitonic strings and BPS saturated dyonic black holes," *Phys. Rev.* **D53** (1996) 5619–5633, `arXiv:hep-th/9512031 [hep-th]`. [Erratum: Phys. Rev.D55,3907(1997)].

[23] D. D. K. Chow and G. Compère, "Black holes in N=8 supergravity from SO(4,4) hidden symmetries," *Phys. Rev.* **D90** (2014) no. 2, 025029, `arXiv:1404.2602 [hep-th]`.

[24] E. Newman and R. Penrose, "An approach to gravitational radiation by a method of spin coefficients," *J. Math. Phys.* **3** (1962) 566–578.

[25] T. Damour, "Analytical calculations of gravitational radiation," in *4th Marcel Grossmann meeting*, R. Ruffini, ed. 1985.

[26] D. Christodoulou, "The global initial value problem in general relativity," in *The 9th Marcel Grossmann meeting*, V. G. Gurzadyan, R. T. Jantzen, and R. Ruffini, eds., pp. 44–54. Dec., 2002.

[27] F. Pretorius and W. Israel, "Quasi-spherical light cones of the Kerr geometry," *Classical and Quantum Gravity* **15** (1998) no. 8, 2289.

[28] N. T. Bishop and L. R. Venter, "Kerr metric in Bondi-Sachs form," *Phys. Rev. D* **73** (Apr, 2006) 084023.

[29] S. J. Fletcher and A. W. C. Lun, "The Kerr spacetime in generalized Bondi-Sachs coordinates," *Classical and Quantum Gravity* **20** (2003) no. 19, 4153.

[30] H. Lu, Y. Pang, and C. N. Pope, "AdS dyonic black hole and its thermodynamics," *JHEP* **11** (2013) 033, arXiv:1307.6243 [hep-th].