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Phys. Rev. D **96**, 074020 — Published 17 October 2017

DOI: 10.1103/PhysRevD.96.074020

Elastic scattering of a quark from a color field: longitudinal momentum exchange

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Abstract

Perturbative QCD in the small Bjorken x limit can be formulated as an effective theory known as the Color Glass Condensate (CGC) formalism. The CGC formalism takes into account the dynamics of large gluon densities at small x and has been successfully applied to Deep Inelastic Scattering (DIS) and particle production in high energy hadronic and nuclear collisions in the small x kinematic region. The effective degrees of of freedom in CGC are Wilson lines which enter in the effective quark (and gluon) propagators and re-sum multiple soft scatterings from the small x gluon field of the target. It is however known that the CGC effective theory breaks down when one probes the moderately large x (high p_t) kinematics where collinear factorization and DGLAP evolution of parton distribution functions should be the right framework. Here we propose a general framework which may allow one to eventually unify the two approaches and to calculate pQCD cross sections in both small and large Bjorken x regions. We take the first step towards this goal by deriving an expression for the quark propagator in a background field which includes scatterings from both small and large x modes of the gluon field of the target. We describe how this quark propagator can be used to calculate QCD structure functions F_2 and F_L at all x and thus generalize the dipole model of DIS. We outline this approach can also be used to extend the so-called hybrid approach to particle production in the forward rapidity region of high energy hadronic and nuclear collisions to all x and p_t regions and speculate on how one may apply the same techniques to extend the McLerran-Venugopalan effective action used in high energy heavy ion collisions to include high p_t physics.

1 Introduction

Parton (gluon) distribution functions are known to grow very fast at small Bjorken x which would lead to a violation of the Froissart bound on growth of physical cross sections with energy. Gluon saturation was proposed [1, 2] as a dynamical mechanism which can tame this growth and restore perturbative unitarity. The Color Glass Condensate (CGC) formalism [3] is an effective action approach to gluon saturation in QCD that includes coherent multiple scatterings from the gluon fields of the target, which are essential when gluon density of the target is large, i.e. at small x. It also includes high energy effects by re-summing large logs of $1/x$ via the JIMWLK/BK evolution equation [4, 5] and as such it differs from the collinear factorization approach to particle production in pQCD where parton distribution functions evolve according to the DGLAP evolution equation [6].

The CGC formalism has been developed significantly since its inception both in terms of precision (higher order corrections) as well as the range of physical processes considered (see [7] for references). Perhaps the most important aspect of the CGC formalism is the emergence of a dynamically generated and potentially large scale, the saturation scale $Q(x, A, b_t)$, which allows one to understand a wide range of phenomena in high energy QCD by making controlled approximations and quantitatively reliable calculations.

Despite many successes of the CGC formalism, its kinematic domain of applicability remains limited; at best it may be applied to processes where the cross section is dominated by small x gluons (as a rough guide one can take small x to mean $x < 0.01$). Recalling that in particle production in high energy hadronic/nuclear collisions x and p_t are kinematically correlated, the dominant contribution to high p_t processes is from the not so small x region. Therefore high p_t particle production is currently not computable in CGC formalism where one must have $\log 1/x \gg \log p_t^2$. Furthermore it's been realized quite recently that CGC calculations of higher order corrections to particle production cross sections are not stable and can become negative [8] at transverse momenta slightly higher than the saturation scale $Q_s(x)$. While there has been various remedies proposed to cure this problem [8] in case of single inclusive hadron production in asymmetric collisions (such as proton-proton and proton-nucleus collisions in the forward rapidity region in the hybrid approach [9]), the general situation remains unsatisfactory and is likely an indication that at high p_t important physics is missing. One such deficiency is the physics of DGLAP evolution which re-sums potentially large logs of p_t^2 in parton distribution functions in the context of collinearly factorized cross sections in pQCD in the leading twist approximation. This is the dominant effect at high p_t where $\log p_t^2 \gg \log 1/x$ and is not included in the CGC formalism. The two approaches coincide when $\log p_t^2 \sim \log 1/x$, known as the double log limit.

Ideally one would like to have a unified formalism for particle production in QCD which reduces to collinear factorization and DGLAP evolution equation at intermediate/large x and/or p_t and to CGC and JIMWLK equation at small x, p_t (see [10] for instance). In collinear factorization/DGLAP formalism quarks and gluons and their distribution functions are the degrees of freedom used while in the small x limit it is most efficient to use (fundamental or adjoint) Wilson lines, path ordered exponentials of gluon field which re-sum coherent multiple scatterings of a quark or gluon on a soft color field. This Wilson line is very closely connected to the quark/gluon propagator in a background color field [11]. One may then naturally ask, what would be the effective degrees of freedom in a unified approach? In other words, what could be the building blocks of a physical cross section in a more general approach which has both small x and large x combined?

In the small x limit one makes a drastic approximation by taking the $x \to 0$ limit, for example, in parton splitting functions whereas in DGLAP approach the full splitting function is kept. Clearly to have any hope of a unified approach, one must relax the $x \to 0$ limit. As x is the ratio of longitudinal momenta or energies of the partons in the parent proton/nucleus, this means that one must consider scattering of a projectile parton not only from the small x modes of the target proton/nucleus but also from the more energetic modes at large x. Furthermore, DGLAP is a leading twist formalism so that multiple exchanges that involve extra powers of the hard scale are dropped. Keeping these two points in mind, we propose a new approach which may enable us to eventually "unify" the two formalisms such that one recovers the CGC formalism and JIMWLK evolution equation at small x and pQCD collinear factorization and DGLAP evolution equation at large x.

Toward this end we consider here scattering of a quark from a background color field which is, unlike in the small x limit, allowed to have both large and small x modes. We re-sum the multiple scatterings from the fields with small x modes to all orders while keeping only one scattering from large x modes of the field. Therefore the scattered quark can in general carry any energy and transverse momentum and be deflected by large angles, unlike the small x limit where the scattering is eikonal (energy of the quark remains the same) and limited to small transverse momentum exchanges. From the calculated quark scattering amplitude we extract the effective quark propagator in this more general background field. We show how this new quark propagator can be used to generalize the dipole model of the QCD structure functions F_2, F_L at small x. We expect that the resulting expressions would form the starting point for calculating the one-loop correction to the structure functions which would then result in a more general evolution equation containing JIMWLK evolution at small x and DGLAP evolution at intermediate/large x. We outline how this new approach may be used to generalize the CGC framework to include particle production at high p_t.

2 Multiple scattering at small x: eikonal approximation

In this section we briefly review multiple scattering in the eikonal limit and its use in small x physics. There are already excellent reviews covering eikonal scattering [12] so everything in this section is already well-known. Therefore here we use the example of eikonal scattering to remind the reader of the approximations and assumptions made at small x, and to set up our formalism. In this section we closely follow the approach of Casalderrey-Solana and Salgado in section 4.1 of [12] (see also section 4 of the first reference in [11]).

The light cone coordinates are defined as

$$x^+ \equiv \frac{t+z}{\sqrt{2}}\ ,\ x^- \equiv \frac{t-z}{\sqrt{2}} \tag{1}$$

and similarly for momenta and fields. We start by considering scattering of a high energy quark moving in the positive z direction, on a high energy target (proton/nucleus) moving to the left as shown in Fig. (1). The quark has momentum $p^\mu = (p^+ \sim \sqrt{s},\, p^- = 0,\, p_t = 0)$ whereas the target has momentum $P^\mu = (P^+ = 0,\, P^- \sim \sqrt{s},\, P_t = 0)$ even though what's really meant in the eikonal approximation is that $p^+ \gg p^-, p_t$ and likewise for the target momentum.

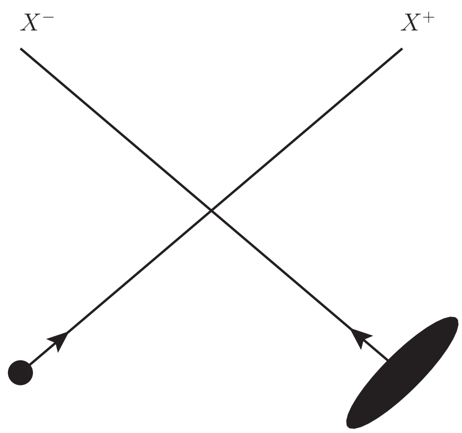

Figure 1: *Quark scattering on a target nucleus or proton*

In the CGC formalism one thinks of the target as a current of color charges which has only one large component, $J^\mu_a \simeq \delta^{\mu -}\rho_a$. The color current is covariantly conserved and satisfies

$$D_\mu\, J^\mu = D_-\, J^- = 0 \tag{2}$$

If we now choose to work in the light cone gauge $A^+ = 0$, we get $\partial_- J^- = 0$ which means the color current (charge) is independent of the coordinate x^-,

$$J^- = J^-(x^+, x_t) \tag{3}$$

To find the color fields generated by this color current one can solve the classical equations of motion

$$D_\mu\, F^{\mu\nu} = J^\nu \tag{4}$$

in the light cone gauge $A^+ = 0$ and find the classical solution A^μ in terms of the color charge ρ. The only non-zero component of the field is A^- which is independent of the coordinate x^-,

$$A^- = A^-(x^+, x_t) \tag{5}$$

so that in momentum space

$$A^- = A^-(p^+ \sim 0, p^-, p_t). \tag{6}$$

This is important in eikonal scattering, the target field carries almost no plus momentum so that it can not impart any plus momentum to the projectile. As a result the plus momentum of the projectile is conserved as we will see in detail below. Also related, the small x momentum modes of the target (moving in the negative z direction) carry negligible P^- momentum and therefore can not impart any sizable minus component of momentum to the projectile. So basically the only momentum exchanged in the eikonal limit is small transverse momenta.

An equivalent way to understand this to consider the coupling of the gauge field A^μ to the fermion current $\bar{u}(p_f)\,\gamma_\mu\,u(p_i)$. In the eikonal approximation one takes $p_f \simeq p_i$ and using $\bar{u}(p)\,\gamma_\mu\,u(p) \sim p_\mu$ we see that if the projectile quark has a large p^+ it couples only to A^- component of the gauge field in eikonal scattering. This is a general result; the quark moving along the " $+$ " direction in its light cone frame will couple only to the " $-$ " component of the filed in that frame. This is extremely important and will be used extensively here.

Clearly to include large x modes of the target one needs to allow exchange of longitudinal momentum which amounts to scattering of a quark not only from the small x modes of the target but also from the large x modes carrying a finite fraction of the target energy P^-. We will consider this in the next section. Therefore to study scattering of a quark on a target at small x we consider scattering (propagation) of a quark in a background field A^- which is independent of coordinate x^-. Each scattering costs a power of g so that if $A^- \sim 1/g$ then $g\,A^- \sim \mathcal{O}\,(1)$ and one needs to re-sum all such scatterings. Note that we haven't made any assumption about the x^+ dependence of the fields which in the extreme limit is usually taken to be a delta function due to Lorentz contraction of the target. Here we will keep the x^+ dependence general to allow for an extended target. To proceed it is useful to define a light-like vector n^μ which points in the x^- direction so that $n^\mu = (n^+ = 0, n^- = 1, n_t = 0)$ and $n^2 = 0$ and that $n \cdot A = 0$ defines our gauge with $\not{n} = \gamma^+$. Using this light-like vector one can also write $A^-_a(x^+, x_t) = n^-\,S_a(x^+, x_t)$ so that $\not{A} = \not{n}\,S = S\,\not{n}$. So basically the Lorentz index of the field A^- is carried by the vector n^μ.

To start we consider multiple scattering of a quark on the target color fields one scattering at a time; momentum of the incoming quark is denoted p, while q is its outgoing momentum. To make the approximations made clear (and so that we don't have to repeat the details in the next section) we will show all the details of the calculation here. The amplitude for one scattering is shown in Fig. (2) where solid line denotes a quark while the dashed line denotes the soft gluon field of the target at coordinate $x_1^\mu = (x_1^+, x_{1t})$. The target is represented by point x_1 and not shown for

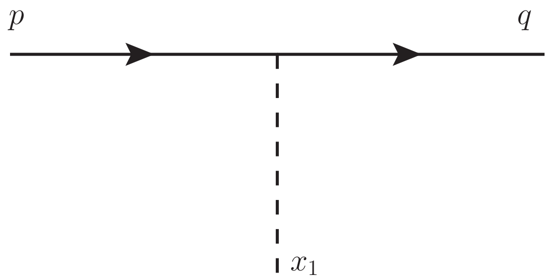

Figure 2: *Scattering of a quark from a soft field at position x_1^μ. The target is not drawn explicitly for brevity.*

brevity. The amplitude is

$$\begin{aligned} i\mathcal{M}_1 &= (ig)\int d^4x_1\, e^{i(q-p)x_1}\, \bar{u}(q)\, \left[\not{n}\, S(x_1)\right]\, u(p) \\ &= (ig)(2\pi)\delta(p^+ - q^+)\int d^2x_{1t}\, dx_1^+\, e^{i(q^- - p^-)x_1^+}\, e^{-i(q_t - p_t)x_{1t}}\, \bar{u}(q)\, \left[\not{n}\, S(x_1^+, x_{1t})\right] u(p)\,. \end{aligned} \tag{7}$$

Since the field S does not depend on coordinate x^- the integration over x^- is trivial and gives an overall delta function $2\pi\delta(p^+ - q^+)$ so that the plus momentum of the quark is conserved. Including a second scattering is depicted in Fig. (3) and gives

$$i\mathcal{M}_2 = (ig)^2\int d^4x_1\, d^4x_2\, \int \frac{d^4p_1}{(2\pi)^4}\, e^{i(p_1-p)x_1}\, e^{i(q-p_1)x_2}\, \bar{u}(q)\, \left[\not{n}\, S(x_2)\, \frac{i\not{p}_1}{p_1^2 + i\epsilon}\, \not{n}\, S(x_1)\right]\, u(p)\,.$$

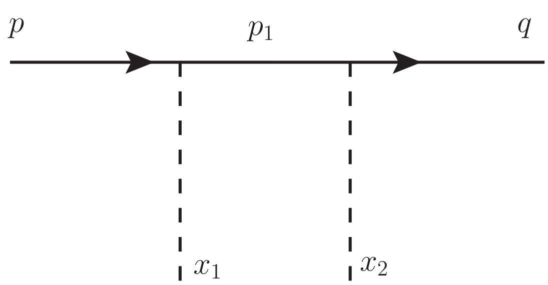

Figure 3: *scatterings of a quark on two soft fields.*

Once again one can perform the x_1^-, x_2^- integration giving two delta functions $2\pi\delta(p_1^+ - q^+)\, 2\pi\delta(q^+ - p_1^+)$ one of which can be used to perform the p_1^+ integration setting $p_1^+ = p^+ = q^+$ with an overall $2\pi\delta(p^+ - q^+)$ remaining. We get

$$\begin{aligned} i\mathcal{M}_2 &= (ig)^2\, 2\pi\delta(p^+ - q^+)\int d^2x_{1t}\, dx_1^+ d^2x_{2t}\, dx_2^+ \int \frac{d^2p_{1t}}{(2\pi)^2}\, \frac{dp_1^-}{(2\pi)}\, e^{i(p_1^- - p^-)x_1^+}\, e^{-i(p_{1t} - p_t)\cdot x_{1t}} \\ & \quad e^{i(q^- - p_1^-)x_2^+}\, e^{-i(q_t - p_{1t})\cdot x_{2t}}\, \bar{u}(q)\, \left[S(x_2^+, x_{2t})\, \not{n}\, \frac{i\not{p}_1}{p_1^2 + i\epsilon}\, \not{n}\, S(x_1^+, x_{1t})\right]\, u(p) \end{aligned} \tag{8}$$

and $p_1^+ = p^+ = q^+$ in the propagator. The next step is to perform the integral over p_1^-. This can be done via contour integration noticing that $\not{p}_1$ in the numerator is next to $\not{n}$ and that $\not{n}\, p_1^- \gamma^+ = p_1^- \not{n}\not{n} = 0$. Therefore the integration over p_1^- can be performed by closing the contour below the real axis and gives

$$\int \frac{dp_1^-}{(2\pi)} \frac{e^{ip_1^-(x_1^+ - x_2^+)}}{2p^+\left[p_1^- - \frac{p_{1t}^2 - i\epsilon}{2p^+}\right]} = \frac{-i}{2p^+}\, \theta(x_2^+ - x_1^+)\, e^{i\frac{p_{1t}^2}{2p^+}(x_1^+ - x_2^+)} \tag{9}$$

There are two points which we should mention here, first an incoming quark with positive p^+ forces ordering along the direction of motion x^+, enforced via the theta function, and second, in the strict eikonal approximation one disregards the exponential on the right hand side by using the fact that $\frac{p_{1t}}{p^+} \ll 1$. It is possible to go beyond this strict eikonal limit and keep these exponentials factors (for example see [12]), however we will not do that here. We then have

$$\begin{aligned} i\mathcal{M}_2 \;\; = \;\; & (ig)^2\, (-i)\, 2\pi\delta(p^+ - q^+) \int d^2x_{1t}\, dx_1^+ d^2x_{2t}\, dx_2^+\, \theta(x_2^+ - x_1^+)\, e^{-ip^- x_1^+}\, e^{iq^- x_2^+} \\ & \int \frac{d^2p_{1t}}{(2\pi)^2}\, e^{-i(p_{1t} - p_t)\cdot x_{1t}}\, e^{-i(q_t - p_{1t})\cdot x_{2t}}\, \bar{u}(q) \left[S(x_2^+, x_{2t})\, \not{n}\, \frac{i\not{p}_1}{2p^+}\, \not{n}\, S(x_1^+, x_{1t})\right] u(p) \end{aligned} \tag{10}$$

There are two further approximations we need to make before we get the final result, first, we note that $p^- = \frac{p_t^2}{2p^+} \ll 1$ and $q^- = \frac{q_t^2}{2q^+} \ll 1$ and one therefore can drop the exponential terms involving p^-, q^- above. More importantly, there is still a p_{1t} term in the intermediate propagator which we also neglect since it is of the form $\frac{p_{1t}}{p^+} \ll 1$ and can be dropped (it is possible to keep these terms, order by order, see [13] for example). After this last step, one can perform the integration over transverse momentum p_{1t} which gives

$$\int \frac{d^2p_{1t}}{(2\pi)^2}\, e^{-ip_{1t}\cdot(x_{1t} - x_{2t})} = (2\pi)^2 \delta^2(x_{1t} - x_{2t}) \tag{11}$$

and enables one to perform one of the transverse coordinate integrations to get

$$\begin{aligned} i\mathcal{M}_2 \;\; = \;\; & (ig)^2\, (-i)(i)\, 2\pi\delta(p^+ - q^+) \int dx_1^+\, dx_2^+\, \theta(x_2^+ - x_1^+) \int d^2x_{1t}\, e^{-i(q_t - p_t)\cdot x_{1t}} \\ & \bar{u}(q) \left[S(x_2^+, x_{1t})\, \not{n}\, \frac{\not{p}_1}{2p^+}\, \not{n}\, S(x_1^+, x_{1t})\right] u(p) \end{aligned} \tag{12}$$

where $\not{p}_1$ is now understood to be $p_1^+ \gamma^-$. The last step is to use $\not{n}\, \frac{\not{p}_1}{2p^+} \not{n} = \not{n}$ since $p_1^+ = p^+$. We note that the two small x fields S are at the same transverse coordinate x_{1t} which was made possible due to neglecting terms $\frac{p_{1t}}{p^+}$ in both the exponentials and in the intermediate propagator. The second point to be kept in mind is the ordering of the scatterings which was caused by positivity of p^+ of the incoming quark and that contour integration over the propagator pole puts the intermediate quark line on shell.

The approximations and steps needed to extend this procedure to include any number of scatterings from the small x fields S should now be clear; the energetic projectile moving in a given direction will couple to the component of the gluon field Lorentz-conjugate to that direction, for example $p^+ : A^-$. One neglects terms of the order $\frac{p_t}{p^+}$ everywhere and that the p^+ component of the projectile momentum is conserved. The generic diagram for n soft scatterings is shown in Fig. (4), The amplitude is then

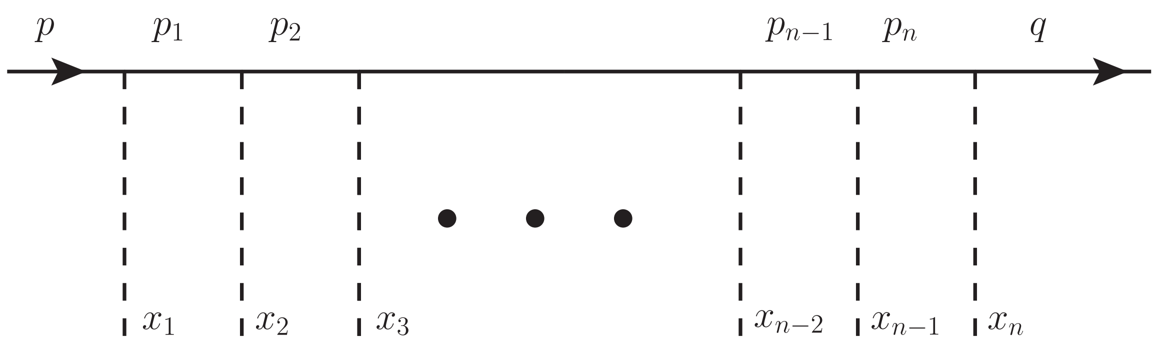

Figure 4: *N soft scatterings of a quark.*

$$\begin{aligned} i\mathcal{M}_n &= 2\pi\delta(p^+ - q^+)\,\bar{u}(q)\,\not{n} \int d^2x_t\, e^{-i(q_t-p_t)\cdot x_t} \\ &\quad \Big\{ (ig)^n\,(-i)^n (i)^n \int dx_1^+\, dx_2^+ \,\cdots dx_n^+\, \theta(x_n^+ - x_{n-1}^+)\,\cdots \theta(x_2^+ - x_1^+) \\ &\quad \big[S(x_n^+, x_t)\, S(x_{n-1}^+, x_t)\, \cdots S(x_2^+, x_t) S(x_1^+, x_t) \big] \Big\}\, u(p) \end{aligned} \tag{13}$$

which after summing over all n scatterings ($i\mathcal{M} = \sum_{n=1}^{\infty} i\mathcal{M}_n$) gives the standard result for quark-target scattering amplitude in the eikonal limit [14],

$$i\mathcal{M}(p,q) = 2\pi\delta(p^+ - q^+)\,\bar{u}(q)\,\not{n} \int d^2x_t\, e^{-i(q_t-p_t)\cdot x_t}\, [V(x_t) - 1]\; u(p) \tag{14}$$

where $V(x_t)$ is the Wilson line in the fundamental representation, defined as [1]

$$V(x_t) \equiv \hat{P}\, \exp\left\{ ig \int_{-\infty}^{+\infty} dx^+\, S_a^-(x^+, x_t)\, t_a \right\} \tag{15}$$

and $S^-(x^+, x_t) = n^-\, S(x^+, x_t)$ is our small x gluon field.

[1]Here we are being sloppy about path ordering vs anti-path ordering of the fields. The Wilson lines appearing in the propagators are path ordered, i.e., the soft fields increase in their x^+ argument from left to right, whereas the Wilson lines in the amplitudes are anti-path ordered. As introducing different symbols for path-ordered vs anti path-ordered will result in clutter and because it will not have any bearing on our result we will ignore this distinction.

One can extract an effective quark propagator using the result for scattering amplitude. To do so we recall that the incoming projectile was a quark with positive p^+ which is all we need in case of scattering of a physical particle. However, to construct a Feynman propagator one also needs to consider the case when a particle with $-p^+$ is moving backward (what we have computed so far would give us the retarded and not the Feynman propagator). This is practically trivial to implement since the only change in the derivation above is the locations of p^- poles of the intermediate propagators which will be above the real axis as compared to before where the poles were below the real axis. This reverses the ordering of projectile propagation in x^+ direction. It also results in a relative minus sign between the positive and negative p^+ contributions due to the reversal of the direction of the integration contour when the contour is closed above vs below the real axis.

Defining (recall that in calculating the amplitude one goes from right to left in a Feynman diagram while for the propagator we go from left to right)

$$\tau_F(p,q) \equiv 2\pi\delta(p^+ - q^+)\, \not{n} \int d^2x_t\, e^{-i(q_t - p_t)\cdot x_t}\, \left\{\theta(p^+)\, [V(x_t) - 1] - \theta(-p^+)\, \left[V^\dagger(x_t) - 1\right]\right\} \tag{16}$$

the quark propagator is then given by

$$S_F(p,q) = (2\pi)^4 \delta^4(p-q)\, S_F^0(p) + S_F^0(p)\, \tau_F(p,q)\, S_F^0(q) \tag{17}$$

where we have added the free propagator for completeness and the scattering amplitude is related to τ_F via

$$i\mathcal{M}(p,q) = \bar{u}(q)\, \tau_F(p,q)\, u(p)\,. \tag{18}$$

3 Multiple scattering at small and large x

It should be clear from the derivation in the last section that the target fields denoted by S have only small P^+ modes. Furthermore, since we are considering scattering from the small x fields of the target, they carry small fraction of the target energy P^- and therefore can not impart large P^- into the projectile. So the only momentum exchanged is transverse momentum which is $\mathcal{O}(\frac{p_t}{p^+})$. Therefore these soft multiple scatterings can not cause a big deflection of the projectile which continues to travel along a straight line in x^+ direction and remains at the same transverse coordinate x_t.

This fact allows one to re-sum the soft multiple scatterings into a Wilson line. Physical observables in DIS and the hybrid formulation of particle production in proton-proton and proton-nucleus collisions involve multi-point correlators of these Wilson lines; the most prominent one being the dipole scattering cross section which is the trace of two Wilson lines in the fundamental representation and satisfies the JIMWLK/BK evolution equation. In this sense Wilson lines are the effective degrees of freedom at small x.

Clearly to have any hope of merging the full DGLAP physics, i.e. exact splitting functions, with the JIMWLK/BK evolution which takes the small x limit of the splitting functions, one needs to relax the small x approximation and include scattering from the large x modes of the target. One would also need to figure out what the effective degrees of freedom are in this more general kinematics where large x modes of the target are included. This is our main goal in this paper. To this end we propose to go beyond scattering from small x fields by including one general field which can carry any momentum fractions x. Such a general field will clearly depend on all 4-coordinates and all of its components (subject to the gauge condition) will participate in the scattering. Therefore we consider scattering of a quark on a target when in addition to the small x fields one "all x" field participates. Since this general field will carry large (as well as small) P^-, it can impart a large p^- momentum to the projectile and cause a large angle deflection so that the scattered quark may have parametrically large transverse momentum ($p^- = \frac{p_t^2}{2p^+}$) and lose some (or even all) of its p^+ momentum (rapidity). We expect that we may not have to include more than one scattering from an all x field and that higher number of such scatterings would be suppressed by the coupling constant. In this paper we will ignore higher number of scatterings from the all x field but intend to come back to this point in the future as it may be relevant for gauge invariance at large x.

3.1 A hard scattering after multiple soft scatterings

We therefore consider scattering of a quark from an all x field which we denote $A^\mu = A^\mu(x^+, x^-, x_t)$, the single scattering amplitude is shown in Fig. (5). From now on we will refer to the all x field as hard and the small x fields as soft so that hard or soft refers to momentum fraction x carried by a field.

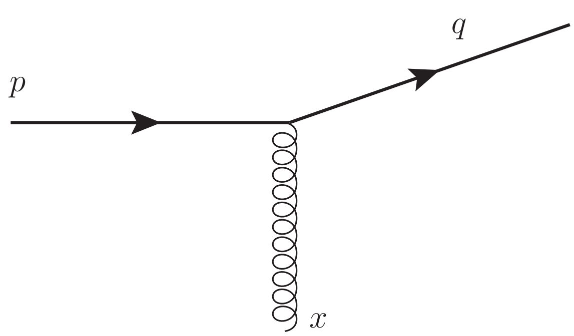

Figure 5: *One hard scattering of a quark.*

The result is

$$\begin{aligned} i\mathcal{M}_1 &= (ig) \int d^4x \, e^{i(q-p)x} \, \bar{u}(q) \, \left[\not{A}(x)\right] \, u(p) \\ &= (ig) \int d^4x \, e^{i(q-p)x} \, \bar{u}(q) \, \left[\not{A}(x) \, \frac{\not{p}}{2p^+} \, \not{n}\right] \, u(p) \end{aligned} \tag{19}$$

where we have used the Dirac equation to arrive at the second line. We can now successively include more scatterings from the small x fields S. Let us consider the case that all such soft scatterings happen before the hard scattering. Including one such soft scattering gives

$$i\mathcal{M}_2 = (ig)^2 \int d^4x\, d^4x_1 \int \frac{d^4p_1}{(2\pi)^4}\, e^{i(p_1-p)x_1}\, e^{i(q-p_1)x}\, \bar{u}(q) \left[\not{A}(x) \frac{i\not{p}_1}{p_1^2 + i\epsilon} \not{n}\, S(x_1) \right] u(p) \tag{20}$$

shown in Fig. (6).

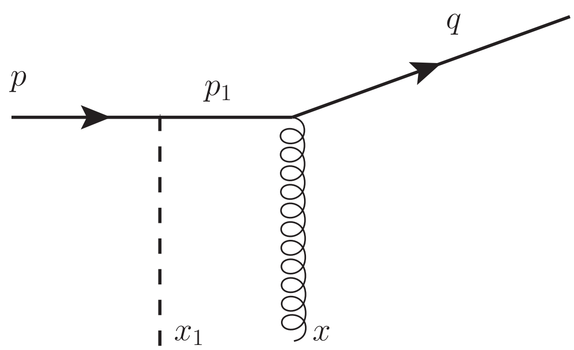

Figure 6: *One soft scattering before a hard one.*

As before we can perform the x_1^- integration since the soft field S does not depend on it, and use the resulting delta function to set $p_1^+ = p^+$. Next integration over p_1^- can be done using contour integration which gives

$$(-i)\, \theta(x^+ - x_1^+)\, \frac{1}{2p^+}\, e^{i\frac{p_{1t}^2}{2p^+}(x_1^+ - x^+)}\,. \tag{21}$$

We get

$$\begin{aligned} i\mathcal{M}_2 &= (ig)^2 \int d^4x\, d^2x_{1t}\, dx_1^+\, \theta(x^+ - x_1^+) \int \frac{d^2p_{1t}}{(2\pi)^2}\, e^{i(q-p_1)x}\, e^{-i(p_{1t}-p_t)\cdot x_{1t}} \\ &\quad \bar{u}(q) \left[\not{A}(x) \frac{\not{p}_1}{2p^+} \not{n}\, S(x_1^+, x_{1t}) \right] u(p) \end{aligned} \tag{22}$$

where $p_1^- = \frac{p_{1t}^2}{p^+}$ and $p_1^+ = p^+$. This procedure can now be repeated to include any number of soft scatterings which happen before the hard one. For example, inserting two soft scatterings before the hard one is shown in Fig. (7) and gives

$$\begin{aligned} i\mathcal{M}_3 &= (ig)^3 \int d^4x\, d^2x_{1t}\, dx_1^+\, dx_2^+\, \theta(x^+ - x_2^+)\, \theta(x_2^+ - x_1^+) \int \frac{d^2p_{2t}}{(2\pi)^2}\, e^{i(q-p_2)x}\, e^{-i(p_{2t}-p_{1t})\cdot x_{1t}} \\ &\quad \bar{u}(q) \left[\not{A}(x) \frac{\not{p}_2}{2p_2^+} \not{n}\, S(x_2^+, x_{1t}) \frac{\not{p}_1}{2p_1^+} \not{n}\, S(x_1^+, x_{1t}) \right] u(p) \end{aligned} \tag{23}$$

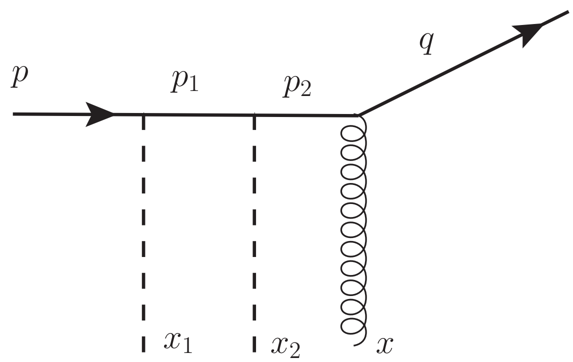

Figure 7: *Two soft scatterings before a hard one.*

where the soft fields are now at the same transverse coordinate as before. It should be now clear how to repeat this procedure by adding more and more soft scatterings. The amplitude for n soft scatterings followed by a hard one is shown in Fig. (8), After re-summing all the contributions from n soft scatterings followed by a hard one we get

$$i\mathcal{M} = (ig) \int d^4x\, d^2z_t \int \frac{d^2k_t}{(2\pi)^2}\, e^{i(q-k)x}\, e^{-i(k_t-p_t)\cdot z_t} \bar{u}(q) \left[\not{A}(x)\, \frac{\not{k}}{2k^+}\, \not{n}\, V(z_t, x^+) \right] u(p)\,. \tag{24}$$

where $k^+ = p^+, k^- \equiv \frac{k_t^2}{k^+}$ (momentum k is on shell) and the Wilson line now extends to x^+,

$$V(z_t, x^+) \equiv \hat{P}\, \exp\left\{ ig \int_{-\infty}^{x^+} dz^+\, S_a^-(z_t, z^+)\, t_a \right\} \tag{25}$$

and $S^-(z_t, z^+) = n^-\, S(z_t, z^+)$ as before.

This has a simple interpretation; the quark propagates through the soft field and scatters eikonally from the target starting from $z^+ = -\infty$ until the point x^+ where the hard scattering takes place. As before multiple soft scatterings do not change the transverse coordinate of the projectile quark and re-sum into a Wilson line.

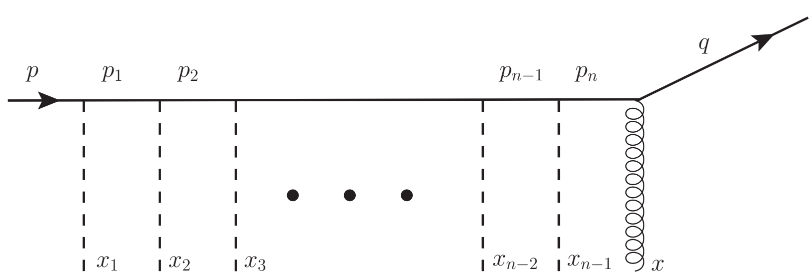

Figure 8: *N soft scatterings before a hard one.*

3.2 Multiple soft scatterings after a hard one

In the eikonal approximation used for the soft multiple scatterings the final state quark can gain some transverse momentum but not much as $p_t \ll p^+$. Furthermore its p^+ is conserved and does not change which also means it has negligibly small p^-. However, a hard scattering such as the one considered above can in principle impart large p^- and p_t to the projectile (and P^+, p_t to the target) so that after the hard scattering the quark will not necessarily propagate along its original trajectory but can be deflected by a large angle and lose some or even all of its p^+. This means we must take into account the possibility that the scattered quark is not moving along the original (positive) z direction but can now move in an arbitrary direction with projections on all x, y, z coordinates. Equivalently the scattered quark will have momenta p^+, p^-, p_t without any restriction on their magnitude (still subject to being on shell). For example, in the extreme case of back scattering $p^+ \to 0$ while p^- will be very large. Or in the case of scattering at right angle we will have $p^+ \sim p^- \sim p_t$. So it seems that we can not use our eikonal methods after the hard scattering any more.

However one can still our eikonal results *if we work in the light cone frame of the scattered quark* so that light cone frame of the scattered quark is rotated (in 3 dimensions) with respect to the light cone frame of the incoming quark. We define the direction of motion of the scattered quark to be the new positive z axis and label it as $\bar{z}$. The direction of propagation of the scattered quark $\bar{z}$ will now be related to it x, y, z in the original frame (defined by the incoming quark) via the standard rotation matrix $\mathcal{O}$ in 3 dimensions.

$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \mathcal{O} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{26}$$

The same rotation matrix will allow us to express the momentum vector of the scattered quark in the rotated frame in terms of quantities in the original frame. The rotation matrix $\mathcal{O}$ will depend on azimuthal and polar angles of the scattered quark with respect to the original frame which can be related to energy and rapidity of the final state quark.

In the new coordinate system the scattered quark is moving along positive $\bar{z}$ so that it will have a large $p_{\bar{z}}$ but no $p_{\bar{x}}$ or $p_{\bar{y}}$. However it will possibly have all momentum components when expressed in terms old frame quantities p_x, p_y, p_z. One can then define the new light cone coordinates $\bar{x}^+, \bar{x}^-, \bar{x}_t$ and light cone momenta $\bar{p}^+, \bar{p}^-, \bar{p}_t$ in the new frame in the standard way. To facilitate this we define a new light cone vector $\bar{n}$ which projects out the plus component in the new frame, so that $\bar{n} \cdot \bar{p} = \bar{p}^+$.

The scattered quark now moves along the new plus direction $\bar{x}^+$ and has only one large momentum component $\bar{p}^+$. Therefore it will couple only to the minus component of the field expressed in the new coordinates $\bar{S}^- = \bar{n}^- \, S(\bar{x}^+, \bar{x}_t)$. It is important to realize that even though only the $-$ component of the field in the new coordinate system is involved, the rotation matrix will generate the other Lorentz components if one expresses the soft field $\bar{S}^-$ in terms of the fields in the old coordinate system via $\bar{S}^\mu = \Lambda^\mu_{\ \nu} \, S^\nu$ where the only non-zero components of the Lorentz transformation matrix

$\Lambda^{\mu}{}_{\nu}$ are the elements of the 3-dimensional rotation sub-group.

To illustrate this we consider the case when there is a hard scattering first, followed by multiple soft scatterings. The amplitude for one soft scattering after the hard one, shown in Fig. (9), can be written as

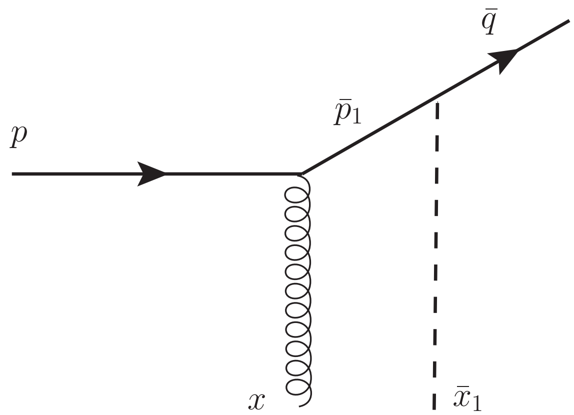

Figure 9: *One soft scattering after a hard one.*

$$i\mathcal{M}_2 = (ig)^2 \int d^4x\, d^4\bar{x}_1 \int \frac{d^4\bar{p}_1}{(2\pi)^4}\, e^{i(\bar{p}_1 - p)x}\, e^{i(\bar{q}-\bar{p}_1)\bar{x}_1}\, \bar{u}(\bar{q}) \left[\not{n}\, S(\bar{x}_1)\, \frac{i\not{\bar{p}}_1}{\bar{p}_1^2 + i\epsilon} \not{A}(x) \right] u(p)\,. \tag{27}$$

All the steps we went through in section (2) can now be repeated; the field $S(\bar{x}_1)$ is independent of $\bar{x}_1^-$ so that integration over $\bar{x}_1^-$ can be performed leading to a delta function of $2\pi\,\delta(\bar{p}_1^+ - \bar{q}^+)$ which allows one to perform the $\bar{p}_1^+$ integration setting $\bar{p}_1^+ = \bar{q}^+$. One then performs the contour integration over $\bar{p}_1^-$ noticing that it is next to $\not{n}$ which puts $\bar{p}_1$ on shell and gives a factor of $\theta(\bar{x}_1^+ - x^+)$, we get

$$\begin{aligned} i\mathcal{M}_2 &= (ig)^2 \int d^4x\, d^2\bar{x}_{1t}\, d\bar{x}_1^+\, \theta(\bar{x}_1^+ - x^+) \int \frac{d^2\bar{p}_{1t}}{(2\pi)^2}\, e^{i(\bar{p}_1 - p)x}\, e^{-i(\bar{q}_t - \bar{p}_{1t})\cdot\bar{x}_{1t}} \\ &\quad \bar{u}(\bar{q}) \left[\not{n}\, S(\bar{x}_1^+, \bar{x}_{1t})\, \frac{\not{\bar{p}}_1}{2\bar{p}_1^+} \not{A}(x) \right] u(p) \end{aligned} \tag{28}$$

with $\bar{p}_1^+ = \bar{q}^+$ and $\bar{p}_1^- = \frac{\bar{p}_{1t}^2}{2\bar{p}_1^+}$. By repeating this procedure for further soft scatterings as shown in Fig. (10), and re-summing them we get

$$i\mathcal{M} = (ig) \int d^4x\, d^2\bar{z}_t \int \frac{d^2\bar{k}_t}{(2\pi)^2}\, e^{i(\bar{k} - p)x}\, e^{-i(\bar{q}_t - \bar{k}_t)\cdot\bar{z}_t} \bar{u}(\bar{q}) \left[\overline{V}(x^+, \bar{z}_t)\, \not{n}\, \frac{\not{\bar{k}}}{2\bar{k}^+} \not{A}(x) \right] u(p) \tag{29}$$

where $\bar{k}_1^+ = \bar{q}^+$ and $\bar{k}_1^- = \frac{\bar{k}_t^2}{2\bar{k}_1^+}$ and the Wilson line is now

$$\overline{V}(x^+, \bar{z}_t) \equiv \hat{P}\, \exp\left\{ ig \int_{x^+}^{+\infty} d\bar{z}^+\, S_a^-(\bar{z}_t, \bar{z}^+)\, t_a \right\} \tag{30}$$

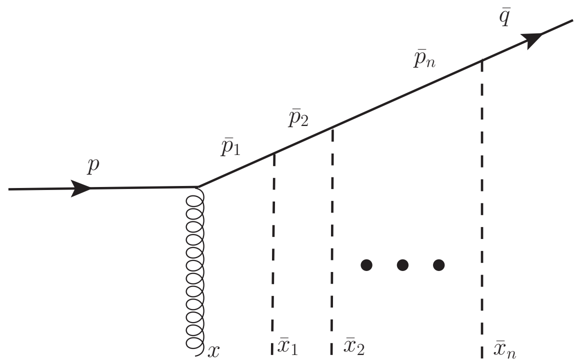

Figure 10: *N soft scatterings after a hard one.*

Again this expression has a simple interpretation; the projectile quark undergoes a hard scattering which changes its direction (possibly in all 3 dimensions) after which it undergoes multiple soft scatterings when expressed in terms of coordinates in the new frame.

3.3 Multiple soft scatterings before and after a hard scattering

We now have all the ingredients to write the general expression for scattering of a projectile quark from the target including soft (small x) scatterings to all order and hard (all x) to the first order. This is shown in Fig. (11) where the final state quark may now have a large q^- or q_t as measured in the incoming quark's frame (but has only a large momentum in $\bar{z}^+$ direction in the rotated frame).

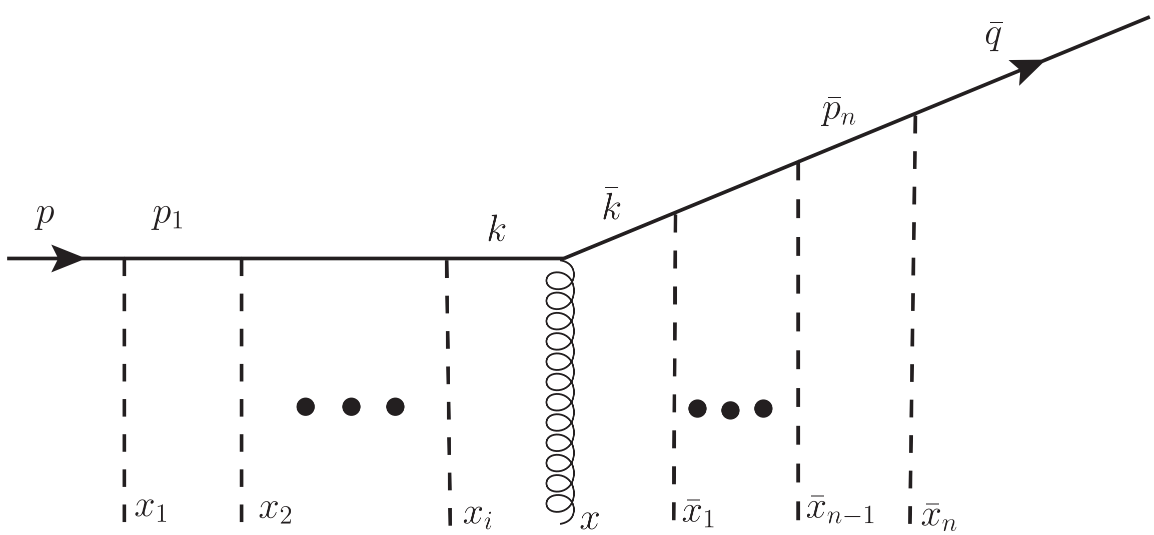

Figure 11: *N soft scatterings before and after a hard one at x.*

The procedure is the same as above, one considers multiple scatterings from the soft fields which are independent of the " − " coordinate leading to conservation of the " + " momentum across a soft scattering, and then doing the contour integration over the " − " component of the intermediate momentum which puts the quark line on shell. All the terms of the form $\frac{p_t}{p^+}\,x^+$ or $\frac{\bar{p}_t}{\bar{p}^+}\,\bar{x}^+$ are neglected which allows one to do the transverse coordinate integration over the soft fields. The only exception is if there is a "mis-match" of barred and un-barred momenta and coordinates in the exponentials which are then kept, for example exponents like $\bar{p}^-\,x^+$ may have large components when expressed fully in terms of quantities in the old frame. The rest of details are the same as before; during soft multiple scatterings the transverse coordinate of the quark remains the same and there is ordering in the direction of x^+ or $\bar{x}^+$ which allows one to re-sum the multiple scatterings into a Wilson line. As a result we have $x_{1t} = x_{2t} = \cdots = x_{it} = z_t$ and $\bar{x}_{1t} = \bar{x}_{2t} = \cdots = \bar{x}_{nt} = \bar{z}_t$. Therefore the full amplitude can be re-summed and written as

$$\begin{aligned} i\mathcal{M} \;&=\; (ig) \int d^4x\, d^2z_t\, d^2\bar{z}_t \int \frac{d^2k_t}{(2\pi)^2}\, \frac{d^2\bar{k}_t}{(2\pi)^2}\, e^{i(\bar{k}-k)x}\, e^{-i(\bar{q}_t-\bar{k}_t)\cdot\bar{z}_t}\, e^{-i(k_t-p_t)\cdot z_t} \\ &\bar{u}(\bar{q}) \left[\overline{V}(x^+,\bar{z}_t)\, \not{\bar{n}}\, \frac{\not{\bar{k}}}{2\bar{k}^+} \not{A}(x)\, \frac{\not{k}}{2k^+}\, \not{n}\, V(z_t,x^+)\right] u(p) \end{aligned} \tag{31}$$

with $k^+ = p^+, k^- = \frac{k_t^2}{2k^+}$ and $\bar{k}^+ = \bar{q}^+, \bar{k}^- = \frac{\bar{k}_t^2}{2\bar{k}^+}$.

As before the scattering amplitude can be written in terms of the interaction part of the propagator via

$$i\mathcal{M}(p,\bar{q}) = \bar{u}(\bar{q})\, \tau_F(p,\bar{q})\, u(p) \tag{32}$$

in terms of which the full (Feynman) propagator is

$$S_F(p,\bar{q}) = (2\pi)^4\delta^4(p-\bar{q})\, S_F^0(p) + S_F^0(p)\, \tau_{hard}(p,\bar{q})\, S_F^0(\bar{q}) \tag{33}$$

with interacting part of the propagator is then given by (including the contribution where quark is moving backward in x^+ which is needed for the Feynman propagator)

$$\begin{aligned} \tau_{hard}(p,\bar{q}) \;&\equiv\; (ig) \int d^4x \int \frac{d^2k_t}{(2\pi)^2}\, \frac{d^2\bar{k}_t}{(2\pi)^2}\, d^2z_t\, d^2\bar{z}_t\, e^{i(\bar{k}-k)x}\, e^{-i(\bar{q}_t-\bar{k}_t)\cdot\bar{z}_t}\, e^{-i(k_t-p_t)\cdot z_t} \\ &\Big\{\theta(p^+)\,\theta(\bar{q}^+)\, V(z_t,x^+)\, \not{n}\, \frac{\not{k}}{2k^+} \not{A}(x)\, \frac{\not{\bar{k}}}{2\bar{k}^+}\, \not{\bar{n}}\, \overline{V}(x^+,\bar{z}_t) - \\ &\theta(-p^+)\,\theta(-\bar{q}^+)\, V^\dagger(z_t,x^+)\, \not{n}\, \frac{\not{k}}{2k^+} \not{A}(x)\, \frac{\not{\bar{k}}}{2\bar{k}^+}\, \not{\bar{n}}\, \overline{V}^\dagger(x^+,\bar{z}_t)\Big\}. \end{aligned} \tag{34}$$

It is straightforward to check that this expression re-sums all the diagrams shown in Fig. (12) by expanding the Wilson lines. The dashed lines correspond to scatterings from the soft fields carrying small x fractions of the target energy whereas the wavy line corresponds to scattering from a hard field carrying arbitrary x fraction of the target energy. It describes propagation of an energetic quark along longitudinal direction z^+ and undergoing multiple soft interactions described by $V(z_t,x^+)$ until the location x^μ at which point it undergoes a potentially hard (large x) scattering which changes it direction. After the hard scattering it propagates again along the new longitudinal direction $\bar{z}^+$ and undergoing multiple scatterings described by $\overline{V}(x^+, \bar{z}_t)$. The location of the hard scattering x^μ is integrated from $-\infty$ to $+\infty$ (or from $-R$ to R where $2R$ is the diameter of the nucleus/size of the medium) so that the hard scattering can happen at any x^+, at the front, back or middle of the nucleus/medium, followed or preceded by multiple soft scatterings.

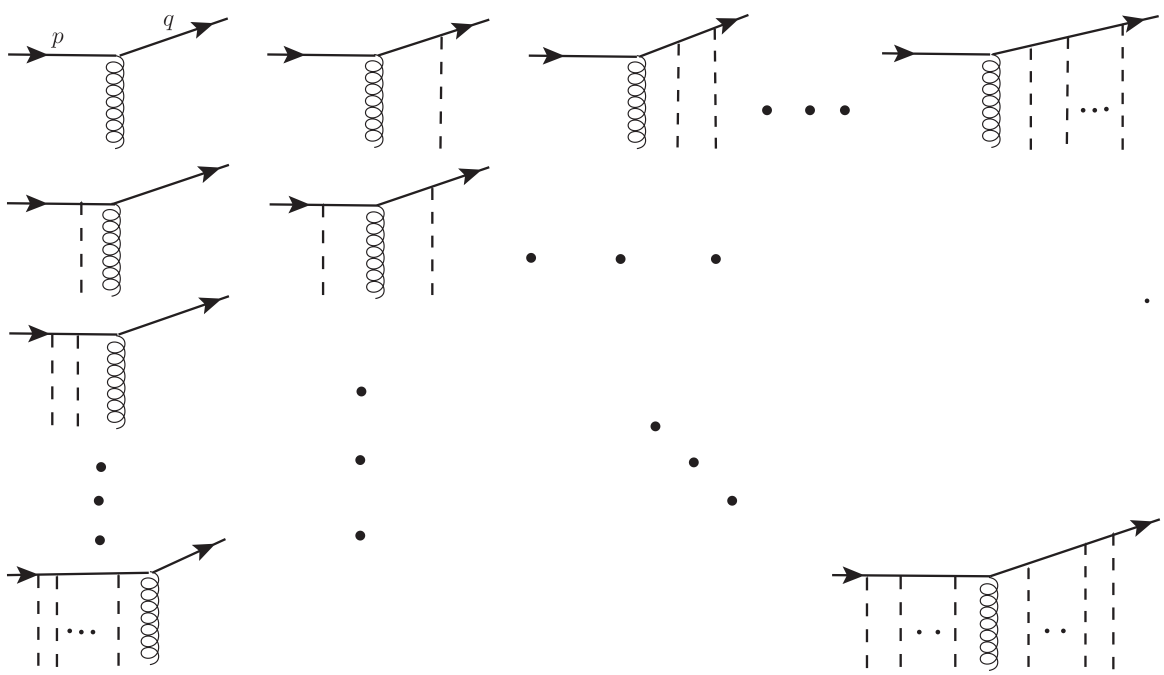

Figure 12: *Diagrams re-summed by eq. (34).*

There is one more technical but essential point that we need to address; the coupling between the quark and gluon fields we have considered is of the form $\bar{\Psi}\not{A}\Psi$ which is not gauge invariant. Instead one needs to use the full covariant coupling $\bar{\Psi}\not{D}\Psi$ in the gauge invariant Lagrangian. This does not matter in the eikonal limit since replacing any of the soft fields by the normal derivative gives a vanishingly small correction. However in our more general case it can act on the Wilson line in the rotated frame and give contributions which are of the same order as considered here. Therefore to have the interacting part of the effective propagator transform covariantly one needs to include it. To do so is trivial and amounts to replacing the hard field $\not{A}$ in our step by step derivation by the covariant derivative $\not{D}$. Therefore our expression in eq. (34)

is modified to

$$\tau_{hard}(p,\bar{q}) \equiv \int d^4x \int \frac{d^2k_t}{(2\pi)^2}\frac{d^2\bar{k}_t}{(2\pi)^2}\, d^2z_t\, d^2\bar{z}_t\, e^{i(\bar{k}-k)x}\, e^{-i(\bar{q}_t-\bar{k}_t)\cdot\bar{z}_t}\, e^{-i(k_t-p_t)\cdot z_t}$$
$$\Big\{\theta(p^+)\,\theta(\bar{q}^+)\,V(z_t,x^+)\,\not{n}\,\frac{\not{k}}{2k^+}\not{D}_x\,\frac{\not{\bar{k}}}{2\bar{k}^+}\,\not{\bar{n}}\,\overline{V}(x^+,\bar{z}_t) -$$
$$\theta(-p^+)\,\theta(-\bar{q}^+)\,V^\dagger(z_t,x^+)\,\not{n}\,\frac{\not{k}}{2k^+}\not{D}_x\,\frac{\not{\bar{k}}}{2\bar{k}^+}\,\not{\bar{n}}\,\overline{V}^\dagger(x^+,\bar{z}_t)\Big\}. \tag{35}$$

This is our final result for the interacting part of the quark propagator. This expression has the interesting property that it includes both partially and fully coherent multiple scatterings of the quark from the target and as such generalizes the fully coherent multiple scattering in eikonal approximation.

It is very interesting to consider the soft limit of this expression, i.e. the limit when the all x field $\not{A}$ carries a small x fraction of the target energy. To do so we need to replace $\not{A}(x^+,x^-,x_t) \to \not{n}\,S(x^+,x_t)$ and take all barred quantities to be un-barred, for example, $\not{\bar{n}} \to \not{n}$ and $\bar{k} \to k$. It is easy then to show eq. (35) vanishes in this limit due to action of the covariant derivative on a Wilson lines under a parallel transport along the direction of motion ($n\cdot D_x\, V(x^+,x_t) = 0$). This property of eq. (35) suggests that to have a quark propagator which can be used for construction of physical quantities such as structure functions F_2 and F_L valid for any x one should include both contributions from purely eikonal scattering as given by eq. (16) and hard scattering given by eq. (35). Therefore we take the most general interaction part of the effective propagator to be

$$\tau_{all\,x} \equiv \tau_F + \tau_{hard} \tag{36}$$

which is shown symbolically in Fig. (13) and where the thick solid lines denote a

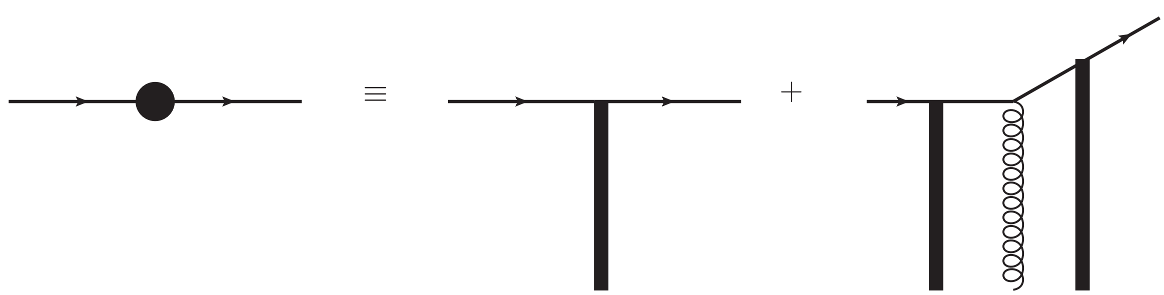

Figure 13: *Interaction part of the effective propagator in eq. (36).*

Wilson line representing multiple soft scatterings from the target to all orders.

It should be noted that while the projectile quark scatters from both low and high x modes of the target color fields, we have not included interactions between the different x modes. In this sense the small and large x fields are non-interacting with respect to each other. For instance it should be possible for a soft field to couple to a hard field as shown in Fig. (14)

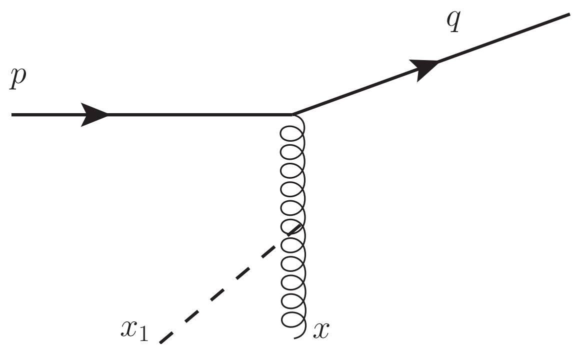

Figure 14: *Interaction of a soft field with a hard one.*

A preliminary study shows that the first coupling of the soft field to the hard field forces the two fields to be at the same position in coordinate space. Contributions of diagrams like this are currently under investigation and will be reported elsewhere.

4 Discussion and summary

We have constructed an effective quark propagator in the background of color fields which can carry both small and large x energy fractions of the target from which the quark scatters. This generalizes the known results for the quark propagator in the background of a color field carrying small x fraction of the target energy based on eikonal approximation.

The constructed effective propagator can be useful in many ways; first, it includes the possibility that a projectile quark can scatter from the more energetic (large x) partons of the target and get deflected by a large angle (high p_t). This may already give significant contributions to single inclusive particle production in high energy collisions [15] when one uses the hybrid formalism. It may also lead to significant final state momentum anisotropy at high p_t if one couples it to realistic nuclear geometries that include fluctuations in the target nucleus/medium [16].

Furthermore, one can use this propagator to calculate the QCD structure functions F_2 and F_L following the procedure suggested in [17]. This would generalize the standard dipole picture of DIS at small x by including the contributions of the large x gluons in the target. This work is in progress and will be reported elsewhere [18].

To proceed further, one can use this method to calculate the effective gluon propagator in a similar fashion. One can then generalize the hybrid formalism for particle production in asymmetric high energy proton-proton and proton-nucleus collisions and extend the validity of the hybrid formalism to high p_t, keeping in mind that high p_t in this context is equivalent to large x. Hence it would include contributions of both fully and partially coherent multiple scatterings.

It will be interesting to see if one can use our results to generalize and extend the McLerran-Venugopalan (small x) effective action [3,19] to include contributions of large

x gluons. If so one would then be able to apply the MV model to not only low [20] but also high p_t particle production in high energy heavy ion collisions. This would allow us to investigate and quantify the contributions of early time dynamics to jet energy loss in Quark-Gluon Plasma formed in high energy heavy ion collisions.

Perhaps most significantly, we hope the physical quantities computed using this effective propagator will serve as the Leading Order expressions which can then be used for deriving Leading Log evolution equations which would have DGLAP and JIMWLK evolution equations in appropriate limits [18].

Acknowledgments

We acknowledge support by the DOE Office of Nuclear Physics through Grant No. DE-FG02-09ER41620 and by the Idex Paris-Saclay though a Jean d'Alembert grant. We would like to thank N. Armesto, F. Gelis, E. Iancu, C. Lorcé, C. Marquet, A.H. Mueller, S. Munier, B. Pire, C. Salgado, G. Soyez and B. Xiao for critical questions, illuminating discussions and helpful suggestions.

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