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Atreya Chatterjee* and David A. Lowe†
Department of Physics, Brown University, Providence, RI 02912, USA

Global conformal invariance determines the form of two and three-point functions of quasi-primary operators in a conformal field theory, and generates nontrivial relations between terms in the operator product expansion. These ideas are generalized to the principal and complementary series representations, which play an important role in the conjectured dS/CFT correspondence. The conformal partial wave expansions are constructed for these representations which in turn determine the operator product expansion. This leads us to conclude that conformal field theories containing such representations have essential singularities, so cannot be realized as conventional field theories.

* atreya_chatterjee@brown.edu
† lowe@brown.edu

I. INTRODUCTION

The success of the anti-de Sitter/conformal field theory correspondence (AdS/CFT) has inspired applications to de Sitter spacetime (dS) [1]. This leads one to try to find conformal field theories of relevance to this correspondence which appear to exhibit novel properties, and many have questioned whether such theories can be defined at all.

In the context of AdS/CFT detailed dictionaries relating the bulk and boundary variables [2] were found at the free level. These ideas were generalized to a precision boundary/bulk correspondence, order by order in a $1/N$ expansion in HKLL [3–6]. Our present goal is to attempt to extend such ideas to the dS/CFT correspondence.

However, it has been difficult to produce examples in Minkowski space and De-Sitter space because of various issues. Instability of string theory in de-Sitter background [7], and the compact spacelike boundary has made holography challenging in de-Sitter [8]. We have discussed many of these problems, in our previous papers [9, 10]. There we have shown how to extend the HKLL dictionary to dS space for the case of non-interacting bulk theory. Many people have contributed in understanding dS-holography including higher-spin holography for dS[1, 11–20].

In section II we introduce principal series and discrete series representation and give a short description of the earlier work in maths literature. Then we show how the generators act on the bulk fields. We derive the bulk fields by solving the appropriate wave equation. Section III, is devoted to the massless scalar field. We find that modes of the massless scalar include both the discrete series and a limit of the complementary series, which is an indecomposable representation of the conformal group. This work makes contact with recent work by Ashtekar et al. [21, 22] on the asymptotic boundary conditions in de Sitter spacetime. In particular the discrete series modes carry vanishing energy, while the indecomposable mode can carry energy, but changes the conformal structure of the boundary. Both sets of operators are needed in the CFT to reproduce a complete set of bulk modes.

It is the main goal of the present paper to construct the operator product expansion in the conformal field theory for operators dual to massive modes in the bulk. As is usual in conformal field theory, the two and three-point functions of quasi-primary operators are determined by conformal invariance. However when we explore the implications of this for the operator product expansion, some surprising results emerge, including the fact that the expansion involves terms with arbitrarily rapid short distance singularities determined by a seemingly infinite number of free parameters. This is in constrast to the more ordinary CFTs appearing in the AdS/CFT correspondence, where the most singular terms in the operator product expansion are determined by the weights of the operators, and conformal invariance implies a single parameter determines the full set of descendent couplings via conformal partial waves. This leads us to conclude that such conformal field theories do not exist in the space of ordinary renormalizable quantum field theories, but rather share many of the features of non-renormalizable field theories. For concreteness, many of our results are stated for three-dimensional de Sitter spacetime. However since we only use the global conformal group, the results are easily generalized to higher dimensions.

II. PRINCIPAL, COMPLEMENTARY AND DISCRETE SERIES REPRESENTATIONS OF BULK STATES.

The isometries of 3-dimensional dS form the group $SO(1,3)$. This spacetime may be viewed as a hyperboloid embedded in 4-dimensional Minkowski spacetime. The generators are given by J_i, K_i for $i = 1,2,3$. J_i are the generators of rotation mixing three spacelike embedding dimensions. K_i are the generators of boost mixing three spacelike dimension with the timelike dimension. There are various Cartan sub-algebras of $SO(1,3)$. Depending on which Cartan subgroup we choose, we get a different basis for the representations. One can choose $SO(3) = \{J_i\}$ as the Cartan subgroup. Most papers in 1950-70 by Naimark, Tagirov, Chernikov, Raczka et al[23–27] do that. So mode functions were labelled by quantum numbers l, m (Eigenvalue of $\{J^2, J_3\}$ respectively). $SO(3)$ (compact group) has only finite dimensional representations $l = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, ..., m = -l, -l+1, ..., l$. So range of m is bounded for a given l.

On the CFT side, states are usually chosen as eigenstates of the $SU(1,1)$ Cartan sub-group. So it is useful to write the bulk generators $SO(1,3)$ as $SU_L(1,1) \otimes SU_R(1,1)$. (Just like $SL(2,C) \cong SU(1,1) \otimes SU(1,1)$.) Combine the generators in the following way

$$K_{1L} = \frac{1}{2}\left(-K_1 + iJ_1\right) \qquad K_{2L} = \tfrac{1}{2}\left(-K_2 + iJ_2\right) \qquad J_L = \frac{1}{2}\left(J_3 + iK_3\right)$$
$$K_{1R} = \frac{1}{2}\left(K_1 + iJ_1\right) \qquad K_{2R} = \tfrac{1}{2}\left(K_2 + iJ_2\right) \qquad J_R = \frac{1}{2}\left(J_3 - iK_3\right) .$$

Then

$$[J_{L(R)}, K_{1L(R)}] = iK_{2L(R)}$$
$$[J_{L(R)}, K_{2L(R)}] = -iK_{1L(R)}$$
$$[K_{1L(R)}, K_{2L(R)}] = -iJ_{L(R)} .$$

Left and right sectors commute. We can also form the raising and lowering operators $K_{\pm L(R)} = K_{1L(R)} \pm iK_{2L(R)}$.

$$[J_R, K_{\pm R}] = \pm K_{\pm R} \qquad [J_L, K_{\pm L}] = \pm K_{\pm L} \,.$$

Thus $\{J_{L(R)}, K_{\pm L(R)}\}$ form $SU_{L(R)}(1,1)$ group.

Now let us discuss unitary irreducible representations of $SU(1,1)$. States are labelled by eigenvalues of $\{C_L = J_L^2 - K_{1L}^2 - K_{2L}^2, J_L, J_R\}$

$$C_L|h,l\rangle = h(h-1)|h,l\rangle$$
$$J_L|h,l\rangle = l|h,l\rangle \,.$$

Irreducible representations split into discrete series and continuous series (principal and complementary series) [25, 26, 28–34]. In the discrete series $h = -n/2, n \in N$. $l = -h, -h+1, ...$ for positive discrete series D^+(lowest weight) and $l = h, h-1, ...$ for negative discrete series D^-(highest weight). For continuous series $h = -\frac{1}{2} + i\rho$, $0 < \rho < \infty$ and $l = 0, \pm 1, \pm 2, ...$ or $l = \pm\frac{1}{2}, \pm\frac{3}{2}, ...$ corresponding to C_ρ^0 or $C_\rho^{1/2}$ respectively.For complementary series $h = -\frac{1}{2} + \rho, 0 < \rho < \frac{1}{2}$ and $l = 0, \pm 1, \pm 2, ...$.

Similarly, $SU_R(1,1)$ sector can be constructed. For scalar fields $h_L = h_R = h$. Casimir of $SO(1,3)$ is then given by

$$\begin{aligned} C &= C_L + C_R \\ &= 2h(h-1) \,. \end{aligned}$$

For the discrete series

$$C = -\frac{1}{2}n(2-n),\, n \in N \,.$$

For the continuous series

$$C = -2\rho^2 - \frac{1}{2},\, 0 < \rho < \infty \,.$$

As we will see, some modes of the massless scalar correspond to the $n = 2$ discrete series. There $l = \pm 1, \pm 2, ...$ for $D^\pm$ respectively.

A. Action of the generators on the states

Let us write below action of all the generators on the state $|h, l, r\rangle$

$$J_R|h,l,r\rangle = r|h,l,r\rangle \tag{1}$$
$$J_L|h,l,r\rangle = l|h,l,r\rangle \tag{2}$$
$$C_L|h,l,r\rangle = h\,(h-1)\,|h,l,r\rangle$$
$$C_R|h,l,r\rangle = h\,(h-1)\,|h,l,r\rangle$$
$$C|h,l,r\rangle = (C_L + C_R)\,|h,l,r\rangle = 2h\,(h-1)\,|h,l,r\rangle \tag{3}$$
$$K_{\pm L}|h,l,r\rangle = i\,(\pm\,(h-1) - l)\,|j_L, l \pm 1, r\rangle \tag{4}$$
$$K_{\pm R}|h,l,r\rangle = i\,(\mp\,(h-1) - r)\,|j_L, l, r \pm 1\rangle \,. \tag{5}$$

For a scalar field of mass m, $4h\,(h-1) = m^2 \implies h_\pm = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{m^2}{4}}$ and $l(r) = 0, \pm 1, \pm 2, ...$ or $l(r) = \pm\frac{1}{2}, \pm\frac{3}{2}, ...$. The principal series corresponds to $m > 1$ and the complementary series corresponds to $1 > m > 0$. A component of the massless scalar behaves like a discrete series with $h = 1$. Figure 1 and 2 show the weight space diagram for principal series and discrete series. Similar weight space diagrams for representation in Anti-de Sitter space was given by Dusedau and Freedman[35].

B. States in coordinate space

Now we know how the generators act on the states. To explore bulk-boundary correspondence, we want to see how the states behave close to the boundary. It is convenient to transform to a basis of eigenstates in coordinate space.

De Sitter space can be described by the flat slicing coordinates $\eta, z, \bar{z}$

$$ds^2 = \frac{1}{\eta^2}\left(-d\eta^2 + dzd\bar{z}\right) \,.$$

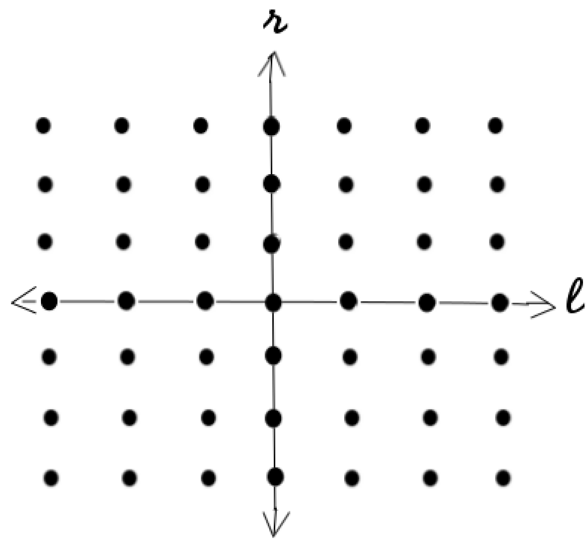

Figure 1. Weight space diagram for principal series. x-y axes are the l, r values. Solid dots represent states for all $l, r \in \mathbb{Z}$. These states have both growing and decaying modes. $K_{\pm L}$ shift the states right and left respectively. Similarly $K_{\pm R}$ shift the states up and down respectively.

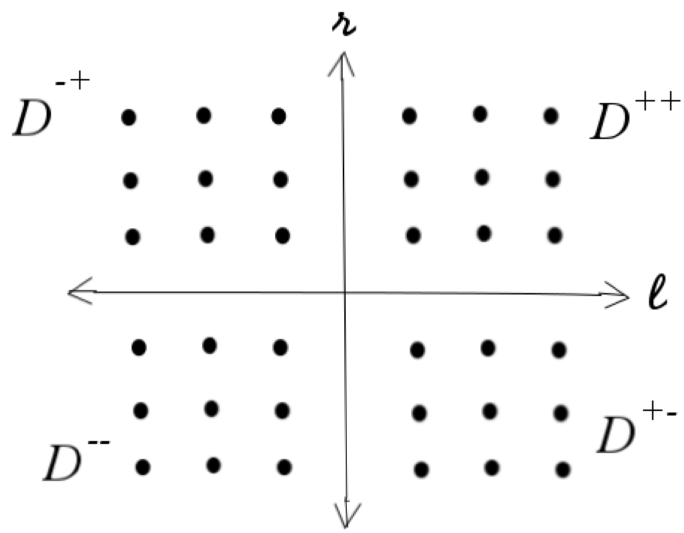

Figure 2. Weight space diagram for discrete series x-y axes are the l, r values. Solid dots represent states for all non zero l, r. These states contain only decaying modes. $K_{\pm L}$ shift the states right and left respectively. $K_{\pm L}$ annihilates $l = \mp 1$ states respectively. Similarly $K_{\pm R}$ shift the states up and down respectively. $K_{\pm R}$ annihilates $r = \mp 1$ states respectively.

There are many nice reviews of de-Sitter space [17]. z is complexified spacelike coordinate. η is timelike coordinate. De-Sitter has boundary at future and past infinity $\eta \to 0$. Bulk isometry generators are

$$J_L = z\partial_z + \frac{\eta}{2}\partial_\eta \quad , \quad K_{+L} = i\left(z^2\partial_z + \eta^2\partial_{\bar{z}} + z\eta\partial_\eta\right) \quad , \quad K_{-L} = -i\partial_z$$
$$J_R = -\bar{z}\partial_{\bar{z}} - \frac{\eta}{2}\partial_\eta \quad , \quad K_{-R} = -i\left(\bar{z}^2\partial_{\bar{z}} + \eta^2\partial_z + \bar{z}\eta\partial_\eta\right) \quad , \quad K_{+R} = i\partial_{\bar{z}}\,.$$

Note that if we put $\eta \to 0$ and $\eta\partial_\eta \to 2h$ as we approach the boundary then

$$\begin{array}{lll} J_L \to -L_0 & K_{+L} \to -iL_1 & K_{-L} \to iL_{-1} \\ J_R \to \bar{L}_0 & K_{+R} \to -i\bar{L}_{-1} & K_{-R} \to i\bar{L}_1 \end{array}$$

as shown in the appendix. Casimir operator is given by

$$C = C_L + C_R\,.$$

Simultaneous eigenstates of J_L, J_R, C with eigenvalues $l, r, \frac{m^2}{4}$ (m is mass) respectively, form the principal series representation. Solving the differential equations (1), (2) and (3) we get

$$\phi_{l,r}(z,\bar{z},\eta) = \left(\frac{z}{\bar{z}}\right)^{\frac{l+r}{2}} \eta^{l-r}\Big[A_1 i^{-l-r}\left(\frac{z\bar{z}}{\eta^2}\right)^{-\frac{l+r}{2}} {}_2F_1\left(\frac{1-2l-\sqrt{1-m^2}}{2}, \frac{1-2l+\sqrt{1-m^2}}{2}, 1-l-r, \frac{z\bar{z}}{\eta^2}\right)$$
$$+A_2 i^{l+r}\left(\frac{z\bar{z}}{\eta^2}\right)^{\frac{l+r}{2}} {}_2F_1\left(\frac{1+2r-\sqrt{1-m^2}}{2}, \frac{1+2r+\sqrt{1-m^2}}{2}, 1+l+r, \frac{z\bar{z}}{\eta^2}\right)\Big]. \tag{6}$$

Near the boundary ($\eta \to 0$) it behaves like

$$\phi_{l,r}(z,\bar{z},\eta\to 0) = \left(\frac{z}{\bar{z}}\right)^{\frac{l+r}{2}} \eta^{l-r}\Big[A_1 i^{-l-r}\left(\frac{z\bar{z}}{\eta^2}\right)^{-\frac{l+r}{2}}\left(a_1\left(\frac{\eta^2}{z\bar{z}}\right)^{-l+h_-} + a_2\left(\frac{\eta^2}{z\bar{z}}\right)^{-l+h_+}\right)$$
$$+A_2 i^{l+r}\left(\frac{z\bar{z}}{\eta^2}\right)^{\frac{l+r}{2}}\left(a_1\left(\frac{\eta^2}{z\bar{z}}\right)^{r+h_-} + a_2\left(\frac{\eta^2}{z\bar{z}}\right)^{r+h_+}\right)\Big]$$
$$= b_-\eta^{2h_-}\left(\frac{1}{z^{h_--l}\bar{z}^{h_-+r}}\right) + b_+\eta^{2h_+}\left(\frac{1}{z^{h_+-l}\bar{z}^{h_++r}}\right)$$
$$= b_-\eta^{2h_-}O_{l,r,h_-}(z,\bar{z}) + b_+\eta^{2h_+}O_{l,r,h_+}(z,\bar{z})$$

where $b_\pm$ are some constants and $h_\pm = \frac{1\pm\sqrt{1-m^2}}{2}$. Here $-\infty \le l, r \le \infty$. Another important thing to note is that $\phi_{l,r} \sim z^l \bar{z}^r$ (power law).

$$\phi_{l,r}(z,\bar{z},\eta\to 0) = \left(b_-\left(\frac{\eta^2}{z\bar{z}}\right)^{h_-} + b_+\left(\frac{\eta^2}{z\bar{z}}\right)^{h_+}\right) z^l\bar{z}^{-r}.$$

For principal series h_-, h_+ are complex conjugate of each other. So the modes oscillate close to the boundary. For complementary series $0 < h_- < \frac{1}{2} < h_+$. So half of the modes grow $\left(\eta^{2h_-}\right)$ and other half of the modes decay $\left(\eta^{2h_+}\right)$ near the boundary. They are respectively called growing and decaying mode. For massless case $h_- = 0$. So these modes stay constant near, hence are called constant modes.

III. MASSLESS SCALAR FIELD

Now we are going to look into the massless case. There are two ways that representations contribute to the massless scalar.

A. Limit of complementary series

One is the $m \to 0$ limit of equation (6). This is the limit of complementary series representation.

$$\phi_{l,r}(z,\bar{z},\eta) = \left(\frac{z}{\bar{z}}\right)^{\frac{l+r}{2}} \eta^{l-r}\Big[A_1 i^{-l-r}\left(\frac{z\bar{z}}{\eta^2}\right)^{-\frac{l+r}{2}} {}_2F_1\left(-l, 1-l, 1-l-r, \frac{z\bar{z}}{\eta^2}\right)$$
$$+A_2 i^{l+r}\left(\frac{z\bar{z}}{\eta^2}\right)^{\frac{l+r}{2}} {}_2F_1\left(r, 1+r, 1+l+r, \frac{z\bar{z}}{\eta^2}\right)\Big]. \tag{7}$$

Close to the boundary it goes like

$$\phi_{l,r}(z,\bar{z},\eta\to 0) = \left(b_- + b_+\frac{\eta^2}{z\bar{z}}\right) z^l\bar{z}^{-r}.$$

Note that, it has both the decaying mode and the constant mode.

B. Discrete series

Second is the Discrete series representation. There are two ways of deriving discrete series. Let us first see how it is derived in earlier math papers [23–25]. First find the eigenstates of $\Box|l,m\rangle = -m^2|l,m\rangle$ in the $|l,m\rangle$ basis (eigenstate of $\{J^2, J_3\}$). $\Box$ is second order differential equation and we get two independent solutions. Then choose only the decaying modes. This removes half of the solutions. This condition results in discrete eigenvalues ($-m^2$) of $\Box$. Hence the representation is called Discrete series. Note that this is in agreement with the previous section where we said that for massless discrete series $h = 1$.

Now let us derive the discrete series in another way. Diagonalize the Hilbert space in the eigenstates of J_L, J_R. In addition to equations (1), (2) and (3)(with $h = 1$) states have to satisfy equations

$$\begin{aligned} C_L|1,l,r\rangle &= 0 \\ C_R|1,l,r\rangle &= 0 \\ K_{\pm L}|1,\mp 1,r\rangle &= 0 \quad (8)\\ K_{\pm R}|1,l,\mp 1\rangle &= 0\,. \quad (9) \end{aligned}$$

There are four sectors as shown in figure 2. $D^{m_L m_R}$ where $l(r) = -1$ is the lowest weight and $l(r) = 1$ is the highest weight state. Thus(8) and (9). In this basis, highest and lowest weight states are manifest. This is over-constrained set of equations. Equation (8) and (9) are first order differential equation which has only one solution. As a result, half of the general solution of equation (3) is removed. We find that eigenstates decay near boundary. To see this consider the following states

$$\begin{aligned} \phi^D_{-1,r}(z,\bar z,\eta) &= A\left(\frac{z}{\eta}\right)^{r-1}\left(\frac{z\bar z}{\eta}-\eta\right)^{-1-r} \\ \phi^D_{1,r}(z,\bar z,\eta) &= A\bar z^{-\frac{r+1}{2}}\eta^2 \\ \phi^D_{l,1}(z,\bar z,\eta) &= A\left(\frac{\bar z}{\eta}\right)^{-l-1}\left(\frac{z\bar z}{\eta}-\eta\right)^{-1+l} \\ \phi^D_{l,-1}(z,\bar z,\eta) &= Az^{\frac{l-1}{2}}\eta^2\,. \end{aligned}$$

Note that close to boundary all the above solutions go like η^2. All other states can be obtained by acting with $K_{\pm L}, K_{\pm R}$. Since $K_{\pm L}, K_{\pm R}$ do not decrease the power of η, all the states will have same η dependence. Hence all the modes of the discrete series decay near the boundary. This also shows that $h = 1$. This suggests that these states are a linear combination of states found in the previous approach.

So either imposing regularity of the modes in the $|m,l\rangle$ basis is equivalent to requiring the existence of a highest weight state in $|1,l,r\rangle$ basis. It removes the half of the modes which stay constant near the boundary. On the other hand, the limit of the complementary series has both growing and decaying modes.

C. Discrete series cannot carry energy in dS.

Now that we have understood discrete series and limit of complementary series in more detail, what are the physical consequences? Does graviton belong to discrete series or complementary series? Ashtekar et al., in a series of papers [21, 22], has shown that gravity waves in de-Sitter cannot carry energy if the constant modes of the gravitons are removed. In light of this,

1. If the gravitons are described by discrete series then the constant modes are absent. Then gravity waves cannot carry energy.

2. If we want graviton modes to carry energy and a complete set of modes, the gravitons must contain modes from the limit of complementary series.

D. Indecomposiblity of limit of complementary series

In this section we will show that limit of complementary series is indecomposible. A representation is indecomposible[36] if it cannot be separated into two or more irreducible representations. We have already shown that decaying modes form the irreducible discrete series representation. Then the question is: Does the remaining constant mode also form irreducible representation?

To establish this we show that constant modes turn into decaying modes under the action of generators. Schematically,

$$\begin{aligned} K|decay\rangle &\to |decay\rangle \\ K|constant\rangle_{m_l\neq 0} &\to |constant\rangle \\ K|constant\rangle_{m_l=0} &\to |decay\rangle \end{aligned}$$

where K is some ladder operator. Equation (7) is the general solution of the massless scalar field. Schematically the two independent solutions are

$$\begin{aligned} \phi_{l,r}(z,\bar{z},\eta\to 0) &= \left(b_- + b_+\frac{\eta^2}{z\bar{z}}\right) z^l \bar{z}^{-r} \\ &= \left(b_-|constant\rangle + b_+\frac{|decay\rangle}{z\bar{z}}\right) z^l \bar{z}^{-r} \\ |decay\rangle_{\eta\to 0} &= \eta^2 \\ |constant\rangle_{\eta\to 0} &= 1 \end{aligned}$$

where b_-, b_+ are some constants. $|decay\rangle$ modes form the irreducible discrete series. They are either highest or lowest weight representations. This we have discussed in previous section.

To understand the issue let us see the general $|-1, l=0, r\rangle$ mode

$$\begin{aligned} \phi_{0,r}(z,\bar{z},\eta) = \left(\frac{z}{\bar{z}}\right)^{\frac{r}{2}} \eta^{-r}\Big[A_1 i^{-r}\left(\frac{z\bar{z}}{\eta^2}\right)^{-\frac{r}{2}} {}_2F_1\left(0,1,1-r,\frac{z\bar{z}}{\eta^2}\right) \\ +A_2 i^r \left(\frac{z\bar{z}}{\eta^2}\right)^{\frac{r}{2}} {}_2F_1\left(r,1+r,1+r,\frac{z\bar{z}}{\eta^2}\right)\Big] \\ = A_1 i^{-r}\bar{z}^{-r} + A_2 i^r \left(\frac{z}{\eta^2}\right)^r \left(1-\frac{z\bar{z}}{\eta^2}\right)^{-r}. \end{aligned}$$

We see that there is only a constant part. Now let us apply K_{-L}, K_{+L}

$$\begin{aligned} K_{-L}|0,r\rangle &= A_2 r i^{-1+r} z^{-1+r}\eta^{2r}\left(1-\frac{z\bar{z}}{\eta^2}\right)^{-r-1} = |-1,r\rangle_{decay} \\ K_{+L}|0,r\rangle &= A_1 r i^{-1-r}\bar{z}^{-1-r}\eta^2 = |1,r\rangle_{decay}\,. \end{aligned}$$

Thus we get only the decaying modes. This shows that the growing modes convert into decaying modes and proves that limit of complementary series is indecomposible representation. Figure 3 gives the weight space diagram for the limit of complementary series to illustrate this point.

IV. TRANSFORMATION FROM $|l,r\rangle$ BASIS TO MOMENTUM BASIS

In this section we derive the transformation from a momentum basis (eigenstate of L_{-1} operator) to the l,r basis (eigenstates of $L_0, \bar{L}_0$ operator with eigenvalue l,r respectively.). One reason to do it is that the scalar field in the bulk is generally written in momentum basis but boundary operators are generally expressed in l,r basis. Subsection (IV A) gives in detail the calculations for principal series. In subsection (IV B) we summarize the main results and compare the differences between the two representations.

A. Principal series

Momentum basis are eigenstates of $L_{-1} = -\partial_z, \bar{L}_{-1} = -\partial_{\bar{z}}$. l,r basis are eigenstate of $L_0 = -(z\partial_z + h), \bar{L}_0 = -\left(\bar{z}\partial_{\bar{z}} + \bar{h}\right)$ respectively. We want to find the coefficients $c_{k,l,r}$ of the relation

$$|k,\bar{k}\rangle = \sum c_{k,l,r}|l,r\rangle \tag{10}$$

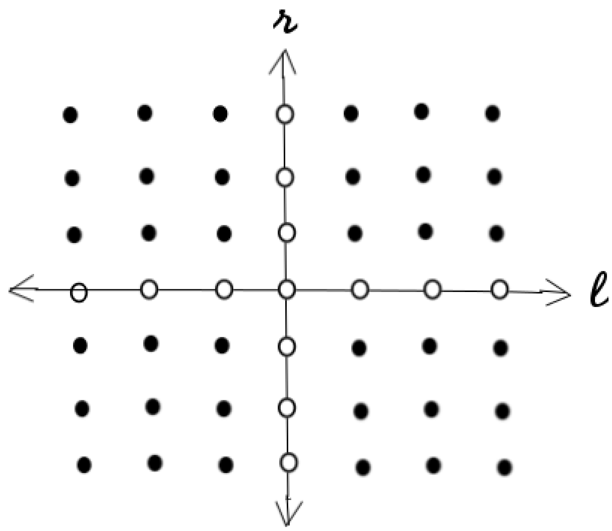

Figure 3. Weight space diagram for limit of complementary series. x-y axes are the l, r values. Solid dots represent states for all non-zero l, r. These states have both constant and decaying modes. Empty dots for $l = 0$ or $r = 0$ represent states which have only constant modes. $K_{\pm L}$ shift the states right and left respectively.$K_{\pm L}$ acting on constant modes of $l = 0$ states, convert them to decaying modes. Similarly $K_{\pm R}$ shift the states up and down respectively. $K_{\pm R}$ acting on constant modes of $r = 0$ states, convert them to decaying modes.

Our approach is similar to what Lindbad et al do in section (4A) of [28]. From commutation relation $[L_n, \phi_l] = ((h-1)\,n - l)\,\phi_{n+l}$ we get

$$L_0|l,r\rangle = -l|l,r\rangle \tag{11}$$
$$L_1|l,r\rangle = (h-1-l)\,|l+1,r\rangle \tag{12}$$
$$L_{-1}|l,r\rangle = (1-h-l)\,|l-1,r\rangle \tag{13}$$
$$L_{-1}|k,\bar{k}\rangle = \left(\frac{i\bar{k}}{2}\right)|k,\bar{k}\rangle\,. \tag{14}$$

To find $c_{k,l,r}$ we act with L_{-1} on both the side of equation (10)

$$\begin{aligned} L_{-1}|k,\bar{k}\rangle &= \sum c_{k,l,r} L_{-1}|l,r\rangle \\ \left(\frac{i\bar{k}}{2}\right)\langle l',r'|k,\bar{k}\rangle &= \sum c_{k,l,r}(-l+1-h)\langle l',r'|l-1,r\rangle \\ \left(\frac{i\bar{k}}{2}\right)c_{k,l',r'} &= c_{k,l'+1,r'}(-l'-h)\,. \end{aligned}$$

Solving the recurrence relation we get

$$\begin{aligned} c_{k,l,r} &= c_{k,l-1,r}\left(\frac{-i\bar{k}}{2}\right)\frac{1}{(h+l-1)} \\ \implies c_{k,l,r} &= \left(\frac{-i\bar{k}}{2}\right)^l \frac{h!}{(h+l-1)!}c_{k,0,r} \\ &= \left(\frac{-i\bar{k}}{2}\right)^l \frac{\Gamma(h+1)}{\Gamma(h+l)}c_{k,0,r} \\ &= \left(\frac{-i\bar{k}}{2}\right)^l \frac{\sin(\pi(h+l))}{\pi}\Gamma(h+1)\Gamma(1-h-l)c_{k,0,r} \\ &= \left(\frac{-i\bar{k}}{2}\right)^l \frac{\sin(\pi h)\cos(\pi l)+\cos(\pi h)\sin(\pi l)}{\pi}\Gamma(h+1)\Gamma(1-h-l)c_{k,0,r} \\ &= \left(\frac{i\bar{k}}{2}\right)^l \frac{\sin(\pi h)}{\pi}\Gamma(h+1)\Gamma(1-h-l)c_{k,0,r} \end{aligned}$$

In the third step, we have used the identity

$$\Gamma(h+l)\Gamma(1-h-l) = \frac{\pi}{\sin(\pi(h+l))}$$

Then, since l is integer

$$\sin(\pi l) = 0 \quad \cos(\pi l) = (-1)^l$$

Similarly one can derive $c_{k,0,r}$ by the action of $\bar{L}_{-1}$. Finally one gets

$$c_{k,l,r} = \left(\frac{ik}{2}\right)^r \left(\frac{i\bar{k}}{2}\right)^l \left(\frac{\sin(\pi h)}{\pi}\right)^2 \Gamma(h+1)^2\Gamma(1-h-l)\Gamma(1-h-r)c_{k,0,0}$$

We choose normalization $c_{k,0,0} = \left(\frac{\pi}{\sin(\pi h)(-2i)^h\Gamma(h+1)}\right)^2 \frac{i}{|k|}$. Plugging this back into equation (10) we get

$$|k,\bar{k}\rangle = \sum \left(\frac{ik}{2}\right)^{r-1/2} \left(\frac{i\bar{k}}{2}\right)^{l-1/2} (-2i)^{-2h-1}\Gamma(1-h-l)\Gamma(1-h-r)|l,r\rangle\,. \tag{15}$$

We can now invert equation (15).

$$|l,r\rangle = \frac{1}{(2\pi i)^2}\oint dk d\bar{k} \left(\frac{ik}{2}\right)^{-r-1/2} \left(\frac{i\bar{k}}{2}\right)^{-l-1/2} \frac{(-2)^{2h+1}}{\Gamma(1-h-l)\Gamma(1-h-r)}|k,\bar{k}\rangle\,. \tag{16}$$

One can check that this is consistent with equation (15). To see that start with the RHS of the above equation, substitute $|k,\bar{k}\rangle$ from equation (15) and we get the LHS of above equation

$$\begin{aligned}
&\frac{1}{(2\pi i)^2}\oint dk d\bar{k} \left(\frac{ik}{2}\right)^{-r'-1/2} \left(\frac{i\bar{k}}{2}\right)^{-l'-1/2} \frac{(-2i)^{2h-1}}{\Gamma(1-h-l')\Gamma(1-h-r')}|k,\bar{k}\rangle \\
&= \sum_{l.r} \frac{1}{(2\pi i)^2}\oint dk d\bar{k} \left(\frac{ik}{2}\right)^{-r'+r-1} \left(\frac{i\bar{k}}{2}\right)^{-l'+l-1} \frac{-\Gamma(1-h-l)\Gamma(1-h-r)}{4\Gamma(1-h-l')\Gamma(1-h-r')}|l,r\rangle \\
&= \sum_{l.r} \delta_{ll'}\delta_{rr'} \frac{\Gamma(1-h-l)\Gamma(1-h-r)}{\Gamma(1-h-l')\Gamma(1-h-r')}|l,r\rangle \\
&= |l',r'\rangle\,.
\end{aligned}$$

Now we know the basis transformations each way $|k,\bar{k}\rangle \leftrightarrow |l,r\rangle$, we can write this as a boundary operator/state correspondence as follows

$$O(z,\bar{z})|0\rangle = |z,\bar{z}\rangle = \sum_{l,r} \frac{1}{z^{h+l}\bar{z}^{h+r}}|l,r\rangle$$

$$O(z,\bar{z})|0\rangle = |z,\bar{z}\rangle = \oint \frac{dk d\bar{k}}{(2\pi i)^2} e^{\frac{i}{2}\left(k\bar{z}+\bar{k}z\right)}|k|^{2h-1}|k,\bar{k}\rangle\,.$$

So RHS of the above two equations must be equal. That is

$$\sum_{l,r} \frac{1}{z^{h+r}\bar{z}^{h+l}}|l,r\rangle = \oint \frac{dk d\bar{k}}{(2\pi i)^2} e^{\frac{i}{2}\left(k\bar{z}+\bar{k}z\right)}|k|^{2h-1}|k,\bar{k}\rangle\,. \tag{17}$$

To verify that, substitute $|l,r\rangle$ from equation (16) in LHS to get the RHS.

$$\begin{aligned}
|z,\bar{z}\rangle &= \sum_{l,r} \frac{1}{z^{h+l}\bar{z}^{h+r}}|l,r\rangle \\
&= \frac{1}{(2\pi i)^2}\oint dk d\bar{k} \sum_{l,r} \left(\frac{ik\bar{z}}{2}\right)^{-r-1/2} \left(\frac{i\bar{k}z}{2}\right)^{-l-1/2} \frac{(-2i)^{2h-1}}{\Gamma(1-h-l)\Gamma(1-h-r)}|k,\bar{k}\rangle \\
&= \frac{1}{(2\pi i)^2}\oint dk d\bar{k}|k|^{2h-1} \left(\sum_{l,r} \left(\frac{ik\bar{z}}{2}\right)^{-h-r} \left(\frac{i\bar{k}z}{2}\right)^{-h-l} \frac{1}{\Gamma(1-h-l)\Gamma(1-h-r)}\right)|k,\bar{k}\rangle \\
&= \frac{1}{(2\pi i)^2}\oint dk d\bar{k}|k|^{2h-1} e^{\frac{i}{2}\left(k\bar{z}+\bar{k}z\right)}|k,\bar{k}\rangle\,.
\end{aligned}$$

Here we have used the identity

$$e^z = \sum_{n \in Z} \frac{z^{h+n}}{\Gamma(h+n+1)} \, .$$

When h is integer, $\frac{1}{\Gamma(h+n+1)} = 0$ for $n < -h$. Thus

$$\begin{aligned} e^z &= \sum_{n \geq -h} \frac{z^{h+n}}{\Gamma(h+n+1)} \\ &= \sum_{m \geq 0} \frac{z^m}{\Gamma(m+1)} \end{aligned}$$

coincides with the usual definition of exponential function. When h is non integer, negative powers of z appear in the sum. Each such term diverges at the the origin but the sum is finite.

B. Summary

The boundary operator/state correspondence is

$$O(z,\bar{z})|0\rangle = \begin{cases} \sum_{l,r \leq 0} \frac{1}{z^l \bar{z}^r} |l-h, r-h\rangle & \text{Highest Weight} \\ \sum_{l,r} \frac{1}{z^{h+l} \bar{z}^{h+r}} |l, r\rangle & \text{Principal Series} \end{cases}$$

Transforming to the momentum basis we get

$$O(z,\bar{z})|0\rangle = \frac{1}{(2\pi i)^2} \oint dk d\bar{k} |k|^{2h-1} e^{\frac{i}{2}\left(k\bar{z}+\bar{k}z\right)} |k, \bar{k}\rangle \, .$$

We can also transform back. As we have already stated in subsection (IV A), the key identity is

$$e^z = \sum_{n \in Z} \frac{z^{h+n}}{\Gamma(h+n+1)} = \sum_{m \geq 0} \frac{z^m}{\Gamma(m+1)}$$

where h is integer.

In the momentum basis, the expansion

$$O(z,\bar{z})|0\rangle = \oint \frac{dk d\bar{k}}{(2\pi i)^2} e^{\frac{i}{2}\left(k\bar{z}+\bar{k}z\right)} |k|^{2h-1} |k, \bar{k}\rangle$$

takes the same form for both the highest weight and principal series representation. So any two or three-point correlation function in the momentum or position basis is going to have the same scaling form for principal series and highest weight representation since the form is fixed by conformal symmetry. For example,

$$\begin{aligned} \langle O(z)O(w)\rangle_{\text{principal-series}} &= \frac{1}{(z-w)^{2h}} \\ \langle O(z)O(w)\rangle_{\text{highest-weight}} &= \frac{1}{(z-w)^{2h}} \end{aligned}$$

where h is the weight of operator. It is real for highest weight rep but complex for principal series. Now consider the following 2-point function $\langle \oint w^{k+h} \phi(w) \oint z^k \phi(z) \rangle$ (h is the weight of the operator). For highest weight representation

$$\langle \oint w^{k+h} \phi_H(w) \oint z^k \phi_H(z) \rangle = 0 \, (\text{for } k \in \mathbb{Z}_+)$$

because $\oint z^k \phi_H(z)|0\rangle = 0$ for $k \in \mathbb{Z}_+$.

For principal series representation it gives

$$\begin{aligned}
&\langle \oint w^{k+h}\phi_{NH}(w) \oint z^k \phi_{NH}(z)\rangle \\
&= \langle \sum_{-\infty<m<\infty} \oint w^{k-m} dw \phi_m \sum_{-\infty<n<\infty} \oint z^{k-n-h} dz \phi_n \rangle \\
&= \langle \phi_{k+1} \sum_{-\infty<n<\infty} \frac{r^{k-n-h+1}\left(e^{2\pi i(k-n-h+1)}-1\right)}{i(k-n-h+1)} \phi_n \rangle \\
&= \frac{i\left(e^{-2\pi h i}-1\right)}{h r^h}
\end{aligned}$$

where r is the radius of the circular loop around the origin. So we have constructed an observable which vanishes for highest weight CFT but does not vanish for non-highest weight CFT.

Another way to distinguish them is to compute the correlation function in l, r basis

$$\begin{aligned}
\langle (L_0 O(l,r))\,(L_0 O(l,r))\rangle_{\text{principal-series}} &= \langle l, r|L_0^\dagger L_0|l.r\rangle = l^2 \\
\langle (L_0 O(l,r))\,(L_0 O(l,r))\rangle_{\text{highest-weight}} &= \langle l-h, r-h|L_0^\dagger L_0|l-h.r-h\rangle = (l-h)^2 \,.
\end{aligned}$$

For principal series, we get integer squared and is independent of the weight. Whereas for highest weight, it is non-integer and depends on the weight of the operator.

V. OPE OF CONTINUOUS (PRINCIPAL AND COMPLEMENTARY) SERIES REPS.

In this section we derive operator product expansion (OPE) for the principal series. First we will review the calculation for highest weight CFT from [37]. Then we will extend the derivation for principal series with suitable modification.

A. Highest weight OPE

Start with an ansatz

$$O_1(z)O_2(0) = \sum_{k\geq 0} \beta_k z^{\Delta_3-\Delta_1-\Delta_2+k} \left(\frac{\partial}{\partial\zeta}\right)^k O_3(\zeta)|_{\zeta\to 0} \,. \tag{18}$$

Then using symmetry we can determine the coefficients. Commute left side with L_1. Using the relation

$$[L_1, O_\Delta(z)] = \left[z^2\frac{\partial}{\partial z} + 2\Delta z\right] O_\Delta(z)$$

we get

$$[L_1, O_1(z)O_2(0)] = \left[z^2\frac{\partial}{\partial z} + 2\Delta_1 z\right] O_1(z)O_2(0) \,.$$

Substituting the ansatz from equation (18) in the right side we get

$$[L_1, O_1(z)O_2(0)] = \sum_{k\geq 0} \beta_k z^{\Delta_3-\Delta_1-\Delta_2+k+1} \left(\Delta_3+\Delta_1-\Delta_2+k\right) \left(\frac{\partial}{\partial\zeta}\right)^k O(\zeta)|_{\zeta\to 0} \,. \tag{19}$$

Now commuting L_1 with the right side of equation (18) we get

$$\sum_{k\geq 0} \beta_k z^{\Delta_3-\Delta_1-\Delta_2+k} \left(\frac{\partial}{\partial\zeta}\right)^k [L_1, O(\zeta)]\,|_{\zeta\to 0} = \sum_{k\geq 0} \beta_k z^{\Delta_3-\Delta_1-\Delta_2+k} \left(\frac{\partial}{\partial\zeta}\right)^k \left(\zeta^2\frac{\partial}{\partial\zeta} + 2\zeta\Delta_3\right) O(\zeta)|_{\zeta\to 0} \,. \tag{20}$$

We can now match the coefficient of power series of equations (19) and (20). Let us set $\Delta_1 = \Delta_2$ for simplicity. As an example, let us match the coefficient of $z^{\Delta_3-\Delta_1-\Delta_2+1}$

$$\beta_0 \frac{\Gamma\left(\Delta_3+\Delta_1-\Delta_2+1\right)}{\Gamma\left(\Delta_3+\Delta_1-\Delta_2\right)} O(\zeta)|_{\zeta\to 0} = \beta_1 \left(\frac{\partial}{\partial\zeta}\right)\left(\zeta^2\frac{\partial}{\partial\zeta}+2\zeta\Delta_3\right) O(\zeta)|_{\zeta\to 0}$$

$$\beta_0\Delta_3 O(\zeta)|_{\zeta\to 0} = \beta_1\left(2\Delta_3+2\zeta\frac{\partial}{\partial\zeta}+\zeta^2\left(\frac{\partial}{\partial\zeta}\right)^2\right) O(\zeta)|_{\zeta\to 0}\,. \tag{21}$$

For highest weight $O(\zeta)|_{\zeta\to 0}$ is finite and $\zeta\frac{\partial}{\partial\zeta}O(\zeta)|_{\zeta\to 0}=0$. Thus we get

$$\beta_1 = \frac{\beta_0}{2}\,.$$

Similarly matching all the terms, we get

$$O_1(z)O_2(0) = \beta_{123}\sum z^{\Delta_3-2\Delta_1}\;{}_1F_1\left(\Delta_3, 2\Delta_3, z\frac{\partial}{\partial\zeta}\right) O_3(\zeta)|_{\zeta\to 0}\,. \tag{22}$$

The above equality can also be derived, starting from the 3 point function

$$\langle O(z_1)O(z_2\to 0)O(z_3)\rangle = \frac{\beta_{123}}{z_1^h (z_3-z_1)^h z_3^h} = \beta_{123} z_1^{-h}\left(1+h\frac{z_1}{z_3}+\ldots\right)\frac{1}{z_3^{2h}}\,. \tag{23}$$

B. Principal series

For the principal series the OPE will take the form

$$O_1(z)O_2(0) = \textstyle\sum_{k>0}\beta_{-k} z^{\Delta_3-\Delta_1-\Delta_2-k}\left(L_1\right)^k O_3(\zeta)|_{\zeta\to 0} + \displaystyle\sum_{k\geq 0}\beta_k z^{\Delta_3-\Delta_1-\Delta_2+k}\left(L_{-1}\right)^k O_3(\zeta)|_{\zeta\to 0}\,. \tag{24}$$

Here we have also added terms with $L_1 O$ because for principal series $L_1 O \neq 0$ in general. Again we commute with L_1 to determine β_k. Commuting the left side of equation (24) we get

$$[L_1, O_1(z)O_2(0)] = \sum_{k>0}\beta_k z^{\Delta_3-\Delta_1-\Delta_2+k}\left(z\left(\Delta_3+\Delta_1-\Delta_2+k\right)\left(L_{-1}\right)^k + \left(\zeta^2\frac{\partial}{\partial\zeta}+2\Delta_2\zeta\right)\left(L_{-1}\right)^k\right) O(\zeta)\,.$$

Commuting the right side of equation (24) we get

$$[L_1, O_1(z)O_2(0)] = \sum_{k>0}\beta_k z^{\Delta_3-\Delta_1-\Delta_2+k}\left[L_1, \left(L_{-1}\right)^k O(\zeta)\right]\,.$$

Equating the above two equations gives

$$\begin{aligned}&\beta_{k+1}\left(L_{-1}^{k+1}\left(\zeta^2\frac{\partial}{\partial\zeta}+2\Delta_3\zeta\right)-\left(\zeta^2\frac{\partial}{\partial\zeta}+2\Delta_2\zeta\right)\left(L_{-1}\right)^{k+1}\right)O(\zeta)\\ &=\beta_k\left(\Delta_3+\Delta_1-\Delta_2+k\right)\left(L_{-1}\right)^k O(\zeta)\,.\end{aligned} \tag{25}$$

Similarly, to determine β_{-k} we can commute both sides with L_{-1}

$$[L_{-1}, O_1(z)O_2(0)] = \sum_{k>0}\beta_{-k} z^{\Delta_3-\Delta_1-\Delta_2-k-1}\left(\left(\Delta_3-\Delta_1-\Delta_2-k\right)\left(L_1\right)^k + z\left(L_1\right)^k L_{-1}\right) O(\zeta)\,.$$

Commuting the right side of equation (24) we get

$$[L_{-1}, O_1(z)O_2(0)] = \sum_{k<0}\beta_{-k} z^{\Delta_3-\Delta_1-\Delta_2-k}\left[L_{-1}, \left(L_1\right)^k O(\zeta)\right]\,.$$

Equating the two sides we get

$$\beta_{-k-1}\left[L_{-1}, L_1^{k+1}\right] O(\zeta)|_{\zeta\to 0} = \beta_{-k}\left(\Delta_3-\Delta_1-\Delta_2-k\right)\left(L_1\right)^k O(\zeta)\,. \tag{26}$$

Simplifying equation (25) gives

$$\beta_{k+1}\left(2\left(k+1+\Delta_3-\Delta_2\right)\zeta\left(\frac{\partial}{\partial\zeta}\right)^{k+1}+\left(k+1\right)\left(k+2\Delta_3\right)\left(\frac{\partial}{\partial\zeta}\right)^{k}\right)O(\zeta)$$
$$=\beta_k\left(\Delta_3+\Delta_1-\Delta_2+k\right)\left(\frac{\partial}{\partial\zeta}\right)^{k}O(\zeta)\,. \tag{27}$$

An important thing to note is that recursion relations explicitly depend on $O(\zeta)$. Now we substitute the expansion

$$O_3(\zeta)=\sum_j\frac{O_{3j}}{\zeta^{h+j}}$$

and compare the coefficient of same power of ζ, we get

$$\beta_{k+1}=\beta_k\frac{\left(\Delta_3+\Delta_1-\Delta_2+k\right)}{\left(2\left(k+1+\Delta_3-\Delta_2\right)\left(-h-j-k\right)+\left(k+1\right)\left(k+2\Delta_3\right)\right)}\,.$$

We find that β_k depends on j. This suggests that we must start with an OPE of the form

$$O_1(z)O_2(\zeta)=\sum_j\left(\sum_{k>0}\beta_{-k,j}z^{\Delta_3-\Delta_1-\Delta_2-k}\left(L_1\right)^k\frac{O_{3j}}{\zeta^j}+\sum_{k>0}\beta_{k,j}z^{\Delta_3-\Delta_1-\Delta_2+k}\left(L_{-1}\right)^k\frac{O_{3j}}{\zeta^j}\right)\,. \tag{28}$$

Then going through the above derivation we get

$$\beta_{k+1,j}=-\beta_{k,j}\frac{\left(\Delta_3+\Delta_1-\Delta_2+k\right)}{\left(k-K_+\right)\left(k-K_-\right)}$$
$$\beta_{-k-1,j}=\beta_{-k,j}\frac{\left(\Delta_3-\Delta_1-\Delta_2-k\right)}{(k+2\Delta_3+2j+2h)(k+1)}$$

where

$$K_\pm=\frac{1}{2}\left(1+2\left(\Delta_2-j-h\right)\pm\sqrt{\left(1+\Delta_2\right)^2+4\left(2\Delta_3-j-h\right)\left(1-j-h\right)}\right)\,.$$

Calculations are shown in the appendix.

Then equation (28) can be written in terms of hypergeometric functions

$$O_1(z)O_2(\zeta)=\sum_j z^{\Delta_3-\Delta_1-\Delta_2}\beta_{0,j}\left({}_1F_1\left(\Delta_1+\Delta_2-\Delta_3;2\Delta_3+2j+2h;-\frac{1}{z}\left(\zeta^2\frac{\partial}{\partial\zeta}+2\Delta_3\zeta\right)\right)\frac{O_{3j}}{\zeta^{h+j}}\right.$$
$$\left.+\,{}_2F_2\left(\Delta_3+\Delta_1-\Delta_2,1;K_+,K_-;-z\frac{\partial}{\partial\zeta}\right)\frac{O_{3j}}{\zeta^{h+j}}\right)\,.$$

This is the conformal partial wave expansion for the principal series.

Using equation (17), we can show that above equation is equivalent to three point function

$$\langle O(z_1)O(z_2\to 0)O(z_3)\rangle=\frac{\beta_{123}}{z_1^h\left(z_3-z_1\right)^h z_3^h}$$

given $\beta_{0,j}=\beta_{123}$.

The main conclusion is that there are infinitely many singular terms coming from terms like L_1^kO in the OPE. The OPE therefore has an essential singularity, unlike any known conformal field theory that may be viewed as arising from a renormalizable field theory. This puts the set of interacting conformal field theories based on representations containing the principal series well outside the class of conventional quantum field theories. The OPE also depends on an infinite number of parameters that are free at this level of analysis, compared to the single parameter one normally encounters in CFT. If these CFTs of relevance for de Sitter space exist, it seems they have more in common with non-renormalizable theories, than with conventional CFTs.

Finally we note all the conclusions of the present section carry over to the complementary series, provided we take h in the appropriate range $1>h>1/2$.

VI. CONCLUSION

de-Sitter holography implies that bulk and boundary states should be in principal, complementary, discrete series and indecomposible representations. Some of the details of these representations were studied from the conformal field theory perspective. In particular, we analyzed the implications of global conformal invariance for the operator product expansion. Because the weights of the principal and complementary series are unbounded, there end up being infinitely many singular terms in the operator product expansion. Nevertheless, this is compatible with the usual simplifications of the two and three-point functions of quasi-primary operators. The essential singularity present in these operator product expansions is not reproducible from conventional quantum field theories. The essential singularity and the infinite number of free parameters is related to our earlier result in [10], that in dS we do not have an ordinary CFT on the boundary but a theory of conformal gravity.

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APPENDIX

A. Bulk isometries

Bulk $SO(3,1)$ isometries can be expressed in terms of embedding coordinates $X_A = (Y_1, Y_2, Z, T)$

$$z_{AB} = i\left(X_B \partial_A - X_A \partial_B\right) \tag{29}$$

where de Sitter spacetime is the hyperboloid

$$R^2 = Y_1^2 + Y_2^2 + Z^2 - T^2 \,.$$

Poincare coordinates (y_1, y_2, η) are given by

$$\begin{aligned} T &= \frac{R}{2}(\eta - \frac{1}{\eta}) - \frac{1}{2R\eta}(y_1^2 + y_2^2) \\ Y_1 &= \frac{y_1}{\eta} \\ Y_2 &= \frac{y_2}{\eta} \\ Z &= \frac{R}{2}(\eta + \frac{1}{\eta}) - \frac{1}{2R\eta}(y_1^2 + y_2^2) \,. \end{aligned}$$

With inverse relations

$$\begin{aligned} \eta &= \frac{\sqrt{Y_1^2 + Y_2^2 + Z^2 - T^2}}{Z - T} \\ y_1 &= \frac{Y_1}{Z - T} \\ y_2 &= \frac{Y_2}{Z - T} \,. \end{aligned}$$

Equation (29) in R,η,y_1,y_2 coordinates becomes

$$\begin{aligned}
J_3 \equiv J_{Y_1Y_2} &= i\left(y_2\partial_{y_1} - y_1\partial_{y_2}\right)\\
J_2 \equiv J_{ZY_1} &= -i\left(\frac{1+y_1^2-y_2^2+\eta^2}{2}\partial_{y_1} + y_1y_2\partial_{y_2} + y_1\eta\partial_\eta\right)\\
-J_1 \equiv J_{ZY_2} &= -i\left(\frac{1-y_1^2+y_2^2+\eta^2}{2}\partial_{y_2} + y_1y_2\partial_{y_1} + y_2\eta\partial_\eta\right)\\
K_1 \equiv K_{Y_1T} &= -i\left(\frac{-1+y_1^2-y_2^2+\eta^2}{2}\partial_{y_1} + y_1y_2\partial_{y_2} + y_1\eta\partial_\eta\right)\\
K_2 \equiv K_{Y_2T} &= -i\left(\frac{-1-y_1^2+y_2^2+\eta^2}{2}\partial_{y_2} + y_1y_2\partial_{y_1} + y_2\eta\partial_\eta\right)\\
K_3 \equiv K_{ZT} &= -i\left(y_1\partial_{y_1} + y_2\partial_{y_2} + \eta\partial_\eta\right)\,.
\end{aligned}$$

We can go to the complex coordinate $z = y_1 + iy_2$ and define

$$\begin{aligned}
J_L = z\partial_z + \frac{\eta}{2}\partial_\eta \quad &, \quad K_{+L} = i\left(z^2\partial_z + \eta^2\partial_{\bar z} + z\eta\partial_\eta\right) \quad , \quad K_{-L} = -i\partial_z\\
J_R = -\bar z\partial_{\bar z} - \frac{\eta}{2}\partial_\eta \quad &, \quad K_{-R} = -i\left(\bar z^2\partial_{\bar z} + \eta^2\partial_z + \bar z\eta\partial_\eta\right) \quad , \quad K_{+R} = i\partial_{\bar z}\,.
\end{aligned}$$

We see that they take very simple form in Poincare coordinates compared to spherical coordinates.

B. OPE calculation for principal series

Calculation of $\beta_{k,j}$ is same as in equation (27).

$$\begin{aligned}
&\beta_{k+1,j}\left(2\left(k+1+\Delta_3-\Delta_2\right)\zeta\left(\frac{\partial}{\partial\zeta}\right)^{k+1} + (k+1)\left(k+2\Delta_3\right)\left(\frac{\partial}{\partial\zeta}\right)^{k}\right)\zeta^{-h-j}\\
&= \beta_{k,j}\left(\Delta_3+\Delta_1-\Delta_2+k\right)\left(\frac{\partial}{\partial\zeta}\right)^{k}\zeta^{-h-j}\\
&\beta_{k+1,j}\left(2\left(k+1+\Delta_3-\Delta_2\right)(-h-j)...(-h-j-k) + (k+1)\left(k+2\Delta_3\right)(-h-j)...(-h-j-k+1)\right)\zeta^{-h-j-k}\\
&= \beta_{k,j}\left(\Delta_3+\Delta_1-\Delta_2+k\right)(-h-j)...(-h-j-k+1)\zeta^{-h-j-k}\\
&\beta_{k+1,j}\left(2\left(k+1+\Delta_3-\Delta_2\right)(-j-h-k) + (k+1)\left(k+2\Delta_3\right)\right)\zeta^{-h-j-k}\\
&= \beta_{k,j}\left(\Delta_3+\Delta_1-\Delta_2+k\right)\zeta^{-h-j-k}\\
&\beta_{k+1,j}\\
&= -\beta_{k,j}\frac{\left(\Delta_3+\Delta_1-\Delta_2+k\right)}{\left(k-K_+\right)\left(k-K_-\right)}
\end{aligned}$$

where

$$K_\pm = \frac{1}{2}\left(1 + 2\left(\Delta_2-j-h\right) \pm \sqrt{\left(1+\Delta_2\right)^2 + 4\left(2\Delta_3-j-h\right)\left(1-j-h\right)}\right)\,.$$

Calculation of $\beta_{-k,j}$ is

$$\begin{aligned}
&\beta_{-k-1,j}\left[L_{-1},L_1^{k+1}\right]\zeta^{-h-j}\\
&= \beta_{-k,j}\left(\Delta_3-\Delta_1-\Delta_2-k\right)\left(L_1\right)^k\zeta^{-h-j}\\
&\beta_{-k-1,j}\left(\left(2\Delta_3+k-h-j-1\right)...\left(2\Delta_3-h-j-1\right)(-h-j) - \left(2\Delta_3+k-h-j\right)...\left(2\Delta_3-h-j\right)(k-h-j+1)\right)\zeta^{-h-j+k}\\
&= \beta_{-k,j}\left(\Delta_3-\Delta_1-\Delta_2-k\right)\left(2\Delta_3+k-1-h-j\right)...\left(2\Delta_3-h-j\right)\zeta^{-h-j+k}\\
&\beta_{-k-1,j}\left(\left(2\Delta_3-h-j-1\right)(-h-j) - \left(2\Delta_3+k-h-j\right)(k-h-j+1)\right)\zeta^{-h-j+k}\\
&= \beta_{-k,j}\left(\Delta_3-\Delta_1-\Delta_2-k\right)\zeta^{-h-j+k}\\
&\beta_{-k-1,j}\\
&= \beta_{-k,j}\frac{\left(\Delta_3-\Delta_1-\Delta_2-k\right)}{(k+2\Delta_3+2j+2h)(k+1)}\,.
\end{aligned}$$

[1] A. Strominger, "The dS / CFT correspondence," *JHEP* **0110** (2001) 034, `arXiv:hep-th/0106113 [hep-th]`.

[2] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, "Large N field theories, string theory and gravity," *Phys. Rept.* **323** (2000) 183–386, `arXiv:hep-th/9905111 [hep-th]`.

[3] A. Hamilton, D. N. Kabat, G. Lifschytz, and D. A. Lowe, "Local bulk operators in AdS/CFT: A Holographic description of the black hole interior," *Phys.Rev.* **D75** (2007) 106001, `arXiv:hep-th/0612053 [hep-th]`.

[4] A. Hamilton, D. N. Kabat, G. Lifschytz, and D. A. Lowe, "Local bulk operators in AdS/CFT and the fate of the BTZ singularity," `arXiv:0710.4334 [hep-th]`.

[5] D. A. Lowe, "Some comments on embedding inflation in the AdS/CFT correspondence," *Phys.Rev.* **D77** (2008) 066003, `arXiv:0710.3564 [hep-th]`.

[6] D. Kabat, G. Lifschytz, and D. A. Lowe, "Constructing local bulk observables in interacting AdS/CFT," *Phys.Rev.* **D83** (2011) 106009, `arXiv:1102.2910 [hep-th]`.

[7] S. Kachru, R. Kallosh, A. D. Linde, and S. P. Trivedi, "De Sitter vacua in string theory," *Phys. Rev.* **D68** (2003) 046005, `arXiv:hep-th/0301240 [hep-th]`.

[8] E. Witten, "Quantum gravity in de Sitter space," in *Strings 2001: International Conference Mumbai, India, January 5-10, 2001*. 2001. `arXiv:hep-th/0106109 [hep-th]`.

[9] A. Chatterjee and D. A. Lowe, "Holographic operator mapping in dS/CFT and cluster decomposition," `arXiv:1503.07482 [hep-th]`.

[10] A. Chatterjee and D. A. Lowe, "de Sitter gravity/Euclidean conformal gravity correspondence," *Phys. Rev.* **D93** no. 2, (2016) 024038, `arXiv:1508.04421 [hep-th]`.

[11] D. Anninos, T. Hartman, and A. Strominger, "Higher Spin Realization of the dS/CFT Correspondence," `arXiv:1108.5735 [hep-th]`.

[12] D. Anninos, F. Denef, G. Konstantinidis, and E. Shaghoulian, "Higher Spin de Sitter Holography from Functional Determinants," *JHEP* **1402** (2014) 007, `arXiv:1305.6321 [hep-th]`.

[13] V. Balasubramanian, J. de Boer, and D. Minic, "Notes on de Sitter space and holography," *Class. Quant. Grav.* **19** (2002) 5655–5700, `arXiv:hep-th/0207245 [hep-th]`. [Annals Phys.303,59(2003)].

[14] S. Banerjee, A. Belin, S. Hellerman, A. Lepage-Jutier, A. Maloney, *et al.*, "Topology of Future Infinity in dS/CFT," *JHEP* **1311** (2013) 026, `arXiv:1306.6629 [hep-th]`.

[15] D. Harlow and D. Stanford, "Operator Dictionaries and Wave Functions in AdS/CFT and dS/CFT," `arXiv:1104.2621 [hep-th]`.

[16] D. Sarkar and X. Xiao, "Holographic Representation of Higher Spin Gauge Fields," *Phys. Rev.* **D91** no. 8, (2015) 086004, `arXiv:1411.4657 [hep-th]`.

[17] M. Spradlin, A. Strominger, and A. Volovich, "Les houches lectures on de sitter space," `hep-th/0110007`.

[18] X. Xiao, "Holographic Representation of Local Operators In De Sitter Space," *Phys.Rev.* **D90** (2014) 024061, `arXiv:1402.7080 [hep-th]`.

[19] D. Kastor and J. H. Traschen, "A Positive energy theorem for asymptotically de Sitter space-times," *Class. Quant. Grav.* **19** (2002) 5901–5920, `arXiv:hep-th/0206105 [hep-th]`.

[20] A. Castro and A. Maloney, "The Wave Function of Quantum de Sitter," *JHEP* **11** (2012) 096, `arXiv:1209.5757 [hep-th]`.

[21] A. Ashtekar, B. Bonga, and A. Kesavan, "Asymptotics with a positive cosmological constant: I. Basic framework," *Class. Quant. Grav.* **32** no. 2, (2015) 025004, `arXiv:1409.3816 [gr-qc]`.

[22] A. Ashtekar, B. Bonga, and A. Kesavan, "Asymptotics with a positive cosmological constant: II. Linear fields on de Sitter space-time," `arXiv:1506.06152 [gr-qc]`.

[23] N. A. Chernikov and E. A. Tagirov, "Quantum theory of scalar fields in de Sitter space-time," *Annales Poincare Phys. Theor.* **A9** (1968) 109.

[24] E. Tagirov, "Consequences of field quantization in de Sitter type cosmological models," *Annals Phys.* **76** (1973) 561–579.

[25] R. Raczka, N. Limić, and J. Niederle, "Discrete degenerate representations of noncompact rotation groups. i," *Journal of Mathematical Physics* **7** no. 10, (1966) 1861–1876, `http://aip.scitation.org/doi/pdf/10.1063/1.1704835`. `http://aip.scitation.org/doi/abs/10.1063/1.1704835`.

[26] N. Limić, J. Niederle, and R. Raczka, "Continuous degenerate representations of noncompact rotation groups. ii," *Journal of Mathematical Physics* **7** no. 11, (1966) 2026–2035, `http://aip.scitation.org/doi/pdf/10.1063/1.1704885`. `http://aip.scitation.org/doi/abs/10.1063/1.1704885`.

[27] M. A. Naimark, *Linear Representation of the Lorentz Group*. Pergamon Press, New York, 1964.

[28] G. Lindblad and B. Nagel, "Continuous bases for unitary irreducible representations of SU(1,1).," *Ann. Inst. Henri Poincaré, Nouv. Sér., Sect. A* **13** (1970) 27–56.

[29] M. Huszár, "Unitary representations of the sl (2, c) group in horospheric basis," *Journal of Mathematical Physics* **14** no. 11, (1973) 1620–1625, `http://aip.scitation.org/doi/pdf/10.1063/1.1666233`. `http://aip.scitation.org/doi/abs/10.1063/1.1666233`.

[30] N. Mukunda, "Unitary representations of the group o(2, 1) in an o(1, 1) basis," *Journal of Mathematical Physics* **8** no. 11, (1967) 2210–2220, `http://aip.scitation.org/doi/pdf/10.1063/1.1705143`. `http://aip.scitation.org/doi/abs/10.1063/1.1705143`.

[31] N. Mukunda, "Unitary representations of the homogeneous lorentz group in an o(2, 1) basis," *Journal of Mathematical*

Physics **9** no. 1, (1968) 50–61, http://aip.scitation.org/doi/pdf/10.1063/1.1664476. http://aip.scitation.org/doi/abs/10.1063/1.1664476.

[32] G. Lindblad, "Eigenfunction expansions associated with unitary irreducible representations of su (1,1)," *Physica Scripta* **1** no. 5-6, (1970) 201.

[33] E. Joung, J. Mourad, and R. Parentani, "Group theoretical approach to quantum fields in de sitter space ii. the complementary and discrete series," *Journal of High Energy Physics* **2007** no. 09, (2007) 030.

[34] E. Joung, J. Mourad, and R. Parentani, "Group theoretical approach to quantum fields in de sitter space, i. the principal series," *Journal of High Energy Physics* **2006** no. 08, (2006) 082.

[35] D. W. Dusedau and D. Z. Freedman, "Lehmann Spectral Representation for Anti-de Sitter Quantum Field Theory," *Phys. Rev.* **D33** (1986) 389.

[36] B. G. Wybourne, *Classical group for physicists.* Wiley Intersicence Publication, 1974.

[37] A. M. Polyakov, A. A. Belavin, and A. B. Zamolodchikov, "Infinite Conformal Symmetry of Critical Fluctuations in Two-Dimensions," *J. Statist. Phys.* **34** (1984) 763.