Conformal basis for flat space amplitudes

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A Conformal Basis for Flat Space Amplitudes

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Abstract

We study solutions of the Klein-Gordon, Maxwell, and linearized Einstein equations in $\mathbb{R}^{1,d+1}$ that transform as d-dimensional conformal primaries under the Lorentz group $SO(1,d+1)$. Such solutions, called conformal primary wavefunctions, are labeled by a conformal dimension Δ and a point in $\mathbb{R}^d$, rather than an on-shell $(d+2)$-dimensional momentum. We show that the continuum of scalar conformal primary wavefunctions on the principal continuous series $\Delta \in \frac{d}{2} + i\mathbb{R}$ of $SO(1,d+1)$ spans a complete set of normalizable solutions to the wave equation. In the massless case, with or without spin, the transition from momentum space to conformal primary wavefunctions is implemented by a Mellin transform. As a consequence of this construction, scattering amplitudes in this basis transform covariantly under $SO(1,d+1)$ as d-dimensional conformal correlators.

1 Introduction

Scattering problems are conventionally studied in momentum space where translation symmetry is manifest. However, not all properties of scattering amplitudes are emphasized in this choice of basis. One famous alternative basis is twistor space [1, 2] where many remarkable properties of perturbative gauge theory amplitudes are naturally explained.

The Lorentz group in $\mathbb{R}^{1,d+1}$ is identical to the Euclidean d-dimensional conformal group $SO(1, d+1)$. It is then natural to ask if there is a basis of wavefunctions where scattering amplitudes in $\mathbb{R}^{1,d+1}$ admit interpretations as Euclidean d-dimensional conformal correlators.[1] For massless scalar and spin-one fields, such wavefunctions were constructed in [3, 4], while the massive scalar wavefunctions were introduced in [5] for $(3 + 1)$ spacetime dimensions. However, it had not been established whether these conformal wavefunctions form a complete set of normalizable solutions to the wave equation in each case. In this paper we study the completeness of these wavefunctions with and without spin and extend the construction to arbitrary spacetime dimensions.

The search for conformal bases of wavefunctions has its roots in the study of two-dimensional conformal symmetries in four-dimensional scattering amplitudes. In [3], de Boer and Solodukhin approached the problem of flat space holography from a hyperbolic slicing of Minkowski space. Since each slice is a copy of a three-dimensional hyperbolic space H_3, the two-dimensional conformal symmetry naturally arises on the boundary via AdS holography. It was then conjectured [6–9] that in any four-dimensional quantum gravity, the Lorentz group $SL(2, \mathbb{C})$ is enhanced to the full Virasoro symmetry. This conjecture was later refined and verified [4,10,11] for tree-level $\mathcal{S}$-matrices following a new subleading soft graviton theorem [12]. In particular, the authors of [11] gave an explicit construction of a two-dimensional stress-tensor that generates a complex Virasoro symmetry acting on the celestial sphere at null infinity. The one-loop correction to the Virasoro stress tensor was recently discussed in [13, 14] from an anomaly [15–18] of this subleading soft graviton theorem. Furthermore, it was observed that insertions of soft photons in the amplitude resemble the Kac-Moody algebra in two dimensions [19–23]. See [24] for a comprehensive review of this subject.

In this paper we introduce a basis of flat space wavefunctions that is natural for the study of the d-dimensional conformal structure of $(d+2)$-dimensional scattering amplitudes beyond the soft limit. We consider on-shell wavefunctions in $\mathbb{R}^{1,d+1}$ with spin that are $SO(1, d + 1)$ conformal primaries, extending the construction in [4, 5] beyond $(3 + 1)$ dimensions. These solutions, called conformal primary wavefunctions, are labeled by a conformal dimension Δ

[1] By a conformal correlator we mean a function of n points on $\mathbb{R}^d$ that transforms covariantly as in (2.26) (or the spin version thereof), i.e. it is a function with the same conformal covariance as an n-point function of primaries in conformal field theory (CFT).

and a point $\vec{w}$ in $\mathbb{R}^d$, as opposed to a $(d+2)$-dimensional on-shell momentum. Crucially, the conformal dimension Δ should be thought of as a continuous label for solutions in this basis, and is *not* fixed by the mass of the bulk field. Rather, we require a continuum of conformal primary wavefunctions to span the solution space of a single bulk field.[2]

One immediate puzzle about the massive conformal primary wavefunction is that, as opposed to the massless case, there does not appear to be a canonical way to associate a point $\vec{w}$ in $\mathbb{R}^d$ to the trajectory of a massive particle in $\mathbb{R}^{1,d+1}$. It turns out that the label $\vec{w}$ is not a point in position space, but in momentum space of the massive particle. More precisely, $\vec{w}$ is a boundary point of the space of $(d+2)$-dimensional on-shell momenta, which is a copy of a $(d+1)$-dimensional hyperbolic space H_{d+1}. See (2.14) for details.

The main objective of this paper is to determine the range of the conformal dimension Δ for these conformal primary wavefunctions to form a basis of on-shell wavefunctions in $\mathbb{R}^{1,d+1}$. If such range of Δ exists, then we can safely translate every scattering amplitude into this conformal primary basis without loss of information. In both the massive and the massless scalar cases, we show that the continuum of conformal primary wavefunctions with[3]

$$\Delta \in \frac{d}{2} + i\mathbb{R}\,, \tag{1.1}$$

spans the complete set of delta-function-normalizable solutions with respect to the Klein-Gordon norm (3.6). This range of Δ is known as the principal continuous series of irreducible unitary representations of $SO(1,d+1)$, which plays a central role in the harmonic analysis of the conformal group (see, for example, [25]). We contrast the scalar conformal primary basis with the momentum basis in Tables 1 and 2. In the massless case, the change of basis is given by a Mellin transform (or plus a shadow transform). In the massive, on the other hand, it is implemented by an integral over all the on-shell momenta with the bulk-to-boundary propagator in H_{d+1} being the Fourier coefficient.

We then discuss massless conformal primary wavefunctions with spin. More specifically, we will construct solutions to the $(d+2)$-dimensional Maxwell and vacuum linearized Einstein equation that transform as d-dimensional spin-one and spin-two conformal primaries, respectively. This extends the study of spin-one conformal primary wavefunctions in $(3+1)$ dimensions of [4]. One qualitative difference between the massless spinning wavefunctions and the scalar wavefunctions is the presence of gauge or diffeomorphism symmetry. It turns out that conformal covariance of the on-shell wavefunction selects a particular gauge. We will also discuss spinning conformal primary wavefunctions that are pure gauge/diffeomorphism

[2] Therefore in any flat space holographic duality formulated through this construction, the dual putative conformal theory will be non-compact (like a Liouville theory). This point was also advocated in [3, 4].

[3] For massive scalar conformal primary wavefunctions, we only require half of the principal continuous series, i.e. $\Delta \in \frac{d}{2} + i\mathbb{R}_{\geq 0}$ or $\Delta \in \frac{d}{2} + i\mathbb{R}_{\leq 0}$. See Section 3 for more details.

Bases	Plane Waves	Conformal Primary Wavefunctions
Notations	$\exp\left[\pm ip\cdot X\right]$	$\phi^{\pm}_{\Delta}(X^\mu;\vec{w})$
Labels	$p^\mu\quad(p^2=-m^2, p^0>0)$	$\Delta\in\frac{d}{2}+i\mathbb{R}_{\geq 0}\ ,\ \ \vec{w}\in\mathbb{R}^d$

Table 1: A comparison between the plane wave basis and the conformal primary basis $\mathcal{B}^{\pm}$ for normalizable, outgoing/incoming solutions to the massive Klein-Gordon equation. The plane wave is labeled by an on-shell momentum p^μ with $p^2=-m^2$, whereas the conformal primary wavefunction is labeled by $\Delta\in\frac{d}{2}+i\mathbb{R}_{\geq 0}$ and $\vec{w}\in\mathbb{R}^d$. Here the plus and minus superscripts denote outgoing and incoming wavefunctions, respectively. There is another basis $\widetilde{\mathcal{B}}^{\pm}$ that is shadow to $\mathcal{B}^{\pm}$.

Bases	Plane Waves	Conformal Primary Wavefunctions
Notations	$\exp\left[\pm ik\cdot X\right]$	$\varphi^{\pm}_{\Delta}(X^\mu;\vec{w})=\left(-q(\vec{w})\cdot X\mp i\epsilon\right)^{-\Delta}$
Labels	$k^\mu\ (k^2=0,\ k^0>0)$	$\Delta\in\frac{d}{2}+i\mathbb{R}\ ,\ \ \vec{w}\in\mathbb{R}^d$

Table 2: A comparison between the plane wave basis and the conformal primary basis $\mathcal{B}^{\pm}_{m=0}$ for normalizable, outgoing/incoming solutions to the massless Klein-Gordon equation. The plane wave is labeled by a null momentum k^μ, whereas the conformal primary wavefunction is labeled by $\Delta\in\frac{d}{2}+i\mathbb{R}$ and $\vec{w}\in\mathbb{R}^d$. There is another basis $\widetilde{\mathcal{B}}^{\pm}_{m=0}$ that is shadow to $\mathcal{B}^{\pm}_{m=0}$.

in diverse spacetime dimensions. Finally we show that the plane waves are spanned by spinning conformal primary wavefunctions on the principal continuous series (1.1). The transition from momentum space to the space of massless spinning conformal primary wavefunctions is implemented by a Mellin transform (or plus a shadow transform).

Scattering amplitudes written in the conformal primary basis manifestly enjoy the conformal covariance of d-dimensional conformal correlators. For example, the three-point decay amplitude of a four-dimensional ϕ^3 theory written in this basis was shown to take the form of a two-dimensional CFT three-point function in a special mass limit [5]. The soft limit and collinear singularities of gluon and graviton amplitudes were analyzed in this basis in [4]. The factorization singularity of scattering amplitudes has also been studied in the CFT language [26, 27]. It will be interesting to explore how other conformal structures can be translated into statements about scattering amplitudes in this basis.

The construction of conformal primary wavefunctions proceeds naturally via the embedding formalism in CFT [28–35]. Our flat space conformal wavefunctions are expressed in terms of the hyperbolic space H_{d+1} bulk-to-boundary propagators lifted to the embedding Minkowski space. In the CFT context, the embedding Minkowski space $\mathbb{R}^{1,d+1}$ is merely a fictitious space one introduces to realize the conformal transformation linearly. By contrast, in the current setting the embedding $\mathbb{R}^{1,d+1}$ is the spacetime where physical scattering processes take place.

The rest of the paper is organized as follows. In Section 2, we review and extend the definition of massive scalar conformal primary wavefunctions in general spacetime dimensions. In Section 3, we determine the range of the conformal dimension to be the principal continuous series of $SO(1, d+1)$. In Section 4, we consider massless scalar conformal primary wavefunctions and determine the range of their conformal dimensions. In particular, we show that the change of basis from momentum space to conformal primary wavefunctions is implemented by a Mellin transform in the massless case. In Sections 5 and 6 we discuss massless spin-one and spin-two conformal primary wavefunctions, respectively.

2 Massive Conformal Primary Wavefunctions

In this section we construct massive scalar wavefunctions in $(d+2)$-dimensional Minkowski spacetime $\mathbb{R}^{1,d+1}$ with coordinates X^μ, $\mu = 0, 1, \cdots, d+1$.

2.1 Massive Scalar Conformal Primary Wavefunctions in General Dimensions

Let us review the massive scalar conformal primary wavefunction defined in [5]. The *massive scalar conformal primary wavefunction* $\phi_\Delta(X^\mu; \vec{w})$ of mass m in $\mathbb{R}^{1,d+1}$ is a wavefunction labeled by a "conformal dimension" Δ and a point $\vec{w}$ in $\mathbb{R}^d$. It satisfies the following two defining properties:[4]

[4]We omit the mass m dependence of the conformal primary wavefunction in the notation $\phi_\Delta(X^\mu; \vec{w})$.

- It satisfies the $(d+2)$-dimensional massive Klein-Gordon equation of mass m,[5]

$$\left(\frac{\partial}{\partial X^\nu}\frac{\partial}{\partial X_\nu} - m^2\right)\phi_\Delta(X^\mu;\vec{w}) = 0\,. \tag{2.1}$$

- It transforms covariantly as a scalar conformal primary operator in d dimensions under an $SO(1,d+1)$ transformation,

$$\phi_\Delta\left(\Lambda^\mu{}_\nu X^\nu;\vec{w}\,'(\vec{w})\right) = \left|\frac{\partial\vec{w}\,'}{\partial\vec{w}}\right|^{-\Delta/d}\phi_\Delta(X^\mu;\vec{w})\,, \tag{2.2}$$

  where $\vec{w}\,'(\vec{w})$ is an $SO(1,d+1)$ transformation that acts non-linearly on $\vec{w}\in\mathbb{R}^d$ and $\Lambda^\mu{}_\nu$ is the associated group element in the $(d+2)$-dimensional representation. More explicitly, $\vec{w}\,'(\vec{w})$ is generated by:

$$\begin{aligned}
&\mathbb{R}^d\text{ translation}: && \vec{w}\,' = \vec{w}+\vec{a}\,,\\
&SO(d)\text{ rotation}: && \vec{w}\,' = M\cdot\vec{w}\,,\\
&\text{dilation}: && \vec{w}\,' = \lambda\vec{w}\,,\\
&\text{special conformal transformation}: && \vec{w}\,' = \frac{\vec{w}+|\vec{w}|^2\vec{b}}{1+2\vec{b}\cdot\vec{w}+|\vec{b}|^2|\vec{w}|^2}\,.
\end{aligned} \tag{2.3}$$

Being a solution to the Klein-Gordon equation, the conformal primary wavefunction can be expanded on the plane waves. The Fourier expansion takes the form of an integral over all the possible outgoing or incoming on-shell momenta, each of which is a copy of the $(d+1)$-dimensional hyperbolic space H_{d+1}. To be more concrete, let $y,\vec{z}$ be the coordinates of H_{d+1} with $y>0$ and $\vec{z}\in\mathbb{R}^d$. The H_{d+1} metric is

$$ds^2_{H_{d+1}} = \frac{dy^2+d\vec{z}\cdot d\vec{z}}{y^2}\,, \tag{2.4}$$

with $y=0$ being the boundary. This geometry has an $SO(1,d+1)$ isometry $\vec{z}\to\vec{z}\,'(y,\vec{z})\,,y\to y'(y,\vec{z})$ that is generated by

$$\begin{aligned}
&\mathbb{R}^d\text{ translation}: && y'=y\,, && \vec{z}\,'=\vec{z}+\vec{a}\,,\\
&SO(d)\text{ rotation}: && y'=y\,, && \vec{z}\,'=M\cdot\vec{z}\,,\\
&\text{dilation}: && y'=\lambda y\,, && \vec{z}\,'=\lambda\vec{z}\,,\\
&\text{special conformal transformation}: && y'=\frac{y}{1+2\vec{b}\cdot\vec{z}+|\vec{b}|^2(y^2+|\vec{z}|^2)}\,, && \vec{z}\,'=\frac{\vec{z}+(y^2+|\vec{z}|^2)\vec{b}}{1+2\vec{b}\cdot\vec{z}+|\vec{b}|^2(y^2+|\vec{z}|^2)}\,.
\end{aligned} \tag{2.5}$$

[5] Our convention for the spacetime signature in $\mathbb{R}^{1,d+1}$ is $(-++\cdots+)$.

We can then parametrize a unit timelike vector $\hat{p}(y,\vec{z})$ satisfying $\hat{p}^2=-1$ in terms of the H_{d+1} coordinates as,

$$\hat{p}(y,\vec{z}) = \left(\frac{1+y^2+|\vec{z}|^2}{2y}\,,\,\frac{\vec{z}}{y}\,,\,\frac{1-y^2-|\vec{z}|^2}{2y}\right)\,. \tag{2.6}$$

The map $\hat{p}(y,\vec{z})$ defines an embedding of the H_{d+1} into the upper branch ($\hat{p}^0>0$) of the unit hyperboloid in $\mathbb{R}^{1,d+1}$. We will henceforth use $\hat{p}$ and $(y,\vec{z})$ interchangeably to parametrize a point in H_{d+1}. The advantage of working with $\hat{p}^\mu$ is that the non-linear $SO(1,d+1)$ action (2.5) on $y,\vec{z}$ now becomes linear on $\hat{p}^\mu$,

$$\hat{p}^\mu(y',\vec{z}\,') = \Lambda^\mu{}_\nu \hat{p}^\nu\,, \tag{2.7}$$

where $\Lambda^\mu{}_\nu$ is the associated group element of $SO(1,d+1)$ in the $(d+2)$-dimensional representation.

One last ingredient we need is the scalar bulk-to-boundary propagator $G_\Delta(\hat{p};\vec{w})$ in H_{d+1} [36],

$$G_\Delta(\hat{p};\vec{w}) = \left(\frac{y}{y^2+|\vec{z}-\vec{w}|^2}\right)^\Delta\,, \tag{2.8}$$

where $\vec{w}\in\mathbb{R}^d$ is a point on the boundary of H_{d+1}. Let us define a map from $\mathbb{R}^d$ to a "unit" null momentum q^μ in $\mathbb{R}^{1,d+1}$ as

$$q^\mu(\vec{w}) = \left(1+|\vec{w}|^2\,,\,2\vec{w}\,,\,1-|\vec{w}|^2\right)\,. \tag{2.9}$$

We will use $\vec{w}$ and q^μ interchangeably to parametrize a point in $\mathbb{R}^d$. While $\vec{w}$ transforms non-linearly under $SO(1,d+1)$ as in (2.3), its embedding q^μ into $\mathbb{R}^{1,d+1}$ transforms linearly,

$$q^\mu(\vec{w}\,') = \left|\frac{\partial\vec{w}\,'}{\partial\vec{w}}\right|^{1/d}\,\Lambda^\mu{}_\nu q^\nu(\vec{w})\,. \tag{2.10}$$

We will often use

$$\partial_a q^\mu \equiv \frac{\partial}{\partial w^a}q^\mu(\vec{w}) = 2(w^a\,,\,\delta^{ba}\,,\,-w^a)\,. \tag{2.11}$$

In terms of the coordinates $\hat{p}^\mu(y,\vec{z})$ and $q^\mu(\vec{w})$, the bulk-to-boundary propagator can be written succinctly as [35]

$$G_\Delta(\hat{p};q) = \frac{1}{(-\hat{p}\cdot q)^\Delta}\,. \tag{2.12}$$

Under an $SO(1,d+1)$ transformation $\hat{p}(y,\vec{z}) \to \hat{p}' = \hat{p}(y',\vec{z}\,')$ and $q(\vec{w}) \to q' = q(\vec{w}\,')$, the G_Δ transforms covariantly as

$$G_\Delta(\hat{p}';q') = \left|\frac{\partial \vec{w}\,'}{\partial \vec{w}}\right|^{-\Delta/d} G_\Delta(\hat{p};q)\,. \tag{2.13}$$

With the above preparation, we can now write down the Fourier expansion of the scalar conformal primary wavefunction on the plane waves,

$$\boxed{\phi^{\pm}_\Delta(X^\mu;\vec{w}) = \int_{H_{d+1}} [d\hat{p}]\, G_\Delta(\hat{p};\vec{w})\, \exp\left[\, \pm im\hat{p}\cdot X\,\right]}\,, \tag{2.14}$$

with Fourier coefficients being the scalar bulk-to-boundary propagator in H_{d+1}. We use a plus (minus) sign for an outgoing (incoming) wavefunction. Here $[d\hat{p}]$ is the $SO(1,d+1)$ invariant measure on H_{d+1}:

$$\int_{H_{d+1}} [d\hat{p}] \equiv \int_0^\infty \frac{dy}{y^{d+1}} \int d^d\vec{z} = \int \frac{d^{d+1}\hat{p}^i}{\hat{p}^0}\,, \tag{2.15}$$

where $i = 1,2,\cdots,d+1$ and $\hat{p}^0 = \sqrt{\hat{p}^i\hat{p}^i+1}$. The conformal primary wavefunction given in (2.14) satisfies the defining property (2.2) thanks to the conformal covariance of the bulk-to-boundary propagator (2.13).

Importantly, the conformal dimension Δ of the conformal primary wavefunction $\phi^{\pm}_\Delta(X^\mu;\vec{w})$ is *not* related to the mass m. Indeed, Δ together with $\vec{w} \in \mathbb{R}^d$ should be thought of as the dual variables to an on-shell momentum p^μ that label the space of solutions to the Klein-Gordon equation. In particular, we require a continuum of conformal primary wavefunctions $\phi_\Delta(X^\mu;\vec{w})$ to form a basis of normalizable wavefunctions. We will determine the range of Δ in Section 3.

2.2 Closed-Form Expression

In (2.14) we have provided an integral representation for the massive scalar conformal primary wavefunction. In this section we will write its closed-form expression in terms of Bessel functions by directly solving the Klein-Gordon equation.

Let us consider the following ansatz for the wavefunction:

$$\phi_\Delta(X^\mu;\vec{w}) = \frac{f(X^2)}{(-q\cdot X)^\Delta}\,, \tag{2.16}$$

with $q^\mu = q^\mu(\vec{w})$ given in (2.9). The factor $1/(-q\cdot X)^\Delta$ solves the massless Klein-Gordon equation and has the desired conformal covariance (2.2) following from (2.10),[6] while the numerator is invariant under $SO(1,d+1)$. Hence the ansatz (2.16) already obeys (2.2) and we only need to solve for $f(X^2)$ such that ϕ_Δ is a solution to the massive Klein-Gordon equation. The massive Klein-Gordon equation gives the following differential equation for $f(X^2)$:

$$0 = 4X^2 f''(X^2) - 2(2\Delta - d - 2) f'(X^2) - m^2 f(X^2)\,, \tag{2.17}$$

from which we obtain

$$f(X^2) = (\sqrt{-X^2})^{\Delta - \frac{d}{2}} \left[c_1\, I_{\Delta - \frac{d}{2}}(m\sqrt{X^2}) + c_2\, I_{-\Delta + \frac{d}{2}}(m\sqrt{X^2}) \right] \tag{2.18}$$

where $I_\alpha(x)$ is the modified Bessel function of the first kind. For large spacelike X^μ, the Bessel function $I_\alpha(m\sqrt{X^2})$ grows exponentially as $e^{m\sqrt{X^2}}$, so a generic solution (2.18) will not give rise to normalizable wavefunctions. By requiring a finite Klein-Gordon norm (3.6) of the wavefunction, we select out a particular linear combination that is proportional to a modified Bessel of the second kind, $K_\alpha(x) = \frac{\pi}{2}\frac{I_{-\alpha}(x) - I_\alpha(x)}{\sin(\alpha\pi)}$, which dies off exponentially as $e^{-m\sqrt{X^2}}$ for large X^2. We can fix the overall constant by comparing with the integral expression (2.14) and find[7]

$$\phi^\pm_\Delta(X^\mu; \vec{w}) = \frac{2^{\frac{d}{2}+1}\pi^{\frac{d}{2}}}{(im)^{\frac{d}{2}}} \frac{(\sqrt{-X^2})^{\Delta - \frac{d}{2}}}{(-q(\vec{w})\cdot X \mp i\epsilon)^\Delta} K_{\Delta - \frac{d}{2}}(m\sqrt{X^2})\,. \tag{2.19}$$

We have introduced an $i\epsilon$ prescription for the denominator.

2.3 Shadow Transform

In this section we show that the conformal primary wavefunction $\phi^\pm_\Delta$ is the shadow transform of $\phi^\pm_{d-\Delta}$.[8]

Given a d-dimensional scalar conformal primary operator $\mathcal{O}_\Delta(\vec{w})$, its *shadow* [37–40] $\widetilde{\mathcal{O}}_\Delta(\vec{w})$ is a non-local operator defined as

$$\widetilde{\mathcal{O}}_\Delta(\vec{w}) \equiv \frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}}\Gamma(\Delta - \frac{d}{2})} \int d^d\vec{w}\,' \frac{1}{|\vec{w} - \vec{w}\,'|^{2(d-\Delta)}}\, \mathcal{O}_\Delta(\vec{w}\,')\,. \tag{2.20}$$

[6]This implies that $1/(-q\cdot X)^\Delta$ is a *massless* conformal primary wavefunction, which will be discussed in full detail in Section 4.

[7]To obtain the above closed-form expression, we should analytically continue to imaginary m as in [5] and perform the integral.

[8]We would like to thank Andy Strominger for pointing this out to us.

The shadow operator $\widetilde{\mathcal{O}}_\Delta$ transforms as a scalar conformal primary operator with conformal dimension $d-\Delta$. The normalization constant is chosen so that our conformal primary wavefunctions transform nicely under (2.20).

More generally, given a d-dimensional conformal primary $\mathcal{O}_{a_1\cdots a_J}(\vec{w})$ in the symmetric traceless rank-J representation of $SO(d)$ with dimension Δ, its shadow $\widetilde{\mathcal{O}}_{a_1\cdots a_J}(\vec{w})$ can be most conveniently computed in terms of its uplift $\mathcal{O}_{\mu_1\cdots\mu_J}(\vec{w})$ to the embedding space $\mathbb{R}^{1,d+1}$ [34]:

$$\widetilde{\mathcal{O}}_{\mu_1\cdots\mu_J}(\vec{w}) = \frac{\Gamma(\Delta+J)}{\pi^{\frac{d}{2}}(\Delta-1)_J\Gamma(\Delta-\frac{d}{2})}\int d^d\vec{w}\,'\,\frac{\prod_{n=1}^J\left[\,\delta^{\nu_n}_{\mu_n}(-\frac{1}{2}q\cdot q')+\frac{1}{2}q'_{\mu_n}q^{\nu_n}\right]}{(-\frac{1}{2}q\cdot q')^{d-\Delta+J}}\mathcal{O}_{\nu_1\cdots\nu_J}(\vec{w}\,')\,, \tag{2.21}$$

where $(a)_J\equiv\Gamma(a+J)/\Gamma(a)$ and $q^\mu=q^\mu(\vec{w})$ as in (2.9). The uplifted operator $\mathcal{O}_{\mu_1\cdots\mu_J}(\vec{w})$ is transverse to q^μ and is defined modulo terms of the form $q^{\mu_i}\Lambda^{\mu_1\cdots\hat{\mu}_i\cdots\mu_J}(\vec{w})$. Note that $-\frac{1}{2}q\cdot q'=|\vec{w}-\vec{w}\,'|^2$. We recover the d-dimensional primary $\mathcal{O}_{a_1\cdots a_J}(\vec{w})$ via the projection:

$$\mathcal{O}_{a_1\cdots a_J}(\vec{w}) = \frac{\partial q^{\mu_1}}{\partial w^{a_1}}\cdots\frac{\partial q^{\mu_J}}{\partial w^{a_J}}\mathcal{O}_{\mu_1\cdots\mu_J}(\vec{w})\,, \tag{2.22}$$

and similarly for its shadow $\widetilde{\mathcal{O}}_{a_1\cdots a_J}(\vec{w})$. The shadow operator $\widetilde{\mathcal{O}}_{a_1\cdots a_J}(\vec{w})$ transforms as a spin-J conformal primary with dimension $d-\Delta$ under $SO(1,d+1)$.

To study the shadow transform of our conformal primary wavefunction, let us first note a useful identity (see, for example, [34])

$$\int d^d\vec{z}\frac{1}{|\vec{z}-\vec{w}|^{2(d-\Delta)}}\frac{1}{(-q(\vec{z})\cdot X)^\Delta} = \frac{\pi^{\frac{d}{2}}\Gamma(\Delta-\frac{d}{2})}{\Gamma(\Delta)}\frac{(-X^2)^{\frac{d}{2}-\Delta}}{(-q(\vec{w})\cdot X)^{d-\Delta}}\,. \tag{2.23}$$

Now the shadow transform of our conformal primary wavefunction (2.19) directly follows from (2.23):

$$\boxed{\widetilde{\phi^\pm_\Delta}(X;\vec{w}) = \phi^\pm_{d-\Delta}(X;\vec{w})}\,. \tag{2.24}$$

Thus the conformal primary wavefunctions $\phi^\pm_\Delta$ and $\phi^\pm_{d-\Delta}$ should not be counted as linearly independent solutions to the massive Klein-Gordon equation as they are related by a shadow transform.

2.4 Integral Transform: From Amplitudes to Conformal Correlators

In (2.14) we determined the change of basis from the plane wave $e^{\pm ip\cdot X}$ to $\phi^\pm_\Delta(X^\mu;\vec{w})$ for a single wavefunction. This change of basis can be imminently extended to any n-point scattering amplitude of massive scalars. Let $\mathcal{A}(p_i^\mu)$ be such an amplitude in momentum space, including the momentum conservation delta function $\delta^{(d+2)}(\sum_i p_i^\mu)$. We can then define an integral transform that takes this amplitude to the basis of conformal primary wavefunctions,

$$\widetilde{\mathcal{A}}(\Delta_i, \vec{w}_i) \equiv \prod_{k=1}^{n} \int_{H_{d+1}} [d\hat{p}_k]\, G_{\Delta_k}(\hat{p}_k; \vec{w}_k)\ \ \mathcal{A}(\pm m_i \hat{p}_i^\mu)\,, \tag{2.25}$$

where we have parametrized an outgoing (incoming) on-shell momentum as $p_i^\mu = m_i \hat{p}^\mu$ ($p_i^\mu = -m_i \hat{p}^\mu$) with $\hat{p}_i^2 = -1$ as in (2.6). Thanks to the conformal covariance (2.2) of the wavefunctions $\phi^\pm_\Delta(\vec{w})$, the scattering amplitude in this basis transforms covariantly as a d-dimensional CFT n-point function of scalar primaries with dimensions Δ_i,

$$\widetilde{\mathcal{A}}(\Delta_i, \vec{w}_i'(\vec{w}_i)) = \prod_{k=1}^{n} \left|\frac{\partial \vec{w}_k'}{\partial \vec{w}_k}\right|^{-\Delta_k/d} \widetilde{\mathcal{A}}(\Delta_i, \vec{w}_i)\,. \tag{2.26}$$

Hence the change of basis (2.25) is implemented as an integral transformation that takes a scattering amplitude $\mathcal{A}(p_i^\mu)$ to a d-dimensional conformal correlator $\widetilde{\mathcal{A}}(\Delta_i, \vec{w}_i)$.

3 A Conformal Primary Basis

It is natural to ask for what, if any, range of the conformal dimension Δ will the set of conformal primary wavefunctions $\phi^\pm_\Delta(X^\mu; \vec{w})$ form a basis for delta-function-normalizable, outgoing/incoming solutions of the Klein-Gordon equation.[9] In this section we show that the range of Δ can be chosen to be the principal continuous series of $SO(1, d+1)$.

3.1 Inverse Transform and Principal Continuous Series

We begin by seeking the inverse transform of (2.14), i.e. the expansion of plane waves into the conformal primary wavefunctions.[10] Since the plane waves form a basis, if we can expand them on a certain set of conformal primary wavefunctions, then the latter also forms a (possibly over-complete) basis. This is possible if the bulk-to-boundary propagator, which is the Fourier coefficient in (2.14), satisfies certain orthonormality conditions for some range of

[9]Throughout this paper, we will consider the space of *complex* solutions to the wave equation. As usual, a reality condition is needed if one wants to perform a mode expansion of a real field on these complex solutions.

[10]We would like to thank H.-Y. Chen, X. Dong, J. Maldacena, and H. Ooguri for discussions on this point.

the conformal dimension Δ. Indeed, this is the case if Δ belongs to the *principal continuous series* of the irreducible unitary $SO(1,d+1)$ representations,[11]

$$\boxed{\Delta \in \frac{d}{2} + i\mathbb{R}}\,. \tag{3.1}$$

One orthonormality condition we need for the H_{d+1} scalar bulk-to-boundary propagator is [35]:

$$\int_{-\infty}^{\infty} d\nu\, \mu(\nu) \int d^d\vec{w}\, G_{\frac{d}{2}+i\nu}(\hat{p}_1;\vec{w}) G_{\frac{d}{2}-i\nu}(\hat{p}_2;\vec{w}) = \delta^{(d+1)}(\hat{p}_1,\hat{p}_2)\,, \tag{3.2}$$

where $\delta^{(d+1)}(\hat{p}_1,\hat{p}_2)$ is the $SO(1,d+1)$ invariant delta function in H_{d+1}. The measure factor $\mu(\nu)$ is

$$\mu(\nu) = \frac{\Gamma(\frac{d}{2}+i\nu)\Gamma(\frac{d}{2}-i\nu)}{4\pi^{d+1}\Gamma(i\nu)\Gamma(-i\nu)}\,, \tag{3.3}$$

which is an even, non-negative function of ν. The second orthonormality condition is [35]:

$$\int_{H_{d+1}} [d\hat{p}]\, G_{\frac{d}{2}+i\nu}(\hat{p};\vec{w}_1) G_{\frac{d}{2}+i\bar{\nu}}(\hat{p};\vec{w}_2) = \tag{3.4}$$

$$2\pi^{d+1}\frac{\Gamma(i\nu)\Gamma(-i\nu)}{\Gamma(\frac{d}{2}+i\nu)\Gamma(\frac{d}{2}-i\nu)}\delta(\nu+\bar{\nu})\delta^{(d)}(\vec{w}_1-\vec{w}_2) + 2\pi^{\frac{d}{2}+1}\frac{\Gamma(i\nu)}{\Gamma(\frac{d}{2}+i\nu)}\delta(\nu-\bar{\nu})\frac{1}{|\vec{w}_1-\vec{w}_2|^{2(\frac{d}{2}+i\nu)}}\,.$$

Now we are ready to write down the inverse transform of (2.14). Combining (2.14) and (3.2), we immediately obtain:

$$\boxed{e^{\pm im\hat{p}\cdot X} = 2\int_0^{\infty} d\nu\, \mu(\nu) \int d^d\vec{w}\; G_{\frac{d}{2}-i\nu}(\hat{p};\vec{w})\; \phi^{\pm}_{\frac{d}{2}+i\nu}(X^\mu;\vec{w})}\,, \tag{3.5}$$

where we have used (2.23) and (2.24) to rewrite the expansion only on wavefunctions with non-negative ν.

Given that the plane waves form a basis for the normalizable solutions of the Klein-Gordon equation, it is tempting to conclude from (3.5) that the conformal primary wavefunctions on the principal continuous series with non-negative ν form a basis too. We have to check the following two conditions, however, in order to prove the above assertion:

[11]More precisely, the principal continuous series representations are labeled by a conformal dimension $\Delta \in \frac{d}{2} + i\mathbb{R}$ and a representation, the spin, of $SO(d)$. In this section we will consider the spin-zero principal continuous series representations, while in Section 5 and 6 we will encounter spin-one and spin-two (i.e. symmetric traceless rank-two tensors of $SO(d)$) representations, respectively.

- Are the conformal primary wavefunctions $\phi^{\pm}_{\Delta}(X^\mu;\vec{w})$ with $\Delta\in\frac{d}{2}+i\mathbb{R}_{\geq 0}$ and $\vec{w}\in\mathbb{R}^d$ linearly independent of each other?

- Are the conformal primary wavefunctions delta-function-normalizable with respect to the Klein-Gordon norm?

We will shortly give positive answers to both questions in Section 3.2 by explicit computation of the Klein-Gordon inner product between conformal primary wavefunctions.

3.2 Klein-Gordon Inner Product

In studying the solution space for the massive Klein-Gordon equation, we focus on wavefunctions that are (delta-function-)normalizable with respect to a certain inner product. A natural inner product between complex wavefunctions is the Klein-Gordon inner product defined as

$$(\Phi_1,\Phi_2)=-i\int d^{d+1}X^i\ \left[\,\Phi_1(X)\,\partial_{X^0}\Phi_2^*(X)-\partial_{X^0}\Phi_1(X)\,\Phi_2^*(X)\right]\,, \tag{3.6}$$

where $i=1,2,\cdots,d+1$ is an index for the spatial directions in $\mathbb{R}^{1,d+1}$ and $*$ stands for complex conjugation. Using the Klein-Gordon equation, one can show that the above inner product does not depend on the choice of the Cauchy surface we integrate over. The plane waves, for example, are delta-function-normalizable with respect to this inner product:[12]

$$(e^{\pm ip\cdot X},e^{\pm ip'\cdot X})=\pm 2(2\pi)^{d+1}\,p^0\,\delta^{(d+1)}(p^i-p^{i\prime})\,. \tag{3.7}$$

Furthermore, they form a basis of normalizable solutions to the Klein-Gordon equation.

Let us compute the Klein-Gordon inner product of two conformal primary wavefunctions:

$$\begin{aligned}\Big(\phi^{\pm}_{\frac{d}{2}+i\nu_1}(X^\mu;\vec{w}_1),\phi^{\pm}_{\frac{d}{2}+i\nu_2}(X^\mu;\vec{w}_2)\Big)&\\ =\pm\frac{2^{d+3}\pi^{2d+2}}{m^d}&\frac{\Gamma(i\nu_1)\Gamma(-i\nu_1)}{\Gamma(\frac{d}{2}+i\nu_1)\Gamma(\frac{d}{2}-i\nu_1)}\,\delta(\nu_1-\nu_2)\,\delta^{(d)}(\vec{w}_1-\vec{w}_2)\\ \pm\frac{2^{d+3}\pi^{\frac{3d}{2}+2}}{m^d}&\frac{\Gamma(i\nu_1)}{\Gamma(\frac{d}{2}+i\nu_1)}\delta(\nu_1+\nu_2)\frac{1}{|\vec{w}_1-\vec{w}_2|^{2(\frac{d}{2}+i\nu_1)}}\,,\end{aligned} \tag{3.8}$$

[12]Strictly speaking, while (3.6) is a positive-definite inner product on the space of outgoing (i.e. positive energy) wavefunctions, we should use the minus of (3.6) as a positive-definite inner product on the space of incoming (i.e. negative energy) wavefunctions.

where we have used the second orthonormality condition (3.4). As we saw in the last section, conformal primary wavefunctions with negative ν are linearly related to those with positive ν by the shadow transform. Therefore we only need to consider the inner product between those with positive ν, in which case the second term in (3.8) drops out. Note that the coefficient of the surviving term (i.e. the first term) is positive (negative) definite for outgoing (incoming) wavefunctions, as it should be for an inner product on single particle solutions.

From (3.8), we then conclude that the conformal primary wavefunctions are delta-function-normalizable with respect to the Klein-Gordon inner product. Furthermore, the set of conformal primary wavefunctions $\phi^{\pm}_{\frac{d}{2}+i\nu}(X^\mu;\vec{w})$ with non-negative ν and $\vec{w}\in\mathbb{R}^d$ are orthogonal to each other, and hence are linearly independent.

3.3 The Massive Scalar Conformal Primary Basis

We are finally ready to write down the complete set of linearly independent conformal primary wavefunctions that span the space of outgoing or incoming solutions to the massive Klein-Gordon equation. We will call such bases of wavefunctions the outgoing (+) and incoming (−) *conformal primary bases* $\mathcal{B}^{\pm}$.

Let us recap the logic. In (3.5) we showed that the plane waves can be expanded upon conformal primary wavefunctions on the principal continuous series with non-negative ν, so the latter must span the whole solution space. In (3.8) we showed that the conformal primary wavefunctions with non-negative ν and $\vec{w}\in\mathbb{R}^d$ are delta-function-normalizable and linearly independent of each other. Thus we conclude that the outgoing/incoming conformal primary bases $\mathcal{B}^{\pm}$ for the massive Klein-Gordon equation can be chosen to be

$$\boxed{\mathcal{B}^{\pm}=\left\{\,\phi^{\pm}_{\frac{d}{2}+i\nu}(X;\vec{w})\,\middle|\,\nu\geq 0,\vec{w}\in\mathbb{R}^d\right\}}\,,\tag{3.9}$$

where we recall that the plus and minus superscripts denote outgoing and incoming wavefunctions, respectively. We contrast the massive conformal primary bases with the plane wave bases in Table 1.

Alternatively, the shadows of the bases (3.9) are equally good conformal primary bases for the massive Klein-Gordon solutions:

$$\boxed{\widetilde{\mathcal{B}}^{\pm}=\left\{\,\phi^{\pm}_{\frac{d}{2}+i\nu}(X;\vec{w})\,\middle|\,\nu\leq 0,\vec{w}\in\mathbb{R}^d\right\}}\,.\tag{3.10}$$

To sum up, we have identified a pair of bases $\mathcal{B}^{+}$ and $\widetilde{\mathcal{B}}^{+}$ for the outgoing, normalizable solutions to the massive Klein-Gordon equation. Similarly, we have identified a pair of bases

$\mathcal{B}^-$ and $\widetilde{\mathcal{B}}^-$ for the incoming, normalizable solutions to the massive Klein-Gordon equation. The four bases are related by complex conjugation and shadow transformation as below:

$$\begin{array}{cccc} \text{Outgoing}: & \mathcal{B}^+ & \underset{\text{shadow}}{\longleftrightarrow} & \widetilde{\mathcal{B}}^+ \\ & \big\updownarrow{\scriptstyle \text{c.c}} & & \big\updownarrow{\scriptstyle \text{c.c}} \\ \text{Incoming}: & \widetilde{\mathcal{B}}^- & \underset{\text{shadow}}{\longleftrightarrow} & \mathcal{B}^- \end{array} \tag{3.11}$$

For odd d, there are also principal discrete series irreducible unitary $SO(1,d+1)$ representations with conformal dimension $\Delta = \frac{d}{2} + \mathbb{Z}_+$. The conformal primary wavefunctions for the discrete series, however, are *not* normalizable with respect to the Klein-Gordon inner product (3.6). To see this, note that the Klein-Gordon inner product between two conformal primary wavefunctions with conformal dimensions Δ_1 and Δ_2 is proportional to

$$\int_{H_{d+1}} [d\hat{p}]\, G_{\Delta_1}(\hat{p};\vec{w}_1)\, G_{\Delta_2^*}(\hat{p};\vec{w}_2)\,, \tag{3.12}$$

which diverges for positive Δ_1 and Δ_2. On the other hand, the wavefunction is delta-function-normalizable if Δ is on the principal continuous series. Therefore we will not consider the discrete series representations in this paper.

Recently, a simple solution to the conformal crossing equation of $SO(1,d+1)$ was found [41] for all d.[13] The spectrum of conformal primaries consists of the whole continuum of principal continuous series representations plus the principal discrete series. It will be interesting to explore the connection between this conformal crossing solution and scattering amplitudes in $\mathbb{R}^{1,d+1}$.

4 Massless Scalars

So far we have been studying the conformal primary basis of the massive Klein-Gordon equation. In this section we show how the massless limit of our massive conformal primary wavefunction reduces to a combination of the Mellin transform of the plane wave and its shadow. We will then determine the conformal primary basis of solutions to the massless Klein-Gordon equation with nonvanishing inner product.

4.1 Mellin Transform

Let us consider the massless limit of the massive conformal primary wavefunction written in the integral representation (2.14). It will be convenient to change the integration variable

[13] See also [42] for a crossing solution in one dimension on the principal continuous series.

from y to

$$\omega \equiv \frac{m}{2y}\,. \tag{4.1}$$

The massless limit is taken by sending $y = \frac{m}{2\omega} \to 0$ (while holding ω fixed) in the bulk-to-boundary propagator,

$$G_\Delta(y,\vec{z};\vec{w}) \underset{m\to 0}{\longrightarrow} \pi^{\frac{d}{2}}\frac{\Gamma(\Delta-\frac{d}{2})}{\Gamma(\Delta)}y^{d-\Delta}\delta^{(d)}(\vec{z}-\vec{w}) + \frac{y^\Delta}{|\vec{z}-\vec{w}|^{2\Delta}} + \cdots\,, \tag{4.2}$$

where $\cdots$ are higher order terms in the small y expansion. Note that for our purpose $\Delta = \frac{d}{2}+i\nu$,[14] so the second term above is not smaller than the first one.

The massless limit of the conformal primary wavefunction is

$$\begin{aligned}\phi^\pm_{\frac{d}{2}+i\nu}(X;\vec{w}) \underset{m\to 0}{\longrightarrow} &\left(\frac{m}{2}\right)^{-\frac{d}{2}-i\nu}\frac{\pi^{\frac{d}{2}}\Gamma(i\nu)}{\Gamma(\frac{d}{2}+i\nu)}\int_0^\infty d\omega\,\omega^{\frac{d}{2}+i\nu-1}e^{\pm i\omega q(\vec{w})\cdot X}\\ &+\left(\frac{m}{2}\right)^{-\frac{d}{2}+i\nu}\int d^d\vec{z}\frac{1}{|\vec{z}-\vec{w}|^{2(\frac{d}{2}+i\nu)}}\int_0^\infty d\omega\omega^{\frac{d}{2}-i\nu-1}\,e^{\pm i\omega q(\vec{z})\cdot X}\,,\end{aligned} \tag{4.3}$$

where $q^\mu(\vec{w}) = (1+|\vec{w}|^2, 2\vec{w}, 1-|\vec{w}|^2)$ as in (2.9). We see that the massive conformal primary wavefunction does not have a well-defined massless limit because of the phases $m^{\pm i\nu}$. Nonetheless, we can extract the massless scalar conformal primary wavefunction from the coefficients of these phases.

The first term in (4.3) takes the form of a Mellin transform of the plane wave. This is the *massless scalar conformal primary wavefunction*, which, up to an overall constant, can be regularized as [3–5,43,44]

$$\boxed{\varphi^\pm_\Delta(X^\mu;\vec{w}) \equiv \int_0^\infty d\omega\,\omega^{\Delta-1}\,e^{\pm i\omega q\cdot X-\epsilon\omega} = \frac{(\mp i)^\Delta\Gamma(\Delta)}{(-q(\vec{w})\cdot X\mp i\epsilon)^\Delta}}\,, \tag{4.4}$$

with $\epsilon>0$. Since we obtain $\varphi^\pm_\Delta(X^\mu;\vec{w})$ from the massless limit of the conformal primary wavefunction, it automatically satisfies the defining properties (2.1) (with $m=0$) and (2.2). The second term in (4.3) is the shadow of $\varphi^\pm_\Delta(X^\mu;\vec{w})$, and so is not a linearly independent wavefunction.

In fact, the massless scalar conformal primary wavefunction (4.4) is, up to a normalization constant, nothing but the H_{d+1} bulk-to-boundary propagator $G_\Delta(\hat{p};\vec{w})$ (2.8), with the unit timelike vector $\hat{p}$ extended to a generic point X^μ in $\mathbb{R}^{1,d+1}$ [4]. Indeed, the bulk-to-boundary

[14] Strictly speaking we have not justified that the massless conformal primary wavefunctions on the principal continuous series form a basis of wavefunctions. We will show this in Section 4.2.

propagator satisfies the two defining properties (2.1) and (2.2) of the conformal primary wavefunction. First, from (2.13), it manifestly has the desired conformal covariance (2.2). Second, when extended to a generic point in $\mathbb{R}^{1,d+1}$, it satisfies the massless Klein-Gordon equation via a hyperbolic slicing of the d'Alembert operator in flat space [4, 43, 45, 46]. We will make use of this fact for the spinning conformal primary wavefunctions in later sections.

Let us elaborate more on the properties of the massless conformal primary wavefunction. For a fixed null momentum q^μ, the Lorentz boost along the spatial direction of q^μ acts as

$$\text{Boost}: \ q \cdot X \to \lambda \, (q \cdot X) \ \Rightarrow \ \varphi_\Delta(X^\mu; \vec{w}) \to \lambda^{-\Delta} \varphi_\Delta(X^\mu; \vec{w}) \,, \tag{4.5}$$

where we have used (2.10). Hence the dilation of the conformal primary wavefunction $\varphi^\pm_\Delta$ is nothing but a Lorentz boost in $\mathbb{R}^{1,d+1}$.

The change of basis from plane waves to conformal primary wavefunctions is physically more intuitive in the massless case. Given a null momentum q^μ in $\mathbb{R}^{1,d+1}$, it can be parametrized by a scale ω and a point $\vec{w} \in \mathbb{R}^d$ as in (2.9). The change of basis is implemented by a Mellin transform on this scale ω, while the point $\vec{w} \in \mathbb{R}^d$ is directly identified as the position of the d-dimension conformal primary. The conformal dimension Δ is the dual variable for the scale ω. By contrast, there is no direct way to relate a timelike momentum $\hat{p}^\mu$ to a point in $\mathbb{R}^d$ in the massive case. The change of basis is done by a Fourier transform integrating over all $\hat{p}$ to conformal primary wavefunctions labeled by the dual variables $\Delta, \vec{w}$ as in (2.14).

4.2 The Massless Scalar Conformal Primary Basis

Since the massive conformal primary wavefunction does not have a well-defined massless limit, we should study the completeness question separately for the massless case.

Before we dive into the space of conformal primary wavefunctions, let us note a qualitative difference between the massless and massive solution spaces to the Klein-Gordon equation. In the massless case, the constant wavefunction is a solution to the massless Klein-Gordon equation, which sits at the intersection between the outgoing and the incoming solution spaces. On the other hand, the massive outgoing and incoming solution spaces are disjoint from each other. The constant wavefunction has strictly zero-energy and thus vanishing Klein-Gordon norm (3.7). *We will exclude the constant wavefunction from our definition of either the outgoing or the incoming solution space.* In fact, we will see that the conformal primary wavefunctions do not cover the constant wavefunction.

Let us start with the inverse transform of (4.4). The inverse Mellin transform of the plane wave is (see, for example, [3]),

$$e^{\pm i\omega q\cdot X-\epsilon\omega}=\int_{-\infty}^{\infty}\frac{d\nu}{2\pi}\,\omega^{-c-i\nu}\frac{(\mp i)^{c+i\nu}\Gamma(c+i\nu)}{(-q\cdot X\mp i\epsilon)^{c+i\nu}}\,,\qquad \omega>0\,, \tag{4.6}$$

where c can be any positive number. Hence all the massless plane waves *except for the constant wavefunction* can be expanded on the conformal primary wavefunctions. In other words, for any positive c, the massless conformal primary wavefunctions (4.4) with $\Delta\in c+i\mathbb{R}$ form a (possibly non-normalizable) basis of nonzero energy solutions to massless Klein-Gordon equations.

Next we need to determine for what value of $c>0$ will the massless conformal primary wavefunction be delta-function-normalizable with respect to the Klein-Gordon inner product (3.6). The Klein-Gordon inner product of the massless conformal primary wavefunctions with $\Delta=c+i\nu$ is

$$\begin{aligned}&\big(\varphi^{\pm}_{c+i\nu_1}(X^\mu;\vec{w}_1)\,,\,\varphi^{\pm}_{c+i\nu_2}(X^\mu;\vec{w}_2)\big)\\&=\pm2(2\pi)^{d+1}\int_0^\infty d\omega_1\,\omega_1^{c-1+i\nu_1}\int_0^\infty d\omega_2\,\omega_2^{c-1-i\nu_2}\,\omega_1(1+|\vec{w}_1|^2)\,\delta^{(d+1)}(\omega_1q^i(\vec{w}_1)-\omega_2q^i(\vec{w}_2))\\&=\pm4\pi^{d+1}\delta^{(d)}\left(\vec{w}_1-\vec{w}_2\right)\int_0^\infty d\omega_2\,\omega_2^{2c-d+i\nu_1-i\nu_2-1}\,.\end{aligned} \tag{4.7}$$

The ω_2 integral is divergent unless $c=\frac{d}{2}$, in which case,

$$\int_0^\infty d\omega\,\omega^{i\nu-1}=2\pi\delta(\nu)\,. \tag{4.8}$$

The Klein-Gordon inner product when $c=\frac{d}{2}$ is (with $\nu_1,\nu_2\in\mathbb{R}$)

$$\Big(\varphi^{\pm}_{\frac{d}{2}+i\nu_1}(X^\mu;\vec{w}_1)\,,\,\varphi^{\pm}_{\frac{d}{2}+i\nu_2}(X^\mu;\vec{w}_2)\Big)=\pm8\pi^{d+2}\,\delta(\nu_1-\nu_2)\,\delta^{(d)}\left(\vec{w}_1-\vec{w}_2\right)\,. \tag{4.9}$$

Thus, the massless conformal primary wavefunctions $\varphi^{\pm}_{\Delta}$ are delta-function-normalizable if the conformal dimensions are chosen to be $\Delta=\frac{d}{2}+i\nu$ with $\nu\in\mathbb{R}$, which are again the principal continuous series representations of $SO(1,d+1)$. The same conclusion was reached from studying the AdS holography on each hyperbolic slice of Minkowski space [3, 4].

Notice that since the constant wavefunction is not spanned by the outgoing (incoming) conformal primary wavefunctions, the inner products of the latter are strictly positive- (negative-) definite as shown in (4.9).

The Klein-Gordon inner product further implies that the massless conformal primary wavefunctions with different $\nu\in\mathbb{R}$ are orthogonal to each other. In particular, $\varphi_{\frac{d}{2}+i\nu}$ is *not* linearly related to $\varphi_{\frac{d}{2}-i\nu}$, in contrast to the massive case (2.24). Instead, using (2.23), the shadow of $\varphi_{\frac{d}{2}+i\nu}$ is

$$\begin{aligned}\widetilde{\varphi^{\pm}_{\frac{d}{2}+i\nu}}(X^\mu;\vec{w}) &= \frac{\Gamma(\frac{d}{2}+i\nu)}{\pi^{\frac{d}{2}}\Gamma(i\nu)}\int d^d\vec{z}\,\frac{1}{|\vec{z}-\vec{w}|^{2(\frac{d}{2}-i\nu)}}\varphi^{\pm}_{\frac{d}{2}+i\nu}(X^\mu;\vec{z})\\ &= (\mp i)^{\frac{d}{2}+i\nu}\Gamma(\frac{d}{2}+i\nu)\,\frac{(-X^2)^{-i\nu}}{(-q(\vec{w})\cdot X\mp i\epsilon)^{\frac{d}{2}-i\nu}}\,,\end{aligned} \tag{4.10}$$

which, up to a normalization constant, is $(-X^2)^{-i\nu}\varphi^{\pm}_{\frac{d}{2}-i\nu}(X^\mu;\vec{w})$. The Klein-Gordon inner product between $\varphi^{\pm}_{\frac{d}{2}+i\nu}$ and its shadow $\widetilde{\varphi^{\pm}_{\frac{d}{2}+i\nu}}$ is then a power law term as a CFT two-point function:

$$\left(\varphi^{\pm}_{\frac{d}{2}+i\nu_1}(X^\mu;\vec{w}_1)\,,\,\widetilde{\varphi^{\pm}_{\frac{d}{2}+i\nu_2}}(X^\mu;\vec{w}_2)\right) = \pm 8\pi^{\frac{d}{2}+2}\frac{\Gamma(\frac{d}{2}-i\nu_1)}{\Gamma(-i\nu_1)}\,\delta(\nu_1-\nu_2)\,\frac{1}{|\vec{w}_1-\vec{w}_2|^{2(\frac{d}{2}+i\nu_1)}}\,. \tag{4.11}$$

We conclude that the massless conformal primary bases $\mathcal{B}^{\pm}_{m=0}$ for the outgoing (+) and incoming (−), delta-function-normalizable solutions of the massless Klein-Gordon equation are

$$\boxed{\mathcal{B}^{\pm}_{m=0} = \left\{\varphi^{\pm}_{\frac{d}{2}+i\nu}(X^\mu;\vec{w})\,\middle|\,\nu\in\mathbb{R}\,,\vec{w}\in\mathbb{R}^d\right\}}\,. \tag{4.12}$$

We compare the plane wave bases with the massless conformal primary bases in Table 2. We emphasize again that the constant wavefunction is excluded from our definition of either the outgoing or the incoming solution space. In particular, the constant wavefunction is not spanned by the conformal primary wavefunctions.

Alternatively, the shadows of the bases (4.12) are equally good conformal primary bases for the outgoing/incoming massless Klein-Gordon solutions:

$$\boxed{\widetilde{\mathcal{B}}^{\pm}_{m=0} = \left\{\widetilde{\varphi^{\pm}_{\frac{d}{2}+i\nu}}(X^\mu;\vec{w})\,\middle|\,\nu\in\mathbb{R},\vec{w}\in\mathbb{R}^d\right\}}\,. \tag{4.13}$$

We have thus identified a pair of bases $\mathcal{B}^{+}_{m=0}$ and $\widetilde{\mathcal{B}}^{+}_{m=0}$ for the outgoing, normalizable solutions of the massless Klein-Gordon equation. Similarly, we have identified a pair of bases $\mathcal{B}^{-}_{m=0}$ and $\widetilde{\mathcal{B}}^{-}_{m=0}$ for the incoming, normalizable solutions of the massless Klein-Gordon equation. The four bases are related by complex conjugation and shadow transform as below:

$$\begin{array}{llcl}\text{Outgoing}: & \mathcal{B}^{+}_{m=0} & \underset{\text{shadow}}{\longleftrightarrow} & \widetilde{\mathcal{B}}^{+}_{m=0}\\ & \quad\updownarrow_{\text{c.c}} & & \quad\updownarrow_{\text{c.c}}\\ \text{Incoming}: & \mathcal{B}^{-}_{m=0} & \underset{\text{shadow}}{\longleftrightarrow} & \widetilde{\mathcal{B}}^{-}_{m=0}\end{array} \tag{4.14}$$

5 Photons

In this and the following section we present a detailed discussion of massless conformal primary wavefunctions with spin. A massless state in $\mathbb{R}^{1,d+1}$ sits in a representation of the massless little group $SO(d)$, the spin or helicity. For a spinning conformal primary wavefunction, this spin gets interpreted as that of a conformal primary in d dimensions. For example, an outgoing conformal primary wavefunctions in $\mathbb{R}^{1,3}$ with positive helicity transforms as a spin $+1$ conformal primary in two dimensions under $SL(2,\mathbb{C})$.

The spin-one massless conformal primary wavefunction in $(3+1)$ dimensions has been constructed in [4]. In Section 5.1 we review this construction and extend it to general spacetime dimensions. In Section 5.2 we discuss how conformal covariance fixes a particular gauge choice for the conformal primary wavefunctions. We also discuss conformal primary wavefunctions that happen to be pure gauge. In Section 5.3, we show that spin-one conformal primary wavefunctions on the principal continuous series $\Delta \in \frac{d}{2} + i\mathbb{R}$ are normalizable and span the space of plane wave solutions to the Maxwell equation.

Since there are no propagating degrees of freedom for a spin-one field in $(1+1)$ spacetime dimensions, we will assume $d \geq 1$ in this section.

5.1 Massless Spin-One Conformal Primary Wavefunctions in General Dimensions

The defining properties of the outgoing $(+)$ and incoming $(-)$ *massless spin-one conformal primary wavefunction* $A^{\Delta\pm}_{\mu a}(X^\mu;\vec{w})$ in $\mathbb{R}^{1,d+1}$ are ($\mu = 0, 1, \cdots, d+1$ and $a = 1, \cdots, d$):

1. It satisfies the $(d+2)$-dimensional Maxwell equation,

$$\left(\frac{\partial}{\partial X^\sigma}\frac{\partial}{\partial X_\sigma}\delta^\mu_\nu - \frac{\partial}{\partial X^\nu}\frac{\partial}{\partial X_\mu}\right) A^{\Delta\pm}_{\mu a}(X^\rho;\vec{w}) = 0\,. \tag{5.1}$$

2. It transforms both as a $(d+2)$-dimensional vector and a d-dimensional spin-one conformal primary with conformal dimension Δ under an $SO(1,d+1)$ Lorentz transformation:

$$A^{\Delta\pm}_{\mu a}\left(\Lambda^\rho{}_\nu X^\nu;\vec{w}\,'(\vec{w})\right) = \frac{\partial w^b}{\partial w'^a}\left|\frac{\partial \vec{w}'}{\partial \vec{w}}\right|^{-(\Delta-1)/d} \Lambda_\mu{}^\sigma A^{\Delta\pm}_{\sigma b}(X^\rho;\vec{w})\,, \tag{5.2}$$

where $\vec{w}\,'(\vec{w})$ is an element of $SO(1,d+1)$ defined in (2.3) and $\Lambda^\mu{}_\nu$ is the associated group element in the $(d+2)$-dimensional representation.

As usual, a solution to the Maxwell equation is subject to the ambiguity of gauge transformations. We will return to this in Section 5.2.

Similar to the scalar massless conformal primary wavefunction, its spin-one analog has been obtained [4] from the spin-one bulk-to-boundary propagator in H_{d+1} [36,47]. The uplift of the latter in the embedding space $\mathbb{R}^{1,d+1}$ with conformal dimension Δ is [35]:

$$G^{\Delta}_{\mu;\nu}(\hat{p};q) = \frac{(-q\cdot\hat{p})\eta_{\mu\nu} + q_\mu \hat{p}_\nu}{(-q\cdot\hat{p})^{\Delta+1}}\,, \tag{5.3}$$

where $\hat{p}^\mu$ is a unit timelike vector and q^ν is a null vector (2.9), both living in $\mathbb{R}^{1,d+1}$. The uplifted bulk-to-boundary propagator satisfies the following two transversality conditions:

$$\hat{p}^\mu G^{\Delta}_{\mu;\nu}(\hat{p};q) = 0\,, \qquad q^\nu G^{\Delta}_{\mu;\nu}(\hat{p};q) = 0\,. \tag{5.4}$$

Under $SO(1,d+1)$, $G^{\Delta}_{\mu;\nu}$ transforms as a $(d+2)$-dimensional rank-two tensor but a scalar conformal primary in d dimensions:

$$G^{\Delta}_{\mu;\nu}(\Lambda\hat{p};\Lambda q) = \left|\frac{\partial \vec{w}\,'}{\partial \vec{w}}\right|^{-\Delta/d} \Lambda^{\rho}{}_{\mu}\Lambda^{\sigma}{}_{\nu}\, G^{\Delta}_{\rho;\sigma}(\hat{p};q)\,. \tag{5.5}$$

This is not quite what we want for the conformal primary wavefunction (5.2), but its projection

$$\frac{\partial q^\nu}{\partial w^a} G^{\Delta}_{\mu;\nu}(\hat{p};q) \tag{5.6}$$

on the second index ν does have the desired conformal covariance (5.2). This can be shown by the second transversality condition in (5.4) and the $SO(1,d+1)$ transformation of $\partial_a q^\mu \equiv \frac{\partial}{\partial w^a} q^\mu(\vec{w})$:

$$\partial_{a'} q^\mu(\vec{w}\,') = \frac{\partial w^b}{\partial w'^a}\left|\frac{\partial \vec{w}\,'}{\partial \vec{w}}\right|^{1/d} \Lambda^{\mu}{}_{\nu}\partial_b q^\nu(\vec{w}) + \frac{\partial w^b}{\partial w'^a}\,\partial_b\left(\left|\frac{\partial \vec{w}\,'}{\partial \vec{w}}\right|^{1/d}\right)\Lambda^{\mu}{}_{\nu} q^\nu(\vec{w})\,. \tag{5.7}$$

We therefore identify the massless spin-one conformal primary wavefunction as the bulk-to-boundary propagator (5.6) with $\hat{p}$ replaced by X^μ [4]:

$$\boxed{A^{\Delta,\pm}_{\mu a}(X^\mu;\vec{w}) = \frac{\partial_a q_\mu}{(-q\cdot X \mp i\epsilon)^{\Delta}} + \frac{\partial_a q\cdot X}{(-q\cdot X \mp i\epsilon)^{\Delta+1}} q_\mu}$$
$$= -\frac{1}{(-q\cdot X \mp i\epsilon)^{\Delta-1}}\frac{\partial}{\partial X^\mu}\frac{\partial}{\partial w^a}\log(-q\cdot X \mp i\epsilon)\,. \tag{5.8}$$

It is straightforward to check that (5.8) indeed satisfies the Maxwell equation.

An equally good spin-one conformal primary wavefunction would be the shadow $\widetilde{A^{\Delta,\pm}_{\mu a}}$ of (5.8). The shadow transform (2.21) for the uplifted wavefunction $A^{\Delta}_{\mu\nu}(X^\mu;\vec{w}) = G^{\Delta}_{\mu;\nu}(X;q)$ is[15]

$$\widetilde{A^{\Delta}_{\mu\nu}}(X;\vec{w}) = \frac{\Gamma(\Delta+1)}{\pi^{\frac{d}{2}}(\Delta-1)\Gamma(\Delta-\frac{d}{2})}\int d^d\vec{w}\,'\frac{(-\frac{1}{2}q\cdot q')\delta^\rho_\nu + \frac{1}{2}q'_\nu q^\rho}{(-\frac{1}{2}q\cdot q')^{d-\Delta+1}}\frac{(-q'\cdot X)\eta_{\mu\rho} + q'_\mu X_\rho}{(-q'\cdot X)^{\Delta+1}}\,. \quad (5.9)$$

Using (2.23), the above integral (excluding the prefactor $\frac{\Gamma(\Delta+1)}{\pi^{\frac{d}{2}}(\Delta-1)\Gamma(\Delta-\frac{d}{2})}$) can be computed to be

$$\begin{aligned}&\frac{\pi^{\frac{d}{2}}\Gamma(\Delta-\frac{d}{2})}{\Gamma(\Delta+1)}\frac{(-X^2)^{\frac{d}{2}-\Delta}}{(-q\cdot X)^{d-\Delta+2}}\\ &\times\left[(\Delta-1)(-q\cdot X)\left(\eta_{\mu\nu}(-q\cdot X)+q_\mu X_\nu\right)-(d+1-\Delta)q_\nu\left(q_\mu X^2 - X_\mu(q\cdot X)\right)\right]\,.\end{aligned} \quad (5.10)$$

The second term in the bracket drops out after the projection (2.22). In the end, the shadow wavefunction $\widetilde{A^{\Delta}_{\mu a}} = \partial_a q^\nu \widetilde{A^{\Delta}_{\mu\nu}}$ is simply:

$$\widetilde{A^{\Delta,\pm}_{\mu a}}(X^\mu;\vec{w}) = (-X^2)^{\frac{d}{2}-\Delta}A^{d-\Delta,\pm}_{\mu a}(X^\mu;\vec{w})\,. \quad (5.11)$$

One can easily show that the shadow wavefunction satisfies the two defining properties of massless spin-one conformal primary wavefunctions as well. Similar to the massless scalar case, for general Δ, the shadow transform does not take the conformal primary wavefunction $A^{\Delta,\pm}_{\mu a}$ back to itself with the shadow dimension $d-\Delta$, but to a different wavefunction.

5.2 Gauge Symmetry

Let us discuss the role of gauge symmetry in $(d+2)$ dimensions. A general gauge transformation has no nice conformal property and thus spoils the conformal covariance of conformal primary wavefunctions. In other words, conformal covariance (5.2) fixes a gauge for the conformal primary wavefunction (5.8). Indeed, (5.8) satisfies both the radial gauge and the Lorenz gauge conditions:

$$X^\mu A^{\Delta,\pm}_{\mu a}(X^\mu;\vec{w}) = 0\,, \qquad \partial^\mu A^{\Delta,\pm}_{\mu a}(X^\mu;\vec{w}) = 0\,. \quad (5.12)$$

The radial gauge condition $X^\mu A_\mu = 0$ comes from the first transversality condition for the bulk-to-boundary propagator in (5.4). Note that for any *on-shell* massless spin-one wavefunction, i.e. a solution to the Maxwell equation, it is always possible to perform

[15]For notational simplicity, we drop the $i\epsilon$ terms and the $\pm$ labels for the outgoing/incoming wavefunctions in this calculation.

a gauge transformation such that it satisfies *both* the radial gauge and the Lorenz gauge conditions [48].[16] Under these gauge conditions, the Maxwell equation simplifies to

$$\partial_\rho \partial^\rho A^{\Delta,\pm}_{\mu a}(X^\mu;\vec{w}) = 0\,. \tag{5.13}$$

All these properties also apply to the shadow wavefunction (5.11).

One natural question is whether the conformal primary wavefunction (5.8) or its shadow is ever a pure gauge in $(d+2)$ dimensions. A short calculation of their field strengths shows that, for any d, $A^{\Delta,\pm}_{\mu a}(X^\mu;\vec{w})$ is a pure gauge only if $\Delta = 1$:

$$A^{\Delta=1,\pm}_{\mu a} = -\frac{\partial}{\partial X^\mu}\frac{\partial}{\partial w^a}\log(-q\cdot X \mp i\epsilon)\,. \tag{5.14}$$

This pure gauge has been studied in the context of soft theorems in $(3+1)$ dimensions in [4]. Incidentally, $\Delta = 1$ is the shadow dimension (i.e. $\Delta \to d-\Delta$) of a conserved spin-one current in a d-dimensional CFT.

What about the shadow conformal primary wavefunctions (5.11)? The shadow transform (2.21) commutes with the X^μ derivative, and hence, at least naively, should map one pure gauge to another. This expectation, however, suffers from an important subtlety that we will describe below. The shadow transform (2.21) is strictly speaking not defined for $\Delta = 1$, in which case (5.8) is a pure gauge. Indeed, from (5.10) with $\Delta = 1$, we see that the shadow integral after the projection (2.22) is either zero or singular for integer d, and is accompanied by a singular prefactor $\Gamma(\Delta+1)/\pi^{\frac{d}{2}}(\Delta-1)\Gamma(\Delta-\frac{d}{2})$ in our normalization for the shadow transform. In practice, we first obtain the expression (5.11) by assuming a generic value of Δ, and only then we analytic continue it to all Δ.

Even though the shadow wavefunctions (5.11) satisfy all the required properties of conformal primary wavefunctions, the analytic continuation in Δ mentioned above generally does not preserve the pure gauge condition. The naive expectation that the shadow wavefunction with conformal dimension $d-1$ is a pure gauge in general spacetime dimensions is falsified by a direct calculation of the field strength of (5.11). Furthermore, one can show that the shadow wavefunction (5.11) is never a pure gauge if $d \neq 2$.

There is one exception in $(3+1)$ spacetime dimensions where the shadow wavefunction can be a pure gauge. When $d = 2$, the shadow wavefunction (5.11) happens to be the same as the pure gauge wavefunction (5.14):

$$\widetilde{A^{\Delta=1,\pm}_{\mu a}}(X;\vec{w}) = A^{\Delta=1,\pm}_{\mu a}(X;\vec{w}) \qquad (d=2)\,, \tag{5.15}$$

and is thus a pure gauge as well. We summarize spin-one conformal primary wavefunctions $A^{\Delta,\pm}_{\mu a}$ and their shadow $\widetilde{A^{\Delta,\pm}_{\mu a}}$ that are pure gauge in Table 3.

[16] This is similar to the fact that temporal gauge $A^0 = 0$ and Coulomb gauge $\partial^i A_i = 0$ can be imposed at the same time for a solution to the Maxwell equation.

	$d=2$	$d\neq 2$
$A^{\Delta,\pm}_{\mu a}$	$\Delta=1$	$\Delta=1$
$\widetilde{A^{d-\Delta,\pm}_{\mu a}}$		$\times$

Table 3: Spin-one conformal primary wavefunctions and their shadows that are pure gauge in $\mathbb{R}^{1,d+1}$. For $d=2$, the conformal primary wavefunction $A^{\Delta=1,\pm}_{\mu a}$ with $\Delta=1$ is identical to its (formal) shadow $\widetilde{A^{\Delta=1,\pm}_{\mu a}}$ (5.15), so we place $\Delta=1$ in the middle between the two rows.

5.3 Mellin Transform

In this section we will determine the range of Δ such that the conformal primary wavefunctions (5.8) are delta-function-normalizable with respect to a certain norm and span the plane wave solutions to the Maxwell equation.

Let us first review massless spin-one on-shell wavefunctions in momentum space. We will be working in Lorenz gauge:

$$\partial^\mu A_\mu(X)=0\,. \tag{5.16}$$

In this gauge, the Maxwell equation reduces to $\partial^2 A_\mu=0$. The outgoing and incoming plane waves are $\epsilon_\mu(k)e^{\pm ik\cdot X}$ where k is a null momentum and $\epsilon_\mu(k)$ is a polarization vector satisfying $k^\mu\epsilon_\mu(k)=0$. The residual gauge symmetry preserving the Lorenz gauge condition is

$$A_\mu(X)\to A_\mu(X)+\partial_\mu\alpha(X)\,,\qquad \partial^2\alpha(X)=0\,. \tag{5.17}$$

In momentum space, this residual gauge symmetry shifts the polarization vector by $\epsilon_\mu(k)\to\epsilon_\mu(k)+Ck_\mu$, where C is any constant. We can fix this residual gauge symmetry by choosing the polarization vectors to be $\partial q_\mu(\vec{w})/\partial w^a$ (2.11):

$$\partial_a q_\mu\, e^{\pm i\omega q\cdot X}\,, \tag{5.18}$$

where we have parametrized a null vector k^μ as $k^\mu=\omega q^\mu(\vec{w})$ with $q^\mu(\vec{w})$ given in (2.9) and $\omega>0$. Here $a=1,\cdots,d$.

There is an inner product on the space of complex solutions, modulo gauge transformations that fall off sufficiently fast at infinity, to the Maxwell equation [49–52]:

$$(A_\mu,A'_{\mu'})=-i\int d^{d+1}X^i\,\left[A^\rho F'^{*}_{0\rho}-A'^{\rho *}F_{0\rho}\right]\,. \tag{5.19}$$

Using the Maxwell equation $\partial^\nu F_{\mu\nu} = 0$, one can show that the above inner product does not depend on the choice of the Cauchy surface we integrate over. Furthermore, the integrand is gauge invariant up to a total derivative [53–55].[17] The plane waves (5.18) are delta-function-normalizable with respect to this inner product:

$$\left(\partial_a q_\mu \, e^{\pm i\omega q\cdot X}, \partial_b q'_\mu \, e^{\pm i\omega' q'\cdot X}\right) = \pm 8(2\pi)^{d+1}\, \delta_{ab}\, \omega q^0 \, \delta^{(d+1)}(\omega q^i - \omega' q'^i)\,. \tag{5.20}$$

Let us now switch gears to conformal primary wavefunctions. It will prove convenient to choose a particular gauge representative of (5.8). As discussed in Section 5.2, conformal covariance (5.2) fixes the conformal primary wavefunctions (5.8) to be in radial gauge $X^\mu A_\mu = 0$ and Lorenz gauge $\partial^\mu A_\mu = 0$ at the same time. However, for the purpose of computing any gauge invariant physical observables such as scattering amplitudes, we can work with any wavefunction that is equivalent to (5.8) by a gauge transformation, and still obtain a conformally covariant answer at the end of the day.

A convenient gauge representative of the conformal primary wavefunction is:

$$\varphi^{\Delta,\pm}_{\mu a}(X^\mu; \vec{w}) = (\mp i)^\Delta \Gamma(\Delta)\, \frac{\partial_a q_\mu}{(-q\cdot X \mp i\epsilon)^\Delta}\,, \tag{5.21}$$

which satisfies the Lorenz gauge condition but not the radial gauge condition. Up to a normalization factor, $\varphi^{\Delta,\pm}_{\mu a}$ is gauge equivalent to the conformal primary wavefunction (5.8) by the following pure gauge:

$$\frac{\partial}{\partial X^\mu}\left(\frac{\partial_a q\cdot X}{(-q\cdot X \mp i\epsilon)^\Delta}\right)\,. \tag{5.22}$$

In fact, since the gauge parameter $\alpha = \frac{\partial_a q\cdot X}{(-q\cdot X\mp i\epsilon)^\Delta}$ satisfies $\partial^2\alpha = 0$, it is a residual gauge transformation (5.17) preserving the Lorenz gauge condition. It follows that even though (5.21) does not transform covariantly under $SO(1, d+1)$ as in (5.2), the non-covariant terms are pure residual gauge (5.17). For this reason we will still call $\varphi^{\Delta,\pm}_{\mu a}$ a spin-one conformal primary wavefunction.

The particular gauge representative (5.21) is chosen such that it is related to the plane wave (5.18) by a Mellin transform as in the case of massless scalars,

$$\varphi^{\Delta,\pm}_{\mu a}(X^\mu; \vec{w}) = \int_0^\infty d\omega \omega^{\Delta-1}\, \left(\partial_a q_\mu \, e^{\pm i\omega q\cdot X - \epsilon\omega}\right)\,. \tag{5.23}$$

[17] Large gauge transformations generally have nontrivial inner products with other on-shell wavefunctions because of this boundary term, and thus should be regarded as nontrivial elements in the solution space of the Maxwell equation. In the following we will only construct conformal primary wavefunctions that span the non-zero energy plane waves (5.18).

It follows that the same argument in Section 4.2 can be directly borrowed for the spin-one case. We conclude that spin-one conformal primary wavefunctions on the principal continuous series $\Delta \in \frac{d}{2}+i\mathbb{R}$ are delta-function-normalizable with respect to (5.19) and span the plane wave solutions (5.18) of the Maxwell equation. Similar to the massless scalar case, another equally interesting space of on-shell wavefunctions with the same property is the shadow of (5.21) on the principal continuous series. Given a gluon scattering amplitude, the transition from momentum space to the space of conformal primary wavefunctions is then implemented by a Mellin transform (5.23) (or plus a shadow transform (2.21)) on each external gluon particle.

6 Gravitons

We now turn to massless spin-two conformal primary wavefunctions. In Section 6.1 we construct solutions to the $(d+2)$-dimensional vacuum linearized Einstein equation that transform as spin-two conformal primaries in d dimensions. In Section 6.2 we identify pure diffeomorphisms that are also conformal primary wavefunctions. In Section 6.3 we show that again the spin-two conformal primary wavefunctions on the principal continuous series are normalizable (with respect to (6.16)) and span the plane wave solutions of the linearized Einstein equation.

Since there are no propagating degrees of freedom in flat space for gravitons below $(3+1)$ dimensions, we will assume $d \geq 2$ in this section.

6.1 Massless Spin-Two Conformal Primary Wavefunctions in General Dimensions

The defining properties for the outgoing $(+)$ and incoming $(-)$ *massless spin-two conformal primary wavefunction* $h^{\Delta,\pm}_{\mu_1\mu_2;a_1a_2}(X^\mu;\vec{w})$ in $\mathbb{R}^{1,d+1}$ are

- It is symmetric both in the $(d+2)$- and d-dimensional vector indices and traceless in the latter:

$$\begin{aligned} h^{\Delta,\pm}_{\mu_1\mu_2;a_1a_2} &= h^{\Delta,\pm}_{\mu_2\mu_1;a_1a_2}\,, \\ h^{\Delta,\pm}_{\mu_1\mu_2;a_1a_2} &= h^{\Delta,\pm}_{\mu_1\mu_2;a_2a_1}\,, \qquad \delta^{a_1a_2}h^{\Delta,\pm}_{\mu_1\mu_2;a_1a_2} = 0\,. \end{aligned} \tag{6.1}$$

- It is a solution to the vacuum linearized Einstein equation in flat space:[18]

$$\partial_\sigma\partial_\nu h^\sigma{}_{\mu;a_1a_2} + \partial_\sigma\partial_\mu h^\sigma{}_{\nu;a_1a_2} - \partial_\mu\partial_\nu h^\sigma{}_{\sigma;a_1a_2} - \partial^\rho\partial_\rho h_{\mu\nu;a_1a_2} = 0\,. \tag{6.2}$$

[18]For notational simplicity, we omit the superscript $\Delta,\pm$ of the wavefunction in this equation.

- It transforms both as a $(d+2)$-dimensional rank-two tensor and a d-dimensional spin-two conformal primary with conformal dimension Δ under an $SO(1,d+1)$ Lorentz transformation:

$$h^{\Delta,\pm}_{\mu_1\mu_2;a_1a_2}\left(\Lambda^\rho{}_\nu X^\nu;\vec{w}\,'(\vec{w})\right)=\frac{\partial w^{b_1}}{\partial w'^{a_1}}\frac{\partial w^{b_2}}{\partial w'^{a_2}}\left|\frac{\partial \vec{w}'}{\partial \vec{w}}\right|^{-(\Delta-2)/d}\Lambda_{\mu_1}{}^{\sigma_1}\,\Lambda_{\mu_2}{}^{\sigma_2}h^{\Delta,\pm}_{\sigma_1\sigma_2;b_1b_2}(X^\rho;\vec{w})\,, \tag{6.3}$$

  where $\vec{w}\,'(\vec{w})$ is an element of $SO(1,d+1)$ defined in (2.3) and $\Lambda^\mu{}_\nu$ is the associated group element in the $(d+2)$-dimensional representation.

With our experiences from the scalar and spin-one wavefunctions, we can immediately write down the massless spin-two conformal primary wavefunctions from the H_{d+1} spin-two bulk-to-boundary propagator in the embedding formalism:

$$\boxed{h^{\Delta,\pm}_{\mu_1\mu_2;a_1a_2}(X;\vec{w})=P^{b_1b_2}_{a_1a_2}\frac{[(-q\cdot X)\partial_{b_1}q_{\mu_1}+(\partial_{b_1}q\cdot X)q_{\mu_1}]\,[(-q\cdot X)\partial_{b_2}q_{\mu_2}+(\partial_{b_2}q\cdot X)q_{\mu_2}]}{(-q\cdot X)^{\Delta+2}}}$$
$$=P^{b_1b_2}_{a_1a_2}\frac{1}{(-q\cdot X\mp i\epsilon)^{\Delta-2}}\partial_{b_1}\partial_{\mu_1}\log(-q\cdot X\mp i\epsilon)\;\partial_{b_2}\partial_{\mu_2}\log(-q\cdot X\mp i\epsilon)\,, \tag{6.4}$$

where $P^{b_1b_2}_{a_1a_2}$ projects a rank-two tensor to its symmetric traceless components:[19]

$$P^{b_1b_2}_{a_1a_2}\equiv\delta^{b_1}{}_{(a_1}\delta^{b_2}{}_{a_2)}-\frac{1}{d}\delta_{a_1a_2}\delta^{b_1b_2}\,. \tag{6.5}$$

It is then straightforward to check that (6.4) satisfies the vacuum linearized Einstein equation. An equally interesting spin-two conformal primary wavefunction is the shadow transform (2.21) of (6.4). A direct calculation shows that the shadow primary is:

$$\widetilde{h^{\Delta,\pm}_{\mu_1\mu_2;a_1a_2}}(X;\vec{w})=(-X^2)^{\frac{d}{2}-\Delta}h^{d-\Delta,\pm}_{\mu_1\mu_2;a_1a_2}(X;\vec{w})\,. \tag{6.6}$$

One can straightforwardly show that (6.6) satisfies all the required properties of massless spin-two conformal primary wavefunctions. Again for general Δ, the shadow transform does not take the conformal primary wavefunction $h^{\Delta,\pm}_{\mu_1\mu_2;a_1a_2}$ to itself with the shadow conformal dimension $d-\Delta$, but to a different wavefunction.

6.2 Diffeomorphism

As in the photon case, conformal covariance (6.3) picks a particular diffeomorphism choice for the conformal primary wavefunction. It is easy to check (6.4) is traceless and satisfies

[19] Our convention for symmetrization of indices is $T_{(ab)}\equiv\frac{1}{2}(T_{ab}+T_{ba})$.

the Lorenz[20] as well as the radial gauge conditions:

$$\eta^{\mu_1\mu_2} h^{\Delta,\pm}_{\mu_1\mu_2;a_1a_2} = 0\,, \qquad \partial^{\mu} h^{\Delta,\pm}_{\mu\mu_2;a_1a_2} = 0\,, \qquad X^{\mu} h^{\Delta,\pm}_{\mu\mu_2;a_1a_2} = 0\,. \tag{6.7}$$

In fact, any solution to the vacuum linearized Einstein equation is diffeomorphic to another solution that satisfies all three conditions in (6.7) [48]. In this gauge (6.7), the vacuum linearized Einstein equation becomes

$$\partial^{\rho}\partial_{\rho}\, h^{\Delta,\pm}_{\mu_1\mu_2;a_1a_2}(X^{\mu};\vec{w}) = 0\,. \tag{6.8}$$

All these properties also apply to the shadow wavefunction (6.6).

While a general diffeomorphism has no nice conformal covariance, the massless spin-two conformal primary wavefunction $h^{\Delta,\pm}_{\mu_1\mu_2;a_1a_2}$ happens to be a pure diffeomorphism if $\Delta = 0$ or $\Delta = 1$ for any d. In these two cases they can be written as

$$\begin{aligned} &h^{\Delta=0,\pm}_{\mu_1\mu_2;a_1a_2}(X^{\mu};\vec{w}) = \partial_{\mu_1}\xi^{0}_{\mu_2;a_1a_2} + \partial_{\mu_2}\xi^{0}_{\mu_1;a_1a_2}\,, \\ &\xi^{0}_{\mu;a_1a_2} = \frac{1}{2}P^{b_1b_2}_{a_1a_2}\,(q\cdot X \pm i\epsilon)\,\partial_{b_1}\left[\, q_{\mu}\partial_{b_2}\log(-q\cdot X \mp i\epsilon)\,\right]\,, \end{aligned} \tag{6.9}$$

and

$$\begin{aligned} &h^{\Delta=1,\pm}_{\mu_1\mu_2;a_1a_2}(X^{\mu};\vec{w}) = \partial_{\mu_1}\xi^{1}_{\mu_2;a_1a_2} + \partial_{\mu_2}\xi^{1}_{\mu_1;a_1a_2}\,, \\ &\xi^{1}_{\mu;a_1a_2} = -\frac{1}{4}P^{b_1b_2}_{a_1a_2}\,\partial_{b_1}\partial_{b_2}\left[\, q_{\mu}\log(-q\cdot X \mp i\epsilon)\,\right]\,. \end{aligned} \tag{6.10}$$

Incidentally, $\Delta = 0$ is the shadow dimension (i.e. $\Delta \to d - \Delta$) of the stress-tensor in a d-dimensional CFT.

The shadows of these two wavefunctions (6.9) and (6.10), however, are not pure diffeomorphisms in general. This subtlety is parallel to the one we encountered in the spin-one case in Section 5.2. In the spin-two case, the shadow transform (2.21) is not defined for $\Delta = 0, 1$, in which cases the conformal primary wavefunctions (6.4) reduce to pure diffeomorphisms. Nonetheless, the shadow wavefunction can be analytically continued to any value of Δ from the expression (6.6). The subtlety is that this analytic continuation spoils the pure diffeomorphism condition. Indeed, by a direct computation of the linearized Riemann curvature tensor, one can check that the shadow wavefunctions (6.6) with conformal dimensions d and $d-1$, i.e. the shadow dimensions of 0 and 1, are never pure diffeomorphisms if $d \neq 2$.

There is again an exception for spin-two wavefunctions in $(3+1)$ spacetime dimensions. When $d = 2$, the $\Delta = 1$ shadow wavefunction (6.6) is identical to the pure diffeomorphism

[20] In general relativity, Lorenz gauge is usually defined as $\partial^{\mu} h_{\mu\nu} - \frac{1}{2}\partial_{\nu} h^{\rho}{}_{\rho} = 0$. Since our wavefunction is traceless, the Lorenz gauge condition reduces to $\partial^{\mu} h_{\mu\nu} = 0$.

wavefunction (6.10),

$$\widetilde{h^{\Delta=1,\pm}_{\mu_1\mu_2;a_1a_2}}(X^\mu;\vec{w}) = h^{\Delta=1,\pm}_{\mu_1\mu_2;a_1a_2}(X^\mu;\vec{w}) \qquad (d=2)\,, \tag{6.11}$$

and is thus a pure diffeomorphism as well. Additionally, the shadow wavefunction (6.6) with conformal dimension 2 is also a pure diffeomorphism:

$$\widetilde{h^{\Delta=0,\pm}_{\mu_1\mu_2;a_1a_2}}(X^\mu;\vec{w}) = \partial_{\mu_1}\xi^2_{\mu_2;a_1a_2} + \partial_{\mu_2}\xi^2_{\mu_1;a_1a_2}\,, \qquad (d=2) \tag{6.12}$$

$$\xi^2_{\mu;a_1a_2} = -\frac{1}{24}P^{b_1b_2}_{a_1a_2}\left[\partial_{b_1}\partial_{b_2}\partial^c\left(X^\rho f_{\rho\mu;c}\log(-q\cdot X \mp i\epsilon)\right) - \frac{1}{2}\partial^c\partial_c\partial_{b_1}\left(X^\rho f_{\rho\mu;b_2}\log(-q\cdot X\mp i\epsilon)\right)\right]\,,$$

where $f_{\rho\mu;c} \equiv q_\rho\partial_c q_\mu - q_\mu\partial_c q_\rho$. In [4], this $\Delta = 2$ pure diffeomorphism was discussed in the context of soft graviton theorem in $(3+1)$ spacetime dimensions. We summarize spin-two conformal primary wavefunctions $h^{\Delta,\pm}_{\mu_1\mu_2;a_1a_2}$ and their shadows $\widetilde{h^{\Delta,\pm}_{\mu_1\mu_2;a_1a_2}}$ that are pure diffeomorphisms in Table 4.

	$d=2$		$d\geq 2$
$h^{\Delta,\pm}_{\mu_1\mu_2;a_1a_2}$	$\Delta=1$	$\Delta=0$	$\Delta=0,1$
$\widetilde{h^{d-\Delta,\pm}_{\mu_1\mu_2;a_1a_2}}$		$\Delta=2$	$\times$

Table 4: Spin-two conformal primary wavefunctions and their shadows that are pure diffeomorphisms in $\mathbb{R}^{1,d+1}$. For $d=2$, the conformal primary wavefunction $h^{\Delta=1,\pm}_{\mu_1\mu_2;a_1a_2}$ with $\Delta=1$ is identical to its (formal) shadow $\widetilde{h^{\Delta=1,\pm}_{\mu_1\mu_2;a_1a_2}}$ (6.11), so we place $\Delta=1$ in the middle between the two rows.

6.3 Mellin Transform

Finally, let us determine the range of the conformal dimension Δ for the spin-two conformal primary wavefunctions so that they are normalizable with respect to a certain norm and span the plane wave solutions of the linearized Einstein equation.

As before, we first review solutions in momentum space. We will be working in Lorenz gauge:

$$\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h^\rho{}_\rho = 0\,, \tag{6.13}$$

in which the vacuum linearized Einstein equation reduces to $\partial^\rho\partial_\rho h_{\mu\nu} = 0$. The outgoing and incoming plane waves are $\epsilon_{\mu\nu}(k)e^{\pm ik\cdot X}$ where k^μ is null and $\epsilon_{\mu\nu}(k)$ is a symmetric polarization tensor satisfying $k^\mu\epsilon_{\mu\nu} = \frac{1}{2}k_\nu\epsilon^\mu{}_\mu$. The residual diffeomorphisms preserving the Lorenz gauge condition are

$$h_{\mu\nu}(X) \to h_{\mu\nu}(X) + \partial_\mu\xi_\nu(X) + \partial_\nu\xi_\mu(X)\,, \qquad \partial^\rho\partial_\rho\xi_\mu(X) = 0\,. \tag{6.14}$$

In momentum space, the residual diffeomorphisms shift the polarization tensor by $\epsilon_{\mu\nu}(k) \to \epsilon_{\mu\nu}(k) + k_\mu r_\nu + k_\nu r_\mu$ for any vector r_μ. Using these residual diffeomorphisms, we can bring the polarization tensor to a rank-two symmetric traceless tensor of the following form:

$$g^{\pm}_{\mu\nu;a_1a_2}(X;\omega,q) = P^{b_1b_2}_{a_1a_2}\,\partial_{b_1}q_\mu\partial_{b_2}q_\nu\, e^{\pm i\omega q\cdot X}\,, \tag{6.15}$$

with $\partial_a q^\mu$ given by (2.11). Again we have parametrized a null vector k^μ as $k^\mu = \omega q^\mu(\vec{w})$ with $\omega > 0$.

The inner product on the space of complex solutions to the vacuum linearized Einstein equation, modulo diffeomorphisms that fall off sufficiently fast at infinity, is [13, 49–52]

$$\begin{aligned}\left(h_{\mu\nu}, h'_{\mu'\nu'}\right) = -i\int d^{d+1}X^i\Big[& h^{\mu\nu}\partial_0 h'^*_{\mu\nu} - 2h^{\mu\nu}\partial_\mu h'^*_{0\nu} + h\partial^\mu h'^*_{0\mu} - h\partial_0 h'^* + h_{0\mu}\partial^\mu h'^* \\ &- (h \leftrightarrow h'^*)\Big]\,,\end{aligned} \tag{6.16}$$

where $h = h^\rho{}_\rho$. Using the linearized Einstein equation we can show that the inner product does not depend on the choice of Cauchy surface. Further, it is invariant under diffeomorphism up to a boundary term [13, 53–55].[21] The inner products between two plane waves are

$$\begin{aligned}&\left(g^{\pm}_{\mu\nu;a_1a_2}(X;\omega,q), g^{\pm}_{\mu'\nu';a'_1a'_2}(X;\omega',q')\right) \\ &\qquad = \pm 32(2\pi)^{d+1}\left(\delta_{a_1(a'_1}\delta_{a'_2)a_2} - \frac{1}{d}\delta_{a'_1a'_2}\delta_{a_1a_2}\right)\omega q^0\,\delta^{(d+1)}(\omega q^i - \omega' q'^i)\,.\end{aligned} \tag{6.17}$$

We now wish to find the space of conformal primary wavefunctions that spans the plane wave solutions (6.15). We start by considering a particular diffeomorphism representative of the conformal primary wavefunction. Let us consider one of the terms in the conformal primary wavefunction (6.4):

$$\varphi^{\Delta,\pm}_{\mu_1\mu_2;a_1a_2}(X^\mu;\vec{w}) = (\mp i)^\Delta\Gamma(\Delta)\,P^{b_1b_2}_{a_1a_2}\,\frac{\partial_{b_1}q_{\mu_1}\partial_{b_2}q_{\mu_2}}{(-q\cdot X \mp i\epsilon)^\Delta}\,. \tag{6.18}$$

[21] Similar to the spin-one case, large diffeomorphisms generally have nonzero inner products with other on-shell wavefunctions and should therefore be included as nontrivial elements in the solution space. We will focus on the non-zero energy plane wave solutions (6.15) to the linearized Einstein equation below.

It is traceless and satisfies the Lorenz gauge condition $\partial^\mu \varphi^{\Delta,\pm}_{\mu\mu_2;a_1a_2} = 0$, but not the radial gauge condition. It also satisfies the linearized Einstein equation in this gauge, $\partial^\rho\partial_\rho\varphi^{\Delta,\pm}_{\mu_1\mu_2;a_1a_2} = 0$. One can straightforwardly show that (6.18) differs from the spin-two conformal primary wavefunction (6.4) by a pure residual diffeomorphism (6.14). Indeed, even though (6.18) does not transform covariantly as in (6.3), the non-covariant terms of the transformation are pure residual diffeomorphisms of the Lorenz gauge condition:

$$\frac{\partial_b q_{(\mu_1} q_{\mu_2)}}{(-q\cdot X)^\Delta} = \frac{1}{\Delta-1}\partial_{(\mu_1}\left(\frac{\partial_b q_{\mu_2)}}{(-q\cdot X)^{\Delta-1}}\right), \qquad \frac{q_{\mu_1}q_{\mu_2}}{(-q\cdot X)^\Delta} = \frac{1}{\Delta-1}\partial_{(\mu_1}\left(\frac{q_{\mu_2)}}{(-q\cdot X)^{\Delta-1}}\right). \tag{6.19}$$

Hence for the purpose of computing any diffeomorphism invariant observables, we can use the wavefunction $\varphi^{\Delta,\pm}_{\mu_1\mu_2;a_1a_2}$ and still reproduce a conformally covariant answer in the end.

The advantage of this diffeomorphism representative (6.18) for the conformal primary wavefunction is that it is simply the Mellin transform of the pane wave (6.15) as in the massless scalar case:

$$\varphi^{\Delta,\pm}_{\mu_1\mu_2;a_1a_2}(X^\mu;\vec{w}) = \int_0^\infty d\omega\omega^{\Delta-1}\, g^{\Delta,\pm}_{\mu_1\mu_2;a_1a_2}(X^\mu;\omega,q^\mu)\,. \tag{6.20}$$

Following the same argument used in Section 4.2, we conclude that spin-two conformal primary wavefunctions on the principal continuous series $\Delta \in \frac{d}{2} + i\mathbb{R}$ are delta-function-normalizable (with respect to (6.16)) and span the plane wave solutions (6.15) of the vacuum linearized Einstein equation. Another equally interesting set of on-shell wavefunctions with the same property is the shadow of (6.18). Given a graviton scattering amplitude, the transition from momentum space to the space of conformal primary wavefunctions is again implemented by a Mellin transform (6.20) (or plus a shadow transform (2.21)) on each external graviton particle.

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