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Abstract

We consider actions for particles and strings, including twistorial descriptions on 4d Minkowski and AdS_5 spacetimes from the point of view of co-adjoint orbits for the isometry group. We also consider the collective coordinate dynamics of singular solutions in Chern-Simons (CS) theories and CS theories of gravity. This is a generalization of the work of Einstein, Infeld and Hoffmann and also has potential points of contact with fluid-gravity correspondence.

1 Introduction

The first part of this article is about exploring interconnections between somewhat different formulations of the actions for point-particles and strings. Point-particles can be identified as unitary representations of the Poincaré group, or, more generally, of the appropriate isometry group. It has been rather well known for a long time that one can use a co-adjoint orbit action for describing such representations and, in fact, it is the basic paradigm for the whole idea of geometric quantization [1]. Actions in terms of the spacetime coordinates, in terms of twistor variables, etc., have also been used. More recently, there has been some work on particle dynamics on AdS spacetimes, including a twistor description for massive particles [2]. We analyze many of these actions from a single point of view, namely, in terms of the co-adjoint orbit action for the isometry group.

In the second part of this article, we consider singular vortex or instanton solutions in Chern-Simons (CS) theories, including the special important case of CS theories of gravity [3]. Removing the locations of the singularities from the manifold under consideration, one can obtain a nonsingular description. The CS action for such solutions is shown to reduce to an appropriate co-adjoint orbit action, thus recovering the results of the earlier sections from a different point of view. Conceptually, this is similar to the work of Einstein, Infeld and Hoffmann (EIH), who considered point-particles as singularities of the gravitational field and then showed that the multiparticle dynamics is determined by the field equations of general relativity [4]. Our approach is similar in spirit, and, in fact, may be considered as the EIH idea applied to CS gravity. Our analysis also falls within the circle of ideas related to recent work on fluid-gravity correspondence [5]. Here one considers a general diffeomorphism of special solutions and then, viewing the diffeomorphism as providing collective degrees of freedom, one obtains the evolution equations for these collective modes, which are seen to be essentially the equations of fluid dynamics. Since one can define fluid dynamics in terms of Poincaré representations as well [6, 7], we expect that there will be further linkages of our work to the fluid-gravity correspondence.

This paper is organized as follows. In section 2, we will consider the actions for massive and massless particles, the Nambu-Goto string and the null string in 4d Minkowski space in terms of the Poincaré group. We will relate our results to the actions which have been suggested for these cases in different contexts in the literature. In section 3, we will do a similar analysis for the AdS_5 spacetime. recovering among other things, the twistor description of massive particles obtained in [2]. We will also consider the twistor version for massless particles and null strings in AdS_5. In section 4, we will consider the Chern-Simons theory for an arbitrary gauge group G to show how singular solutions can lead to the co-adjoint orbit action. A similar analysis for Chern-Simons gravity in 3 and 5 dimensions will also be discussed.

2 Poincaré orbits

Free particles and free strings can be described in terms of representations of the Poincaré group. For particles, the action is given by the integral of the symplectic potential on a co-adjoint orbit of the spacetime isometry group. So we start with a statement of this method. Let g be a matrix representing a general element of a Lie group G (in some particular matrix representation). The symplectic potential on an orbit is given by

$$\mathcal{A} = i \sum_a w_a \, \mathrm{Tr}(t_a \, g^{-1} dg) \tag{1}$$

where t_a give a basis of the diagonal generators of the Lie algebra (the Cartan subalgebra) and w_a are a set of numbers characterizing the chosen orbit. We take t_a to be normalized as $\mathrm{Tr}(t_a \, t_b) = \delta_{ab}$. The action for free particle dynamics can then be taken to be

$$S = i \sum_a w_a \int d\tau \; \mathrm{Tr}(t_a \, g^{-1} \dot{g}), \qquad \dot{g} = \frac{dg}{d\tau} \tag{2}$$

where the integral is along some path $g(\tau)$ in the group parametrized by τ. Upon quantization, the theory defined by (2) leads to a Hilbert space which carries a unitary irreducible representation (UIR) of G, this UIR being specified by the highest weight $(w_1, w_2, \cdots, w_r)$. Here r is the rank of the group, which is also the range of summation for the subscript a. The canonical one-form associated to (2) is evidently $\mathcal{A}$. Under transformations $g \to g \, \exp(-it_a \varphi^a)$, we find $\mathcal{A} \to \mathcal{A} + df$, $f = \sum w_a \varphi^a$. Thus the symplectic two-form $\Omega = d\mathcal{A}$ is invariant under the transformation $g \to g \, \exp(-it_a \varphi^a)$ and hence Ω is defined on $G/H_{\rm C}$, $H_{\rm C}$ being the Cartan subgroup. Further, the transformation $\mathcal{A} \to \mathcal{A} + df$ shows that in the quantum theory, where wave functions transform as e^{iS}, there can be quantization conditions on w_α. These will turn out to be the required conditions for $(w_1, w_2, \cdots, w_r)$ to qualify as the highest weight of a UIR. The existence of such conditions will, of course, depend on whether the corresponding directions in $H_{\rm C}$ are compact or not.

We will now apply this, taking G to be the Poincaré group in 4-dimensional Minkowski space and in AdS$_5$, to obtain point-particle and string actions. We will also relate this to the twistor actions for particles and strings.

For particle dynamics in 4-dimensional Minkowski space, it is simplest to consider the Poincaré group as a contraction of the de Sitter group. We can use the standard spinorial representation with the generators

$$P_\mu = \frac{\gamma_\mu}{l}, \qquad J_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu] \tag{3}$$

where γ_μ are the 4×4 Dirac matrices and l is a parameter with the dimension of length. The limit $l \to \infty$ is the contraction giving the Poincaré algebra with commuting translation generators P_μ. For a massive point-particle, we need an orbit that corresponds to a time-like momentum vector, so we can take this as $t_1 \sim \gamma_0$. There are three generators J_{12}, J_{23}, J_{31} which commute with γ_0. We can take t_2 as one of these, say, $\gamma_1\gamma_2$. The symplectic potential for a massive particle in 4d Minkowski spacetime is thus

$$\mathcal{A} = \left[-i\frac{ml}{4}\mathrm{Tr}(\gamma_0\, g^{-1}dg) + \frac{s}{2}\mathrm{Tr}(\gamma_1\gamma_2\, g^{-1}dg)\right]_{l\to\infty} \tag{4}$$

A general element of the de Sitter group can be parametrized as

$$g = \exp\left(i\frac{\gamma_\mu x^\mu}{l}\right)\, \Lambda \tag{5}$$

where Λ denotes an element of the Lorentz group, of the form $\Lambda = \exp(iJ_{\mu\nu}\theta^{\mu\nu})$. Using this parametrization, the symplectic potential (4) reduces to

$$\mathcal{A} = p_\mu\, dx^\mu + \frac{s}{2}\mathrm{Tr}(\gamma_1\gamma_2\,\Lambda^{-1}d\Lambda) \tag{6}$$

where $p_\mu = m\Lambda^\alpha_{\ 0}\eta_{\mu\alpha}$ and $\Lambda^\alpha_{\ \beta}$ is the vector representation of the Lorentz group defined by $\Lambda\gamma_\beta\Lambda^{-1} = \gamma_\alpha\,\Lambda^\alpha_{\ \beta}$. Notice that by construction $p^2 = m^2$. $\mathcal{A}$ given in (6) is the standard and rather well-known form used for describing point-particles with mass and spin. The second term in $\mathcal{A}$ describes the spin degrees of freedom. A variant of this formalism is to consider p_μ as four independent variables to begin with, i.e., not given in terms of $\Lambda^\alpha_{\ 0}$, and then impose the condition $p^2 = m^2$ as a constraint. Further points about the dynamics, including coupling to external fields, the emergence of the wave equation in the quantum theory, magnetic moment and spin-orbit interactions, extensions to fluids, etc. can be found in [7] as well as in earlier references cited there.

For a massless point-particle, we need a null orbit. This can be obtained by the choice $t_1 \sim \gamma_0 + \gamma_3$ and $t_2 \sim \gamma_1\gamma_2$. Thus

$$\begin{aligned} \mathcal{A} &= \left[-i\frac{\mu l}{4}\mathrm{Tr}\big[(\gamma_0+\gamma_3)\, g^{-1}dg\big] + \frac{s}{2}\mathrm{Tr}(\gamma_1\gamma_2\, g^{-1}dg)\right]_{l\to\infty} \\ &= \mu\,\eta_{\mu\alpha}(\Lambda^\alpha_{\ 0} + \Lambda^\alpha_{\ 3})\, dx^\mu + \frac{s}{2}\mathrm{Tr}(\gamma_1\gamma_2\,\Lambda^{-1}d\Lambda) \end{aligned} \tag{7}$$

where μ is a scale parameter with the dimensions of energy. Notice that the momentum $p_\mu = \mu\,\eta_{\mu\alpha}(\Lambda^\alpha_{\ 0} + \Lambda^\alpha_{\ 3})$ satisfies $p^2 = 0$. Infinitesimal Lorentz transformations acting on the left of Λ are given by

$$\Lambda \to (1 + \tfrac{1}{4}\omega^{\alpha\beta}[\gamma_\alpha, \gamma_\beta])\,\Lambda \tag{8}$$

while the vectors transform as $p_\mu \to p_\mu + \omega_{\mu\nu}p^\nu$. The canonical generator of this transformation given by the symplectic form in (7) is

$$M_{\alpha\beta} = -(x_\alpha p_\beta - x_\beta p_\alpha) + \frac{s}{8}\mathrm{Tr}(\gamma_1\gamma_2\Lambda^{-1}[\gamma_\alpha,\gamma_\beta]\Lambda) \tag{9}$$

The Pauli-Lyubanski spin vector can now be calculated as

$$W^\mu \equiv -\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\, M_{\nu\alpha}\, p_\beta = s\, p^\mu \tag{10}$$

This identifies s as the helicity of the massless particle.

It is also useful to obtain the twistor action for a massless particle from this description. For simplicity, we will consider the spinless case first; in this case, the symplectic form is just the first term of $\mathcal{A}$ in (7). We can reduce this further. Since γ_5 is invariant under Lorentz transformations, it is possible to consider the projections $\frac{1}{2}(1 \pm i\gamma_5)$ separately. We then take $P_\mu = \frac{1}{2}(1 - i\gamma_5)\gamma_\mu / l$. This will give $P^2 = 0$; for a massless particle this is acceptable. We choose a representation of γ-matrices as

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix}, \qquad \gamma_5 = -i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{11}$$

(Our choice of an antihermitian γ_5 is convenient for later discussions.) The parametrization of the group element may be taken as

$$g = \begin{pmatrix} 1 & 0 \\ i\,X & 1 \end{pmatrix} \Lambda, \qquad X = \frac{x^0 + \sigma \cdot \vec{x}}{l} \tag{12}$$

The symplectic form is then obtained as

$$\mathcal{A} = -\frac{i}{2}\, \mu l \, \mathrm{Tr}\big[(\gamma_0 + \gamma_3)\, g^{-1} dg\big] \tag{13}$$

We have changed the normalization, absorbing a factor of 2 into the parameter μ. The limit $l \to \infty$ is also not needed in this expression. For the group element g, we also have $g^\dagger \gamma_0 = \gamma_0\, g^{-1}$. Using this relation and the representation (11, 12), we find

$$\mathcal{A} = \mu l \, (\Lambda_L)_{A2}\, dX^{A\dot{A}}\, (\Lambda_L)_{\dot{A}2} \tag{14}$$

where $A,\ \dot{A} = 1, 2$. We can now define $\pi_{\dot{A}} = \sqrt{\mu l}\, (\Lambda_L)_{\dot{A}2}$, $\omega^A = -iX^{A\dot{A}} \pi_{\dot{A}}$. Further, let

$$Z = \begin{pmatrix} \omega^A \\ \pi_{\dot{A}} \end{pmatrix}, \qquad \bar{Z} = Z^\dagger\, \gamma_0 = (\bar{\pi}_A \ \ \bar{\omega}^{\dot{A}}) \tag{15}$$

The action, which is the integral of $\mathcal{A}$, can now be written as

$$S = \int \mathcal{A} = i \int (\bar{\pi}_A d\omega^A + \bar{\omega}^{\dot{A}} d\pi_{\dot{A}}) = i \int d\tau\, \bar{Z}\, \dot{Z} \tag{16}$$

From their definition, $Z,\ \bar{Z}$ are seen to obey the condition $\bar{Z}Z = 0$. The strategy now is to regard all four components of Z_s, $s = 1, 2, 3, 4$, to be independent *a priori* and impose the condition $\bar{Z}Z = 0$ as a constraint on the phase space variables for the action (16). This will eliminate the arbitrary parameter μl and one of the phases in Z. Classically, the solution of the constraint will lead us back to the expression in terms of the group elements. Quantum theoretically, $\bar{Z}$ is the canonically conjugate variable and the constraint generates the transformation $Z \to \lambda\, Z$, $\lambda \in \mathbb{C} - \{0\}$, so that we get a reduction of the phase space to the projective twistor space. Particles with nonzero spin can also be described in the same formalism by relaxing the constraint to some nonzero constant value for $\bar{Z}Z$.

We now turn to actions for strings which can be viewed as tracing out a two-dimensional surface in spacetime. Since we can regard Minkowski spacetime as the Poincaré group modulo the Lorentz group, we can view strings as maps of a two-dimensional worldsheet into the Poincaré group subject to certain conditions. The surface area may be regarded as the (wedge) product of two one-forms, one of them being time-like and the other space-like. With no additional spin variables, the timelike one-form can be taken as the first term of $\mathcal{A}$ in (4),

$$\mathcal{A} = \left[-i\frac{ml}{4}\mathrm{Tr}(\gamma_0\, g^{-1}dg) \right]_{l\to\infty} = m\, \eta_{\mu\alpha}\Lambda^{\mu}{}_{0}\, dx^{\alpha} \tag{17}$$

The spacelike one can be chosen to be along any one of the other directions; we make the choice

$$\mathcal{B} = \left[-i\frac{\tilde{m}l}{4}\mathrm{Tr}(\gamma_3\, g^{-1}dg) \right]_{l\to\infty} = \tilde{m}\, \eta_{\mu\alpha}\Lambda^{\mu}{}_{3}\, dx^{\alpha} \tag{18}$$

The action, which is the integral of the area element or the product of these two one-forms is then

$$\begin{aligned} S &= \int V_{\alpha\beta}\, dx^{\alpha} \wedge dx^{\beta} \\ V_{\alpha\beta} &= M^2 \eta_{\mu\alpha}\eta_{\nu\beta}(\Lambda^{\mu}{}_{0}\Lambda^{\nu}{}_{3} - \Lambda^{\nu}{}_{0}\Lambda^{\mu}{}_{3}) \end{aligned} \tag{19}$$

with $M^2 = m\tilde{m}/2$. $V_{\alpha\beta}$ obeys the constraints

$$V_{\alpha\beta}\, V^{\alpha\beta} = -2\, M^4, \qquad \epsilon^{\mu\nu\alpha\beta}\, V_{\mu\nu}\, V_{\alpha\beta} = 0 \tag{20}$$

As in the case of point-particles, it is possible to treat $V_{\alpha\beta}$ as *a priori* independent variables, enforcing the constraints via Lagrange multipliers. We can pull back the two-form in (19) to the worldsheet to write the action as an integral over the worldsheet coordinates ξ^1, ξ^2. With a Lagrange multiplier for the first of the constraints in (20), this leads to the action

$$S = \int V_{\alpha\beta}\, \partial_a x^{\alpha}\, \partial_b x^{\beta}\, d\xi^a \wedge d\xi^b - \frac{1}{2}\int d^2\xi \sqrt{-g}\, \left[V_{\alpha\beta}V^{\alpha\beta} + 2\, M^4 \right] \tag{21}$$

where g_{ab} is the worldsheet metric. The equation of motion for $\sqrt{-g}$ gives the constraint on $V_{\alpha\beta}$. Eliminating them by their equations of motion leads to the Nambu-Goto action

$$S = -2\, M^2 \int d^2\xi\, \sqrt{-\det \rho} \tag{22}$$

where $\rho_{ab} = \eta_{\alpha\beta}\partial_a x^{\alpha}\, \partial_b x^{\beta}$ is the induced metric on the worldsheet.

For a null string, we need one lightlike direction and a spacelike direction. So we make the choice

$$\begin{aligned} \mathcal{A} &= -\frac{i}{2}\,\mu l\, \mathrm{Tr}\big[(\gamma_0 + \gamma_3)\, g^{-1}dg\big] = \mu l\, \eta_{\mu\alpha}(\Lambda^{\mu}{}_{0} + \Lambda^{\mu}{}_{3})\, dx^{\alpha} \\ \mathcal{C} &= -\frac{i}{2}\tilde{m}\sqrt{\mu l}\, \mathrm{Tr}(\gamma_1\, g^{-1}dg) = \tilde{m}\sqrt{\mu l}\, \eta_{\mu\alpha}\Lambda^{\mu}{}_{1} dx^{\alpha} \end{aligned} \tag{23}$$

The action is proportional to $\int \mathcal{A} \wedge \mathcal{C}$ and has the form

$$\begin{aligned} S &= M^2 \int V_{\alpha\beta}\, dx^\alpha \wedge dx^\beta \\ V_{\alpha\beta} &= \eta_{\mu\alpha}\eta_{\nu\beta}\left[(\Lambda^\mu{}_0 + \Lambda^\mu{}_3)\Lambda^\nu{}_1 - (\Lambda^\nu{}_0 + \Lambda^\nu{}_3)\Lambda^\mu{}_1\right] \end{aligned} \tag{24}$$

where M^2 includes factors of $\tilde{m}$ and μl. $V_{\alpha\beta}$ now obey the constraints

$$V_{\alpha\beta}\, V^{\alpha\beta} = 0, \qquad \epsilon^{\mu\nu\alpha\beta}\, V_{\mu\nu}\, V_{\alpha\beta} = 0 \tag{25}$$

This is the form of the action obtained in [8].

The null string can also be described using twistors, just like the massless particle. We have already written $\mathcal{A}$ as $i\bar{Z}dZ$. For $\mathcal{C}$, using the parametrization (12),

$$\begin{aligned} \mathcal{C} &= -\frac{1}{2}\tilde{m}\sqrt{\mu l}\left[(\Lambda_L^*)_{A2}\, dX^{A\dot{A}}\,(\Lambda_L)_{\dot{A}1} + (\Lambda_L^*)_{A1}\, dX^{A\dot{A}}\,(\Lambda_L)_{\dot{A}2}\right] \\ &= -\frac{i}{2}(\bar{Z}dW + \bar{W}dZ) \end{aligned} \tag{26}$$

where we have defined another set of twistor variables

$$W = \tilde{m}\begin{pmatrix} -iX^{A\dot{A}}(\Lambda_L)_{\dot{A}1} \\ (\Lambda_L)_{\dot{A}1} \end{pmatrix}, \qquad \bar{W} = \tilde{m}\,\left((\Lambda_L)^*_{A1}\;\; i(\Lambda_L)^*_{A1}X^{A\dot{A}}\right) \tag{27}$$

The action for the null string may then be taken as

$$S = i\int (\bar{Z}dZ) \wedge (\bar{Z}dW + \bar{W}dZ) \tag{28}$$

From the definitions, we have the constraints

$$\begin{aligned} \bar{Z}Z &= 0, \qquad & \bar{W}W &= 0 \\ \bar{Z}W &= 0, \qquad & \bar{W}Z &= 0 \end{aligned} \tag{29}$$

As before the strategy is to take variables Z, W to be independent *a priori*, and then impose the constraints (29) on the phase space corresponding to the action (28). This twistor version of the null string is then identical to what was obtained in [9].

To recapitulate the results of this section briefly: The action for massive and massless particles , the Nambu-Goto string, the Schild null string are all described in terms of the Poincaré group. The different cases have been separately discussed in the literature before, but they are brought together here to a single point of view. The purpose is primarily to set the stage for the rest of this article.

3 Particle dynamics in AdS$_5$ spacetime

We now turn to particle dynamics in AdS$_5$ spacetime, which may be considered as $SO(4,2)/SO(4,1)$. The $SO(4,1)$ generators are given by

$$\Sigma_{\mu\nu} = \frac{i}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu), \qquad \mu,\nu = 0,1,2,3,5 \tag{30}$$

The remaining generators in the orthogonal complement are γ_μ themselves. Defining $z = e^{2\theta}$, $e^{i\gamma_5\theta}$ is diagonal with eigenvalues $\sqrt{z}$, $1/\sqrt{z}$. A convenient parametrization of a group element $g \in SO(4,2)$ is

$$g = \begin{pmatrix} \sqrt{z} & i\tilde{X}/\sqrt{z} \\ 0 & 1/\sqrt{z} \end{pmatrix} \Lambda \tag{31}$$

where $\Lambda \in SO(4,1)$ is generated by $\Sigma_{\mu\nu}$ in (30) and $\tilde{X} = x^0 - \vec{\sigma}\cdot\vec{x}$. The Cartan-Killing metric on the coset space is given by

$$\begin{aligned} ds^2 &= -\frac{R^2}{4}\eta^{\mu\nu}\,\mathrm{Tr}(\gamma_\mu\, g^{-1}dg)\,\mathrm{Tr}(\gamma_\nu\, g^{-1}dg) \\ &= R^2\,\frac{dx^2 - dz^2}{z^2} \end{aligned} \tag{32}$$

This is one of the usual forms of the metric on AdS$_5$. (Here R is a scale parameter.)

For describing massive particles on this spacetime, we can, as usual, use

$$\mathcal{A} = -i\frac{mR}{2}\mathrm{Tr}(\gamma_0 g^{-1}dg) = mR\,\eta_{\mu\nu}\,\Lambda^\mu{}_0\frac{dx^\nu}{z} \tag{33}$$

where in the second equality we used (31). This identifies the momentum as $p_\nu = \eta_{\mu\nu}\Lambda^\mu{}_0/z$, which is seen to obey the mass-shell condition

$$\eta^{\mu\nu}p_\mu p_\nu = \frac{(mR)^2}{z^2} \tag{34}$$

This agrees with the constraint discussed in [2]. With the inclusion of spin, the symplectic potential is to be taken as

$$\mathcal{A} = -i\frac{mR}{2}\mathrm{Tr}(\gamma_0 g^{-1}dg) + \frac{s_1}{2}\mathrm{Tr}(\gamma_1\gamma_2\, g^{-1}dg) + \frac{s_2}{2}\mathrm{Tr}(\gamma_3\gamma_5\, g^{-1}dg) \tag{35}$$

Since the isotropy group for the orbit is $SO(4)$, which is of rank 2, we need two spin labels s_1 and s_2.

We now consider another representation of the Dirac matrices, given by

$$\Gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \Gamma_5 = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{36}$$

These are related to the γ-matrices in (11) by a similarity transformation, $\Gamma = S\gamma S^{-1}$, with

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{37}$$

Correspondingly, define $S\,g\,S^{-1} = N$. The group element N obeys $N^\dagger\,\Gamma_0\,N = \Gamma_0$. With Γ_0 as given in (36), we see that N can be taken as an element of $SU(2,2)$. The symplectic potential for the spinless case becomes

$$\mathcal{A} = -i\frac{mR}{2}\mathrm{Tr}(\Gamma_0\,N^{-1}dN) = -i\frac{mR}{2}\mathrm{Tr}\big[(1+\Gamma_0)\,N^{-1}dN\big]$$

$$
\begin{aligned}
&= -imR\, N^\dagger_{ar}(\Gamma_0)_{rs}\, dN_{sa} \\
&= -imR\left[N^\dagger_{ab}\, dN_{ba} - N^\dagger_{a\tilde{b}} dN_{\tilde{b}a}\right]
\end{aligned} \tag{38}
$$

where we have used the fact that $N^{-1}dN$ is traceless. The indices a, b take values $1, 2$, while r, s take values 1 to 4, with $\tilde{b} = 1, 2$ corresponding to $r, s = 3, 4$. Defining the 2×2 matrices

$$
\xi_{ba} = \sqrt{2mR}\, N_{ba}, \qquad \zeta_{ba} = \sqrt{2mR}\, N_{\tilde{b}a} \tag{39}
$$

we can write

$$
\mathcal{A} = -\frac{i}{2}\mathrm{Tr}(\xi^\dagger d\xi - \zeta^\dagger d\zeta) \tag{40}
$$

The constraint on ξ, ζ is given by $N^\dagger \Gamma_0 N = \Gamma_0$ and translates to

$$
\xi^\dagger\, \xi - \zeta^\dagger\, \zeta = (2mR)\, \mathbb{1} \tag{41}
$$

This is a 2×2 matrix equation. We can regard ξ and ζ as *a priori* independent variables with the symplectic potential given by $\mathcal{A}$ in (40) and impose (41) as constraints on the phase space. We now introduce two linear combinations of ξ and ζ as

$$
\xi = \frac{\mathbb{U} - i\,\mathbb{W}}{\sqrt{2}}, \qquad \zeta = \frac{\mathbb{W} - i\,\mathbb{U}}{\sqrt{2}} \tag{42}
$$

The action (which is the integral of $\mathcal{A}$) and the constraint become

$$
\begin{aligned}
S &= \frac{1}{2}\int d\tau\, \mathrm{Tr}(\mathbb{W}^\dagger\, \dot{\mathbb{U}} - \mathbb{U}^\dagger\, \dot{\mathbb{W}}) \\
\mathbb{U}^\dagger\, \mathbb{W} - \mathbb{W}^\dagger\, \mathbb{U} &= i\,(2mR)\mathbb{1}
\end{aligned} \tag{43}
$$

We see that we have recovered the twistor description of the massive particle in AdS$_5$ obtained in [2]. (Our $\mathbb{W}$ corresponds to the $\tilde{\mathbb{W}}$ used in that paper.)

It is also useful to consider the massless particle in AdS$_5$. For this, it is easier to go back to the first representation of the Dirac matrices and use the group element g. Then, up to an overall scale factor which will be irrelevant, we can take

$$
\begin{aligned}
\mathcal{A} &= -\frac{i}{2}\,\mathrm{Tr}\big[(\gamma_0 + \gamma_3)\, g^{-1}dg\big] = -\frac{i}{2}\,\mathrm{Tr}\big[(1 + \gamma_3\gamma_0)\, \bar{g}dg\big] \\
&= -i\left[\bar{g}_{2r}dg_{r2} + \bar{g}_{3r}dg_{r3}\right] \\
&= -i(\bar{Z}dZ + \bar{W}dW)
\end{aligned} \tag{44}
$$

where $\bar{g} = g^\dagger\, \gamma_0$ and we have defined

$$
\begin{aligned}
Z_r &= g_{r2}, \qquad & W_r &= g_{r3} \\
\bar{Z}_r &= \bar{g}_{3r}, \qquad & \bar{W}_r &= \bar{g}_{3r}
\end{aligned} \tag{45}
$$

These obey the constraints

$$
\bar{Z}Z = 0, \qquad \bar{W}W = 0
$$

$$\bar{Z}W = 0, \qquad \bar{W}Z = 0 \tag{46}$$

which are identical to (29), but, now, for massless particles in AdS$_5$.

Turning to strings in AdS$_5$, we can use a spacelike one-form of the form $\mathrm{Tr}(\gamma_3 g^{-1}dg) \sim \eta_{\alpha\beta}\Lambda^{\alpha}_{3}dx^{\beta}/z$ along with the timelike one-form in (33). The action is then of the same form as in (21), with $V_{\alpha\beta}$ as in (19), except for the fact that Λ^{α}_{μ} is now an element of $SO(4,1)$ rather than just $SO(3,1)$. For null strings, one can again use the action (24), with $\Lambda \in SO(4,1)$.

One can also write down a twistor description for null strings. Taking the spatial direction to be along γ_1, and the null vector as given in (44), the action is seen to be of the form

$$S = \int (\bar{Z}dZ + \bar{W}dW) \wedge (\bar{W}dW' + \bar{W}'dW - \bar{Z}dZ' - \bar{Z}'dZ) \tag{47}$$

We have introduced two more twistors, corresponding to $Z' \sim g_{r1}$, $W' \sim g_{r4}$. The constraints follow from $\bar{g}\,g = \gamma_0$; explicitly, we have

$$\bar{Z}'W = 1, \qquad \bar{Z}W' = 1 \tag{48}$$

with all other products of the form $(\overline{\text{twistor}}\ \text{twistor})$ vanishing. (The constraints in (48) are complex, the conjugates are obtained as well.) Given the large number of variables and constraints, this is not an economical way to describe null strings.

In this section, in terms of the relevant isometry group $SO(4,2) \sim SU(2,2)$, we have obtained the actions for massive and massless particles and for strings (including the null string) in AdS$_5$. In particular, the standard co-adjoint orbit action is shown to give, in a simple straightforward way, the twistor description of massive particles obtained in [2]. We also give the twistor version for massless particles and null strings in AdS$_5$.

4 A Chern-Simons approach

4.1 Chern-Simons action and co-adjoint orbits

We now come to the theme of the second part of this article, namely, relating the co-adjoint orbit actions to the Chern-Simons theory and to the CS gravity. The basic idea is that, generally for any field theory, the motion of a classical configuration within the gauge group can be described using a group element as the collective coordinates. The quantization of the action for these will generate the dynamics. If we use a Chern-Simons form as the starting action, then, because of its topological nature, one basically gets trivial dynamics except for a full group representation. Thus the co-adjoint orbit action should be obtainable from the CS theory.

To see how this works out in detail, consider $2+1$ dimensional spacetime of the form $M \times \mathbb{R}$, where the spatial manifold M has the topology of a disc. We then consider the action

$$S = \frac{k}{4\pi} \int \mathrm{Tr}(A\, dA + \tfrac{2}{3} A^3) + S_b(A, \psi) \tag{49}$$

Here A belongs to the Lie algebra of some Lie group G, ψ is a field which may be taken as representing degrees of freedom on the boundary of the manifold. $S_b(A, \psi)$ is the action for ψ on $\partial M \times \mathbb{R}$. The variation of the CS term in (49) under a gauge transformation $A \to A^g = g^{-1} A g + g^{-1} dg$ is given by

$$\frac{k}{4\pi} \int \mathrm{Tr}(A^g\, dA^g + \tfrac{2}{3} A^{g3}) = \frac{k}{4\pi} \int \mathrm{Tr}(A\, dA + \tfrac{2}{3} A^3) - \Gamma[g] + \frac{k}{4\pi} \oint_b \mathrm{Tr}(A\, dg g^{-1}) \tag{50}$$

where

$$\Gamma[g] = \frac{k}{12\pi} \int \mathrm{Tr}(dg g^{-1})^3 \tag{51}$$

We take $S_b(A, \psi)$ to have a compensating gauge anomaly; i.e., we choose $S_b(A, \psi)$ so that

$$S_b(A^g, \psi^g) = S_b(A, \psi) - \frac{k}{4\pi} \oint_b \mathrm{Tr}(A\, dg g^{-1}) + \Gamma[g] \tag{52}$$

Thus the action (49) is invariant under all gauge transformations of the fields, *including the transformations g which are not necessarily the identity on the boundary ∂M*. The nature of the field ψ and the action S_b are not important for what follows; we may even replace S_b by an effective action obtained by integrating out ψ,

$$e^{iS_{\mathrm{eff}}(A)} = \int [\mathcal{D}\psi]\, e^{iS_b(A,\psi)} \tag{53}$$

The variation of the action (49) is

$$\delta S = \frac{k}{2\pi} \int \mathrm{Tr}(\delta A\, F) + \frac{k}{4\pi} \oint \mathrm{Tr}\,(\delta A\, A) + \delta S_b(A, \psi) \tag{54}$$

where $F = dA + A^2$. We see that the equation of motion gives $F = 0$ in the bulk; we may extend this to the boundary by continuity, with the total current $(k/4\pi)\epsilon^{ij} A_j + (\delta S_b / \delta A_i) = 0$ as a constraint on the boundary. Since $F = 0$, A is a pure gauge in the bulk.

Our aim now is to consider singular classical solutions on the disc M of the form

$$A_i = a_i, \qquad da + a^2 = \sum_{\alpha=1}^{N} q_\alpha\, \delta^{(2)}(x - x_\alpha), \qquad A_0 = a_0 = 0 \tag{55}$$

This corresponds to N magnetic vortices at the points $\vec{x}_\alpha$, with charges q_α. For simplicity, we will take q_α to be in the Cartan subalgebra of G, so that we may write $da + a^2 = da$. Thus we are considering Abelian magnetic vortices. To discuss this case using the Chern-Simons action, we consider the theory on $\tilde{M}$ which is the disc M with a number of points $\vec{x}_\alpha$ removed; i.e., $\tilde{M} = M - \{\vec{x}_\alpha\}$. The boundary of $\tilde{M}$ is thus the outer boundary (or ∂M)

at $|\vec{x}| \to \infty$ and a set of circles C_α, one around each point $\vec{x}_\alpha$. On $\tilde{M}$, since we have excised small neighborhoods around $\vec{x}_\alpha$, we have $da = 0$. The points $\vec{x}_\alpha$ do not lie on the boundary and hence $F = 0$ on the new boundaries $\{C_\alpha\}$ as well. Thus A is a pure gauge, say, $U^{-1}dU$, and the solution (55) is nonsingular on all of $\tilde{M}$. Notice that even though $a_i = U^{-1}\partial_i U$ on $\tilde{M}$, U is singular at $\vec{x}_\alpha$, since the field strength is nonvanishing at those points; so they have to be excluded from the manifold for a nonsingular description.

The general solution of the equations of motion is a pure gauge on $\tilde{M}$. It may be viewed as the gauge transform of (55) by an element $g \in G$ and can be written as

$$A_i = g^{-1} a_i\, g + g^{-1}\partial_i g = (Ug)^{-1}\partial_i(Ug) \qquad A_0 = g^{-1}\partial_0 g = (Ug)^{-1}\partial_0(Ug) \tag{56}$$

g is to be nonsingular everywhere on M. The value of g on the outer boundary ∂M is compensated for by the boundary action $S_b(A, \psi)$. But the values of g on the new boundaries, i.e., at C_α are edge degrees of freedom which will act as moduli or collective coordinates for the solution (55). The dynamics of these moduli can be analyzed by evaluating the action on the general solutions (56). The result is thus

$$S = S[a] - \frac{k}{4\pi}\sum_\alpha \oint_{C_\alpha} \Big[\mathrm{Tr}(a\, dg\, g^{-1})\Big]_{\vec{x}_\alpha} \tag{57}$$

Since a has only spatial components, dgg^{-1} in (57) must be $\partial_0 g\, g^{-1} dt$. Now consider shrinking the circles C_α to zero radius. Since g is nonsingular at $\vec{x}_\alpha$, $\partial_0 g\, g^{-1} dt$ has a limit which we denote as $\dot{h}_\alpha^{-1}\, h_\alpha dt = -h_\alpha^{-1}\dot{h}_\alpha\, dt$, where $g(\vec{x}_\alpha, t) = h_\alpha^{-1} \in G$; h_α can be taken as a time-dependent group element, one element for each vortex. Further $\oint_{C_\alpha} a = q_\alpha$ and the action for the moduli becomes

$$S = \frac{k}{4\pi}\sum_\alpha \int dt\ \mathrm{Tr}(q_\alpha\, h_\alpha^{-1}\, \dot{h}_\alpha) \tag{58}$$

Writing $q = \sum_a w_a\, t_a$ where t_a are the diagonal generators of the group, and w_a are suitable weights of a representation, we see that we have obtained the co-adjoint orbit action.

A couple of remarks are in order at this point. We used the boundary action $S_b(A, \psi)$ so that we could start with a gauge invariant theory on $M \times \mathbb{R}$. This can be avoided if we take M to be compact without boundary to begin with. For example, we could consider vortices on $M = S^2$ rather than the disc. The special solution (55) corresponds to a certain total flux $\sum_\alpha q_\alpha$ which is split into the vortices at the locations $\vec{x}_\alpha$. (If one considers the sphere as embedded in $\mathbb{R}^3$, this is equivalent to having a monopole of charge $\sum_\alpha q_\alpha$ at the center.) After excising the singular points, the boundary is given by the union of C_α; there is no other boundary to consider. However, since the total flux in nonzero, we will need to consider different gauge potentials on different coordinate patches with suitable transition gauge transformations on the overlaps to get a nonsingular description. There will also be Dirac quantization conditions on the fluxes.

Also, so far we have only considered introducing collective coordinates for the dynamics of the vortices within the group. Generally, one could also consider dynamics or motion of the particles where the locations $\vec{x}_\alpha$ evolve in time. For the Chern-Simons action, the Hamiltonian is zero and the worldlines of the vortices can be chosen freely. (To phrase this another way, one can define a functional integral for the Chern-Simons action with the insertion of Wilson lines corresponding to the vortices; the corresponding Hamiltonian is then precisely what is needed to evolve them along predetermined worldlines. This was how it was done in [10].) The only nontrivial dynamics is then the braiding of the worldlines.

4.2 Particles and dynamics in Chern-Simons gravity

We will now consider using this method for the Poincaré or AdS group. Since the action is a topological one, there is no difficulty in using it for noncompact groups such as these. We will consider two different directions in what follows. First, we simply consider the action as in (49) and take the gauge group to be the Poincaré or AdS group of the appropriate dimension. This will immediately give the co-adjoint orbit action for particles. Thus various actions for particle dynamics considered in the previous sections are obtained for suitable choices of the charges in the Cartan subalgebra.

The second, more natural, choice would be to consider gravity in terms of the Chern-Simons actions. In any odd spacetime dimension, one can write a parity-conserving gravitational action which is the difference of two CS actions. Generally, this does not lead to Einstein gravity, except in 2+1 dimensions where it does indeed describe Einstein gravity. The choice of the group is natural in this approach. Our use of vortex solutions also ties in with the well known observation due to Einstein, Infeld and Hoffmann (EIH), where the field equations for gravity determine the dynamics of particles, the latter being identified as singularities of the gravitational field [4]. In their analysis, EIH used surface conditions around the singularities of the field to obtain the dynamics, with the surface at infinity giving the description of the center of mass motion for the particles. More recently, there have been a number of investigations of the dynamics of a fluid on the boundary of AdS_5 as determined by the bulk field equations [5]. Effectively, one considers diffemorphisms of special solutions, in very much the same way as we consider gauge transformations of the vortex solutions (55). There have also been some recent studies of vortex-particle duality in 2+1 dimensions. In some sense, our use of the CS action is the most elementary prototype of such considerations. In particular, we may think of what follows as the CS gravity analogue of the EIH work. Being a CS version of gravity, there is no real dynamics, so we just end up getting noninteracting particles, each described as a unitary irreducible representation of the corresponding group.

We will first consider gravity in 2+1 dimensions, for which the relevant group is $SO(2,1) \times SO(2,1)$. We will refer to these as $SO(2,1)_L$ and $SO(2,1)_R$ to distinguish them. The generators of the Lie algebras are denoted by M_a for $SO(2,1)_L$ and N_a for $SO(2,1)_R$

with the standard commutation rules. A useful spinorial matrix representation is given by

$$
\begin{aligned}
M_0 &= \begin{bmatrix} -t_3 & 0 \\ 0 & 0 \end{bmatrix}, \qquad M_1 = \begin{bmatrix} it_2 & 0 \\ 0 & 0 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} -it_1 & 0 \\ 0 & 0 \end{bmatrix} \\
N_0 &= \begin{bmatrix} 0 & 0 \\ 0 & -t_3 \end{bmatrix}, \qquad N_1 = \begin{bmatrix} 0 & 0 \\ 0 & -it_2 \end{bmatrix}, \qquad N_2 = \begin{bmatrix} 0 & 0 \\ 0 & it_1 \end{bmatrix}
\end{aligned} \tag{59}
$$

where $t_a = \frac{1}{2}\sigma_a$ and $\mathrm{Tr}(M_a M_b) = \mathrm{Tr}(N_a N_b) = \frac{1}{2}\eta_{ab}$, $\eta_{ab} = diag(1,-1,-1)$. Further, $\mathrm{Tr}(M_a N_b) = 0$. We also introduce the connections

$$
A_L = (-iM_a)\, A_L^a = (-iM_a) \left(\tilde{\omega}^a + \frac{e^a}{l} \right), \qquad A_R = (-iN_a)\, A_R^a = (-iN_a) \left(\tilde{\omega}^a - \frac{e^a}{l} \right) \tag{60}
$$

where we will identify e^a as the frame fields and $\tilde{\omega}^a$ is related to the spin connection ω^{bc} by

$$
\tilde{\omega}^a = -\frac{1}{2} \eta^{ak} \epsilon_{kbc}\, \omega^{bc} \tag{61}
$$

In (60) l is a constant with the dimension of length; it is related to the cosmological constant. In terms of the $SO(2,1)$'s, the generators of Lorentz transformations are $L_a = M_a + N_a$ and translations are generated by $P_a = (M_a - N_a)/l$. The action for gravity is then given by

$$
\begin{aligned}
S &= -\frac{k}{4\pi} \left[\int \mathrm{Tr}\left(AdA + \tfrac{2}{3} A^3 \right)_L - \int \mathrm{Tr}\left(AdA + \tfrac{2}{3} A^3 \right)_R \right] \\
&= -\frac{k}{4\pi\, l} \int d^3x \, \det e \, \left[R - \frac{2}{l^2} \right] + \text{total derivative}
\end{aligned} \tag{62}
$$

where R is the Ricci scalar for the curvature corresponding to ω. This is the Einstein-Hilbert action, with Newton's constant of gravity G being related to the level number of the CS action by $k = (l/4G)$. The standard parity transformation along with $A_L \leftrightarrow A_R$ leaves the action invariant.

Starting with the first line of (62) with the Chern-Simons actions, we can add a boundary action as in (49) and consider vortices. The resulting particle action will be as in (58), and reads

$$
\begin{aligned}
S &= -\frac{k}{4\pi} \sum_\alpha \int dt \, \left[\mathrm{Tr}(q_\alpha\, h_\alpha^{-1}\, \dot{h}_\alpha)_L - \mathrm{Tr}(q_\alpha\, h_\alpha^{-1}\, \dot{h}_\alpha)_R \right] \\
&= -\frac{k}{8\pi} \int dt \, \sum_\alpha \Big[(q_{L\alpha} + q_{R\alpha}) \mathrm{Tr}(M_0 - N_0) g_\alpha^{-1} \dot{g}_\alpha) \\
&\qquad\qquad + (q_{L\alpha} - q_{R\alpha}) \mathrm{Tr}(M_0 + N_0) g_\alpha^{-1} \dot{g}_\alpha) \Big]
\end{aligned} \tag{63}
$$

where

$$
g = \begin{pmatrix} h_L & 0 \\ 0 & h_R \end{pmatrix} \tag{64}
$$

This describes multiparticle dynamics in 2+1 dimensions with the mass and spin of the particle related to the weights q_L, q_R by

$$\begin{aligned} m &= \frac{k}{8\pi\, l}(q_L+q_R) = \frac{q_R+q_L}{32\pi G} \\ s &= \frac{k}{4}(q_L-q_R) \end{aligned} \tag{65}$$

Here we are considering equal number of vortices for the two groups. For unequal numbers of vortices, we will get states which are parity-asymmetric.

A similar analysis can be done in higher dimensions. Here we will consider the 5-dimensional case, starting with the general situation before specializing to CS gravity. The 5d analogue of the action (49) is given by

$$S = CS(A) + S_b \tag{66}$$

where $CS(A)$ is the 5d Chern-Simons term,

$$\begin{aligned} CS(A) &= -\frac{ik}{24\pi^2}\int \mathrm{Tr}\left(AF^2 - \frac{1}{2}A^3F + \frac{1}{10}A^5\right) \\ &= -\frac{ik}{24\pi^2}\int_{M\times\mathbb{R}} \mathrm{Tr}\left(A\,dA\,dA + \frac{3}{2}A^3\,dA + \frac{3}{5}A^5\right) \end{aligned} \tag{67}$$

and S_b is the boundary action (on $\partial M\times\mathbb{R}$) needed to cancel the anomaly from the CS term. The bulk equation of motion is given by $F\,F=0$. This is satisfied by choosing A as a pure gauge (although this may not be the most general solution). Since the spatial manifold is four-dimensional, the natural choice for a singular solution is to take the point-like limit of instantons. The general solution (with $F=0$) is thus given by $A = g^{-1}\,a\,g + g^{-1}dg$ where the potential a is a singular solution (point-like configuration) with nonzero instanton number. The instanton density is concentrated at a set of points $\{\vec{x}_\alpha\}$, so that A is a pure gauge in all of $\tilde{M} \equiv M - \{\vec{x}_\alpha\}$. We excise small balls around each singularity to define $\tilde{M}$. The boundary of the excised ball around $\vec{x}_\alpha$ is C_α which is topologically a three-sphere. Upon using the general solution for A, the action becomes

$$\begin{aligned} S \;=\; & CS(a) + S_b(a^g,\psi^g) - \frac{ik}{240\pi^2}\int \mathrm{Tr}(dg\,g^{-1})^5 \\ & + \frac{ik}{48\pi^2}\oint_{\partial M}\mathrm{Tr}\left(dgg^{-1}(ada+daa+a^3) + a(dgg^{-1})^3 - \frac{1}{2}dgg^{-1}\,a\,dgg^{-1}\,a\right) \\ & - \frac{ik}{48\pi^2}\sum_\alpha \oint_{C_\alpha}\mathrm{Tr}\left(dgg^{-1}(ada+daa+a^3) + a(dgg^{-1})^3 - \frac{1}{2}dgg^{-1}\,a\,dgg^{-1}\,a\right) \end{aligned} \tag{68}$$

The boundary action will cancel the integral around ∂M as well as the term proportional to $\mathrm{Tr}(dgg^{-1})^5$. Now consider evaluating $CS(a)$. Since a is a pure gauge in $\tilde{M}$, we have $da = -a^2$, so that $CS(a) = -(ik/240\pi^2)\int \mathrm{Tr}(a^5)$. The special solution a is at a fixed time t, with $a_0 = 0$, so $a^5 = 0$ and we get $CS(a) = 0$. The action for the moduli thus reduces to

$$S = -\frac{ik}{48\pi^2}\sum_\alpha \oint_{C_\alpha}\mathrm{Tr}\left(-dgg^{-1}\,a^3 + a(dgg^{-1})^3 - \frac{1}{2}dgg^{-1}\,a\,dgg^{-1}\,a\right) \tag{69}$$

To simplify further, we note that the instanton number is given by

$$\nu = -\frac{1}{8\pi^2}\int \mathrm{Tr}(F\,F) = -\frac{1}{8\pi^2}\oint \mathrm{Tr}(ada + \frac{2}{3}a^3) = \frac{1}{24\pi^2}\oint \mathrm{Tr}(a^3) \tag{70}$$

where we have used the fact that a is a pure gauge on $\tilde{M}$ including on C_α. We will consider point-like instantons in an $SU(2)$ subgroup of the gauge group G. Further we consider the standard embedding of $SU(2)$ in G and take $a = U^{-1}dU$, with with

$$U = \phi^0 + i\sigma_i\,\phi^i, \qquad (\phi^0)^2 + (\phi^i\phi^i) = 1 \tag{71}$$

In this case, we can work out a^3 as

$$a^3 = t_1\,\epsilon_{\mu\nu\alpha\beta}\,\phi^\mu d\phi^\nu d\phi^\alpha d\phi^\beta, \qquad t_1 = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{bmatrix} \tag{72}$$

The winding number for the $S^3 \to S^3$ map, corresponding in our case to the map $U : C_\alpha \to SU(2)$, is given by

$$Q[u] = \frac{1}{12\pi^2}\int \epsilon_{\mu\nu\alpha\beta}\,\phi^\mu d\phi^\nu d\phi^\alpha d\phi^\beta \tag{73}$$

We can now start simplifying the terms in (69). Using (72) , (73), the first term of (69) can now be written as

$$\begin{aligned} \frac{ik}{48\pi^2}\sum_\alpha \oint_{C_\alpha} \mathrm{Tr}(dgg^{-1}\,a^3) &= \frac{ik}{48\pi^2}\sum_\alpha \int dt \oint_{C_\alpha} \mathrm{Tr}(\partial_0 gg^{-1}\,a^3) \\ &= \frac{ik}{4}\sum_\alpha \int dt \mathrm{Tr}(t_1\,\partial_0 gg^{-1})Q_\alpha \end{aligned} \tag{74}$$

(A similar result is obtained for other $SU(2)$ subgroups of G as well.) Now consider the next term in (69), namely, $\oint \mathrm{Tr}[a\,(dgg^{-1})^3]$. Separating out the time-derivative part, we are considering terms of the form

$$\oint \mathrm{Tr}[a\,(dgg^{-1})^2 - dgg^{-1}adgg^{-1} + (dgg^{-1})^2 a]\,\partial_0 gg^{-1}dt$$

where the d's denote differentiation with respect to the angular coordinates of the three-sphere surrounding $\vec{x}_\alpha$. These derivatives must go to zero as we shrink the spheres to zero radius to ensure that g is nonsingular. Thus the contribution of this term is zero. A similar argument can be made for the last term in (69). The action for the moduli is thus

$$S = \frac{ik}{2}\sum_\alpha \oint_{C_\alpha} \left[Q_\alpha\,\mathrm{Tr}(t_1\partial_0 g_\alpha\,g_\alpha^{-1})\right] \tag{75}$$

We now apply this line of reasoning to the case of $SO(4,2) \sim SU(2,2)$ which is appropriate for CS gravity with a cosmological constant in 5 dimensions. The relation between the gauge fields and the frame fields and spin connection is given by

$$A_L = -i\left[\tfrac{1}{2}\omega^{ab}\Sigma_{ab} + \tfrac{1}{2}e^a\Gamma_a\right], \qquad A_R = -i\left[\tfrac{1}{2}\omega^{ab}\Sigma_{ab} - \tfrac{1}{2}e^a\Gamma_a\right] \tag{76}$$

where Σ_{ab} generate the $SO(4,1)$ group and Γ_a correspond to the coset directions. The explicit form of these matrices are as given in (11, 30, 36). The action is given by

$$S = CS(A_L) + S_{bL} - \{CS(A_R) + S_{bR}\} \tag{77}$$

Consider first $CS(A_L)$. There are two $SU(2)$ sugbroups generated by $i(\frac{1}{2}\epsilon_{ijk}\Gamma_i\Gamma_j \mp \Gamma_5\Gamma_k)/4$, corresponding to the upper and lower 2×2 block diagonal matrices in the chosen representation. We consider point-like instantons in these two $SU(2)$ subgroups. We can then adapt (74) to the present case; the action for the moduli is then

$$\begin{aligned} S_L &= \frac{ik}{4}\sum_\alpha \int dt\, \mathrm{Tr}\left(\frac{1+\Gamma_0}{2}\,\partial_0 g g^{-1}\, Q_\alpha^{(1)} + \frac{1-\Gamma_0}{2}\,\partial_0 g g^{-1}\, Q_\alpha^{(2)}\right) \\ &= -\frac{ik}{8}\sum_\alpha (Q_\alpha^{(1)} - Q_\alpha^{(2)}) \int dt\;\, \mathrm{Tr}\,(\Gamma_0\, h_\alpha^{-1}\dot{h}_\alpha) \end{aligned} \tag{78}$$

We have made the replacement $g \to h^{-1}$ to agree with the notation of (58) and (61). We can parametrize the group element as in (31), namely,

$$h_L = S^{-1}\, g = S^{-1}\begin{pmatrix} \sqrt{z} & i\tilde{X}/\sqrt{z} \\ 0 & 1/\sqrt{z}\end{pmatrix}\Lambda \tag{79}$$

where S is the matrix relating Γ's to γ's. The action now reduces to

$$S_L = \sum_\alpha \int \frac{k}{4}(Q_\alpha^{(1)} - Q_\alpha^{(2)})\, \eta_{\mu\nu}\Lambda^\mu{}_0\, \frac{dx^\nu}{z} \tag{80}$$

A similar result holds for the $CS(A_R)$ part. In this case, we have

$$h_R = S^{-1}\, g' = S^{-1}\begin{pmatrix} 1/\sqrt{z} & 0 \\ -iX/\sqrt{z} & \sqrt{z}\end{pmatrix}\Lambda, \qquad X = x^0 + \sigma\cdot\vec{x} \tag{81}$$

The action for this part is then

$$S_R = -\sum_\alpha \int \frac{k}{4}(Q_\alpha^{(1)} - Q_\alpha^{(2)})\, \eta_{\mu\nu}\Lambda^\mu{}_0\, \frac{dx^\nu}{z} \tag{82}$$

The total action, given by $S_L - S_R$ is thus of the form (33), as is appropriate for particles in AdS$_5$, with the identification

$$mR = \frac{k}{2}(Q_\alpha^{(1)} - Q_\alpha^{(2)}) \tag{83}$$

This result does not include spin. For spin, we will need to consider multi-instantons taking account of the placement of the $SU(2)$'s within the larger group. A simple way to see this is to consider merging two of the points $\vec{x}_\alpha$, say $\vec{x}_1$ and $\vec{x}_2$, into a single one. This is equivalent to setting $h_1 = h_2$. However, there are different placements of the $SU(2)$ subgroups in $SU(2,2)$, related by Weyl group transformations. In particular, we have

$$T_1^{-1}\,\Gamma_0\, T_1 = i\Gamma_1\Gamma_2, \qquad T_2^{-1}\,\Gamma_0\, T_2 = i\Gamma_3\Gamma_5 \tag{84}$$

where

$$T_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad T_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \tag{85}$$

Thus if we consider merging the two points with the identification $h_1 = h_2$ and $a_{(2)} = T_1^{-1}\, a_{(1)}\, T_1$, we get the action

$$S = -\frac{ik}{8} \int dt \left[(Q_1^{(1)} - Q_1^{(2)})\, \mathrm{Tr}\,(\Gamma_0\, h_1^{-1} \dot{h}_1) + (Q_2^{(1)} - Q_2^{(2)})\, \mathrm{Tr}\,(i\Gamma_1 \Gamma_2\, h_1^{-1} \dot{h}_1) \right] + \cdots \tag{86}$$

By considering a number of such mergers, with different values of $Q^{(1)}$ and $Q^{(2)}$, we can get a general combination of $\mathrm{Tr}\,(\Gamma_0\, h_1^{-1}\dot{h}_1)$, $\mathrm{Tr}\,(i\Gamma_1\Gamma_2\, h_1^{-1}\dot{h}_1)$ and $\mathrm{Tr}\,(i\Gamma_3\Gamma_5\, h_1^{-1}\dot{h}_1)$, thus giving a general action for particle dynamics. While this does show the role of the choice of the $SU(2)$ subgroups, there should be a more satisfactory way of obtaining the general co-adjoint orbit action, following directly from the multi-instanton solutions. This is currently being explored and will be deferred to a future publication.

To summarize the results of this section, we considered the dynamics which follows from the use of singular solutions of the CS theory, where a gauge transformation furnishes the moduli for the singularities. This is very similar to the strategy of Einstein, Infeld and Hoffmann, who derived multiparticle dynamics from the general theory of relativity, treating particles as singularities of the gravitational field. We considered the 3d CS theory. We also applied the same strategy to CS gravity in 3 and 5 dimensions. Clearly there are still many points to be clarified and elaborated on such as the use of more general solutions including various types of gravitational instantons as well as solutions which can lead to interacting particles in five dimensions, issues related to the orientations of various subgroups, as well as the inclusion of nontopological terms, i.e., going beyond the CS action. The connection to fluid dynamics, alluded to earlier, is another interesting avenue to explore. These issues will be taken up in future.

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