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ABSTRACT

Experimental data from hadronic τ decays allow for a precision determination of the slope of the $I = 1$ vacuum polarization at zero momentum. We use this information to provide a value for the next-to-next-to-leading order (NNLO) low-energy constant C_{93} in chiral perturbation theory. The largest systematic error in this determination results from the neglect of terms NNNLO (and higher) in the effective chiral Lagrangian, whose presence in the data will, in general, make the effective C_{93} determined in an NNLO analysis mass dependent. We estimate the size of this effect by using strange hadronic τ-decay data to perform an alternate C_{93} determination based on the slope of the strange vector polarization at zero momentum, which differs from that of the $I = 1$ vector channel only through $SU(3)$ flavor-breaking effects. We also comment on the impact of such higher order effects on ChPT-based estimates for the hadronic vacuum polarization contribution to the muon anomalous magnetic moment.

I. INTRODUCTION

The spin $J = 0 + 1$ polarization sums, $\Pi^{0+1}_{V/A;ud,us}$, of the flavor ud and us vector (V) and axial vector (A) current two-point functions of QCD have been calculated to two-loop order in Chiral Perturbation Theory (ChPT) [1]. It is therefore, in principle, possible to provide estimates for low-energy constants (LECs) appearing in these expressions at next-to-next-to-leading order (NNLO) by comparing the relevant ChPT expressions to either dispersive representations of the subtracted polarizations, $\Pi^{0+1,\text{sub}}_{V/A;ud,us}(Q^2) \equiv \Pi^{0+1}_{V;ud,us}(Q^2) - \Pi^{0+1}_{V;ud,us}(0)$, or inverse-moment finite-energy sum-rule (IMFESR) results for their slopes at $Q^2 = -q^2 = -s = 0$, both of which can be determined from experimental data for the spectral functions in the V and A channels. Since LECs encode the physics of QCD at low energies, their knowledge is indispensable in phenomenological applications of ChPT. Since many applications now employ ChPT to NNLO, it is important to determine the numerical values of as many NNLO LECs as possible.[1]

The LECs that appear in the ChPT expressions for $\Pi^{0+1}_{V/A;ud,us}(Q^2)$ are the next-to-leading order (NLO) LECs L_9 and L_{10} and the NNLO LECs C_{12}, C_{13}, C_{61}, C_{62}, C_{80}, C_{81}, C_{87} and C_{93} in the $SU(3)$-flavor-symmetric limit, and, in addition, the NLO LEC L_5 in flavor-breaking contributions proportional to $m_K^2 - m_\pi^2$. In previous work [2–4], we provided determinations of L_{10} and the linear combinations $C_{12} - C_{61} + C_{80}$, $C_{13} - C_{62} + C_{80}$, C_{61} and C_{87} using dispersive and IMFESR results for the flavor ud $V - A$ polarization and flavor-breaking $ud - us$ V and $V + A$ polarization combinations. For the ud $V - A$ polarization, both lattice results at unphysical quark mass [5] and physical-quark-mass results, obtained using hadronic τ decay data from OPAL [6] and ALEPH [7] for the non-strange spectral functions, were employed. The IMFESRs used to determine the $Q^2 = 0$ values of the flavor-breaking V and $V + A$ polarizations required, in addition, strange hadronic τ-decay data from ALEPH [8], Belle [9–11] and BaBar [12–14], together with 2014 HFAG strange branching fractions [15].

In the present paper, we consider the LEC C_{93}, which can be obtained from a determination of the slope with respect to Euclidean momentum-squared, Q^2, at $Q^2 = 0$, of the V polarization using the ALEPH data. C_{93} is the only NNLO LEC appearing in the NNLO representation of the subtracted polarizations $\Pi^{\text{sub}}_{ud,us}(Q^2)$.[2] The ud representation also depends on the NLO LEC L_9 and the us representation on the NLO LECs L_5 and L_9. With $\Pi^{\text{sub}}_{ud,us}(Q^2)$ both admitting once-subtracted dispersive representations, C_{93} can, in principle, be determined from the experimental spectral data of either channel. As in our previous work, we will take L_5 and L_9 from outside sources [16, 17]. In what follows, in addition to $\Pi^{\text{sub}}_{ud,us}$, we also consider the subtracted version of the V current polarization, $\Pi_\eta(Q^2)$ in the notation of Ref. [1], associated with the neutral octet V current $(\bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d - 2\bar{s}\gamma_\mu s)/\sqrt{6}$. L_9 and C_{93} are the only NLO and NNLO LECs appearing in the NNLO representation of $\Pi^{\text{sub}}_\eta(Q^2)$.

Spectral functions (generically denoted $\rho(s)$) obtained from hadronic τ decays are, of course, limited to $s \leq m_\tau^2$. This limits the radius, s_0, of the circular contour in the complex-s plane used in τ-based IMFESRs to $s_0 \leq m_\tau^2$. Dispersive representations of the subtracted

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[1] For one application, see the discussion in Sec. V of this paper.

[2] Since we will consider only the spin $J = 0 + 1$ V case in this paper, we will drop the superscript $0 + 1$ and the subscript V from now on.

polarizations require the corresponding $\rho(s)$ for all s. In the ud channel, we will use a representation of $\rho(s)$ above the τ mass obtained from sum-rule-based fits employing perturbation theory, augmented by a model for duality-violating (resonance) effects, performed in Ref. [18]. While this introduces an assumption about the validity of this model into our extraction of C_{93}, the low-Q^2 region from which C_{93} is determined is very insensitive to the details of this assumption. Hence, we believe that the associated potential uncertainty is far smaller than the systematic error due to the neglect of orders beyond NNLO in ChPT.

Since we will employ ChPT to NNLO, the value of C_{93} obtained from the $ud\ V$ channel analysis, which we denote by C_{93}^{ud}, will have a residual mass dependence, originating from the effect of beyond-NNLO loop and LEC contributions present in the data, but absent from the NNLO representation of $\Pi_{ud}^{\text{sub}}(Q^2)$. NNNLO loop corrections have not been calculated, so an expanded NNNLO analysis is not possible. We will, therefore, use flavor-breaking IMFESRs to obtain an estimate for the slope of the difference of the us and $ud\ V$ polarizations, hence also of the slope of the $us\ V$ polarization, at $Q^2 = 0$. This latter result provides an alternate NNLO us-V-channel-based determination, C_{93}^{us}, of C_{93}. The difference between C_{93}^{ud} and C_{93}^{us} then provides an estimate of the size of residual mass-dependent effects originating from orders beyond NNLO.

This paper is organized as follows. In Sec. II we summarize the necessary theory, and in Sec. III we present our central ud-V-channel-based result for C_{93}, with the experimental error coming from the ALEPH data. In Sec. IV we analyze the $us - ud$ difference mentioned above, and obtain an estimate of the systematic error due to the neglect of higher orders in ChPT. In Sec. V we comment on the use of NNLO ChPT for estimates of the hadronic vacuum-polarization contribution to the anomalous magnetic moment of the muon. We conclude with a discussion of our results.

II. THEORY SUMMARY

In this section we briefly summarize the necessary theory.

A. ChPT

In the isospin limit, the expression for the subtracted vacuum polarization $\Pi_{ud}^{\text{sub}}(Q^2)$ was calculated to NNLO in ChPT in Ref. [1]. As a function of Euclidean momentum-squared Q^2, it is given by

$$
\begin{aligned}
\Pi_{ud}^{\text{sub}}(Q^2) \ = \ & -8\hat{B}(Q^2, m_\pi^2) - 4\hat{B}(Q^2, m_K^2) \\
& + \frac{16}{f_\pi^2}\, L_9^r\, Q^2 \left(2B(Q^2, m_\pi^2) + B(Q^2, m_K^2) \right) \\
& - \frac{4}{f_\pi^2}\, Q^2 \left(2B(Q^2, m_\pi^2) + B(Q^2, m_K^2) \right)^2 + 8C_{93}^r Q^2 \ ,
\end{aligned}
\tag{2.1}
$$

where $\hat{B}(Q^2, m^2) = B(Q^2, m^2) - B(0, m^2)$ is the subtracted standard, equal-mass, two-propagator, one-loop integral, with

$$
B(0, m^2) \ = \ \frac{1}{192\pi^2} \left(1 + \log \frac{m^2}{\mu^2} \right) \ ,
\tag{2.2}
$$

3

$$\hat{B}(Q^2, m^2) \;=\; \frac{1}{96\pi^2}\left(\left(\frac{4m^2}{Q^2}+1\right)^{3/2}\coth^{-1}\sqrt{1+\frac{4m^2}{Q^2}}-\frac{4m^2}{Q^2}-\frac{4}{3}\right)\,,$$

and the low-energy constants (LECs) L_9^r and C_{93}^r are renormalized at the scale μ, in the "$\overline{MS}+1$" scheme employed in Ref. [1].

From Eq. (2.1) it is clear that C_{93}^r can be determined from the slope of $\Pi_{ud}^{\text{sub}}(Q^2)$ at $Q^2 = 0$. Since we will only use the explicit expression for $\Pi_{us}^{\text{sub}}(Q^2)$ for a systematic error estimate, we do not provide it here, but refer to Ref. [1] for the full expression.[3]

B. Flavor-breaking sum rule

The difference $\Delta\Pi(Q^2)$ of the ud and us spin $J = 0+1$ V unsubtracted two-point functions $\Pi_{ud}(Q^2)$ and $\Pi_{us}(Q^2)$ satisfies the flavor-breaking IMFESR [19]

$$\frac{d\Delta\Pi(Q^2)}{dq^2}\bigg|_{q^2=0} = -\frac{d\Delta\Pi(Q^2)}{dQ^2}\bigg|_{Q^2=0} = \int_{4m_\pi^2}^{s_0} ds\, w_\tau(s/s_0)\frac{\Delta\rho(s)}{s^2} \tag{2.3}$$
$$+\frac{1}{2\pi i}\oint_{|s|=s_0} ds\, w_\tau(s/s_0)\frac{\Delta\Pi(Q^2=-s)}{s^2}\,,$$

where $q^2 = -Q^2$, $w_\tau(x) = (1-x)^2(1+2x)$, and $\Delta\rho(s) = \rho_{ud}(s)-\rho_{us}(s)$. As long as we choose $s_0 \leq m_\tau^2$, the first integral on the right-hand side can be computed using experimentally available spectral functions. We have used that $w_\tau(0) = 1$ and $dw_\tau(s/s_0)/ds|_{s=0} = 0$.

As in other applications of FESRs, we will approximate $\Delta\Pi(s)$ in the second integral by the operator product expansion (OPE),[4] and assume that the contribution from duality violations to this sum rule are negligibly small. In this case, this is reasonable because of the presence of a weight function with a double pinch at $s = s_0$, as well as a further $1/s^2$ suppression of the contribution from higher-s values to the integral. This assumption can be tested for self-consistency by studying the s_0 dependence of the right-hand side of Eq. (2.3). Because the left-hand side is independent of s_0, the individually s_0-dependent ud- and us-spectral integral and OPE integral contributions should combine to produce a right-hand side independent of s_0, within errors.

The sum rule (2.3) gives access to the difference of the slopes of the subtracted polarizations $\Pi_{ud}^{\text{sub}}(Q^2)$ and $\Pi_{us}^{\text{sub}}(Q^2)$ at $Q^2 = 0$. This, together with the independent dispersive determination of the slope of $\Pi_{ud}^{\text{sub}}(Q^2)$, yields the value of the slope of $\Pi_{us}^{\text{sub}}(Q^2)$ at $Q^2 = 0$. The NNLO expression for this slope provides the alternate determination, C_{93}^{us}, of C_{93} already introduced above.

Of course, since LECs are, by definition, mass independent, the NNLO analysis results C_{93}^{ud} and C_{93}^{us} should be the same, provided NNNLO and higher order contributions are negligible. The experimental data used in their determination, however, know about the existence of higher orders in ChPT, and if these are not, in fact, negligible, we expect the

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[3] In the notation of Ref. [1], $\Pi_{us}^{\text{sub}}(Q^2) = \Pi_{VK}^{(1)}(-Q^2) + \Pi_{VK}^{(0)}(-Q^2) - (\Pi_{VK}^{(1)}(0) + \Pi_{VK}^{(0)}(0))$.

[4] Mass-independent, purely perturbative contributions cancel for the flavor-breaking combination considered here. The leading, dimension-2, OPE contribution is thus perturbative in origin, and of order $(m_s-m)^2/s$, where m is the u, d-averaged light quark mass.

two values to be different. The numerical difference provides an indication of the size of higher-order, residual mass-dependent effects.

In ChPT, the leading mass-dependent corrections to the slopes of the V polarizations at $Q^2 = 0$ result from NNNLO operators having a single insertion of the chiral mass operator (χ_+ in the notation of Ref. [20]). A two-trace NNNLO operator of this form in which χ_+ appears through the factor $\text{Tr}\,(\chi_+)$ produces an $SU(3)$-flavor-symmetric contribution proportional to $2m_K^2 + m_\pi^2$ to the slopes of all of the ud, us and η V channel polarizations at $Q^2 = 0$. A single-trace NNNLO operator containing one factor of χ_+, similarly, produces contributions proportional to m_π^2, m_K^2 and $m_\eta^2 = \frac{4}{3}m_K^2 - \frac{1}{3}m_\pi^2$, respectively, to those same slopes.[5] To take these effects into account, we introduce two NNNLO LECs, $\delta C_{93}^{(1)}$ and $\delta C_{93}^{(2)}$, where the (1), (2) superscripts indicates the number of trace factors in the accompanying operators, normalized such that they produce mass-dependent contributions

$$
\begin{aligned}
C_{93}^{ud} &= C_{93}^r + \delta C_{93}^{(2)}(2m_K^2 + m_\pi^2) + \delta C_{93}^{(1)} m_\pi^2 \ , \\
C_{93}^{us} &= C_{93}^r + \delta C_{93}^{(2)}(2m_K^2 + m_\pi^2) + \delta C_{93}^{(1)} m_K^2 \ , \\
C_{93}^{\eta} &= C_{93}^r + \delta C_{93}^{(2)}(2m_K^2 + m_\pi^2) + \delta C_{93}^{(1)} \left(\frac{4}{3}\, m_K^2 - \frac{1}{3}\, m_\pi^2 \right) \ .
\end{aligned}
\tag{2.4}
$$

Only the first of the new NNNLO LECs, $\delta C_{93}^{(1)}$, contributes to the difference $C_{93}^{ud} - C_{93}^{us}$ at NNNLO, and thus if, guided by what is found at NNLO, we assume NNNLO LEC contributions will dominate loop contributions also at NNNLO, the sum rule (2.3) will give us an estimate of $\delta C_{93}^{(1)}$. The observation that $\delta C_{93}^{(2)}$ is suppressed in large N_c then leads to two expectations: one that C_{93}^{ud} should be much closer to the true, mass-independent C_{93} than C_{93}^{us}; the other that the difference $C_{93}^{ud} - C_{93}^{us}$ should give a reasonably conservative estimate of the systematic error associated with the neglect of contributions beyond NNLO in ChPT. We emphasize that loop contributions at NNLO will also produce mass-dependent contributions to the slopes of $\Pi_{ud}^{\text{sub}}(Q^2)$ and $\Pi_{us}^{\text{sub}}(Q^2)$, and thus that this estimate relies on the assumption, also made in the rest of this paper, that such mass-dependent higher-order loop contributions are small compared to the LEC contributions, at the scale $\mu = 0.77$ GeV in the $\overline{MS} + 1$ scheme we will use in this paper.

We note in closing this section that the higher-order, mass-dependent effects discussed above are also included in the phenomenological approach of Ref. [21], where NNLO and higher LEC contributions are modelled by replacing the NNLO LEC contributions proportional to C_{93}^r in the expressions for the subtracted polarizations with the corresponding full vector-meson dominance (VMD) contributions obtained using ρ and ϕ masses in the VMD expressions for the $I = 1$ and strange current channels, respectively. The chiral-limit part of the vector meson mass in this approach produces quark-mass-independent contributions which, in the chiral expansion, would be parametrized by a tower of NNLO and higher LECs, including C_{93}^r and the NNNLO LEC C^r introduced in Ref. [3] (which produces a common $SU(3)$-flavor-symmetric contribution $C^r Q^4$ to the subtracted V polarizations we consider in this paper). The quark-mass-dependent parts of the different vector meson masses used in the different V channels, similarly, generate contributions which would be parametrized by $\delta C_{93}^{(1)}$, $\delta C_{93}^{(2)}$, and yet higher-order LECs. The VMD extension of the NNLO results contains only contributions analytic in Q^2 in the low-Q^2 region and hence also neglects NNNLO and higher loop contributions.

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[5] Here we use the tree-level relations between quark and meson masses.

III. C_{93} FROM ALEPH DATA

The once-subtracted ud V polarization $\Pi_{ud}^{\text{sub}}(Q^2)$ can be defined in terms of the corresponding spectral function $\rho_{ud}(s)$ as a function of the Euclidean momentum-squared Q^2 by the dispersion relation

$$\Pi_{ud}^{\text{sub}}(Q^2) = -Q^2 \int_{4m_\pi^2}^{\infty} ds \, \frac{\rho_{ud}(s)}{s(s+Q^2)} \ . \tag{3.1}$$

For $s < m_\tau^2$, we can use the experimental spectral function provided by Ref. [7], but for $s > m_\tau^2$ we will need a theoretical representation, with parameters fit from the data in the region below m_τ^2. We follow the procedure employed in Refs. [2, 4, 22], using the fitted version of the theoretical representation obtained starting from the rescaled version of the data for the ALEPH ud spectral function, and following the procedure described in detail in Ref. [18]. The theoretical representation is the sum of the QCD perturbation theory (PT) expression $\rho_{ud,\text{PT}}(s)$ and a "duality-violating" (DV) part $\rho_{ud,\text{DV}}(s)$ representing the effects of resonances, with the *ansatz*

$$\rho_{ud,\text{DV}}(s) = e^{-\delta_V - \gamma_V s} \sin\left(\alpha_V + \beta_V s\right) \tag{3.2}$$

used for the DV part. The perturbative expression is known to order α_s^4 [24]. Fits to the weighted integrals of the ALEPH data determining the parameters α_s, α_V, β_V, γ_V and δ_V have been performed in Ref. [18], with a focus on the high-precision determination of α_s from hadronic τ decays. We will use the values obtained from the fixed-order perturbation theory (FOPT) $s_{\text{switch}} = s_{min} = 1.55$ GeV^2 fit of Table 1 of Ref. [18],

$$\begin{aligned}
\alpha_s(m_\tau^2) &= 0.295(10) \ , \\
\alpha_V &= -2.43(94) \ , \\
\beta_V &= 4.32(48) \text{ GeV}^{-2} \ , \\
\gamma_V &= 0.62(29) \text{ GeV}^{-2} \ , \\
\delta_V &= 3.50(50) \ .
\end{aligned} \tag{3.3}$$

The matches between the data and theory representations of both the weighted spectral integrals and the spectral function in the window used in performing the fits are excellent, and there is no discernible effect on $\Pi_{ud}^{\text{sub}}(Q^2)$ for the values of Q^2 smaller than 0.2 GeV^2 of interest in the comparison to ChPT if we vary the point at which we switch from the experimental to the theoretical version of $\rho_{ud}(s)$ within this fit window, use the results of a contour-improved perturbation theory (CIPT)[23] fit instead of an FOPT fit, or employ parameter values from one of the other optimal fits in Ref. [18]. Results for $\Pi_{ud}^{\text{sub}}(Q^2)$ in the region below $Q^2 = 0.2$ GeV^2, at intervals of 0.01 GeV^2, are shown in Fig. 1. The errors shown are fully correlated, taking into account, in particular, correlations between the parameters of Eq. (3.3) and the data. We emphasize again that systematic effects due to the use of the *ansatz* (3.2) can be assumed to be small compared to systematic effects due to the neglect of higher orders in ChPT.

It follows from Eq. (2.1) that the slope of $\Pi_{ud}^{\text{sub}}(Q^2)$ at $Q^2 = 0$ is a linear combination of the LECs L_9^r and C_{93}^r. We will use

$$\begin{aligned}
m_\pi &= 139.57 \text{ MeV} \ , \\
m_K &= 495.65 \text{ MeV} \ , \\
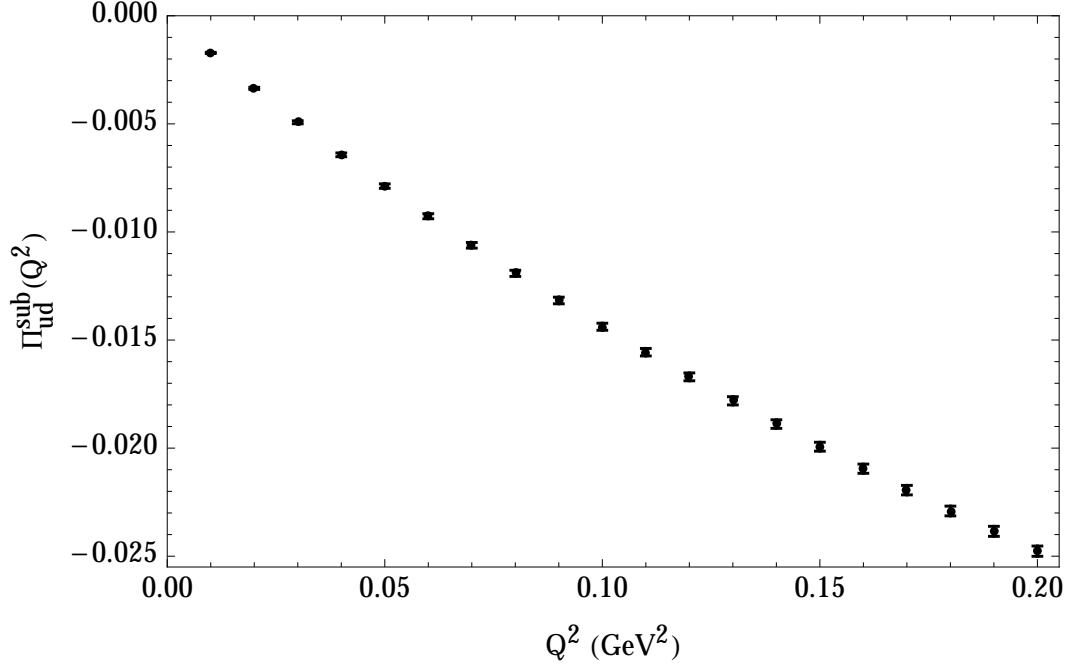f_\pi &= 92.21 \text{ MeV} \ .
\end{aligned} \tag{3.4}$$

FIG. 1: $\Pi_{ud}^{\text{sub}}(Q^2)$ *as a function of* Q^2 *constructed from ALEPH data as explained in Sec. III.*

The errors on these values are so small that they can be ignored in the computation of the error on C_{93}^r. We also use the value [17]

$$L_9^r(\mu = 0.77 \text{ GeV}) = 0.00593(43) \ . \tag{3.5}$$

With these inputs, the NNLO representation of the slope, Eq. (2.1), becomes

$$\frac{d\Pi_{ud}^{\text{sub}}(Q^2)}{dQ^2}\bigg|_{Q^2=0} = (-0.02253 - 0.00291 - 0.02775(201)) \text{ GeV}^{-2} + 8C_{93}^{ud} , \tag{3.6}$$

where the first term is the NLO contribution, the second the NNLO loop contribution involving only LO vertices and the third the NNLO loop contribution proportional to L_9^r. The slope obtained from the results for $\Pi_{ud}^{\text{sub}}(Q^2)$ shown in Fig. 1,[6]

$$\frac{d\Pi_{ud}^{\text{sub}}(Q^2)}{dQ^2}\bigg|_{Q^2=0} = -0.17608 \pm 0.00291 \text{ GeV}^{-2} , \tag{3.7}$$

then yields, for the (potentially mass-dependent) ud channel effective LEC C_{93}^{ud}, the result

$$C_{93}^{ud}(\mu = 0.77 \text{ GeV}) = -0.01536 \pm 0.00036 \pm 0.00025 \text{ GeV}^{-2} , \tag{3.8}$$

where the first error comes from the error in the slope, and the second from the error in L_9^r. As one can see, the result in (3.6) is dominated by the contribution from C_{93}^{ud}. Residual mass-dependent effects causing C_{93}^{ud} to, in principle, differ from C_{93}^r remain to be estimated.

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[6] The error on this value is based on propagation of the full covariance matrix.

IV. ESTIMATE OF RESIDUAL MASS DEPENDENCE

The value for C_{93} obtained in Eq. (3.8) appears to have a very small error, but this is misleading. There are, in fact, other small errors which we neglected in this result, for instance due to the use of our *ansatz* (3.2) and isospin breaking. However, as mentioned already above, there is also a theoretical systematic error due to the neglect of orders in ChPT beyond NNLO, which is likely to be more important, and which we address in this section. We do so by using the IMFESR (2.3) to determine the difference in the slopes at $Q^2 = 0$ of $\Pi_{ud}^{\text{sub}}(Q^2)$ and $\Pi_{us}^{\text{sub}}(Q^2)$, and, from this, using the result for $d\Pi_{ud}^{\text{sub}}(Q^2)/dQ^2|_{Q^2=0}$ from Eq. (3.7), determine $d\Pi_{us}^{\text{sub}}(Q^2)/dQ^2|_{Q^2=0}$, whose NNLO representation then provides us with an alternate, us-channel determination of C_{93}, denoted C_{93}^{us}. C_{93}^{ud} and C_{93}^{us} should be equal within errors if contributions beyond NNLO are negligible. As it turns out, they are not, and this allows us to estimate the effect of the neglect of higher order contributions. While it is difficult to convert this estimate into a reliable systematic error, it will be clear that this is the dominant uncertainty in our result for C_{93}^r.

In order to evaluate the right-hand side of Eq. (2.3), we will need the ud and us spectral functions, as well as the dimension-2 and dimension-4 terms in the OPE.

The ud spectral function is the same as that used to construct $\Pi_{ud}^{\text{sub}}(Q^2)$ above; we refer again to Ref. [18] for a more detailed discussion (*cf.* Sec. III.A, in particular).

The us spectral function we will use is constructed as a sum over exclusive mode contributions, as in Ref. [3], to which we refer for a detailed discussion (*cf.* Sec. III.C, in particular). All 2014 HFAG inputs used previously have been updated to reflect current 2016 HFAG values [25].

One additional issue to consider is the choice of $K\pi$ branching fractions. These provide the overall normalization used to convert the unit-normalized Belle experimental distribution [9] to the actual, physically normalized $K\pi$ contribution to the us spectral function. The first normalization is the one provided by HFAG [25],

$$B[K^-\pi^0] = 0.00433(15) , \qquad (4.1)$$
$$B[\bar{K}^0\pi^-] = 0.00839(14) ,$$

the errors of which are essentially uncorrelated, yielding a 2-mode $K\pi$ branching fraction sum 0.01271(21). The dispersive study of Ref. [26], however, finds clear tension between such branching fraction values, $K_{\ell3}$ results and dispersive constraints on the $K\pi$ form factors. The analysis of Ref. [26] yields slightly higher expectations for these branching fractions,

$$B[K^-\pi^0] = 0.00471(18) , \qquad (4.2)$$
$$B[\bar{K}^0\pi^-] = 0.00857(30) ,$$

this time with the errors essentially 100% correlated and hence a 2-mode $K\pi$ branching fraction sum 0.01327(48). We consider both possibilities in our analysis; the associated $ud - us$ slope uncertainty is found to be about half the size of the error induced by other experimental uncertainties.

We treat the OPE in the same way as in Ref. [3], and refer to Sec. III.B of Ref. [3] for the explicit expressions. We will use the input parameters[7]

$$\alpha_s(m_\tau^2) = 0.3155(90) \qquad \text{(converted from Ref. [27])} , \qquad (4.3)$$

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[7] Because the OPE contribution to Eq. (2.3) is so small, it does not matter whether one uses the value for $\alpha_s(m_\tau^2)$ given below, or the one given in Eq. (3.3).
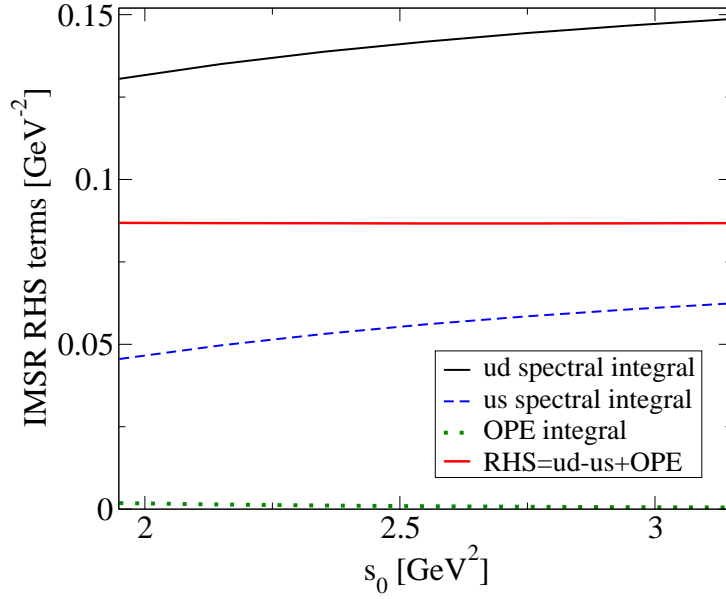
FIG. 2: *Contributions to the right-hand side (RHS) of the IMFESR (2.3) and the resulting ud-us+OPE sum, as a function of s_0. The us spectral integrals are those obtained using the ACLP branching-fraction normalization of the $K\pi$ distribution.*

$$
\begin{aligned}
m_s(2 \text{ GeV}) &= 93.9(1.1) \text{ MeV} & \text{(Ref. [28])} \ , \\
m_\tau &= 1.77686(12) \text{ MeV} & \text{(Ref. [27])} \ , \\
\langle \bar{s}s \rangle / \langle \bar{u}u \rangle &= 1.08(16) & \text{(Ref. [29])} \ , \\
B_e &= 0.17815(23) & \text{(Ref. [25])} \ , \\
V_{ud} &= 0.97417(21) & \text{(Ref. [30])} \ , \\
V_{us} &= 0.22582(91) & \text{(3-family unitarity)} \ , \\
S_{EW} &= 1.0201(3) & \text{(Ref. [31])} \ ,
\end{aligned}
$$

where $\langle \bar{u}u \rangle$ is in the isospin limit, and its value is obtained from the GMOR relation. We find that the OPE contribution to the right-hand side of Eq. (2.3) is less than 1.6% of the total for $s_0 = 2.15$ GeV^2, and decreases for larger values of s_0.

Very good s_0-stability is observed for the slope obtained from this analysis. This is illustrated, for the ACLP choice of the $K\pi$ normalization, in Figure 2. The figure shows the individual terms (OPE integral, ud spectral integral and us spectral integral) appearing on the right-hand side of Eq. (2.3), together with the ud-us+OPE combination which determines $-\frac{d\Delta\Pi(Q^2)}{dQ^2}|_{Q^2=0}$, all as a function of s_0. The corresponding results for the HFAG $K\pi$ normalization choice are essentially identical, and hence not shown explicitly. The excellent s_0-stability provides a self-consistency check on our neglect of duality violations employing the IMFESR (2.3), and confirms the very minor role played by the OPE.

Using the $K\pi$ branching-fraction normalization of Eq. (4.1), and quoting the $s_0 = m_\tau^2$

result to be specific, we find a slope for the $ud - us$ difference

$$\frac{d\Delta\Pi(Q^2)}{dQ^2}\bigg|_{Q^2=0} = -0.0894(35) \text{ GeV}^{-2} , \tag{4.4}$$

which yields

$$\frac{d\Pi_{us}^{\text{sub}}(Q^2)}{dQ^2}\bigg|_{Q^2=0} = -0.0867(46) \text{ GeV}^{-2} \tag{4.5}$$

for the slope in the us channel. The corresponding NNLO representation, with Eq. (3.4) as input, is

$$\frac{d\Pi_{us}^{\text{sub}}(Q^2)}{dQ^2}\bigg|_{Q^2=0} = (-0.000868 - 0.004740 - 0.6836L_5^r - 0.5419L_9^r) \text{ GeV}^{-2} + 8C_{93}^{us} , \tag{4.6}$$

with the first term the NLO contribution, the second the NNLO loop contribution involving only LO vertices and the third and fourth the NNLO one-loop contributions with a single NLO vertex. Using [28]

$$L_5^r(\mu = 0.77 \text{ GeV}) = 0.00119(25) , \tag{4.7}$$

and Eq. (3.5) for L_9^r, we have, adding the first two terms on the right-hand side of Eq. (4.6),

$$\frac{d\Pi_{ud}^{\text{sub}}(Q^2)}{dQ^2}\bigg|_{Q^2=0} = (-0.005606 - 0.000814(171) - 0.003214(230)) \text{ GeV}^{-2} + 8C_{93}^{us} , \tag{4.8}$$

and hence

$$C_{93}^{us}(\mu = 0.77 \text{ GeV}) = -0.00963(58) \text{ GeV}^{-2} , \tag{4.9}$$

where the error is dominated by the experimental error on the $ud - us$ slope. Again, as in the ud channel, the slope (4.8) is dominated by the contribution from the LEC C_{93}^{us}.

Using, instead, the $K\pi$ branching-fraction normalization of Eq. (4.2), we find a slope for the $ud - us$ difference

$$\frac{d\Delta\Pi(Q^2)}{dQ^2}\bigg|_{Q^2=0} = -0.0868(40) \text{ GeV}^{-2} \tag{4.10}$$

at $s_0 = m_\tau^2$, yielding for the slope in the us channel

$$\frac{d\Pi_{us}^{\text{sub}}(Q^2)}{dQ^2}\bigg|_{Q^2=0} = -0.0893(49) \text{ GeV}^{-2} , \tag{4.11}$$

and the result

$$C_{93}^{us}(\mu = 0.77 \text{ GeV}) = -0.00996(61) \text{ GeV}^{-2} . \tag{4.12}$$

The values (4.9) and (4.12) are consistent within errors. Comparing these results with that in Eq. (3.8) shows the existence of significant residual mass-dependent effects. Taking the average of the values (4.9) and (4.12) yields

$$\frac{C_{93}^{ud} - C_{93}^{us}}{C_{93}^{ud}}\bigg|_{\mu=0.77 \text{ GeV}} = 0.36(4) . \tag{4.13}$$

The size of this difference is consistent with the expectation for an $SU(3)$ breaking effect. Finally, from

$$\frac{d\Delta\Pi(Q^2)}{dQ^2}\bigg|_{Q^2=0} = (-0.019832 + 0.6836L_5^r - 4.1376L_9^r) \text{ GeV}^{-2} - 8\delta C_{93}^{(1)}(m_K^2 - m_\pi^2) , \tag{4.14}$$

we find

$$\delta C_{93}^{(1)}(m_K^2 - m_\pi^2) = \begin{cases} 0.00573(49) \text{ GeV}^{-2} \text{ (HFAG)} , \\ 0.00540(55) \text{ GeV}^{-2} \text{ (ACLP)} , \end{cases} \qquad (4.15)$$

for the $K\pi$ branching-fraction normalizations of Ref. [25] and Ref. [26], respectively.

V.   CHPT AND THE MUON ANOMALOUS MAGNETIC MOMENT

The lowest-order hadronic contribution to the muon anomalous magnetic moment is given by an integral over Q^2 of the hadronic vacuum polarization times a weight that causes about 90% of the integral to correspond to the integral between $Q^2 = 0$ and $Q^2 = 0.2$ GeV^2. One may thus hope that ChPT can be used to constrain the low-momentum part of this integral [21, 32–34]. In particular, since it is difficult to compute the quark-disconnected part of the hadronic vacuum polarization on the lattice [35–38], ChPT has been used to estimate the size of the disconnected contribution relative to the connected contribution [21, 33].

In Ref. [37], the disconnected part has been computed on the lattice. In this analysis, an estimate of the systematic uncertainty associated with the inability to accurately resolve the disconnected signal at large Euclidean times was achieved by considering the Fourier transform $\Pi_{uu-ss,dd-ss}(Q^2)$ of $\langle 0|T\left[V_{uu-ss}^\mu(x)V_{dd-ss}^\nu(0)\right]|0\rangle$, which (in the isospin limit) is equal to 9 times the sum of the connected strange and the full disconnected contributions to the electromagnetic vacuum polarization. A physical model for the large-time disconnected contribution was then obtained by subtracting from a fitted two-exponential representation of the strange-connected-plus-full-disconnected sum the well-determined strange connected contribution. Though restricted in Ref. [37] to an investigation of the behavior of the disconnected contribution at large Euclidean times, this strategy is, in principle, usable more generally. Thus, were a reliable continuum representation of the strange-connected-plus-full-disconnected sum to be available, the disconnected contribution to the electromagnetic polarization could be obtained from this simply by subtracting lattice results for the strange connected contribution, which, for example, has been accurately determined in Refs. [39, 40]. The hope is that ChPT might provide such a reliable continuum representation, at least in the low-Q^2 region. To see that this might indeed be possible, note that one has, in terms of the $I = 1$ and $SU(3)$-octet vacuum polarizations $\Pi_{V\pi}^{(1)}$ and $\Pi_{V\eta}^{(1)}$ of Ref. [1],

$$\Pi_{uu-ss,dd-ss}(Q^2) = -\frac{1}{2}\Pi_{V\pi}^{(1)}(Q^2) + \frac{3}{2}\Pi_{V\eta}^{(1)}(Q^2) . \qquad (5.1)$$

The results of Ref. [1] thus provide an NNLO representation of $\Pi_{uu-ss,dd-ss}(Q^2)$. From Eq. (2.4), the "effective" C_{93} contribution, including NNLO residual mass effects, to $\frac{1}{9}\Pi_{uu-ss,dd-ss}(Q^2)$ is equal to

$$\frac{8}{9}Q^2 C_{93}^{\text{eff}} \equiv = \frac{8}{9}Q^2\left(C_{93}^r + \delta C_{93}^{(1)}(2m_K^2 - m_\pi^2) + \delta C_{93}^{(2)}(2m_K^2 + m_\pi^2)\right) \qquad (5.2)$$

$$= \frac{8}{9}Q^2\left(C_{93}^{ud} + 2\delta C_{93}^{(1)}(m_K^2 - m_\pi^2)\right) .$$

Using Eqs. (3.8) and (4.15), we find a significant cancellation between the C_{93}^{ud} and the $\delta C_{93}^{(1)}$ contributions in (5.2) resulting in

$$C_{93}^{\text{eff}} = \begin{cases} -0.0039(11) \text{ GeV}^{-2} \text{ (HFAG)} , \\ -0.0046(12) \text{ GeV}^{-2} \text{ (ACLP)} . \end{cases} \qquad (5.3)$$

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The two estimates are consistent within errors, but very different from our best estimate for the true value of C_{93}^r, given in Eq. (3.8). The strong cancellation between the C_{93}^{ud} and the $\delta C_{93}^{(1)}$ contributions produces a result for the slope of $\frac{1}{9}\left[-\frac{1}{2}\Pi_{V\pi}^{(1)} + \frac{3}{2}\Pi_{V\eta}^{(1)}\right]$ much less strongly dominated by the effective NNLO LEC combination $C_{93}^{ud} + 2(m_K^2 - m_\pi^2)\delta C_{93}^{(1)}$ than is the case for the slopes of either of the individual terms entering the difference. Explicitly, one finds for the slope of this combination

$$[0.00082 + 0.00016 + 0.00189(14)] \text{ GeV}^{-2} + \frac{8}{9}C_{93}^{\text{eff}}$$
$$= 0.00287(14) \text{ GeV}^{-2} + \frac{8}{9}C_{93}^{\text{eff}} , \qquad (5.4)$$

where the first three terms in the first line are the NLO contribution, the NNLO loop contribution with only LO vertices, and the NNLO loop contribution proportional to L_9^r, respectively, all at $\mu = 0.77$ GeV. The results given in Eq. (5.3) yield for the last contribution, $8C_{93}^{\text{eff}}/9$, the values $-0.0035(9)$ and $-0.0041(10)$ GeV^{-2}, for the HFAG and ACLP $K\pi$ normalization choices, respectively.[8] These are only slightly larger in magnitude than the sum of the NLO and other NNLO contributions. In contrast, for $\left.\frac{d\Pi_{V\pi}^{(1)}(Q^2)}{dQ^2}\right|_{Q^2=0}$, the results of Eq. (3.6) show a $\mu = 0.77$ GeV NNLO contribution proportional to C_{93}^{ud} a factor of ~ 5.5 larger than the corresponding NLO contribution and ~ 4.0 larger than the remaining NNLO contributions. The slope $\left.\frac{d\Pi_{V\eta}^{(1)}(Q^2)}{dQ^2}\right|_{Q^2=0}$ is even more strongly dominated by the effective NNLO LEC contribution, with

$$\left.\frac{d\Pi_{V\eta}^{(1)}(Q^2)}{dQ^2}\right|_{Q^2=0} = (-0.00258 - 0.00002 + 0.00210(15)) \text{ GeV}^{-2} + 8C_{93}^\eta , \qquad (5.6)$$

where $C_{93}^\eta \equiv C_{93}^{ud} + \frac{4}{3}\delta C_{93}^{(1)}(m_K^2 - m_\pi^2)$, the first three terms have the same meaning as in Eq. (3.6) and, with the results for C_{93}^{ud} and $\delta C_{93}^{(1)}$ given above, $8C_{93}^\eta = -0.0653(56)$ GeV^{-2} and $-0.0618(53)$ GeV^{-2}, for the ACLP and HFAG $K\pi$ normalization cases, respectively. Moreover, in contrast to the $\Pi_{V\pi}^{(1)}$ and $\Pi_{V\eta}^{(1)}$ cases, where the effective NNLO LEC contributions have the same signs as the NLO and remaining NNLO contributions, the effective NNLO LEC contribution to the slope in Eq. (5.4) has the opposite sign, leading to further cancellation between the effective NNLO LEC and other contributions. The final values for the slope of the $\frac{1}{9}\left[-\frac{1}{2}\Pi_{V\pi}^{(1)} + \frac{3}{2}\Pi_{V\eta}^{(1)}\right]$ combination, $-0.0006(10)$ GeV^{-2} and $-0.012(11)$

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[8] It is worth noting that, though the VMD estimates for C_{93}^{ud} and C_{93}^{us} differ by $\sim 10-30\%$ from the corresponding dispersive and IMFESR determinations (see below for details), the VMD estimate for C_{93}^{eff} works rather well. Explicitly, with $f_{\text{EM},V}$ the vector meson decay constants, $\langle 0|J_\mu^{\text{EM}}|V(q)\rangle = g_{\text{EM},V}m_V\epsilon_\mu(q) = f_{\text{EM},V}m_V^2\epsilon_\mu(q)$, one finds the VMD expectation

$$\frac{8}{9}C_{93}^{\text{eff}} = \frac{1}{9}\frac{f_{\text{EM},\rho}^2}{m_\rho^2} - \frac{f_{\text{EM},\omega}^2}{m_\omega^2} - \frac{f_{\text{EM},\phi}^2}{m_\phi^2} . \qquad (5.5)$$

With PDG values for the masses and $V \to e^+e^-$ decay widths, $g_{\text{EM},\rho} = 156.4$ MeV, $g_{\text{EM},\omega} = 46.6$ MeV and $g_{\text{EM},\phi} = 75.9$ MeV, the VMD estimate yields $C_{93}^{\text{eff}} = -0.0041$ GeV^{-2}, in good agreement with the results of Eq. (5.3).

GeV^{-2} for the HFAG and ACLP $K\pi$ normalization choices, respectively, thus show a further factor of 3 to 6 reduction relative to the already reduced effective NNLO LEC contributions. This raises the question of how safe it is to neglect NNNLO and higher loop contributions for this particular combination.[9]

We emphasize that the mass-dependent NNNLO terms considered here further supplement the mass-independent NNNLO contribution $C^r Q^4$ added to Eq. (2.1) in Refs. [34, 41]. The latter was required to account for the deviation between the Q^2 dependence of the full vacuum polarization and the NNLO ChPT expression, visible already beyond $Q^2 \approx 0.1$ GeV^2. Such a mass-independent term is, of course, also present at NNNLO, but does not contribute to the values of the slopes at $Q^2 = 0$ considered above.

VI. DISCUSSION

We determined the value of the NNLO LEC C_{93} from ALEPH data for the V hadronic ud and us spectral functions. The difference between these two determinations gives an estimate of the systematic uncertainty due to effects beyond NNLO in ChPT, and turns out to dominate the total uncertainty.

One would expect that the value C_{93}^{ud} is closer to the true mass-independent result than C_{93}^{us} since the pion mass is much smaller than the kaon mass. Assuming a mass-dependent contamination linearly dependent on the square of the meson mass, this would lead to an extrapolated value $C_{93}^r = -0.0158$ GeV^2. Such an extrapolation, however, does not take into account the effect of the $1/N_c$-suppressed NNNLO contribution proportional to $\delta C_{93}^{(2)}$, or other higher-order effects. To be conservative in our assessment, we therefore take as our central result for C_{93}^r the value of C_{93}^{ud} given in Eq. (3.8) of Sec. III, and assign to this an uncertainty equal to the difference $C_{93}^{us} - C_{93}^{ud}$ (*cf.* Eq. (4.13) in Sec. IV). This represents our best estimate of the uncertainty associated with the presence of residual higher-order mass-dependent effects. Our final result is then

$$C_{93}^r(\mu = 770 \text{ GeV}) = -0.015(5) \text{ GeV}^{-2} . \tag{6.1}$$

It is interesting to compare the results obtained above with estimates based on VMD. VMD leads to the expectation $C_{93}^{ij} \sim -\frac{f_{\text{EM},V}^2}{4m_V^2}$ [1], with $m_V = m_\rho = 775$ MeV, $f_{\text{EM},V} = f_{\text{EM},\rho} \sim 0.2$ for $ij = ud$ and $m_V = m_{K^*} = 892$ MeV, $f_{\text{EM},V} = f_{\text{EM},K^*} \sim f_{\text{EM},\rho}$ for $ij = us$. The resulting $ij = ud$ estimate, $C_{93}^{ud} \sim -0.017$ GeV^{-2}, agrees at the $\sim 10\%$ level with the result found in Eq. (3.8). For $ij = us$, VMD correctly predicts that $|C_{93}^{us}| < |C_{93}^{ud}|$, though the magnitude in this case agrees with the determinations of Eqs. (4.9) and (4.12) only at the approximately 30% level. As noted already, the VMD estimate for C_{93}^{eff}, where the existence of strong cancellations might lead one to anticipate a much larger fractional error, in fact,

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[9] Significant cancellation in the LEC contributions is, in fact, expected. If one neglects the ρ width and $\rho - \omega$ mass difference, and assumes ideal mixing and negligible flavor-breaking in the vector meson couplings, ρ and ω contributions to the slope of $-\frac{1}{2}\Pi_{V\pi} + \frac{3}{2}\Pi_{V\eta}$ cancel exactly. This cancellation mechanism which, owing to the near degeneracy of the ρ and ω masses, will also be present in the vector meson contributions to the higher derivatives at $Q^2 = 0$, is specific to the vector meson contributions, encoded in the NNLO and higher LECs. There is thus no reason to expect a similar cancellation in the corresponding NNNLO and higher loop contributions.

works very well. The strong cancellation does, however, raise worries about the possible impact of neglected NNNLO and higher loop contributions.

Finally, in Sec. V, we showed that the strong cancellation produced by NNNLO residual-mass-dependent effects in the supplemented NNLO representation of the sum of strange connected and full disconnected contributions calls into question the accuracy with which this sum can be represented by a supplemented NNLO ChPT form neglecting currently unknown NNNLO and higher-order contributions. The slope of this sum at $Q^2 = 0$, in particular, could receive sizeable corrections from such contributions, significantly impacting the accuracy with which the associated low-Q^2 contributions to the muon anomalous magnetic moment can be estimated.

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