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Lichnerowicz Modes and Black Hole Families in Ricci Quadratic Gravity

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ABSTRACT

A new branch of black hole solutions occurs along with the standard Schwarzschild branch in n-dimensional extensions of general relativity including terms quadratic in the Ricci tensor. The standard and new branches cross at a point determined by a static negative-eigenvalue eigenfunction of the Lichnerowicz operator, analogous to the Gross-Perry-Yaffe eigenfunction for the Schwarzschild solution in standard $n = 4$ dimensional general relativity. This static eigenfunction has two rôles: both as a perturbation away from Schwarzschild along the new black-hole branch and also as a threshold unstable mode lying at the edge of a domain of Gregory-Laflamme-type instability of the Schwarzschild solution for small-radius black holes. A thermodynamic analogy with the Gubser and Mitra conjecture on the relation between quantum thermodynamic and classical dynamical instabilities leads to a suggestion that there may be a switch of stability properties between the old and new black-hole branches for small black holes with radii below the branch crossing point.

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1 Introduction

Higher derivative theories are widespread in physics, including string theory where they arise in the low-energy expansion of the effective action. One can take different viewpoints as to how to treat such higher-derivative contributions in the equations of motion. In one viewpoint, since the higher-derivative terms arise in a perturbative α' expansion, one could take the view that only solutions where the effects of the higher-derivative terms act as perturbations to the leading-order form should be considered. It may be argued that if, say, quadratic-curvature terms were to make a significant modification to the form of a leading-order Einstein solution, then terms of cubic and higher order (ad infinitum) would make equally as significant further modifications. From such a viewpoint, focusing on a particular set of higher-derivative corrections, while neglecting the remainder, might seem to be unjustified. On the other hand, one might argue that in the vast landscape of vacua within string theory there could exist regions where certain specific higher-order terms might play an exceptionally dominant rôle. This may particularly be the case of terms quadratic in the curvature tensor. In $n = 4$ spacetime dimensions, there are just two independent such quadratic curvature invariants after excluding the Gauss-Bonnet combination $(R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2)\sqrt{-g}$, whose $n = 4$ spacetime integral is a topological invariant and so does not contribute to the perturbative structure of the theory. Inclusion of the two independent $n = 4$ quadratic curvature invariants together with the Einstein-Hilbert action yields a renormalisable theory [1]. The corresponding perturbative spectrum includes massless spin-two gravity together with ghost massive spin-two excitations and non-ghost massive spin-zero excitations, as well as a rich phase space of new spherically symmetric static classical solutions [2–4]. Asymptotic forms of such solutions obtained from Frobenius analysis at the origin or at horizons need to be numerically linked to linearised theory solutions at spatial infinity. Such analysis has been carried out with increased precision in [5, 6].

In this paper we consider the effects of a restricted set of such higher-derivative modifications to Einstein gravity in a variety of spacetime dimensions $n \geq 4$. We consider the addition to the Einstein-Hilbert action of quadratic invariants in the Ricci tensor and Ricci scalar only. In other words we exclude from consideration invariants built from the full curvature such as $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$. In $n = 4$ dimensions, as noted above, this poses no limitation in generality, owing to the topological nature of the Gauss-Bonnet invariant, but in dimensions $n > 4$ this restriction becomes substantive. The reason for considering only this restricted class of corrections is that Ricci-flat solutions of the leading-order theory then continue to be solutions of the corrected theory. In particular, this means that we can continue to study certain black-hole solutions in the corrected theory. Much of our specific focus will be on the case where just the quadratic invariants $R^{\mu\nu}R_{\mu\nu}$ and R^2 are added, but much of our discussion will extend, with little or no modification, to the case with higher-order Ricci invariant terms as well. In our opinion, it is worthwhile to study the solution space of these theories as extensively as possible, even if they do not necessarily capture all of the features that one might expect to see in a fully generic higher-derivative theory.

We shall see that negative-eigenvalue eigenfunctions of the Lichnerowicz operator play a central rôle in the analysis both of static non-Schwarzschild black-hole solutions such as those found in $n = 4$ dimensions in [3, 4] and also in the analysis of time-dependent black-hole instabilities. The existence of a bifurcation in the set of spherically symmetric solutions to higher-derivative gravity in consequence of the $n = 4$ dimensional Gross-Perry-Yaffe eigenvalue [7] was already presaged by Whitt [8], who also considered the black-hole stability question but did not find any classical instability. The key to finding such instabilities was found by Gregory and Laflamme [9], who considered fluctuations about an $(n + 1)$ dimensional black string solution. This, in turn proves to be related to black-hole instabilities in n dimensional massive gravity theory [10, 11] and thence to black-hole instabilities in n dimensional higher-derivative gravity theory [12].

In this paper, we begin in Section 2 with a study of linearised perturbations of the Ricci tensor about initially Ricci-flat backgrounds for theories including in the action quadratic terms in the Ricci tensor and Ricci scalar. This gives rise to the study of transverse-traceless (TT) eigenfunctions of the Lichnerowicz operator generalising the Gross-Perry-Yaffe eigenfunction [7] in $n = 4$ dimensions, which is carried out in detail in Section 3. Numerical results for the analogous eigenfunctions in dimensions $4 \leq n \leq 11$ are given in Section 4. The implications of these eigenfunctions for the existence of new spherically symmetric and static non-Schwarzschild black holes are analysed in Section 5 and their implications for time-dependent perturbations and stability are analysed in Section 6. A useful reformulation of such perturbation problems for tensor fluctuations about a Schwarzschild background in terms of a "Schrödinger equation" form was given long ago by Zerilli [13]; this is applied to the present case of Ricci quadratic gravity in dimension n in Section 6.1, while application to the analysis of the threshold unstable mode is given in Section 6.2. In Section 6.3, we apply the known results on Gregory-Laflamme instabilities [9] to the various n dimensional cases and establish agreement between the edges of the parametric zones of instability and the Lichnerowicz negative eigenvalues given in Section 4. The relevance of the negative Lichnerowicz eigenmodes both to the static perturbations of black holes at the Lichnerowicz crossing point and to thermodynamic instabilities leads finally in Section 7 to consideration of the implications of relative negative specific-heat values, as found in [3], and similarly of free energies, to suggested ranges of stability and instability for both the Schwarzschild and non-Schwarzschild black hole families.

2 Black Holes in Ricci Quadratic Gravity

We consider the n-dimensional theory described by the Lagrangian[1]

$$\mathcal{L} = \sqrt{-g}\,(R + \xi\, R^{\mu\nu} R_{\mu\nu} + \beta\, R^2)\,. \tag{2.1}$$

Note that in $n \geq 5$ dimensions this is a specialisation compared to the most general quadratic gravity theory, since we do not include a term $R^{\mu\nu\rho\sigma}\, R_{\mu\nu\rho\sigma}$ in the Lagrangian. We make this restriction so that Ricci-flat metrics will always be solutions of the theory, and so, of particular interest for our purposes, the Schwarzschild-Tangherlini (hereafter just referred to as Schwarzschild) black hole will be a solution. This will allow us to study black-hole solutions that are perturbatively close to the Schwarzschild solution. Without this simplifying assumption, it would be very difficult to study black-hole solutions in the quadratic theory.

The equations of motion following from (2.1) are

$$\begin{aligned} &R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + 2\beta R\,(R_{\mu\nu} - \frac{1}{4} R\, g_{\mu\nu}) + (2\beta + \xi)\,(g_{\mu\nu}\,\Box - \nabla_\mu \nabla_\nu)\,R \\ &+\xi\,\Box(R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu}) + 2\xi\,(R_{\mu\rho\nu\sigma} - \frac{1}{4} g_{\mu\nu}\, R_{\rho\sigma})\, R^{\rho\sigma} = 0\,. \end{aligned} \tag{2.2}$$

Taking the trace of(2.2) gives

$$(1 - \frac{1}{2}n)R + (2\beta + \xi)(n-1)\Box R + \xi(1 - \frac{1}{2}n)\Box R + 2(1 - \frac{1}{4}n)(\beta R^2 + \xi R^{\mu\nu} R_{\mu\nu}) = 0\,. \tag{2.3}$$

The case of $n = 4$ dimensions is very special, because the quadratic-curvature terms disappear and one is just left with the equation, linear in R,

$$2(3\beta + \xi)\,\Box R - R = 0\,. \tag{2.4}$$

This was used in [3], following earlier work in [14] to prove that R must in fact vanish for any static spherically-symmetric black-hole solution. Essentially, one uses a Lichnerowicz-type argument, multiplying (2.4) by R and integrating over the spatial 3-volume between

[1] This action for Ricci quadratic gravity is given in terms of the Ricci tensor and the Ricci scalar. It is related to the $\int \sqrt{-g}(R - \alpha C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \beta_{n=4} R^2)$ action of Refs [3, 4] in $n = 4$ dimensions by $\xi = -2\alpha$; $\beta = \beta_{n=4} + \frac{2}{3}\alpha$.

the horizon and infinity, subject to the assumption of appropriate boundary conditions. This leads to a huge simplification in the study of static black-hole solutions, allowing the differential equations for the functions in the metric ansatz to be reduced from 4 to 2, and thus making the use of numerical integration techniques much easier to implement.

The findings in [3] were that in addition to the 1-parameter family of Schwarzschild black holes there exists also a second branch of static spherically-symmetric black holes that bifurcates from the Schwarzschild branch at a specific black-hole mass that is determined in terms of the coefficient ξ in the Lagrangian (2.1).

Since in dimensions $n > 4$ we shall not have the simplification of knowing that $R = 0$ for static black-hole solutions, we shall not attempt to carry out such an extensive construction of black-hole solutions using numerical integration methods. What we shall still be able to do, however, is to carry out a linearised investigation of static black-hole solutions that are perturbatively close to the Schwarzschild solution. These will be the analogues of the four-dimensional black holes on the second branch that were found in [3], in the region very close to the bifurcation point of the two branches. We can study such solutions by looking at the infinitesimal variation of the equations of motion (2.2) around the Ricci-flat background of the Schwarzschild solution. Making the variation, and then setting $R_{\mu\nu} = 0$, we find

$$\delta R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\,\delta R + (2\beta+\xi)(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)\delta R + \xi\Box\,(\delta R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\,\delta R) + 2\xi\, R_{\mu\rho\nu\sigma}\,\delta R^{\rho\sigma} = 0\,. \tag{2.5}$$

Taking the trace of this equation gives

$$(1-\frac{1}{2}n)\,\delta R + [2(n-1)\beta + \frac{1}{2}n\xi]\,\Box\delta R = 0\,. \tag{2.6}$$

We can now again use a Lichnerowicz-type argument, in which we multiply this by δR and integrate over the spatial 3-volume between the black-hole horizon and infinity. With appropriate boundary conditions, and for the relevant sign of the prefactor of the $\Box\delta R$ term, corresponding to a positive mass-squared m_0^2 for the scalar mode, with

$$m_0^2 = \frac{n-2}{n\xi + 4(n-1)\,\beta}\,, \tag{2.7}$$

we deduce that δR must vanish. This is weaker than the $R = 0$ result [3, 14] of four dimensions, but is does allow us to say that any static black hole that is perturbatively close to Schwarzschild must still have vanishing Ricci scalar at linearised order. Feeding the conclusion $\delta R = 0$ back into (2.5), we then find

$$\Big(\Delta_L - \frac{1}{\xi}\Big)\,\delta R_{\mu\nu} = 0\,, \tag{2.8}$$

where

$$\Delta_L\, \delta R_{\mu\nu} \equiv -\Box \delta R_{\mu\nu} - 2R_{\mu\rho\nu\sigma}\, \delta R^{\rho\sigma} \tag{2.9}$$

is the Lichnerowicz operator on a Ricci-flat background. Note that since $\delta R = 0$ we must have $g^{\mu\nu}\, \delta R_{\mu\nu} = 0$ around the Ricci-flat background, and furthermore, from the variation of the contracted Bianchi identity $\nabla^\mu R_{\mu\nu} - \frac{1}{2}\nabla_\nu R = 0$ and using $\delta R = 0$, we have that $\nabla^\mu\, \delta R_{\mu\nu} = 0$, so we conclude that $\delta R_{\mu\nu}$ is a transverse traceless (TT) tensor.

Going back to (2.8), we see that if there exists a TT tensor eigenfunction $\psi_{\mu\nu}$ of the Lichnerowicz operator,

$$\Delta_L\, \psi_{\mu\nu} = \lambda\, \psi_{\mu\nu} \tag{2.10}$$

with eigenvalue given by $\lambda = 1/\xi$, then we can obtain a linearised perturbation away from the Schwarzschild solution. In order for the squared mass of the massive spin-2 mode $m_2^2 = -1/\xi$ in the quadratic theory to be positive (and hence non-tachyonic), we must have $\xi < 0$. Thus we will obtain a linearised perturbation away from Schwarzschild if there exists a negative Lichnerowicz mode, provided that ξ is the inverse of the negative eigenvalue.

As a check, we can look at the situation in four dimensions. It is known from the work of Gross, Perry and Yaffe [7] that the Schwarzschild metric has just one normalisable mode of the Lichnerowicz operator, and that this has $\lambda \approx -0.19\, M^{-2}$, where M is the Schwarzschild mass. Comparing with the numerical results in [3], we find that indeed this corresponds nicely with the point at which the second branch of solutions bifurcates from the Schwarzschild branch.

3 Lichnerowicz Operator in a Schwarzschild Background

Here we give a construction of the negative-eigenvalue mode of the Lichnerowicz operator in the background of n-dimensional Schwarzschild spacetime. Consider the metric

$$ds^2 = -h\, dt^2 + f^{-1}\, dr^2 + \rho^2\, d\Omega_{n-2}^2\,, \tag{3.1}$$

where h, f and ρ are functions of r. Labelling the coordinates $x^\mu = (t, r, y^i)$, where $d\Omega_{n-2}^2 = \gamma_{ij}\, dy^i dy^j$ is the metric on the unit $(n-2)$ sphere, the non-vanishing components of the Christoffel connection are given by

$$\begin{aligned} \Gamma^0{}_{01} &= \frac{h'}{2h}\,, \qquad \Gamma^1{}_{11} = -\frac{f'}{2f}\,, \qquad \Gamma^1{}_{00} = \frac{1}{2} f h'\,, \\ \Gamma^1{}_{ij} &= -\rho\rho'\, f\gamma_{ij}\,, \qquad \Gamma^i{}_{1j} = \frac{\rho'}{\rho}\, \delta^i_j\,, \qquad \Gamma^i{}_{jk} = \bar{\Gamma}^i{}_{jk}\,, \end{aligned} \tag{3.2}$$

where $\bar\Gamma^i{}_{jk}$ is the Christoffel connection for the $(n-2)$-sphere metric γ_{ij}. The non-vanishing components of the Riemann tensor are given by

$$
\begin{aligned}
R_{0101} &= \tfrac12 h'' - \frac{h'^2}{4h} + \frac{h'\,f'}{4f}\,, \qquad R_{0i0j} = \tfrac12 \rho\,\rho'\,f\,h'\,\gamma_{ij}\,,\\
R_{1i1j} &= -\Big(\rho\,\rho'' + \frac{\rho\,\rho'\,f'}{2f}\Big)\,\gamma_{ij}\,, \qquad R_{ijkl} = r^2\,(1-f\,\rho'^2)\,(\gamma_{ik}\,\gamma_{j\ell} - \gamma_{i\ell}\,\gamma_{jk})\,. \qquad (3.3)
\end{aligned}
$$

We now specialise to the case where $f=h$ and $\rho(r)=r$, and consider a symmetric tensor

$$
\psi_{00} = h\,\psi_0\,, \qquad \psi_{11} = h^{-1}\,\psi_1\,, \qquad \psi_{ij} = r^2\,\gamma_{ij}\,\bar\psi\,, \qquad (3.4)
$$

where ψ_0, ψ_1 and $\bar\psi$ are functions of r only. We impose the tracefree condition $g^{\mu\nu}\,\psi_{\mu\nu}=0$, which implies

$$
-\psi_0 + \psi_1 + (n-2)\,\bar\psi = 0\,, \qquad (3.5)
$$

and the transversality condition $\nabla^\mu\psi_{\mu\nu}=0$, which implies

$$
\psi_1' + \frac{h'}{2h}\,(\psi_0+\psi_1) + \frac{n-2}{r}\,(\psi_1 - \bar\psi) = 0\,. \qquad (3.6)
$$

The Lichnerowicz operator in a Ricci-flat background is given by

$$
\Delta_L\,\psi_{\mu\nu} = -\Box\psi_{\mu\nu} - 2R_{\mu\rho\nu\sigma}\,\psi^{\rho\sigma}\,. \qquad (3.7)
$$

With the understanding that $\rho(r)=r$ and $f=h$, with h taken to be

$$
h = 1 - \Big(\frac{r_0}{r}\Big)^{n-3}\,, \qquad (3.8)
$$

giving the Schwarzschild-Tangherlini solution, we find that with $\psi_{\mu\nu}$ having the assumed form (3.4), one has

$$
\begin{aligned}
\Delta_L\psi_{00} &= -h^2\,\psi_0'' - (hh' + \frac{n-2}{r}\,h^2)\,\psi_0' + \tfrac12 h'^2\,\psi_0 + (\tfrac12 h'^2 - hh'')\,\psi_1\\
&\quad -\frac{(n-2)hh'}{r}\,\bar\psi\,, \qquad (3.9)\\
\Delta_L\psi_{11} &= -\psi_1'' - (\frac{h'}{h} + \frac{n-2}{r})\,\psi_1' + (\frac{h'^2}{2h^2} + \frac{2(n-2)}{r^2})\,\psi_1 + (\frac{h'^2}{2h^2} - \frac{h''}{h})\,\psi_0\\
&\quad +\frac{n-2}{r}\,(\frac{h'}{h} - \frac{2}{r})\,\bar\psi\,,\\
\Delta_L\psi_{ij} &= r^2\gamma_{ij}\Big[-h\bar\psi'' - (h' + \frac{n-2}{r}\,h)\bar\psi' + (\frac{2(n-2)}{r^2}\,h - \frac{2(n-3)}{r^2})\,\bar\psi\\
&\qquad -\frac{h'}{r}\,\psi_0 + (\frac{h'}{r} - \frac{2h}{r^2})\,\psi_1\Big]\,.
\end{aligned}
$$

Imposing the eigenfunction condition, where for quadratic Ricci gravity one has $\lambda = 1/\xi$ as in (2.10),

$$
\Delta_L\psi_{\mu\nu} = \lambda\,\psi_{\mu\nu}\,, \qquad (3.10)
$$

together with solving the tracefree (3.5) and transversality (3.6) conditions for ψ_0 and $\bar\psi$ as functions of ψ_1 and ψ_1', gives rise to three ordinary differential equations, coming from the 00, 11 and ij directions. The equation from the 11 direction is of second order in derivatives of ψ_1:

$$-\Big(1-\frac{r_0^{n-3}}{r^{n-3}}\Big)\,\psi_1'' \quad + \quad \frac{(3n^2-11n+18)\,r_0^{n+3}\,r^{n+3}-n(n-1)\,r_0^{2n}\,r^6-2nr_0^6\,r^{2n}}{2r_0^6\,r^{2n+1}-(n-1)\,r_0^{n+3}\,r^{n+4}}\psi_1' \\ + \quad \frac{2n(n-3)^2\,r_0^n\,r}{2r_0^3\,r^n-(n-1)r_0^n\,r^3}\,\psi_1 = \lambda\,\psi_1\,. \tag{3.11}$$

The other two equations are of third order in derivatives of ψ_1, but they are in fact implied by the second-order equation (3.11). Thus it is only necessary to solve (3.11) in order to obtain transverse-traceless Lichnerowicz modes of the form (3.4).

Making a trial leading-order solution of the form $\psi_1 \sim (r-r_0)^c$ in the vicinity of the horizon, one finds $c=0$ or $c=-1$, and thus for regularity we must take $c=0$. The near-horizon expansion, which can be used in order to set data just outside the horizon for a numerical integration out to near infinity, then takes the form

$$\psi_1 = \sum_{m\ge 0} a_m\,(r-r_0)^m\,. \tag{3.12}$$

At large distances, when $r >> r_0$, the equation (3.11) becomes approximately

$$\psi_1'' + \frac{n}{r}\,\psi_1' + \lambda\,\psi_1 \approx 0\,, \tag{3.13}$$

and thus for a negative eigenvalue we have $\psi_1 \sim e^{\pm\sqrt{-\lambda}\,r}$ at large r. The numerical solution of the equation can therefore be performed by the shooting method, seeking a value of the eigenvalue λ for which the exponentially diverging solution $e^{+\sqrt{-\lambda}\,r}$ is absent. As in four dimensions, there appears in a general dimension to be just one negative-eigenvalue normalisable mode.

4 Numerical Results

In this section we universally set $r_0 = 1$, so that the Schwarzschild black hole has unit horizon radius. We constructed a near-horizon Taylor expansion of the form (3.12), up to $(r-1)^8$ order, and used this to set initial data at a radius $r_i = 1 + 1/100$ just outside the horizon, for numerical integration out to large r. If the eigenvalue parameter λ is taken to be positive, one finds as expected oscillatory solutions for all $\lambda > 0$. It appears to be the case for each dimension $n \ge 4$ that there is just one negative value of λ for which one can have a normalisable eigenfunction (i.e. no exponentially-growing term $\sim e^{\sqrt{-\lambda}\,r}$). For dimensions $4 \le n \le 11$ we find

$$\begin{aligned} n=4: &\qquad \lambda \approx -0.7677\,,\\ n=5: &\qquad \lambda \approx -1.610\,,\\ n=6: &\qquad \lambda \approx -2.499\,,\\ n=7: &\qquad \lambda \approx -3.417\,,\\ n=8: &\qquad \lambda \approx -4.356\,,\\ n=9: &\qquad \lambda \approx -5.309\,,\\ n=10: &\qquad \lambda \approx -6.272\,,\\ n=11: &\qquad \lambda \approx -7.242\,. \end{aligned} \tag{4.1}$$

The $n=4$ result accords with the eigenvalue reported by Gross, Perry and Yaffe [7]. (They took $M=1$ and hence $r_0=2$; the eigenvalues scale like $1/r_0^2$ if one removes the $r_0=1$ specialisation we took.) The $n\ge 5$ results are broadly in accordance with the higher-dimensional results reported from previous work in Eqn (34) of Ref. [15].

5 New Black Holes

5.1 Transverse and traceless gauge

The existence of a TT Lichnerowicz mode with negative eigenvalue λ in the Schwarzschild background implies the existence of a new branch of static, spherically-symmetric black-hole solutions that bifurcates from the Schwarzschild black-hole branch at the point where $\xi = 1/\lambda$. To see this, we note that for the transverse and traceless perturbation of the Schwarzschild metric, $g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$ with $g^{\mu\nu} h_{\mu\nu} = 0 = \nabla^\mu h_{\mu\nu}$, we have $\delta R_{\mu\nu} = \frac{1}{2}\Delta_L h_{\mu\nu}$, implying that the equations of motion become

$$(\Delta_L - \lambda)\Delta_L h_{\mu\nu} = 0\,. \tag{5.1}$$

(The analogous equation in AdS black hole background was obtained in [16].) The general solution of this fourth-order equation is a linear superposition of solutions of $\Delta_L h_{\mu\nu} = 0$ and solutions of $(\Delta_L - \lambda) h_{\mu\nu} = 0$.

For perturbations that maintain the spherical symmetry, solutions with $\Delta_L h_{\mu\nu} = 0$ lead again to a Schwarzschild black hole. In fact the Lichnerowicz TT zero modes can easily be solved for using the results presented in Section 3. The general solution of (3.11) for the function ψ_1, in the case $\lambda = 0$, is given by

$$\psi_1 = \frac{c_1\,[2r^{n-3} - (n-1)(n-2)]}{(1 - r^{n-3})} + \frac{c_2\,[(n-1) - 2(n-2)\,r^{n-3}]}{r^{n-1}\,(1 - r^{n-3})}\,, \tag{5.2}$$

where c_1 and c_2 are arbitrary constants. (We have again set $r_0 = 1$ here for convenience.) From this, it follows, in particular, that

$$\psi_{ij} = -2(c_1\,r^2 + c_2\,r^{3-n})\,\gamma_{ij}\,, \tag{5.3}$$

and so the perturbation $g_{\mu\nu} \to g_{\mu\nu} + \psi_{\mu\nu}$ can be re-expressed in terms of a canonical radial variable ρ for which $ds^2 = \cdots + \rho^2\,d\Omega_{n-2}^2$ by means of the redefinition

$$r = \rho + c_1\,\rho + c_2\,\rho^{2-n}\,, \tag{5.4}$$

where c_1 and c_2 are now understood to be infinitesimal. A straightforward calculation of the rest of the perturbed metric shows that the parameter c_2 is trivial (corresponding merely to a diffeomorphism that is absorbed by the redefinition (5.4)), and that the parameter c_1 corresponds to a mass perturbation under which the original Schwarzschild metric function $1 - \rho^{3-n}$ is transformed according to

$$1 - \frac{1}{\rho^{n-3}} \longrightarrow 1 - \frac{1 + c_1\,(n-1)(n-3)}{\rho^{n-3}}\,. \tag{5.5}$$

On the other hand, perturbations with $(\Delta_L - \lambda)h_{\mu\nu} = 0$, with $\xi = 1/\lambda$, describe a branch of new black-hole solutions. To see this concretely, we write the general spherically symmetric ansatz for static black holes

$$ds^2 = -hdt^2 + f^{-1}\,dr^2 + \rho^2 d\Omega_{n-2}^2\,, \tag{5.6}$$

where (h, f, ρ) are functions just of r. It is then straightforward to see that for a transverse and traceless perturbation of the form (3.4), the new black-hole solution is given by

$$h = \bar{f}(1 - \epsilon\psi_0)\,, \qquad f = \bar{f}(1 - \epsilon\psi_1)\,, \qquad \rho = r(1 + \frac{1}{2}\epsilon\bar{\psi})\,, \tag{5.7}$$

with $\bar{f} = 1 - r_0^{n-3}/r^{n-3}$ for the unperturbed background Schwarzschild black hole. In this perturbation, the horizon remains at $r = r_0$, but the area of the horizon is changed, being given by

$$\mathcal{A} = \frac{1}{4} r_0^{n-2}\,\Omega \to \frac{1}{4} r_0^{n-2}(1 + \frac{1}{2}\epsilon\bar{\psi})^{n-2}\,\Omega\,, \tag{5.8}$$

where Ω is the spherical volume factor of the $(n-2)$-dimensional space $d\Omega_{n-2}^2$.

5.2 Fixed horizon area

In an alternative description, we may consider a gauge choice in which the area of the horizon is fixed. This can be done by requiring

$$\rho = r\,, \qquad h = \bar f(1+\epsilon\,\tilde h)\,, \qquad f = \bar f(1+\epsilon\,\tilde f)\,. \tag{5.9}$$

Equations (2.10) taken together with $\delta R = 0$ yield

$$\begin{aligned}
&2r^2\bar f\tilde h'' - r\big((n-5)\bar f - 3(n-3)\big)\tilde h' + r\big((n-3)\bar f + n-3\big)\tilde f' + 2(n-2)(n-3)\tilde f = 0\,,\\
&-4r^2\big(r\bar f\tilde h' + (n-3)\tilde f\big) + \xi\Big(2r^2\bar f\big((n-1)\bar f - n+3\big)\tilde f''\\
&\quad -r\big((n-1)\bar f^2 - 2(n-3)(3n-5)\bar f + 5(n-3)^2\big)\tilde f'\\
&\quad +r(n-3)(\bar f-1)\big((n+1)\bar f - n+3\big)\tilde h'\\
&\quad +2(n-3)\big((n^2-5n+2)\bar f - (n-2)(n-3)\big)\tilde f\Big) = 0\,.
\end{aligned} \tag{5.10}$$

Note that we have fixed the radial coordinate gauge choice by imposing $\rho(r) = r$. This means that the metric perturbation is neither transverse nor traceless in this gauge. In order for the perturbative solution to be regular on the horizon, we consider the Taylor expansion

$$\tilde f = 1 + f_1(r-r_0) + f_2(r-r_0)^2 + \cdots\,, \qquad \tilde h = h_0 + h_1(r-r_0) + h_2(r-r_0)^2 + \cdots\,. \tag{5.11}$$

with

$$f_1 = \frac{1}{r_0}\Big(1 - \tfrac12 n - \frac{3r_0^2}{4(n-3)\xi}\Big)\,, \qquad h_1 = \frac{1}{r_0}\Big(1 - \tfrac12 n + \frac{r_0^2}{4(n-3)\xi}\Big)\,, \qquad etc. \tag{5.12}$$

For each given r_0, there is indeed a value of ξ for which a solution exists. For example, with $r_0 = 1$, we find such a value in $n = 6$ dimensions to be

$$\xi \sim -0.4002\,, \qquad \frac{1}{\xi} = -2.499\,. \tag{5.13}$$

The large-r behaviour of the function $\tilde f$ is

$$\begin{aligned}
\tilde f &= \frac{c_1}{r^3} + \frac{c_2(r^2 + 3\mu_0^2 r + 3\mu_0^2)e^{-r/\mu_0}}{r^3} + \cdots\,,\\
\tilde h &= \frac{c_1}{r^3} + \frac{4c_2\mu_0(r+\mu_0)e^{-r/\mu_0}}{r^3} + \cdots\,.
\end{aligned} \tag{5.14}$$

We find that $h_0 = 1.401$ and $c_1 = 3.602$. The coefficient c_2 is too unstable to be determined accurately owing to the exponential factor.

5.3 Non-Schwarzschild black holes below and above the branch crossing

One feature of the black-hole solution space that becomes manifest from the Lichnerowicz eigenfunction analysis of the static spherically symmetric perturbations away from the Schwarzschild solution is that the non-Schwarzschild solutions (5.7) exist both for smaller as well as larger black holes than the solution sitting precisely at the crossing of the two branches. In other words, the perturbation parameter ϵ controlling location along the non-Schwarzschild branch in (5.7) can be negative as well as positive.

The possibility of such sub-crossing as well as super-crossing non-Schwarzschild black holes was not highlighted in the earlier numerical analysis of Ref. [3], but once their existence is understood from the Lichnerowicz perturbation analysis, it is possible to find them numerically [5,6]. Using the same sort of numerical analysis as in Refs [3,4], one obtains in $n = 4$ spacetime dimensions the branch structure shown in Figure 1.

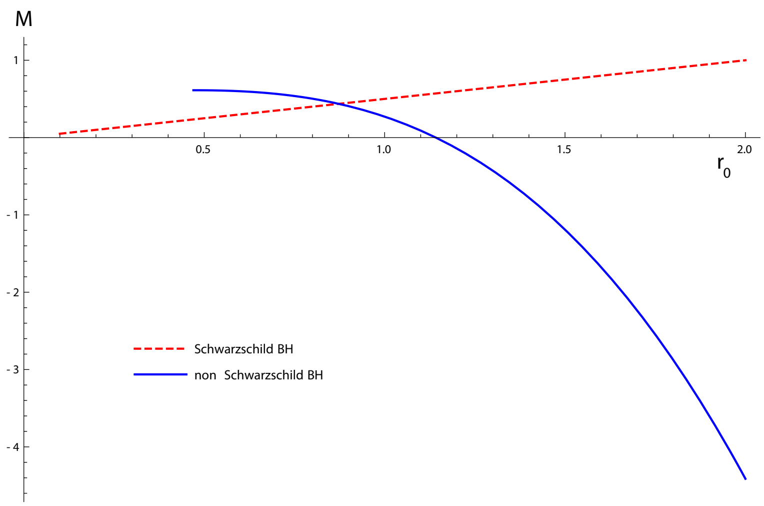

Figure 1: Non-Schwarzschild black-hole solutions crossing the Schwarzschild branch.

For black holes significantly smaller than that at the branch crossing, numerical calculation becomes increasingly unstable as one moves to smaller r_0, and it is not yet clear what happens for very small r_0 although it seems that the non-Schwarzschild branch asymptotes to a limiting mass value.

5.4 The first law of thermodynamics

Having obtained linearised numerical solutions of new black holes that can be viewed as small perturbations from the Schwarzschild black hole, we can establish the first law of black-hole thermodynamics by use of the Wald formalism. The Wald formalism for spherically-symmetric black holes in quadratically extended gravity in general dimensions was derived in Ref. [17]. The entropy can be derived from the Wald entropy formula,[2] giving

$$\begin{aligned} S &= \tfrac{1}{4}\mathcal{A}\Big(1-(n-2)(4\beta+\xi)\frac{f'(r_0)\rho'(r_0)}{\rho(r_0)}+\frac{2(n-2)(n-3)\beta}{\rho(r_0)^2} \\ &\quad +\tfrac{1}{4}(2\beta+\xi)\big(f''(r_0)+\frac{3f'(r_0)h''(r_0)}{h'(r_0)}\big)\Big)\,, \end{aligned} \tag{5.15}$$

where the derivative is with respect to r and $r=r_0$ is the location of the horizon. For the Schwarzschild black hole with horizon radius r_0, the mass, temperature and entropy are

$$M=\frac{(n-2)r_0^{n-3}\,\Omega}{16\pi}\,,\qquad T=\frac{n-3}{4\pi r_0}\,,\qquad S=\tfrac{1}{4}r_0^{n-2}\,\Omega\,, \tag{5.16}$$

If the Schwarzschild solution is perturbed by making an infinitesimal variation of the parameter r_0, the corresponding changes in M and S obey the first law $dM=TdS$.

We now consider the perturbative description of the new branch of black-hole solutions, firstly in the gauge where the metric perturbation is transverse and traceless. In this case, since the normalisable perturbative mode falls off exponentially at large r, the mass is unchanged by the perturbation, and hence $\delta M=0$ as we move along the trajectory of perturbative deformations. The near-horizon expansion of the metric functions are

$$\begin{aligned} h &= h_0(r)(1-\epsilon\psi_0)\sim(n-3)(1+\epsilon)(r-1) \\ &\quad -\Big(\frac{1}{2}(n-2)(n-3)+\frac{1}{2}\epsilon\big((n-3)(3n-2)+\frac{3}{\xi}\big)\Big)(r-1)^2+\cdots\,, \\ f &= h_0(r)(1-\epsilon\psi_1 r)\sim(n-3)(1-\epsilon)(r-1) \\ &\quad -\Big(\frac{1}{2}(n-2)(n-3)-\frac{1}{2}\epsilon\big((n-3)(3n-2)+\frac{3}{\xi}\big)\Big)(r-1)^2+\cdots\,, \\ \rho &= r-\frac{\epsilon}{n-2}+\Big(1+\frac{(n-1)(n-3)\xi+1}{(n-2)(n-3)\xi}\Big)(r-1)+\cdots \end{aligned} \tag{5.17}$$

Substituting these into the entropy formula, we find

$$S\sim\tfrac{1}{4}\Omega\Big(r-\frac{\epsilon}{n-2}\Big)^{n-2}(1+\epsilon)\sim\tfrac{1}{4}\mathcal{A}_0+\mathcal{O}(\epsilon^2)\,. \tag{5.18}$$

Thus $\delta S=0+\mathcal{O}(\epsilon^2)$, implying that S is also unchanged at linear order in ϵ, as we move along the trajectory of deformations. This means that in the transverse-traceless gauge, the first law $\delta M=T\delta S$ is trivially verified as one moves along the new branch of black holes (at the linearised level), since both M and S are unchanged as the perturbation is turned on.

[2] The Wald entropy is given [18, 19] by $S=-\frac{1}{8}\int_+\sqrt{h}\,d^{n-2}x\,\epsilon^{ab}\epsilon^{cd}\,\frac{\partial L}{\partial R^{abcd}}$ where the integration is over a cross-section of the horizon with metric $h_{\mu\nu}$ and binormal ϵ^{ab}.

In the alternative description in the gauge where $\rho(r) = r$ (fixed horizon area), the temperature and entropy of the new black hole are

$$\begin{aligned} T &= \frac{\sqrt{h'(r_0)f'(r_0)}}{4\pi} = \frac{n-3}{4\pi r_0}\Big(1 + \frac{\epsilon(1+h_0)\xi}{2r_0^2}\Big)\,, \\ S &= \tfrac{1}{4}\mathcal{A}\Big[1 - (n-2)(4\beta+\xi)\frac{f'(r_0)}{r_0} + \frac{2(n-2)(n-3)\beta}{r_0^2} \\ &\qquad -\frac{1}{4}(2\beta+\xi)\big(f''(r_0) + \frac{3f'(r_0)h''(r_0)}{h'(r_0)}\big)\Big] \\ &= \tfrac{1}{4}r_0^{n-2}\Omega\Big(1 + \frac{\epsilon(n-2)(n-3)\xi}{r_0^2}\Big)\,. \end{aligned} \tag{5.19}$$

Thus we see that the entropy of the perturbed solution differs from that of the original solution by

$$\delta S = \frac{1}{4}\epsilon\,(n-2)(n-3)\,r_0^{n-4}\xi\Omega\,, \tag{5.20}$$

This implies from the first law that we must have

$$\delta M = \frac{\epsilon\,\Omega}{16\pi}(n-2)(n-3)^2\xi r_0^{n-5}\,. \tag{5.21}$$

We do not have a universal proof, but we can establish the relation by numerical analysis in a case-by-case basis. For the $n = 6$ numerical solution described in the previous section, we find

$$M = \frac{\Omega}{4\pi}(1 - 3.60\,\epsilon)\,, \qquad \delta M = -3.60\,\epsilon\,\frac{\Omega}{4\pi}\,. \tag{5.22}$$

This is precisely the δM from the first law (5.21), with $n = 6$, $\xi = -0.4002$ and $r_0 = 1$.

6 Time-Dependent Perturbations and Stability

So far, we have discussed spherically-symmetric TT perturbations that are time independent. In particular, one finds there is one such negative-eigenvalue mode of the Lichnerowicz operator in the Schwarzschild background. The approximate values for the negative eigenvalue in dimensions $4 \le n \le 11$ are given in Eqn (4.1) for the case where the Schwarzschild radius r_0 is taken to be unity. If we denote the negative eigenvalues λ for such time-independent solutions in n dimensions by $\bar\kappa_n$ when $r_0 = 1$, then for general values of r_0 the negative eigenvalues will be

$$\kappa_n = \frac{\bar\kappa_n}{r_0^2}\,. \tag{6.1}$$

We now consider the stability properties of the black hole solutions. Once again, the Lichnerowicz negative eigenvalue will play a key rôle, this time in delimiting the edge of a domain of instability for the Schwarzschild black-hole family. We begin with a formulation setting up a perturbative approach.

6.1 Time-dependent Fluctuation Problem in Schrödinger Form

In four dimensions, a way to recast into a Schrödinger form for a single field the Lichnerowicz problem of time-dependent tensor fluctuation modes around a Schwarzschild background was given by Zerilli [13]. This formulation has been particularly useful in analysing instabilities in massive gravity theories [11]. Here, we give the analogous results for perturbations about a Schwarzschild background in arbitrary dimensions for Ricci quadratic gravity.

Including time dependence, which we take to be of the form $e^{-i\omega t} = e^{\nu t}$, the spherically-symmetric ansatz for TT modes takes the form

$$\psi_{00} = h\,\psi_0(r)\,e^{\nu t}\,,\quad \psi_{01} = \chi(r)\,e^{\nu t}\,,\quad \psi_{11} = h^{-1}\,\psi_1(r)\,e^{\nu t}\,,\quad \psi_{ij} = r^2\,\bar\psi(r)\,\gamma_{ij}\,e^{\nu t}\,. \tag{6.2}$$

The traceless and transverse conditions imply three non-trivial equations, which can be solved for ψ_0, $\bar\psi$ and χ'. The Lichnerowicz eigenvalue equation

$$\Delta_L\,\psi_{\mu\nu} = \lambda\psi_{\mu\nu} \tag{6.3}$$

implies two two-derivative equations (the (01) and (11) components), and feeding in the expressions for ψ_0, $\bar\psi$ and χ' from the TT conditions, we can solve these for χ itself, and obtain a second-order equation purely for ψ_1. The remaining two non-trivial equations from the Lichnerowicz eigenvalue equation are of third order in derivatives of ψ_1, and these can be seen to be consequences of the second-order equation for ψ_1.

To cast the equation into a Schrödinger form, we follow steps analogous to those that can be used in four dimensions to reproduce the Zerilli form. Thus, we introduce a new variable $\phi(r)$, defined by

$$\phi(r) = \nu^{-1}\,u(r)\,\chi(r) + v(r)\,\bar\psi(r)\,, \tag{6.4}$$

where u and v are functions of r that will be determined shortly. Crucially, we assume that $u(r)$ and $v(r)$ do not depend on ν (although they may depend on the Lichnerowicz eigenvalues λ, as well as r).[3] We solve for u and v by making the following requirements. Firstly, we require that in terms of the "tortoise" coordinate r_*, ranging from the horizon as $r_* \to -\infty$ to spatial infinity as $r_* \to +\infty$ and defined by

$$r_* = \int^r \frac{dr'}{h(r')}\,, \tag{6.5}$$

[3] These requirements are motivated by the specific example of the Zerilli construction in four dimensions, where these properties hold.

the function ϕ should satisfy an equation of the form

$$\frac{d^2\phi}{dr_*^2} = W(r)\,\phi\,, \tag{6.6}$$

where we make use of the second-order equation satisfied by ψ_1 that was determined previously. The requirement that the left-hand side of (6.6) should be expressible purely as a function of r times ϕ (*i.e.* that there should be no term with a first derivative of ϕ) implies one equation with derivatives of u and v (up to second order). The second requirement we make is that $W(r)$ should be of the form

$$W(r) = \nu^2 + V(r)\,, \tag{6.7}$$

where $V(r)$ does not depend on ν. The consequent property that $dV/d\nu = 0$ gives an equation (up to first derivatives only) involving u and v. This equation is exactly integrable, leading to the relation

$$v(r) = \frac{r\,u(r)}{h(r)}\,. \tag{6.8}$$

Feeding this back into the previous second-order equation that followed from the first of the two above requirements results in an integrable equation for u, whose solution is

$$u(r) = \frac{h(r)\,r^{\frac{3}{2}(n-2)}}{\frac{1}{2}(n-2)(n-3) - \lambda r^{n-1}}\,. \tag{6.9}$$

Note that here, and below, we are taking the Schwarzschild radius to be $r_0 = 1$, so

$$h(r) = 1 - \Big(\frac{r_0}{r}\Big)^{n-3} = 1 - \frac{1}{r^{n-3}}\,. \tag{6.10}$$

With u and v now determined, we have arrived at the Schrödinger form of the Lichnerowicz eigenvalue equation

$$-\frac{d^2\phi}{dr_*^2} + \Big[\nu^2 + V(r)\Big]\,\phi = 0\,, \tag{6.11}$$

where the Zerilli-type potential $V(r)$ is given by

$$V(r) = -\frac{h(r)}{r^{n-1}\,[\frac{1}{2}(n-2)(n-3) - \lambda\,r^{n-1}]^2}\,Y(r)\,, \tag{6.12}$$

where

$$\begin{aligned} Y(r) \quad = \quad & -\frac{1}{16}(n-2)^3\,(n-3)^2\,[(n-2) + (n-4)r^{n-3}] \\ & +\frac{1}{4}(n-2)(n-3)\lambda\,r^{n-1}\,[2n^2 - 5n + 6 - 3n(n-2)r^{n-3}] \\ & +\frac{1}{4}(n+2)\,\lambda^2\,r^{2(n-1)}\,[3(n-2) - nr^{n-3}] + \lambda^3\,r^{3(n-1)}\,. \end{aligned} \tag{6.13}$$

6.2 Potentials and Wavefunctions

The Schrödinger form (6.11) for the equation governing the metric perturbations is a convenient starting point for an analysis of the stability of the static spherically-symmetric black-hole solutions. The equation cannot be solved analytically, and so it is necessary to resort to approximation techniques or to numerical analysis. It is however straightforward to obtain solutions numerically, and this seems to be the simplest way to proceed.

First, however, we need to determine the relevant boundary conditions for the Schrödinger wavefunction ϕ solving (6.11). In general, the Zerilli-type potential V (6.12) has a form when expressed in terms of the r_* tortoise coordinates like that shown in Figure 2.

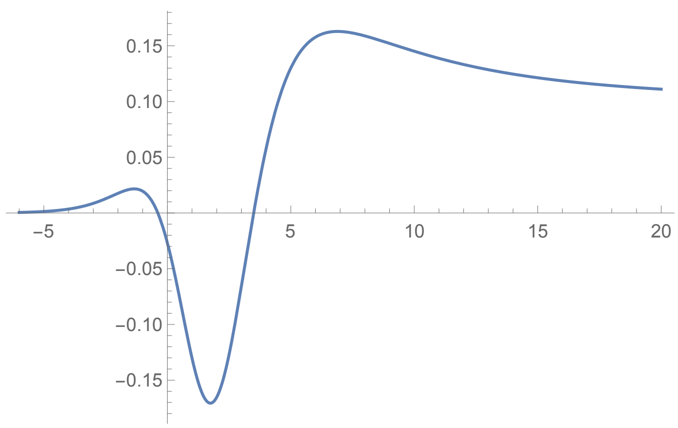

Figure 2: *Zerilli-type potential $V(r_*)$ in $n = 4$ dimensions for the Schrödinger problem.*

Key features of this potential are its asymptotic limits to zero as $r_* \to -\infty$ and to a nonzero constant as $r_* \to +\infty$. The analysis of black-hole quasinormal modes for nonzero frequencies given in References [20, 21] prescribes purely outgoing waves in the left and the right asymptotic regions, *i.e.* asymptotic coordinate dependences of the forms $e^{-i\omega(r_*+t)}$ as $r_* \to -\infty$ and $e^{i\omega(r_*-t)}$ as $r_* \to +\infty$.[4] The limiting wavefunction at the boundary of the domain of instability has $\nu = i\omega = 0$ and so must tend to r_* independent constants as $r_* \to \pm\infty$. Since $V(r_*) \to 0$ as $r_* \to -\infty$, any constant asymptotic value of ϕ will satisfy

[4] For generic values of λ below a certain limiting value, Figure 2 shows a mild "hump" maximum before settling down to its asymptotic $r_* \to +\infty$ constant value. The WKB analysis of quasinormal modes given in Refs [20,21] matches the top of a similar "hump" of a Zerilli-type potential for perturbations of the Kerr solution to a parabola and then uses parabolic cylinder functions to do the WKB matching and so derive the imaginary parts of quasinormal-mode frequencies. Our main concern here, however, is to understand the boundary of the domain of instability for the Schwarzschild solution, *i.e.* the limiting λ value of the solution at which the instability disappears. As it happens, the potential "hump" in the present problem diminishes in this limit and a standard WKB analysis becomes unreliable.

(6.11) in that region, but for $V(r_*) \to \text{const} \neq 0$ as $r_* \to +\infty$, one must have $\phi(r_*) \to 0$ as $r_* \to +\infty$ in order to satisfy (6.11). Accordingly, the tortoise-coordinate outgoing-wave boundary conditions are equivalent to regularity conditions as $r_* \to \pm\infty$. These regular boundary conditions can be satisfied only for specific values of λ in the (6.12) potential $V(r_*)$, depending on the dimension n.

As an illustration, consider the regular $\nu = 0$ solution in $n = 4$ dimensions at the edge of the domain of instability. By means of a simple shooting-method numerical study of the regular solution of Equation (6.11), we find that this is achieved when the constant λ takes the specific value $\lambda = \bar{\kappa}_4 \approx -0.7677$, in full agreement with the results given earlier in Section 4. In Figure 3 we display the potential V, plotted as a function of the redefined radial coordinate r_*, together with a plot of the corresponding wavefunction ϕ. The potentials and wavefunctions are qualitatively similar in all dimensions.

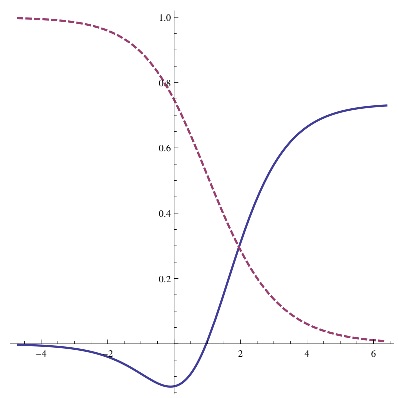

Figure 3: *The Zerilli-type potential* $V(r_*)$ *(solid line), and the wavefunction* $\phi(r_*)$ *(dashed line) in* $n = 4$ *dimensions for the Lichnerowicz negative eigenvalue* $\lambda \approx -0.7677$ *that gives rise to a regular solution.*

6.2.1 Square well approximation

Unlike the Zerilli-type potential for the case of perturbations in standard Einstein gravity, *e.g.* for the Kerr solution, it can be seen that the potential shown in Figure 3 does not readily lend itself to a straightforward WKB approximation. Another approximate approach is to consider a simple idealisation of the potential in which it is represented by a rectangular well, as depicted in Figure 4. The approximation of the true potential (6.12) by a rectangular well provides an instructive qualitative picture of the nature of the regular solutions to the Schrödinger-type equation (6.11). Let the rectangular-potential analogue of the Schrödinger wavefunction ϕ be denoted φ.

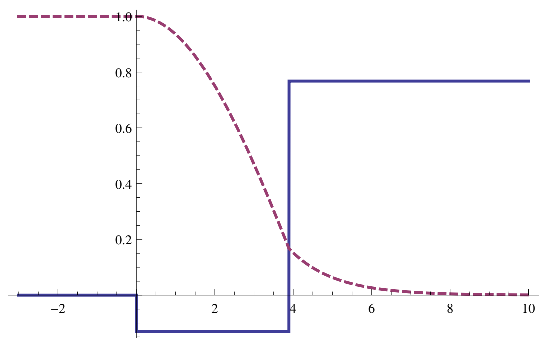

Figure 4: *The rectangular well potential (solid line), and the wavefunction (dashed line), for the regular solution in $n = 4$ dimensions.*

The specific details of the well in this figure have been chosen for the four-dimensional case, but the same general idea can be applied in all dimensions. Thus in general one can define a rectangular potential

$$V(x) = \begin{cases} 0 \text{ for } x < 0\,, \\ V_- \text{ for } 0 \le x \le \ell\,, \\ V_+ \text{ for } x > \ell\,, \end{cases} \tag{6.14}$$

where $V_- < 0$ and $V_+ > 0$. The coordinate x here corresponds to the radial coordinate r_* in the Schrödinger formulation in (6.11), up to an unimportant additive constant. The values of V_- and V_+ are chosen to match roughly the corresponding values of the minimum and maximum of the actual potential given by (6.12) and (6.13).

As we discuss below, for generic values of the quantities V_-, V_+ and ℓ that characterise the rectangular potential, the solution for the exact Schrödinger wavefunction will necessarily be divergent either as x goes to $-\infty$ (corresponding to the horizon) or as x goes to $+\infty$ (corresponding to $r \to \infty$) or both. Only by fine tuning the parameters of the potential can one obtain a solution where the wavefunction is regular everywhere. This is analogous to the way in which the parameter λ in the Zerilli-type potential $V(r)$ in the actual Schrödinger problem $-d^2\phi/dr_*^2 + V\phi = 0$ is determined by the requirement that ϕ should be regular both on the horizon and at infinity.

Concretely, if we define

$$V_- = -k^2\,, \qquad V_+ = -\lambda = \mu^2\,, \tag{6.15}$$

then the general solutions φ_1, φ_2 and φ_3 in the three regions $x < 0$, $0 \le x \le \ell$ and $x > \ell$ respectively are:

$$\begin{aligned}\varphi_1 &= a_1 + b_1\, x\,, \\ \varphi_2 &= a_2\, \cos kx + b_2\, \sin kx\,, \\ \varphi_3 &= a_3\, e^{-\mu x} + b_3\, e^{\mu x}\,. \end{aligned} \tag{6.16}$$

In order for the wave function to be finite as $x \to -\infty$ we must have $b_1 = 0$. Without loss of generality, we shall then choose the normalisation $a_1 = 1$. To match the functions and first derivatives at $x = 0$ we then need $a_2 = 1$ and $b_2 = 0$. For the function φ_3 to be regular as $x \to \infty$, we must have $b_3 = 0$. Finally, to match the functions and first derivatives at $x = \ell$ we require

$$\cos k\ell = a_3\, e^{-\mu\ell}\,, \qquad k \sin k\ell = \mu\, a_3\, e^{-\mu\ell}\,. \tag{6.17}$$

Thus we have

$$\varphi_1 = 1\,, \qquad \varphi_2 = \cos kx\,, \qquad \varphi_3 = a_3\, e^{-\mu x}\,, \tag{6.18}$$

where μ, k and ℓ must satisfy

$$\tan k\ell = \frac{\mu}{k}\,, \tag{6.19}$$

and a_3 is then determined from (6.17). We can see graphically how the solutions to (6.19) arise. Writing (6.19) in terms of V_+, V_- and ℓ, we have

$$V_+ = (-V_-)\, \tan(\sqrt{-V_-}\, \ell)\,. \tag{6.20}$$

This is the condition on the parameters of the rectangular potential that must be satisfied in order to obtain an everywhere-regular wavefunction.

We can now compare the results from this rectangular-well approximation with the actual results from the numerical solutions. Using $V_+ = -\lambda$, and reading off V_- from the plot in Figure 3, we can then use (6.20) to calculate ℓ, the width of the negative part of the rectangular potential, finding

$$V_+ = 0.7677\,, \quad V_- = -0.13\,, \quad \Longrightarrow\ \ell \approx 3.89\,. \tag{6.21}$$

We see that this width for the negative part of the rectangular potential does indeed match reasonably well with the estimate one can make from the plot in figure 3, and that the wave-function in the rectangular approximation reasonably matches that in the true numerical solution.

6.3 Gregory-Laflamme instabilities

The instability of the Schwarzschild solution in n-dimensional quadratic Ricci gravity as discussed in the previous subsection is directly related to a familiar instability of the black string in $n+1$ dimensions, as found by Gregory and Laflamme [9].

The result of the Gregory-Laflamme analysis is that for each dimension n there exists a continuous family of spherically-symmetric time-dependent TT eigenfunctions with negative eigenvalues λ lying in the range

$$\kappa_n < \lambda < 0\,, \tag{6.22}$$

where κ_n is as given in (6.1) and for which the time dependence has a real exponential form $e^{\nu\,t}$, thus corresponding to instabilities growing exponentially in time. That is to say, the static negative eigenmode discussed in Sections 3 and 4 represents also the limiting, *i.e.* most negative, value of a continuous range of less negative eigenvalues for TT modes with real exponential time dependence. For this reason, the static mode may be called a "threshold unstable mode".

Related discussions can be given for quadratic Ricci gravity or for pure massive gravity or for which the same form of equation applies [12]. The black-string metric in $(n+1)$ dimensions is given by simply adding a dz^2 term to the black-hole metric in n dimensions:

$$d\hat{s}^2 = ds^2 + dz^2\,. \tag{6.23}$$

The relevant S-wave in n dimensions fluctuations will be of the form

$$\hat{\psi}_{\mu\nu} = e^{\mathrm{i}\,\mu\,z}\,\psi_{\mu\nu}\,, \tag{6.24}$$

where the TT field $\psi_{\mu\nu}$ depends just on r and t of the the n-dimensional spacetime, with the t dependence assumed to be of the form (6.2). For the $n\times n$ part of the metric $\psi_{\mu\nu}$ we then have the Lichnerowicz condition

$$(\Delta_L + \mu^2)\,\psi_{\mu\nu} = 0\,, \tag{6.25}$$

which clearly shows the identity between the $(n{+}1)$ dimensional black-string time-dependent fluctuation problem and the quadratic Ricci gravity time-dependent fluctuation problem (3.10) with the identification $\mu^2 = m_2^2 = -\lambda = -\frac{1}{\xi}$. Accordingly, the Gregory-Laflamme black-string instabilities of Ref. [9] directly determine the instabilities of the quadratic Ricci gravity theories. They also directly determine the instabilities of black holes in massive gravity theory with spin-two mass m_2. Moreover, the eigenvalue interval (6.22) for instabilities of the $(n+1)$ dimensional black string problem translates directly into the corresponding interval for instabilities of the Schwarzschild solution in quadric Ricci gravity.

Fix a value ξ for the coefficient of the $R^{\mu\nu} R_{\mu\nu}$ term in the Lagrangian (2.1). We then know that there will be an associated critical value r_c of the Schwarzschild radius given by

$$\frac{1}{\xi} = \frac{\bar{\kappa}}{r_c^2}\,, \tag{6.26}$$

corresponding to the radius at which the bifurcation of static black-hole branches occurs and the second branch of non-Schwarzschild black holes crosses the Schwarzschild branch. Now consider a Schwarzschild black hole of radius r_0 in this theory (with ξ related to r_c as in (6.26)). Since the TT fluctuations satisfy $\Delta_L\,\psi_{\mu\nu} = (1/\xi)\,\psi_{\mu\nu}$, it follows that the criterion (6.22) for having an instability implies

$$\frac{\bar{\kappa}_n}{r_0^2} < \frac{\bar{\kappa}_n}{r_c^2}\,. \tag{6.27}$$

Since $\bar{\kappa}_n$ is negative, this means that $1/r_0^2 > 1/r_c^2$, and hence the Schwarzschild black hole will be unstable if its radius r_0 is such that

$$r_0 < r_c\,. \tag{6.28}$$

Thus Schwarzschild black holes whose radius is less than the radius at the bifurcation point become unstable, while those whose radius exceeds that at the bifurcation point are stable.

The black-string analysis also gives directly information about black-string instabilities in massive gravity theories. Consider a massive gravity theory in $(n+1)$ spacetime dimensions for which the TT fluctuations satisfy the Lichnerowicz condition

$$(\hat{\Delta}_L + m_2^2)\,\hat{\psi}_{MN} = 0\,, \tag{6.29}$$

where m_2 is the mass of the spin-2 field in $(n+1)$ dimensions. The relevant fluctuations will again be of the form

$$\hat{\psi}_{\mu\nu} = e^{\mathrm{i}\,\mu\,z}\,\psi_{\mu\nu}\,, \tag{6.30}$$

where the $n\times n$ TT field $\psi_{\mu\nu}$ depends just on r and t of the the n-dimensional spacetime. In n dimensions we then have the condition

$$(\Delta_L + \mu^2 + m_2^2)\,\psi_{\mu\nu} = 0\,. \tag{6.31}$$

If we take ds^2 to be the metric of an n-dimensional Schwarzschild black hole of radius r_0, there will accordingly be exponentially-growing time-dependent modes if

$$\frac{\bar{\kappa}_n}{r_0^2} < -\mu^2 - m_2^2\,. \tag{6.32}$$

In view of the fact that $\bar{\kappa}_n$ is negative, this means that the criterion for instability is $(-\bar{\kappa}_n)/r_0^2 > \mu^2 + m_2^2$, and so conversely, the criterion for *stability* can be written as

$$r_0^2 \geq \frac{(-\bar{\kappa}_n)}{(\mu^2 + m_2^2)} \,. \tag{6.33}$$

Thus we will have complete stability (for all possible μ) if

$$r_0^2 \geq \frac{(-\bar{\kappa}_n)}{m_2^2} \,. \tag{6.34}$$

If we consider ordinary massless gravity with $m_2 = 0$, the criterion (6.34) can never be satisfied, regardless of the size of the black hole. This gives rise to the usual Gregory-Laflamme instability for black strings in $(n+1)$ dimensional Einstein gravity. However, if we consider instead black strings in $(n+1)$ dimensional massive gravity theory, then we see that provided the n dimensional Schwarzschild black hole has a sufficiently large radius such that (6.34) is satisfied, then the black string is protected from a Gregory-Laflamme instability.

7 Thermodynamic Implications for Stability

An important issue for a variety of black-hole systems has been the relation between thermodynamic instability and spacetime dynamical instability [23–27]. This is clearly germane to the context of the present paper as one can see from the rôle of the Gross-Perry-Yaffe Lichnerowicz negative eigenvalue in setting the boundary of the Schwarzschild dynamical S-wave instability, whereas it originally arose in the analysis of thermodynamic instability in Euclideanised quantum gravity.

One message of these studies is that spacetime dynamical instability requires thermodynamic instability, although the converse is not always the case. For systems with extensive quantities including ADM mass M together with other extensive quantities X_i, such as angular momentum J_i, ADM spatial momentum P_i or charges Q_i, dynamical stability is equivalent [27] to thermodynamic stability on the subspace of perturbations satisfying

$$\delta M = \delta X_i = 0 \,. \tag{7.1}$$

In pure Einstein theory, although the static Schwarzschild solution is thermodynamically unstable at the quantum level owing to the existence of the Gross-Perry-Yaffe negative Lichnerowicz eigenvalue, this does not imply classical dynamical instability because this spherically symmetric Schwarzschild solution does not possess any other extensive conserved quantities X_i, and the thermodynamically unstable perturbation is not consistent with the sole $\delta M = 0$ requirement. The situation changes for stationary Kerr black holes where the presence of nonzero angular momentum J_i allows for thermodynamically and dynamically unstable perturbations satisfying (7.1) to exist, leading, *e.g.*, to the phenomenon of superradiance [11, 28].

In systems such as black branes, which in their static configurations are spatially homogeneous, instabilities can occur via perturbations which locally disturb this homogeneity while maintaining overall conservation of extensive quantities such as total mass or angular momentum [26]. Accordingly in systems such as the $n = 5$ black string of Eq. (6.23), instabilities can occur that involve "bunching" of the mass distribution in the z direction while maintaining spherical symmetry in the $n = 4$ dimensional subspace. This "bunching" mechanism does not apply to solutions without such homogeneous structure, but a criterion for stability against perturbations subject to (7.1) can be given generally in terms of the positivity requirement of a "canonical energy" quantity as given in Ref. [26].

The analysis of relations between thermodynamic and dynamical instabilities in Einstein theory, including with matter coupling, has not yet been extended to the cases of massive gravity or of higher-derivative gravity. However the relevance of the negative Lichnerowicz eigenvalue both to the existence of the new family of non-Schwarzschild black holes and to the edge of the zone of Schwarzschild instability as discussed in Section 6 suggests that such a thermodynamic-dynamic instability correspondence may still apply in these cases. A full study of dynamical stability could require a numerical quasinormal mode analysis about both Schwarzschild and non-Schwarzschild families of solutions, which in the non-Schwarzschild case is itself only known numerically. However, one can nonetheless gain a hint of the relative stability properties of the two black hole families from thermodynamic considerations.

Two features of the thermodynamic-dynamic instability correspondence in Einstein theory lead to an analogous suggestion in quadratic curvature gravity.

- One is the existence of negative specific heats for solutions from the thermodynamic point of view. The key relation here is between the absolute values of the specific heats for Schwarzschild and non-Schwarzschild black holes in regions where the classical stability properties of the Schwarzschild solution are known from the Gregory-Laflamme analysis.

- The other is the necessary presence of threshold unstable modes as one crosses from a parametric zone of instability to a zone of stability [25]. Such a necessarily time-independent mode also constitutes the linearised perturbation describing the branching of the non-Schwarzschild black hole family as one moves from $r_0 < r_c$ instability to $r_0 > r_c$ stability (with r_c as given in (6.26)) along the Schwarzschild family trajectory as shown in Figure 1.

Conversely, as one moves along the non-Schwarzschild family trajectory of Figure 1, crossing a boundary between an unstable zone and a stable zone would similarly require a threshold unstable mode for this family from the thermodynamic point of view. From the dynamical point of view, this would correspond simply to a perturbation away from the crossing point onto the Schwarzschild family.

The question then arises as to the direction in which the change from stability to instability might take place. An indication of an answer to this question may be taken from the relative magnitudes of the specific heats.

It was pointed out already in [3] that in Ricci quadratic gravity, both the Schwarzschild and non-Schwarzschild black holes have negative specific heats, *i.e.* negative values of $C = dM/dT$. In that paper, discussion was given only of the portion of the non-Schwarzschild solution branch with masses M less than that of the Lichnerowicz crossing point, and it was pointed out that for such solutions, the non-Schwarzschild black holes have specific heats C that are more negative than those of the corresponding Schwarzschild solution at the same temperature T. This suggests that the non-Schwarzschild black holes lying to the right of the branch crossing point in Figure 1 may be unstable there, given the stability of the Schwarzschild black holes in that range.

What happens, then, in the range of solutions lying to the left of the Figure 1 crossing point? We may gain an idea of the relative susceptibility of the two branches to instabilities by considering a plot of the corresponding specific heats. In Reference [3], for the $n = 4$ dimensional case, the relations between Wald entropy S [18,19] and mass M and temperature T were given for the non-Schwarzschild black holes as

$$\begin{aligned} M_{\rm NSch} &\approx 0.168 + 0.131\,S - 0.00749\,S^2 - 0.000139\,S^3 + \cdots\,, \\ T_{\rm NSch} &\approx 0.131 - 0.0151\,S - 0.000428\,S^2 + \cdots\,. \end{aligned} \tag{7.2}$$

Eliminating S yields the mass versus temperature relation for non-Schwarzschild black holes

$$M_{\rm NSch} = 0.785221 - 3.22043\,T + 7.49256\,T^2 - 144.089\,T^3 + \cdots\,, \tag{7.3}$$

while for Schwarzschild black holes, one has the classic relation

$$M_{\rm Sch} = \frac{1}{8\pi T}\,; \tag{7.4}$$

a plot of the results is shown in Figure 5,

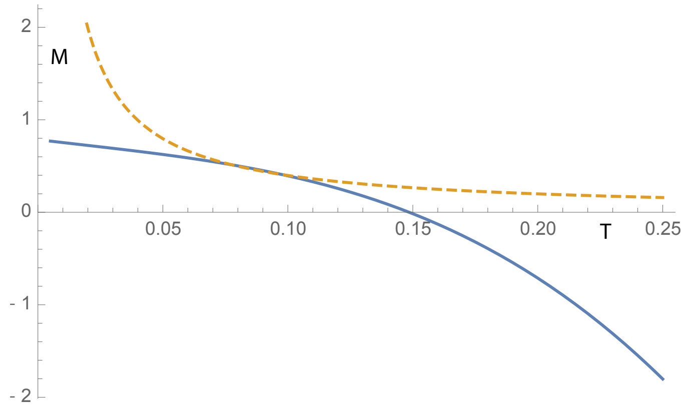

Figure 5: *Mass M versus temperature T relations for Schwarzschild (dashed line) and non-Schwarzschild (solid line) black holes.*

while for the corresponding specific heats $C = dM/dT$ one has the results shown in Figure 6.

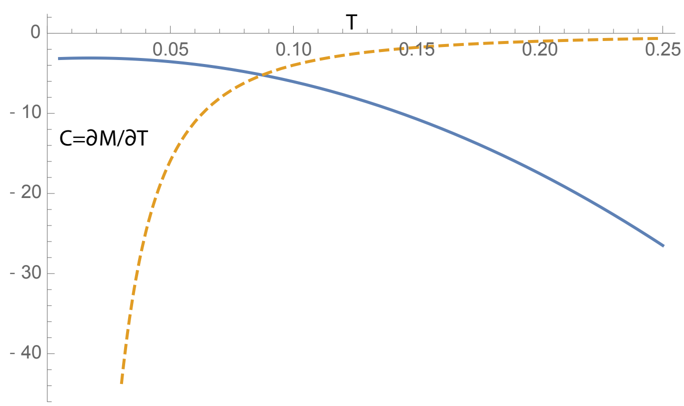

Figure 6: *Specific heat C versus temperature T relations for Schwarzschild (dashed line) and non-Schwarzschild (solid line) black-holes families.*

From Figure 6, one observes that for higher temperatures, which for Schwarzschild black holes correspond to small masses M and small radii r_0 and which for non-Schwarzschild black holes correspond to smaller (and eventually negative) masses but larger radii, the non-Schwarzschild black holes have a more negative specific heat than the Schwarzschild black holes. Accordingly, since Schwarzschild black holes are known to be subject to classical Gregory-Laflamme instabilities in this portion of their family trajectory, the suggestion is that the non-Schwarzschild black holes are more unstable in this "hot" portion of their family trajectory than the Schwarzschild black holes.[5]

Conversely, for lower temperatures, which for Schwarzschild black holes correspond to large masses M and large radii r_0 and for non-Schwarzschild black holes correspond to larger (now all positive) masses but smaller radii, the non-Schwarzschild black holes have a less negative specific heat than the Schwarzschild black holes. Accordingly, since Schwarzschild black holes are known to be classically immune to Gregory-Laflamme instabilities in this portion of their family trajectory, the suggestion is that the non-Schwarzschild black holes are more stable than the in this "cold" portion of their family trajectory than the Schwarzschild black holes.

The same inferences may be drawn by considering a graph of the free energy $F = M - TS$ versus temperature T relations of the two black-hole families, as shown in Figure 7, where again one sees a crossing of the two family curves.[6]

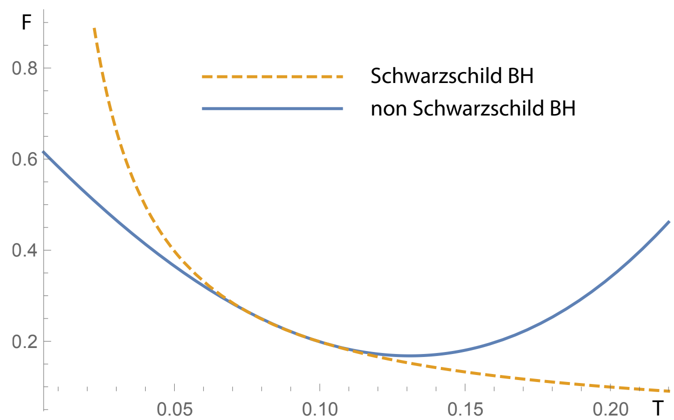

Figure 7: *Free energy $F = M - TS$ versus temperature T relations for Schwarzschild (dashed line) and non-Schwarzschild (solid line) black holes families.*

At temperatures above the crossing point of the two black-hole family curves, where the Schwarzschild solution is known to have Gregory-Laflamme instabilities, the free energy of the non-Schwarzschild solutions is higher, indicating a greater susceptibility to thermodynamic instability. Conversely, at temperatures below the crossing point, where the Schwarzschild solution is classically stable, the free energy of the non-Schwarzschild solutions is lower, indicating greater thermodynamic stability.

[5] In this discussion, we adopt the helpful "hot" and "cold" terminology of Ref. [6].

[6] A graph of the free-energy versus temperature relations for Schwarzschild and "hot" non-Schwarzschild black holes was given in Ref. [3]. Extension of the non-Schwarzschild solution family to the "cold" segment at radii below the crossing point was given in Ref. [5] without focussing on thermodynamic aspects. A graph analogous to Figure 7 showing the crossing of the free-energy curves was recently given in Ref. [6].

Thus, a coherent suggestion emerges for the phase structure of dynamical stability and instability ranges of the Schwarzschild and non-Schwarzschild black hole families, as shown in Figure 8. This suggestion arises from two interrelated observations. The first is the existence of threshold unstable modes in linear perturbations away from the Schwarzschild and the non-Schwarzschild black hole families at the Lichnerowicz crossing point, as shown in the analysis of Sections 2, 3 and 4. The second is the pattern of relative susceptibilities to thermodynamic instability as revealed by study of the specific heats or the free energies of the solution families.

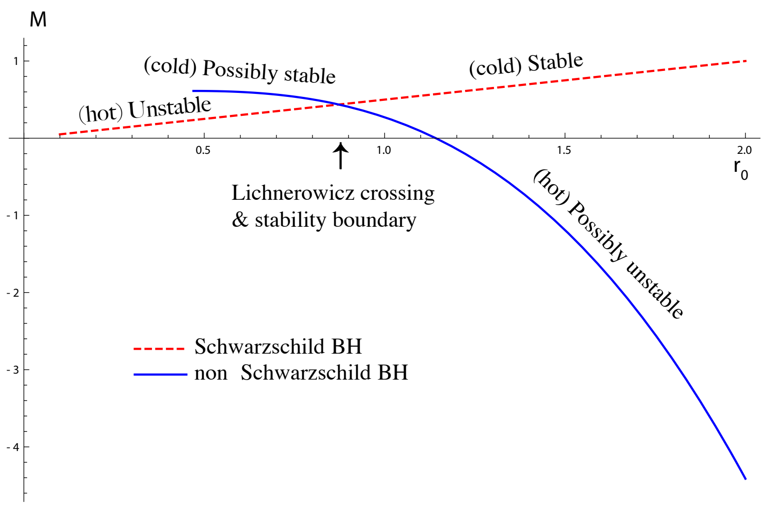

Figure 8: *Classical stability ranges for Schwarzschild (dashed line) and suggested classical stability ranges for non-Schwarzschild (solid line) black holes.*

8 Conclusion

We have seen how the static negative-eigenvalue Lichnerowicz eigenfunction plays an interrelated set of rôles related to black-hole solutions of Ricci quadratic gravity in dimensions $n \geq 4$. Writing the fluctuation equation (2.8) in terms of the metric when expanded about a Ricci-flat background, one has

$$\Delta_L(\Delta_L + m_2^2)\delta g_{\mu\nu} = 0\,, \tag{8.1}$$

so one has both $\Delta_L \delta g_{\mu\nu} = 0$ Lichnerowicz zero-mode solutions and $(\Delta_L + m_2^2)\delta g_{\mu\nu} = 0$ Lichnerowicz negative-mode solutions. These provide leading-order perturbative branching between the family of conventional Schwarzschild black holes and the non-Schwarzschild black holes found in Refs [3,4]. The negative mode also forms a threshold unstable mode boundary between known stable "cold" and unstable "hot" Schwarzschild phases, the unstable phase being subject to S-wave Gregory-Laflamme classical dynamical instabilities.

The suggestion made in Section 7 from analysis of the relative values of specific heats or free energies for the Schwarzschild and non-Schwarzschild solution branches is that the $\Delta_L \delta g_{\mu\nu} = 0$ mode may play a similar dual rôle for the non-Schwarzschild branch: it obviously provides a leading-order branching off the non-Schwarzschild branch onto the conventional Schwarzschild branch, and conjecturally could also form a threshold unstable mode boundary between stable "cold" and unstable "hot" phases of the non-Schwarzschild branch.

What one learns from the thermodynamic analysis is comparative: at "hot" temperatures where the Schwarzschild black holes are classically dynamically unstable and have high free energies, implying thermodynamic instability, the non-Schwarzschild black holes have even higher free energies, suggesting a similar classically dynamically unstable phase for them as well. Conversely, at "cold" temperatures where the Schwarzschild black holes are classically dynamically stable and have low free energies, implying greater thermodynamic stability, the non-Schwarzschild black holes have even lower free energies, suggesting a similar classically dynamically stable phase for them.

Confirmation of this suggested stability phase structure could be made by a direct analysis of the quasinormal modes, or by further development of the relation between quantum thermodynamic stability and classical dynamical stability for higher-derivative gravity theories, along the lines of the canonical energy analysis of References [26, 27].

The present paper has focussed exclusively on Ricci quadratic gravity models (2.1) in various dimensions n, which all admit classic Schwarzschild-Tangherlini black holes. A possible extension of focus would be to consider effective actions involving yet higher $(R_{\mu\nu})^n$ powers of the Ricci tensor. One can directly see, however, that the Lichnerowicz eigenfunctions considered in this paper will continue to play similar dual rôles in such extended models. The fourth-order metric fluctuation equation (8.1) arises from a second variation of the effective action: the first variation yields the field equation, while the second yields (8.1). The presence of $(R_{\mu\nu})^k$ terms with $k \geq 3$ in the effective action would not change anything in such a fluctuation analysis about an initial Ricci-flat background because a second variation would leave untouched $R_{\mu\nu}$ factors in the fluctuation equation, which would consequently all vanish.

Another extension of focus would be to consider solutions to the theories in dimensions $n > 4$ that are asymptotically maximally symmetric but not flat. As one can see from Equation (2.3), solutions with $R = \text{const} \neq 0$ also exist, and one can have black-hole solutions with such asymptotic geometry. Given the multiple vacua, stability of such solutions needs to be investigated in each vacuum separately. This question is likely to be more involved than the known [29] stability properties of the individual vacua themselves.

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