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Phys. Rev. D **96**, 045005 — Published 7 August 2017

DOI: [10.1103/PhysRevD.96.045005](https://doi.org/10.1103/PhysRevD.96.045005)

Fluctuations in quantum mechanics and field theories from a new version of semiclassical theory. II

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This is the second paper on semiclassical approach based on the density matrix given by the Euclidean time path integral with fixed coinciding endpoints. The classical path, interpolating between this point and the classical vacuum, called “flucton”, plus systematic one- and two-loop corrections, has been calculated in the first paper [1] for double-well potential and now extended for a number of quantum-mechanical problems (anharmonic oscillator, sine-Gordon potential). The method is based on systematic expansion in Feynman diagrams and thus can be extended to QFTs. We show that the loop expansion in QM reminds the leading log-approximations in QFT. In this sequel we present complete set of results obtained using this method in unified way. Alternatively, starting from the Schrödinger equation we derive a *generalized* Bloch equation which semiclassical-like, iterative solution generates the loop expansion. We re-derive two loop expansions for all three above potentials and now extend it to three loops, which has not yet been done via Feynman diagrams. All results for both methods are fully consistent with each other. Asymmetric (tilted) double-well potential (non-degenerate minima) is also studied using the second method.

I. INTRODUCTION

Semiclassical approximations are well known tools, both in quantum mechanical and quantum field theory. Standard textbooks of quantum mechanics usually start with Bohr-Sommerfeld quantization conditions, and semiclassical WKB approximation for the wave function. Unfortunately, extending such methods beyond of the first correction for the one-dimensional case or those with separable variables, or to a multidimensional case, proved to be difficult. In mathematical physics a general construction for multidimensional complexified saddle points are related with its extrema and the network of the so called Lefschetz thimbles (solutions of the gradient flow equations) connecting them. Applications of such theory in path integral context have been investigated by Witten [4], now followed by a significant literature. However, most of the paths considered below are real, and we will only turn to complexified paths at the end of the paper, in section V B. We will not discuss solutions of the thimble kind.

Semiclassical methods which start with Feynman path integrals, on the other hand, are applicable not only for many dimensions or many-body applications, but even for quantum field theories. Among early applications of those is e.g. the paper by Rossi and Testa [5] who used expansion around classical paths.

The so called instanton calculus has been developed for gauge theories and other models. Instantons are indeed responsible for many important phenomena in QFTs, which we cannot discuss here. From technical point of view however, so far in none of these QFT applications the calculations progressed beyond one loop.

In our previous paper [1], to be referred below simply as I, we discussed a different version of the semiclassical approach, based on Feynman’s path integral representation of the density matrix [2, 3] analytically continued to imaginary (Euclidean) time. It corresponds to a transition from quantum mechanics to statistical mechanics: If the time is defined as a periodic variable, with period $\beta = \hbar/T$, the density matrix corresponds to quantum ensemble at nonzero temperature T . The paths, classical and quantum to be considered in this settings, thus need to have the same periodicity. However, in this paper we will only focus on the zero temperature limit, in which the period becomes infinite.

At the classical level, the theory is based on a classical (minimal action) periodic path, which extends from some arbitrary point x_0 to the “classical vacuum”, the minimum of the potential, and return. This path has been introduced

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in [6] and was named “flucton”. Only recently, we have learned that essentially the same construction has been proposed later on in [7].

General advantage of this approach is that the path integrals lead to systematic perturbative series, in the form of Feynman diagrams, with clear rules for each order. Text-book perturbative approaches for the wave functions do not have that, and basically are never used beyond say first and second orders.

Of course, the higher level of generality comes with a heavy price. While classical part is relatively simple, already quantum part at one-loop level one needs to calculate determinants of certain differential operators. At two and more loops Feynman diagrams need to be evaluated on top of space-time dependent backgrounds: therefore those should be evaluated in the space-time representation rather than in energy-momentum one mostly used in QFT applications. Most content of the first part of this paper is the explicit demonstration of how one can do all that, in analytic form, for three classical examples – quartic anharmonic and double-well, and sine-Gordon (Mathieu) potentials. Their quantum Hamiltonian is of the standard form

$$\mathcal{H} = -\frac{1}{2m}\partial_x^2 + V(x), \quad \partial_x = \frac{d}{dx}, \quad (1)$$

where we use units $\hbar = 1$ and $m = 1$, without a loss of generality.

Since our ultimate aim remains a generalization of the semiclassical theory to QFT’s, our quantum-mechanical examples should be represented in a certain specific form of anharmonic perturbation of the harmonic oscillator

$$V(x) = \frac{\tilde{V}(gx)}{g^2} = \frac{1}{2}x^2 + a_3 gx^3 + a_4 g^2 x^4 + \dots \quad (2)$$

where \tilde{V} has a minimum at $x = 0$, it always starts from quadratic terms, the frequency of the small oscillation near minimum placed equal to one, $\omega = 1$ and g is the coupling constant. Classical (vacuum) energy is always taken to be zero, $V(0) = 0$ and $a_{2,3,\dots}$ are parameters. We call (gx) the *classical* coordinate, see below. Both the classical coordinate and the Hamiltonian (1) are invariant with respect to simultaneous change

$$x \rightarrow -x \quad , \quad g \rightarrow -g .$$

It implies that the energy is the function of g^2 ,

$$E = E(g^2) . \quad (3)$$

The semiclassical expansion is done in powers of small coupling g . In a way it is similar to the perturbation theory which is also in powers of small coupling g .

Let us indicate potentials we are going to study in the form (2):

Quartic AnHarmonic Oscillator (AHO),

$$V = \frac{1}{2}x^2(1 + g^2x^4), \quad (4)$$

Quartic Double Well Potential (DWP),

$$V = \frac{1}{2}x^2(1 - gx)^2, \quad (5)$$

Sine-Gordon (Mathieu) Potential (SGP),

$$V = \frac{(1 - \cos gx)}{g^2}, \quad (6)$$

and eventually,

Quartic Asymmetric (tilted) Double Well Potential (ADWP):

$$V = \frac{1}{2}x^2(1 + 2tgx + g^2x^2), \quad (7)$$

where the parameter t “measures” the asymmetry of wells, for $t = \pm 1$ the wells are symmetric and we arrive at DWP, while for $t = 0$ there occurs AHO.

In our previous paper [1] we used Feynman diagrams to calculate one and two-loop corrections to classical flucton action for DWP potential. For two other famous quantum-mechanical problems, the AHO and SGP, we had only

presented the derivation of the Green functions and the (one-loop) determinants. For completeness, here we also add the results for the two-loop corrections, calculated from the same set of Feynman diagrams, for those two problems. Like for DWP case the complete set of results looks surprisingly simple and compact. Remarkably, it does not contain any transcendental functions, logs and polylogs, which appear for individual diagrams. Furthermore, the classical (flucton) action is always the WKB action, $\int pdq$, but evaluated at *zero* energy - this general observation was missed before. To clarify a meaning of loop expansion we will be able to derive a certain Bloch-type, Riccati equation the iteration solution of which generates exactly the loop expansion. This equation will be called *the generalized Bloch equation*.

II. FLUCTUATION CORRECTIONS FROM THE FEYNMAN DIAGRAMS

The method has been extensively described in I and there is no need to repeat it here in detail. Its main idea is that quantum fluctuations around classical flucton path x_{flucton} can be described by standard expansion of the action in the powers of $(x - x_{\text{flucton}})$, with quadratic term giving by the Green function while the higher order terms produce vertices of the Feynman diagrams. Let us only remind in brief the main definitions.

A. Generality

By definition the Feynman path integral gives the density matrix in quantum mechanics [2]

$$\rho(x_i, x_f, t_{tot}) = N \int_{x(0)=x_i}^{x(t_{tot})=x_f} Dx(t) e^{iS[x(t)]/\hbar}. \quad (8)$$

Here N is a normalization factor and S is the usual classical action of the problem,

$$S = \int_0^{t_{tot}} dt \left[\frac{m}{2} \left(\frac{dx}{dt} \right)^2 - V(x) \right],$$

for a particle of mass m in a static potential $V(x)$ - it provides the weight of the paths in (8).

As it is well known, see e.g.[3], one can also apply these expressions in statistical mechanics. For this one needs to change time into its Euclidean version $\tau = it$ defined on a circle with circumference $\beta = \tau_{tot}$. Such periodic time is known as the Matsubara time, and the density matrix of quantum system is related to probability for thermal system with temperature

$$T = \hbar/\beta. \quad (9)$$

At $T \rightarrow 0$ the ground state of the quantum system is naturally recovered. Periodicity of the path implies that there is only one endpoint $x_i = x_f = x_0$.

The main object of our study is the diagonal matrix element of the density matrix, giving the probability for the specific coordinate value x_0 (or a particular field configuration $\phi_0(\vec{x})$ in QFT) in this ensemble

$$P(x_0, \beta) = N \int_{x(0)=x_0}^{x(\beta)=x_0} Dx(\tau) e^{-S_E[x(\tau)]/\hbar}. \quad (10)$$

So, we take into account all (closed) trajectories starting and ending at x_0 . Here the weight is defined via the Euclidean action

$$S_E = \int_0^\beta d\tau \left[\frac{m}{2} \left(\frac{dx}{d\tau} \right)^2 + V(x) \right].$$

Using standard definition of the density matrix in terms of stationary states $|n\rangle$ with energy E_n , the sum over states becomes a set of decreasing exponentials

$$P(x_0, \beta) = \sum_n |\psi_n(x_0)|^2 e^{-E_n \beta}. \quad (11)$$

In the limit of large β or low temperature T , in the expression (11) the dominant term

$$P(x_0, \beta \rightarrow \infty) \sim |\psi_0(x_0)|^2 e^{-E_0 \beta}, \quad (12)$$

describes the ground state, the main state we are interested in.

B. The classical path - flucton

We assume for simplicity that the potential (2) has a global minimum at $x = 0$,

$$V(x) \geq 0 \quad \text{and} \quad \frac{d}{dx}V|_{x=0} = 0 .$$

thus, the exponent in (11) is non-negative, $S_E \geq 0$.

In Euclidean time τ the kinetic energy changes sign, which is equivalent to the potential effectively flipping sign, $V(x) \rightarrow -V(x)$, turning a minimum into a maximum. Now, let us ask if there exist a real path, starting at some arbitrary point x_0 at $\tau = 0$ and returning to it after the required time duration, at the Matsubara time $\tau = \beta$. The lowest action path of this kind is the classical path we call *flucton*. Its energy is defined by its period.

Since in this work we deal only with quantum mechanical limit of vanishing temperature $T \rightarrow 0$, the Matsubara time goes to infinity. It is clear then that the particle should spend a divergently long time near the turning point, which is the case when $x_t \rightarrow 0$, the location of the maximum of $-V$, see Fig.1. Evidently, such classical path with infinite period is the one with zero energy $E = 0$. The basic idea is that such classical path with zero energy $E = 0$ "climbs up the hill" to its maximum at $x = 0$.

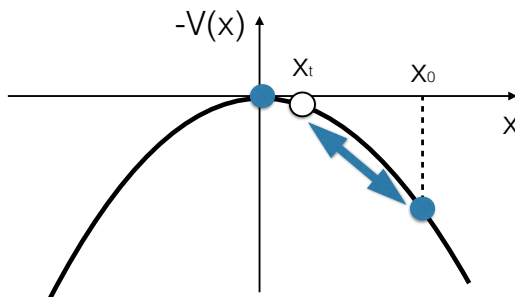


FIG. 1: The sketch of the inverted potential $-V$ versus coordinate x . Flucton is the classical trajectory starting and ending at the same initial point x_0 . At non-zero temperature it goes through the turning point x_t , see text. At zero temperature x_t coincides with the location of the maximum, $x_t = 0$.

Let us find the flucton paths explicitly. They, of course, satisfy the second order classical Equation Of Motion (EOM), but in the one-dimensional case it is much easier to use the energy conservation, at $E = 0$

$$\frac{d}{d\tau}x(\tau) = \sqrt{2V(x)} . \quad (13)$$

The circular trajectory emerging in (13) which starts and ends at x_0 passing through $x = 0$ for the time β is called a *flucton* [6],

$$x_{\text{flucton}}(0) = x_{\text{flucton}}(\beta) = x_0 ,$$

$$x_{\text{flucton}} = x_{\text{flucton}}(\tau; x_0, \beta) .$$

It enables us to evaluate the transition amplitude $P(x_0, \beta)$ (10). Putting

$$\phi(x_0, \beta) \equiv -\log[P(x_0, \beta)] ,$$

in (12) and expanding the classical action around the flucton we obtain

$$\phi(x_0, \beta) =$$

$$S_{\text{flucton}} + \frac{1}{2} \log(N^{-2} \det(O_{\text{flucton}})) + \text{loops} , \quad (14)$$

where $S_{\text{flucton}} \equiv S_E[x_{\text{flucton}}]$ and

$$O_{\text{flucton}} \equiv -\frac{d^2}{d^2\tau} + \frac{\partial^2}{\partial x^2} V(x)|_{x=x_{\text{flucton}}(\tau; x_0, \beta)}, \quad (15)$$

is a Schrödinger-type operator in τ variable with x_0, β as parameters.

In order to construct the loop expansion (14) three building blocks are used:

- (i) The action S_{flucton} of the flucton. In the limit $\beta \rightarrow \infty$, the action S_{flucton} provides the dominant term of the phase of the ground state function (see (12)) and reproduces exactly the WKB result (obtained from the Riccati-Bloch equation for the phase).
- (ii) The determinant of O_{flucton} (15) describes the quadratic quantum fluctuations.
- (iii) The loop-corrections, true quantum corrections decreasing at large distances and in the present formalism given by explicit Feynman diagrams in the flucton-background.

While the computation of S_{flucton} (14) is relatively simple, the evaluation of $\det O_{\text{flucton}}$ already involves the diagonalization of a certain non-trivial second-order differential operator. Usually, it is a highly non-trivial calculation which is enormously simplified by the generalized Riccati-Bloch equation, see below.

While in our paper I we discussed to some extent several quantum mechanical problems, only for the double well potential (DWP) the calculations have been done to two loops. It is clear now that the formalism can be applied for *any* potential of a type of perturbed harmonic oscillator (2). Therefore, without explanations, we now list simply the full set of the results, for AHO, DWP and SGP potentials and also for ADWP. The units used assume particle mass $m = 1$ and the Planck constant $\hbar = 1$.

1. Relating the determinant and the Green function

In this section we calculate the quadratic order quantum oscillations around classical (flucton) path, namely, the determinant (14). For the harmonic oscillator, the potential $\partial^2 V(x)/\partial x^2|_{x=x_{\text{flucton}}}$ in (15) is just a constant, so in this case the fluctuations do not depend on the classical path, and direct diagonalization of the operator (15) [8] shows that

$$N^{-2} \text{Det}(O_{\text{flucton}}) = 2\pi \sinh \beta.$$

In general, a direct diagonalization of (15) is highly-non trivial and analytical results for it are extremely rare.

At 1978 Brown and Creamer [9] invented the way how to relate the determinant and the Green function reducing it to calculation of a symbolic one-loop Feynman diagram. One can apply their procedure to quantum mechanics. When the potential $V_{\text{flucton}} \equiv V(x_{\text{flucton}}(\tau; x_0, \beta))$ depends on some parameter, it can be varied. To this end, we rewrite the potential as

$$V_{\text{flucton}} = 1 + W(\tau; X, \beta),$$

where $X = X(x_0)$. Its variation resulting in an extra potential

$$\delta V_{\text{flucton}} = \frac{\partial W}{\partial X} \delta X \quad (16)$$

is a perturbation: its effect can be evaluated by the following Feynman diagram

$$\frac{\partial \log \text{Det}(O_{\text{flucton}})}{\partial X} = \int d\tau G(\tau, \tau) \frac{\partial V_{\text{flucton}}(\tau)}{\partial X}, \quad (17)$$

containing derivative of the potential as a vertex and the ‘‘loop’’-Green function $G(\tau, \tau)$,

$$O_{\text{flucton}} G(\tau_1, \tau_2) = \delta(\tau_1 - \tau_2), \quad (18)$$

at the same point $\tau_1 = \tau_2 = \tau$, see Fig.2. For simplicity, the dependence of $G(\tau_1, \tau_2)$ on x_0 and β is omitted. The equation (17) relates the determinant and the Green function: if the r.h.s. of it can be calculated, the derivative over X can be integrated back. Hence, if the Green function is known, one can calculate the determinant.

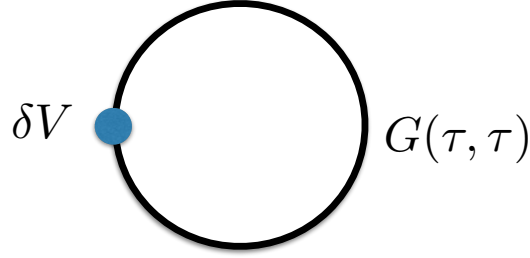


FIG. 2: Symbolic one-loop diagram, including variation of the fluctuation potential δV and the simplified “single-loop” Green function $G(\tau, \tau)$, see [1].

2. Two-loop-correction: Feynman diagrams

The loop expansion for $P(x_0, \beta)$ can be written in the form

$$\text{loops} = 1 + g^2 B_1 + g^4 B_2 + \dots \quad (19)$$

where $B_n = B_n(x_0)$ is the n -loop contribution. Equivalently, in (14) the loop expansion is of the form

$$\text{loops} = g^2 B_1 + g^4 \tilde{B}_2 + \dots \quad (20)$$

where $\tilde{B}_n = \tilde{B}_n(x_0)$, $n > 1$ is made out of loop contributions B_n , e.g. $\tilde{B}_2 = -B_1^2/2 + B_2$ etc.

In general, for any potential of the form (2) the two-loop correction B_1 is given by the sum of three Feynman diagrams, see Fig. 3,

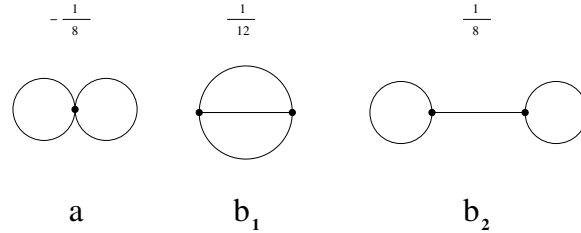


FIG. 3: Diagrams contributing to the two-loop correction $B_1 = a + b_1 + b_2$. The signs of contributions and symmetry factors are indicated.

where diagram a is given by a one-dimensional integral while the diagrams b_1 and b_2 correspond to two-dimensional integrals. Explicitly,

$$a \equiv -\frac{1}{8} \int_0^\infty [v_4(\tau) G^2(\tau, \tau) - v_{4,0} G_0^2(\tau, \tau)] d\tau,$$

$$b_1 \equiv \frac{1}{12} \int_0^\infty \int_0^\infty [v_3(\tau_1) v_3(\tau_2) G^3(\tau_1, \tau_2) - v_{3,0} v_{3,0} G_0^3(\tau_1, \tau_2)] d\tau_1 d\tau_2,$$

$$b_2 \equiv \frac{1}{8} \int_0^\infty \int_0^\infty [v_3(\tau_1) v_3(\tau_2) G(\tau_1, \tau_1) G(\tau_1, \tau_2) G(\tau_2, \tau_2)]$$

$$- v_{3,0} v_{3,0} G_0(\tau_1, \tau_1) G_0(\tau_1, \tau_2) G_0(\tau_2, \tau_2)] d\tau_1 d\tau_2 , \quad (21)$$

where

$$v_k(\tau) = \frac{\partial^k}{\partial x^k} V(x)|_{x=x_{\text{flucton}}} , \quad k = 3, 4$$

are the vertices in the flucton background,

$$v_{k,0} = \frac{\partial^k}{\partial x^k} V(x)|_{x=0} , \quad k = 3, 4$$

denote the ‘‘vacuum vertices’’, the Green function $G(\tau_1, \tau_2)$ is defined in (18) and

$$G_0 = \frac{e^{-|\tau_1 - \tau_2|}}{2} - \frac{e^{-\tau_1 - \tau_2}}{2} , \quad (22)$$

is the ‘‘harmonic propagator’’. Its presence is related with the necessity to subtract (space-time divergent) anharmonic effects, unrelated to the fluctons.

In the case of three-loop contribution for general potential (2) there exist 15 diagrams which contribute, see e.g. [11], while for DWP the number of diagrams drops to 12, see e.g. [10]. Like in instanton calculus we do not hope all these diagrams can be calculated analytically.

C. AHO

This section contains the expansion (14) up to two-loops for AHO.

The *anharmonic oscillator* (AHO) is defined by the potential (4)

$$V(x) = \frac{1}{2} x^2 (1 + g^2 x^2) = \frac{1}{2g^2} u^2 (1 + u^2) , \quad u = gx ,$$

at zero temperature, $T = 0$. The classical flucton path solution with the energy $E = 0$ of (13) is

$$x_{\text{flucton}}(\tau) = \frac{g x_0}{\cosh(|\tau|) + \sqrt{1 + g^2 x_0^2} \sinh(|\tau|)} .$$

The corresponding classical action is given by

$$S_E[x_{\text{flucton}}] = \frac{2}{3} \frac{(1 + g^2 x_0^2)^{\frac{3}{2}} - 1}{g^2} .$$

In the limit $g \rightarrow 0$ we recover the classical action for the harmonic oscillator and at $x_0 \rightarrow \infty$ we obtain the expansion

$$S_E[x_{\text{flucton}}] = \frac{2}{3} g x_0^3 + \frac{1}{g} x_0 + \text{lower order terms} \quad (23)$$

It is convenient to introduce a new variable

$$X_{AHO}(x) = (1 + g^2 x^2)^{\frac{1}{2}} = (1 + u^2)^{\frac{1}{2}} , \quad u = gx . \quad (24)$$

For the anharmonic oscillator, the Green function of the operator O_{flucton} (15) is given by

$$\begin{aligned}
G(\tau_1, \tau_2; X_{AHO}) &= \frac{\operatorname{sech}[\frac{1}{2}(-|\tau_-| + \tau_+)] (X_{AHO} \cosh[\frac{1}{2}(|\tau_-| + \tau_+)] + \sinh[\frac{1}{2}(|\tau_-| + \tau_+)])}{4X_{AHO} (\cosh[\frac{1}{2}(|\tau_-| + \tau_+)] + X_{AHO} \sinh[\frac{1}{2}(|\tau_-| + \tau_+)])^2 (1 + X_{AHO} \tanh[\frac{1}{2}(-|\tau_-| + \tau_+)])^2} \\
&\times \left[X_{AHO} \cosh[|\tau_-| - \tau_+] \left(1 + 3X_{AHO}^2 + X_{AHO}(3 + X_{AHO}^2) \tanh[\frac{1}{2}(-|\tau_-| + \tau_+)] \right) \right. \\
&+ (1 - X_{AHO}^2) (4 - 5X_{AHO}^2 - 3X_{AHO}(|\tau_-| - \tau_+)) \tanh[\frac{1}{2}(-|\tau_-| + \tau_+)] \\
&\left. - X_{AHO} \left(1 + 3X_{AHO} (X_{AHO} + (1 - X_{AHO}^2)(|\tau_-| - \tau_+)) \right) \right]
\end{aligned} \tag{25}$$

where $\tau_- = \tau_2 - \tau_1$, $\tau_+ = \tau_1 + \tau_2$.

Taking the above Green function and the ‘‘vertex’’

$$\frac{\partial V_{\text{flucton}}(\tau)}{\partial X_{AHO}} = \frac{12 (\sinh(\tau) + X_{AHO} \cosh(\tau))}{(\cosh(\tau) + X_{AHO} \sinh(\tau))^3}, \tag{26}$$

($\tau > 0$) to evaluate (17) we obtain analytically

$$\log \operatorname{Det}(O_{\text{flucton}}) = 2 \log[X_{AHO}(1 + X_{AHO})]. \tag{27}$$

As for the next term, the two-loop correction B_1 , the results for all three two-loop Feynman diagrams shown in Fig.3, can be found analytically and they are

$$\begin{aligned}
a &= -\frac{3(32 + 96X + 29X^2 - 74X^3 - 35X^4)}{560X^2(1 + X)^2} \\
&\quad - \frac{9 \left(2(2 - 3X^2) \log(2) - 2(2 - 3X^2) \log[1 + X] - 3X(1 - X^2) \operatorname{PolyLog}\left[2, \frac{X-1}{1+X}\right] \right)}{70X(1 - X^2)}, \\
b_1 &= -\frac{140 + 184X + 272X^2 + 193X^3 - 478X^4 - 455X^5}{840X^3(1 + X)^2} \\
&\quad + \frac{3(2(2 - 3X^2) \log(2) - 2(2 - 3X^2) \log[1 + X] - 3X(1 - X^2) \operatorname{PolyLog}\left[2, \frac{X-1}{1+X}\right])}{35X(1 - X^2)}, \\
b_2 &= -\frac{140 + 528X + 464X^2 - 169X^3 - 626X^4 - 385X^5}{560X^3(1 + X)^2} \\
&\quad + \frac{3(2(2 - 3X^2) \log(2) - 2(2 - 3X^2) \log[1 + X] - 3X(1 - X^2) \operatorname{PolyLog}\left[2, \frac{X-1}{1+X}\right])}{70X(1 - X^2)},
\end{aligned} \tag{28}$$

where $X \equiv X_{AHO}$ and $\operatorname{PolyLog}[n, z] = \sum_{k=1}^{\infty} z^k / k^n$ is the polylogarithm. Each diagram provides a contribution in a form of sum of rational (meromorphic) function and a transcendental function. Functionally, both functions are similar in each diagram. Eventually, after summing all three diagrams a, b_1, b_2 , the two-loop correction B_1 takes an amazingly simple form,

$$B_1^{(AHO)} = \frac{(1 - X_{AHO})(5 + 16X_{AHO} + 25X_{AHO}^2 + 17X_{AHO}^3)}{12X_{AHO}^3(1 + X_{AHO})}, \tag{29}$$

where all transcendental contributions are *cancelled out(!)* and the answer turns out to be the meromorphic function of X_{AHO} only. We will observe similar cancellations for DWP and SGP below.

D. DWP

The *double well potential* we define as in (5)

$$V = \frac{1}{2}x^2(1-gx)^2 = \frac{1}{2g^2}u^2(1-u)^2, \quad u = gx.$$

Its two degenerate minima are situated at $x = 0$ and $x = 1/g$, respectively. At zero temperature limit $T = 1/\beta = 0$, the flucton trajectory is given by [1]

$$x_{\text{flucton}}(\tau) = \frac{x_0(\cosh(|\tau|) - \sinh(|\tau|))}{1 + gx_0(1 - \cosh(|\tau|) + \sinh(|\tau|))},$$

and the corresponding classical action reads

$$S_E[x_{\text{flucton}}] = x_0^2 \left(1 + \frac{2gx_0}{3}\right).$$

For this case it is convenient to introduce the variable

$$X_{DWP}(x) \equiv u = gx, \quad (30)$$

c.f. (24). In this variable the corresponding Green function takes the form

$$\begin{aligned} G(\tau_1, \tau_2) = & \frac{e^{-|\tau_1 - \tau_2|}}{2(e^{\tau_1}(1+X) - X)^2(e^{\tau_2}(1+X) - X)^2} \left[8e^{\frac{1}{2}(\tau_1 + \tau_2 + 3|\tau_1 - \tau_2|)} X^3(1+X) \right. \\ & - 8e^{\frac{1}{2}(3\tau_1 + 3\tau_2 + |\tau_1 - \tau_2|)} X(1+X)^3 + e^{2(\tau_1 + \tau_2)}(1+X)^4 - 6e^{(\tau_1 + \tau_2 + |\tau_1 - \tau_2|)} X^2(1+X)^2 |\tau_1 - \tau_2| \\ & \left. + e^{(\tau_1 + \tau_2 + |\tau_1 - \tau_2|)} \left(6X^4(\tau_1 + \tau_2) + 12X^3(1 + \tau_1 + \tau_2) + 6X^2(3 + \tau_1 + \tau_2) + 4X - 1 \right) \right. \\ & \left. - e^{2|\tau_1 - \tau_2|} X^4 \right], \quad (31) \end{aligned}$$

for $\tau_1, \tau_2 > 0$, where $X = X_{DWP}$.

Substituting (31) and the ‘‘vertex’’

$$\frac{\partial V_{\text{flucton}}(\tau)}{\partial X_{DWP}} = \frac{6e^\tau(X_{DWP} + e^\tau(1 + X_{DWP}))}{(e^\tau(1 + X_{DWP}) - X_{DWP})^3}. \quad (32)$$

in the r.h.s. (17) gives

$$\frac{\partial \log \text{Det}(O_{\text{flucton}})}{\partial X_{DWP}} = \frac{4}{1 + X_{DWP}}, \quad (33)$$

c.f. (27).

The results for all three two-loop Feynman diagrams shown in Fig.3 can be found analytically

$$\begin{aligned} a = & \frac{3}{560X^2(1+X)^4} \times \left(24X - 60X^2 - 520X^3 - 1024X^4 - 832X^5 - 245X^6 \right. \\ & \left. - 24(1+X)^2(1 - 4X - 18X^2 - 12X^3) \log[1+X] + 288X^2(1+X)^4 \text{PolyLog}\left[2, \frac{X}{1+X}\right] \right), \end{aligned}$$

$$\begin{aligned}
b_1 &= -\frac{1}{280X^2(1+X)^4} \times \left(24X - 60X^2 - 520X^3 - 1024X^4 - 832X^5 - 245X^6 \right. \\
&\quad \left. -24(1+X)^2(1-4X-18X^2-12X^3) \log[1+X] + 288X^2(1+X)^4 \text{PolyLog}\left[2, \frac{X}{1+X}\right] \right), \\
b_2 &= -\frac{1}{560X^2(1+X)^4} \times \left(24X - 60X^2 + 1720X^3 + 5136X^4 + 4768X^5 + 1435X^6 \right. \\
&\quad \left. -24(1+X)^2(1-4X-18X^2-12X^3) \log[1+X] + 288X^2(1+X)^4 \text{PolyLog}\left[2, \frac{X}{1+X}\right] \right).
\end{aligned} \tag{34}$$

where $X = X_{DWP}$.

Again, the full two-loop correction takes an amazingly simple form,

$$B_1^{(DWP)} = -X_{DWP} \frac{(4 + 3X_{DWP})}{(1 + X_{DWP})^2}. \tag{35}$$

c.f. (29), and all transcendental contributions are again cancelled out, and the answer is the meromorphic function of X_{DWP} only [1].

E. SGP

In this Section we consider the *sine-Gordon potential* (6)

$$V = \frac{1}{g^2}(1 - \cos(gx)) = \frac{1}{g^2}(1 - \cos u), \quad u = gx,$$

with infinite number of degenerate vacua situated periodically in x .

For this potential, the flucton at $T = 0$ takes the form

$$x_{\text{flucton}}(\tau) = \frac{4 \operatorname{arccot} \left[(\cosh \tau - \sinh \tau) \cot\left(\frac{gx_0}{4}\right) \right]}{g}.$$

The classical flucton action gives

$$S_E[x_{\text{flucton}}] = \frac{16 \sin^2\left(\frac{gx_0}{4}\right)}{g^2}.$$

For the SGP, we introduce the variable

$$X_{SGP}(x) = \frac{gx}{4} = \frac{u}{4}, \quad u = gx, \tag{36}$$

c.f. (24), (30).

Standard construction yields the following Green function

$$\begin{aligned}
G(\tau_1, \tau_2) &= \frac{1}{8(\cosh(\tau_1) + \cos(2X) \sinh(\tau_1))} \times \frac{1}{(\cosh(\tau_2) + \cos(2X_3) \sinh(\tau_2))} \times \\
&\quad \left[2(\tau_1 + \tau_2 - |\tau_2 - \tau_1|) \sin^2(2X) + 8 \cos(2X) \sinh^2\left(\frac{1}{2}(\tau_1 + \tau_2 - |\tau_2 - \tau_1|)\right) \right. \\
&\quad \left. + (3 + \cos(4X)) \sinh(\tau_1 + \tau_2 - |\tau_2 - \tau_1|) \right], \tag{37}
\end{aligned}$$

for $\tau_1, \tau_2 > 0$, here $X = X_{SGP}$.

In this case, the ‘‘vertex’’

$$\frac{\partial V_{\text{flucton}}(\tau)}{\partial X_{SGP}} = -\frac{16e^{2\tau} \sec(X_{SGP})^2 \tan(X_{SGP})(e^{2\tau} - \tan(X_{SGP})^2)}{(e^{2\tau} + \tan(X_{SGP})^2)^3},$$

and direct evaluation of (17) leads to the amazingly simply expression

$$\frac{\partial \log \text{Det}(O_{\text{flucton}})}{\partial X_{SGP}} = 4 \tan[X_{SGP}], \quad (38)$$

c.f. (27), (33).

As for the three two-loop Feynman diagrams shown in Fig.3 we are able to calculate analytically the diagram a only,

$$a = -\text{Re} \left[\frac{1}{640} \left(5 - 2 \sec^2(X_{SGP}) + \sec^4(X_{SGP}) \right. \right. \\ \left. \left. - 8 \text{PolyLog}[2, -\tan^2(X_{SGP})] + 8 [\csc^2(X_{SGP}) - \sec^2(X_{SGP})] \log[\cos(X_{SGP})] \right) \right], \quad (39)$$

here $\text{Re}[x]$ denotes the real part of x . Irrational contributions occur again (see the 2nd line of (39)) like at AHO (28) and DWP (34) cases. In no way we were able to calculate $b_{1,2}$ contributions analytically. However, these two-dimensional Feynman integrals we can calculate numerically. We make two assumptions: (i) all irrational contributions cancel in the sum $B_1 = a + b_1 + b_2$, see the 2nd line of (39), and (ii) the sum $B_1 = a + b_1 + b_2$ is given by a polynomial in $\sec^2(X)$ of degree two,

$$A - D \sec^2(X) + C \sec^4(X),$$

c.f. (39), the 1st line with coefficients A, D, C . Based on these assumptions we fit the numerical data of B_1 and find that these coefficients A, D, C can be calculated explicitly. It leads to a very simple expression for the two-loop correction

$$B_1^{(SGP)} = -\frac{g^2}{16} \tan^2(X_{SGP}). \quad (40)$$

Eventually, this expression is verified numerically. Later on this result will be derived in quantum mechanics using the generalized Bloch equation.

F. The results summarized

The combined results from Sections C, D, E show that the expansion (14) of the phase of the ground state function for AHO, DWP and SGP is given explicitly by

$$2\phi^{(AHO)}(g, x) = \frac{2}{3} \frac{(1 + g^2 x^2)^{\frac{3}{2}} - 1}{g^2} + \log \left[\frac{(1 + g^2 x^2 + \sqrt{1 + g^2 x^2})}{2} \right] \\ + g^2 \frac{(1 - \sqrt{1 + g^2 x^2}) \left(5 + 16\sqrt{1 + g^2 x^2} + 25(1 + g^2 x^2) + 17(1 + g^2 x^2)^{\frac{3}{2}} \right)}{12(1 + g^2 x^2)^{\frac{3}{2}} (1 + \sqrt{1 + g^2 x^2})} + \dots \\ 2\phi^{(DWP)}(g, x) = \frac{1}{g^2} \left((gx)^2 + \frac{2(gx)^3}{3} \right) + 2 \log(1 + gx) - g^2 \frac{(gx)(4 + 3gx)}{(1 + gx)^2} + \dots \\ 2\phi^{(SGP)}(g, x) = \frac{16}{g^2} \sin^2\left(\frac{gx}{4}\right) + 2 \log\left[\cos\left(\frac{gx}{4}\right)\right] - \frac{g^2}{16} \tan^2\left(\frac{gx}{4}\right) + \dots \quad (41)$$

which is the Laurant expansion in powers of g^2 if the variable $u = gx$ is introduced.

III. RE-DERIVATION OF THE LOOP EXPANSION FROM THE SCHRÖDINGER EQUATION

Our ultimate aim remains a generalization of the semiclassical theory to QFT's, thus, all our quantum-mechanical examples should be written as anharmonic perturbation of the harmonic oscillator,

$$V = \frac{\tilde{V}(gx)}{g^2},$$

see (2), where \tilde{V} has a minimum at $x = 0$ and it always starts from quadratic terms. Classical vacuum energy is always taken to be zero, $V(0) = 0$. Note that \hbar was put to one, as it is traditionally done in QFT's. The semiclassical expansion is done in powers of small coupling g instead of the powers of \hbar . It should be in the agreement with the so called non-perturbative normalization of the non-Abelian gauge theory, in where the coupling appears *only* in front of the action.

However, in quantum-mechanical setting traditional units are different, and in the next subsection we show how one can reformulate these results as an expansion in the powers of the Planck constant \hbar .

A. Quantum-mechanical meaning of the loop expansion: generalized Bloch equation

In order to clarify the meaning of the semiclassical loop expansion in quantum mechanics, it is convenient to change notations as follows. Taking the AHO potential (4) as the example, we remind the units used. The mass of the particle $m = 1$, the frequency of the near-minimum oscillations $\omega = 1$ and the Planck constant $\hbar = 1$ are all put to unity. Now we want to restore \hbar in the exponent of the quantum (statistical) weight $\exp(-S_E/\hbar)$, and make a shift to the “classical coordinate” $u = gx$. Now the Euclidean action looks as follows

$$S_E = \frac{1}{\hbar g^2} \int d\tau \left(\frac{\dot{u}^2}{2} + \frac{u^2(1+u^2)}{2} \right). \quad (42)$$

In other words, we have selected different unit of length, eliminating the parameter in the nonlinear, quartic term. The coupling constant now appears only together with the Planck constant. So, one can put it to one, $g = 1$, using only \hbar as a parameter of the loop semiclassical expansion. The classical equation of motion does *not* depend on it, and so is the flucton solution itself. Furthermore, the Green function – inverting the operator of quantum fluctuations around the flucton – depends on the classical coordinate $u = gx$ but does *not* depend on \hbar . Similar consideration can be made for a general potential (2) and we arrive at the Euclidean action

$$S_E = \frac{1}{\hbar g^2} \int d\tau \left(\frac{\dot{u}^2}{2} + \tilde{V}(u) \right). \quad (43)$$

The parameters of the problem can thus be defined as (i) the quantum parameter $\hbar g^2$ (or just \hbar for the $g = 1$ choice) and (ii) the classical coordinate location $u_0 = g x_0$ under consideration. *The loop expansion of the semiclassical theory we discuss is therefore redefined as just the Laurant expansion in powers of $\hbar g^2$* , starting from the classical term $O(1/\hbar g^2)$, the determinant $O((\hbar g^2)^0)$, the two-loop diagrams $O((\hbar g^2)^1)$, and so on. For $g = 1$ the loop expansion appears as the Laurant expansion in \hbar . Naturally, its validity is expected when $S_{flucton}/(\hbar g^2) \gg 1$, thus, at small $\hbar \ll 1$ and/or sufficiently large u_0 . Below we quantify the accuracy of this expansion in details.

B. Iterative solution of the Schrödinger equation

In this section we re-derive first three terms of the loop expansion (14), and derive one more term, based on quantum mechanics, employing the usual Schrödinger equation for the wave function. Let us stress, that this is the only part of our program which cannot be generalized to QFT straightforwardly, at least, so far. It allows us to cross-check of the results obtained in loop expansion.

The first step is standard, we proceed from the Schrödinger equation on the wave function to that of its logarithmic derivative, which eliminates the overall normalization constant from consideration. The second step is that we extract one power of coordinate from the function

$$x z(gx) = -\frac{\psi'(x)}{\psi(x)}. \quad (44)$$

It reflects the fact that since it is assumed the original potential (2) has minimum at $x = 0$ the logarithmic derivative of wavefunction (the derivative of the phase) has to vanish at $x = 0$. Substituting it to the Schrödinger equation

$$\left(-\frac{1}{2} \frac{d^2}{dx^2} + V(x) \right) \psi(x) = E \psi(x) ,$$

where the Planck constant is placed equal to one, $\hbar = 1$, one gets the following equation on $z(gx)$

$$g x z'(gx) + z(gx) - x^2 z(gx)^2 = 2E - \frac{2}{g^2} \tilde{V}(gx) . \quad (45)$$

Now we redefine the coordinate $u = gx$ and obtain the form of the equation we will be solving,

$$g^2 u z'(u) + g^2 z(u) - u^2 z(u)^2 = 2g^2 E - 2\tilde{V}(u) . \quad (46)$$

We call this equation *the generalized Bloch equation*. Here $z(u)$ has a meaning of *reduced* logarithmic derivative. Now we proceed to solving the equation (46).

C. Weak coupling expansions

1. Riccati-type equation

Let us now re-introduce logarithmic derivation

$$y(x) = -\frac{\psi'(x)}{\psi(x)}$$

and write a standard Riccati equation for the potential (2),

$$y' - y^2 = 2E - x^2 - 2a_3 g x^3 - 2a_4 g^2 x^4 - \dots , \quad (47)$$

cf. (45), instead of the Schrödinger equation, as in *non-linearization procedure* [14], [12]. Now we develop a perturbation theory in powers of g ,

$$E = \sum_0^{\infty} e_m g^m , \quad y(x) = \sum_0^{\infty} y_m(x) g^m .$$

It is evident that all e_m with odd m should vanish, see (3), $e_{2k+1} = 0$, $k = 0, 1, 2, \dots$. Unperturbed solution of (47) at $g = 0$ is equal to $E_0 = e_0 = \frac{1}{2}$ and $y_0 = x$.

The equation for the correction of the order g reads,

$$y_1' - 2xy_1 = -2a_3 x^3 ,$$

with solution

$$y_1 = a_3 x^2 + a_3 , \quad e_1 = 0 ,$$

The equation for the correction of the order g^2 ,

$$y_2' - 2xy_2 = e_2 - 2a_4 x^4 + y_1^2 = e_2 + a_3^2 - (2a_4 - a_3^2) x^4 + 2a_3^2 x^2 ,$$

with solution

$$-y_2 = (a_4 - a_3^2/2) x^3 + a_3 x^2 + ax + a , \quad e_1 = 0 ,$$

In general, the equation for m -th correction has the form

$$y_m' - 2y_0 y_m = e_m - Q_m - a_m x^m ,$$

where $Q_m = -\sum_{p=1}^{m-1} y_p y_{m-p}$ for $m > 1$ play a role of effective perturbation potential: it is made from previous iterations. It can be easily demonstrated that *m*-the correction $y_m(x)$ is a finite order polynomial in x and, in principle, it can be found by algebraic means,

$$y_m(x) = A_{m-1}^{(m)} x^{m-1} + \dots + A_k^{(m)} x^k + \dots + A_0^{(m)}. \quad (48)$$

A straightforward analysis leads to the remarkable property

$$A_k^{(m)} \sim \frac{m!}{k!},$$

see [12]. In general, $A_k^{(m)}$ look as generalized Catalan numbers. Hence, the coefficient $A_k^{(m)}$ at fixed $k \sim m$ defines the convergent series in m , while at small fixed k the series is usually divergent. In particular,

$$A_1^{(m)} = e_m.$$

Let us change variable in (48),

$$A_k^{(m)} \rightarrow A_{m-k}^{(m)}, \quad k = 1, 2, \dots, (m-1).$$

It is natural to introduce the generating function

$$\tilde{y}_k(x; \{a\}) = \sum_{m=k}^{\infty} g^m A_{m-k}^{(m)}(\{a\}) x^{m-k}.$$

If the potential (2) is a polynomial, several leading generating functions can be found explicitly, at $k = 1, 2, 3, \dots$. E.g. for AHO ($a_3 = 0, a_4 = 1$ and $a_k = 0, k = 5, 6, \dots$),

$$\tilde{y}_0 = \sum_{m=0}^{\infty} g^m A_m^{(m)} x^m = x (1 + g^2 x^2)^{1/2},$$

see [12], c.f. (41). It can be immediately recognized as the classical momentum at zero energy! Hence, the sum of leading terms of the corrections $y_m(x), m = 0, 1, 2, \dots$ at $x \rightarrow \infty$ is the classical momentum at zero energy: it reminds the leading log approximation in QFT. In the same way one can calculate the sum of next-to-leading terms of the corrections $y_m(x), m = 1, 2, \dots$ at $x \rightarrow \infty$,

$$\tilde{y}_1 = \sum_{m=1}^{\infty} g^m A_{m-1}^{(m)} x^{m-1} =$$

$$g^2 x \frac{1 + \frac{1}{2\sqrt{1+g^2 x^2}}}{(1 + g^2 x^2 + \sqrt{1 + g^2 x^2})},$$

see [12], c.f. (41), which is the logarithmic derivative of the determinant! Hence, the sum of sub-leading (next-after-leading) terms of the corrections $y_m(x), m = 0, 1, 2, \dots$ at $x \rightarrow \infty$ is the logarithmic derivative of the determinant: it reminds the next-to-leading log approximation in QFT. We can move even further and calculate next-after-next-to-leading terms in the corrections $y_m(x), m = 1, 2, \dots$ at $x \rightarrow \infty$ and then sum them up and discover that

$$\tilde{y}_2 = \sum_{m=2}^{\infty} g^m A_{m-2}^{(m)} x^{m-2} = \frac{dB_1}{dx}$$

see [12], c.f. (41). Thus, the result occurs as derivative of two-loop contribution (19), see (29), where $x = X_{AHO}$. It can be checked, see below, that \tilde{y}_3 is the first derivative of three-loop contribution (19).

The property that the first three generating functions $\tilde{y}_{0,1,2}$ are first derivatives of the first three terms in loop expansion holds for DWP. In general, the expansion in such generating functions,

$$y = \tilde{y}_0 + \tilde{y}_1 + \tilde{y}_2 + \dots$$

is a *new* semiclassical expansion, resembling e.g. the logarithmic approximations of QFT.

2. Generalized Bloch equation case

Calculations of the loop expansion for the ground state wave function phase performed earlier for AHO, DWP and SGP show that this expansion looks like a perturbation series in g^2 (starting from $1/g^2$ term) if classical coordinate $u = gx$ is introduced. It is natural to construct this perturbation theory for phase in generalized Bloch equation (46) and compare with the loop calculation.

Solving (46) iteratively we generate the flucton loop expansion. Let us define the series

$$z(u) = \sum_{n=0} g^{2n} z_n(u) , \quad (49)$$

which corresponds to perturbative solution of this equation in powers of g^2 with

$$E = \sum_{n=0} g^{2n} E_n ,$$

given by standard perturbation theory in g^2 .

In the zeroth order, in which all terms proportional to the coupling are ignored, the equation is very simple

$$-u^2 z_0(u)^2 = -2\tilde{V}(u) , \quad (50)$$

leading to

$$z_0(u) = \frac{\sqrt{2\tilde{V}(u)}}{u} . \quad (51)$$

This result (uz_0) is, in fact, the classical momentum at zero energy, and therefore, as one returns to the wave function, the zeroth order term gives the well known semiclassical action. So, at this stage, the result is well known $\psi \sim \exp(-\int^x p(x')dx')$ but at zero energy. It can be immediately checked that classical flucton action for AHO, DWP and SGP, see (41), is nothing but the semiclassical action at zero energy, $\int u z_0(u) du$ with $z_0(u)$ given by (51).

Moving to the next term of the expansion, one finds the following equation $O(g^2)$ for it

$$uz'_0(u) + z_0(u) - 2u^2 z_0(u) z_1(u) = 2E_0 . \quad (52)$$

Note here, that the equation involves the known function z_0 of the previous order, and z_1 just appears linearly. The similar feature takes place in all orders!

Important point of the correct procedure is that the energy needs to be used in the form of perturbative expansion in powers of g^2 as well but for the original potential (2),

$$E = \sum_{n=0} g^{2n} E_n . \quad (53)$$

The zeroth order potential is of the harmonic oscillator, so $2E_0 = 1$. Hence, the first correction

$$z_1(u) = \frac{uz'_0(u) + z_0(u) - 1}{2u^2 z_0(u)} . \quad (54)$$

It can be immediately checked that that the logarithm of determinant $\log \text{Det}(O_{\text{flucton}})$ (17) for all three potentials AHO, DWP and SGP, see (41) is nothing but $\int u z_1(u) du$ with $z_1(u)$ given by (54). Thus, in a very simple way we calculated determinant, $\log \text{Det}(O_{\text{flucton}})$ explicitly (!) in closed analytic form for the general potential $V(x)$ (2). Or, in other words, we calculated explicitly the one-loop diagram of Fig. 2. This result, written in terms of classical flucton action and its derivatives, is one of the central results of this paper.

Moving to the next term of the expansion, one finds the following equation $O(g^4)$ for it

$$uz'_1(u) + z_1(u) - u^2 z_1^2(u) - 2u^2 z_0(u) z_2(u) = 2E_1 . \quad (55)$$

Note here, that the equation involves the known functions of the previous orders $z_{0,1}$ non-trivially, but the new function z_2 appears only linearly. This feature is generic, repeated in each order: so there is no difficulty to find new corrections.

The perturbative coefficient E_1 is the perturbative correction $\sim g^2$ to the ground state energy in the quartic part of the original potential (2),

$$V(x) = \frac{1}{2}x^2 + a_3 gx^3 + a_4 g^2 x^4 ,$$

which can be easily found explicitly e.g. in non-linearization procedure [12],

$$2E_1 = \frac{3}{2}a_4 - \frac{11}{4}a_3^2 .$$

Solving (55), we find the second correction

$$z_2(u) = \frac{uz'_1(u) + z_1(u) - u^2 z_1^2(u) - 2E_1}{2u^2 z_0(u)} , \quad (56)$$

which defines the two-loop contribution B_1 . It can be immediately checked that for AHO, DWP and SGP, see (41) is nothing but $\int u z_2(u) du$ with $z_2(u)$ given by (56). Thus, in a very simple way we calculated two-loop contribution explicitly (!) in closed analytic form for the general potential $V(x)$ (2). Or, in other words, we calculated sum of three two-loop diagrams on Fig. 3, weighted with symmetry factors, explicitly. For AHO, DWP and SGP potentials this sum $\int u z_2(u) du$ with $z_2(u)$ does *not* contain transcendental contributions. It must be emphasized again that it was difficult to guess that such a result can exist in such a generality.

Moving to the next term of the expansion, one finds the following equation $O(g^6)$ for it

$$\begin{aligned} uz'_2(u) + z_2(u) - 2u^2 z_1(u)z_2(u) - 2u^2 z_0(u)z_3(u) \\ = 2E_2 . \end{aligned} \quad (57)$$

Note here, that the equation involves the known functions of the previous orders $z_{0,1,2}$, and again z_3 appears linearly. The perturbative coefficient E_2 is the perturbative correction $\sim g^4$ to the ground state energy in the sextic part,

$$V(x) = \frac{1}{2}x^2 + a_3 gx^3 + a_4 g^2 x^4 + a_5 gx^5 + a_6 g^2 x^6 ,$$

of the general potential (2), which can be straightforwardly found e.g. in non-linearization procedure [12]. Eventually, the third correction

$$z_3(u) = \frac{uz'_2(u) + z_2(u) - 2u^2 z_1(u)z_2(u) - 2E_2}{2u^2 z_0(u)} , \quad (58)$$

defines the three-loop contribution B_2 analytically. In the Feynman diagram technique (the flucton formalism) it corresponds in general to sum of 15 three-loop diagrams on Fig. 2 in [11] weighted with symmetry factors. No single diagram we were able to calculate analytically for AHO, DWP and SGP potentials yet. Needless to say that next iterations will provide higher loop contributions in the same straightforward way.

D. AHO: three-loop correction

Starting from two-loop correction $z_2(u)$ the details of specific example become relevant, since one needs a concrete value for E_1 . So, from this point on, we present for the AHO case (4) one more term, three-loop correction for AHO. In order to do it we repeat consideration of the previous section in brief for the case of AHO.

For convenience, we introduce a new variable $s = u^2$. Then the generalized Bloch equation (46) takes the form

$$2g^2 s z'(s) + g^2 z(s) - s z(s)^2 = 2E g^2 - s(1+s) , \quad (59)$$

where $s \in [0, \infty)$. At zero order $(g^2)^0$ we have the equation

$$z_0(s)^2 = (1+s) , \quad (60)$$

cf. (51) then for normalizability of the wave function it is required to take the positive solution $z_0(s) = \sqrt{1+s}$. The equation to the next order g^2 is given by

$$2s z'_0(s) + z_0(s)(1 - 2s z_1(s)) = 2E_0 , \quad (61)$$

from which it follows that

$$z_1(s) = \frac{1 + 2s - 2E_0\sqrt{1+s}}{2s(1+s)}, \quad 2E_0 = 1,$$

Note that the condition that the function $z_1(s)$ is not singular at the origin, $s = 0$ also implies $2E_0 = 1$. Now, vanishing of the coefficient in (59) of order of g^4 leads to the equation for the second correction $z_2(s)$, which is equal to

$$z_2(s) = \frac{-E_1 + z_1 - sz_1^2 + 2sz_1'}{2sz_0} = \frac{4(-1 + \sqrt{1+s}) + s(-7 - 8s + 8\sqrt{1+s}) - 4s(1+s)^2 E_1}{8s^2(1+s)^{5/2}}, \quad (62)$$

where $z_1(s)$ is already known and $E_1 = \frac{3}{4}$ is well-known first energy correction to the AHO ground state. At small $s \rightarrow 0$ we obtain

$$z_2(s) \approx \frac{3 - 4E_1}{8s} + \frac{1}{4}(-6 + E_1) - \frac{3}{64}s(-63 + 4E_1) + \dots \quad (63)$$

The value $E_1 = \frac{3}{4}$ leads to disappearance of the first (singular) term in this expansion. Similarly, we obtain

$$z_3(s) = \frac{1}{32\sigma^8(1+\sigma)^3} \left[60 + 230\sigma + 346\sigma^2 + 270\sigma^3 + 150\sigma^4 + 108\sigma^5 + 84\sigma^6 + 63\sigma^7 + 21\sigma^8 \right], \quad (64)$$

where $E_2 = -\frac{21}{16}$, and $\sigma = \sqrt{1+s}$. Of course, in the variable σ which is nothing but $z_0(s)$ (60) it can be easily seen that all corrections $z_n(s)$ are meromorphic functions, no transcendental terms occur. From (44) we immediately make the identifications

$$\begin{aligned} z_0(s) &= g^2 \partial_s S_{\text{flucton}}, \\ z_1(s) &= \partial_s \left(\frac{1}{2} \log \text{Det}(\text{flucton}) \right), \\ z_2(s) &= g^{-2} \partial_s (\text{two-loop}), \\ z_n(s) &= g^{-2(n-1)} \partial_s (\text{n-loop}). \end{aligned} \quad (65)$$

E. ADWP: classical action and one-, two-, three-loop corrections

Finally, within the iteration method for the generalized Bloch equation (46), we consider the asymmetric double-well potential (ADWP), see (7),

$$V = \frac{1}{2}x^2(1 + 2tgx + g^2x^2), \quad t \in [0, 1],$$

which is, in fact, general quartic potential.

In this case, one potential minimum is situated at $x = 0$ and $V(0) = 0$, while for the second minimum (when exists) is situated to left from $x = 0$ and $V(x_{\min}) \geq 0$.

The generalized Bloch equation (46) takes the form

$$g^2 uz'(u) + g^2 z(u) - u^2 z(u)^2 - 2Eg^2 + u^2(1 + 2tu + u^2) = 0. \quad (66)$$

After a straightforward calculation, see e.g. [12] we find explicitly the first three coefficients of the perturbative expansion in g^2 of the energy (53),

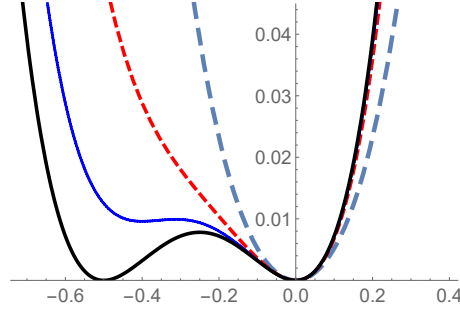


FIG. 4: The ADWP for $t=0$ (solid black), 0.8 (solid blue), 0.95 (short dashed, red), 1 (dashed grey) at the coupling $g = 2$.

$$\begin{aligned}
 E_0 &= 1, \\
 E_1 &= \frac{1}{4} (3 - 11t^2), \\
 E_2 &= -\frac{3}{16} (7 - 114t^2 + 155t^4).
 \end{aligned} \tag{67}$$

These coefficients will be needed to find one-, two-, three-loop contributions in iteration method applied for (66). Zero iteration of (66) gives the classical momentum at zero energy

$$z_0 = u\sqrt{1 + 2tu + u^2} = gx\sqrt{1 + 2tgx + g^2x^2} \equiv (gx)X^{(ADWP)},$$

where for convenience we will denote hereafter $X^{(ADWP)} \equiv X_4$, while the classical flucton action $\sim \int uz_0(u)$ reads,

$$S_{\text{flucton}} = \frac{-2 + 3t^2 + 3t(1 - t^2)(\text{Log}[1 + t] - \text{Log}[F + X_4]) - 3FtX_4 + 2X_4^3}{3g^2}, \tag{68}$$

where $F \equiv \sqrt{-1 + t^2 + X_4^2}$. As for the determinant, $\log \text{Det}(O_{\text{flucton}})$ it is equal to

$$\frac{1}{2} \log \text{Det}(\text{flucton}) = \text{Log} \left[\frac{X_4}{2} \right] + \text{Log} [1 + (F - t)t + X_4]. \tag{69}$$

It can be immediately checked that taking (68) and (69) at $t = 0$ we recover the results for the AHO and at $t = \pm 1$ those of the DWP. As for two-loop correction B_1 it takes the form

$$\begin{aligned}
 B_1^{(ADWP)} &= \frac{1}{12(F - t)^2(-1 + t^2)X_4^3} \left(5(1 + Ft - 2t^2 - Ft^3 + t^4) + 6(1 - t^2)X_4 - 2(1 + Ft - t^2)X_4^2 \right. \\
 &+ (-17 - 34Ft + 134t^2 + 200Ft^3 - 285t^4 - 170Ft^5 + 170t^6)X_4^3 + (-9 + 31Ft - 20t^2 - 33Ft^3 + 33t^4)X_4^4 \\
 &\left. + (17 - 100t^2 + 85t^4)X_4^5 \right).
 \end{aligned}$$

In the limits $t = \pm 1$ and $t = 0$ it coincides with two-loop correction B_1 for DWP and AHO, respectively.

The three-loop correction B_2 , which in the path integral formalism is given by the sum of 15 weighted (with symmetry factors) Feynman integrals (running from one-dimensional up to six-dimensional integrals), in the iterative approach to the generalized Bloch equation (46) can be easily calculated

$$\begin{aligned}
B_2^{(ADWP)} &= \frac{1}{64(F-t)^4} \left[\frac{40(1+2Ft-2t^2)(-1+t^2)}{X_4^6} - \frac{40(1+Ft-t^2)}{X_4^5} \right] \\
&+ \frac{8(8+8Ft-13t^2)}{X_4^4} + \frac{64}{X_4^3} - \frac{8(5-6Ft+17t^2+22Ft^3-22t^4)}{X_4^2} + \frac{8(-5+11Ft-11t^2)}{X_4} \\
&+ \left((207-3598t^2+5639t^4)(F-t)^4 + 8(-3+11t^2) \right) + 4(35+426t^2-1833t^4+1860t^6-3Ft(69-301t^2+620t^4)) X_4 \\
&- 4(21+406t^2-1395t^4+ Ft(-187+465t^2)) X_4^3 \Big].
\end{aligned} \tag{70}$$

It is rather surprising, that $B_2^{(ADWP)}$ is given by so compact expression. In particular,

$$\begin{aligned}
B_2^{(AHO)} &= -\frac{40+120X_{AHO}+136X_{AHO}^2+88X_{AHO}^3+80X_{AHO}^4+112X_{AHO}^5-39X_{AHO}^6-330X_{AHO}^7-207X_{AHO}^8}{64X_{AHO}^6(1+X_{AHO})^2} \\
B_2^{(DWP)} &= \frac{X_{DWP}(128+300X_{DWP}+248X_{DWP}^2+71X_{DWP}^3)}{4(1+X_{DWP})^4}.
\end{aligned} \tag{71}$$

For completeness we present the three-loop correction for the SGP

$$B_2^{(SGP)} = \frac{7 - (6 + \cos(X_{SGP})) \sec^4(X_{SGP})}{1024}. \tag{72}$$

Note it is of the amazingly simple form.

F. Strong coupling expansion

So far, we studied weak coupling expansion for the generalized Bloch equation (46),

$$z(u) = \sum_{n=0} g^{2n} z_n(u), \quad E = \sum_{n=0} g^{2n} E_n,$$

which corresponds to perturbation theory in g^2 . Now we will study the strong coupling expansion in $1/g$. It is convenient to consider a particular potential breaking the idea of generality. We present here the results for the AHO case (4)

$$V = \frac{1}{2} x^2 (1 + g^2 x^4).$$

Let us introduce as first the classical coordinate $u = gx$ and then as second introduce new variable $s = u^2$. Then the generalized Bloch equation (46) takes the form (59),

$$2g^2 s z'(s) + g^2 z(s) - s z(s)^2 = 2E g^2 - s(1+s),$$

where $s \in [0, \infty)$. Since we know (functionally) the strong coupling expansion for energy, see e.g. [13], let us develop perturbation theory

$$E = g^{2/3} \sum b_n g^{-\frac{4n}{3}}, \quad z = g^{2/3} \sum F_n(s) g^{-\frac{4n}{3}}, \tag{73}$$

where $b_n, n = 0, 1, 2 \dots$ are strong coupling coefficients for energy, few of them are found numerically with high accuracy. The equation for finding the zero order $O(g^{2+\frac{2}{3}})$ is of the form

$$2sF'_0 + F_0 = b_0 ,$$

it does *not* depend on the potential explicitly and

$$F_0 = b_0 . \quad (74)$$

The equation for the 1st order correction $O(g^{2-\frac{2}{3}})$,

$$2sF'_1 + F_1 - sF_0^2 = b_1 ,$$

which also does *not* depend on the potential explicitly,

$$F_1 = \frac{b_0^2}{3}s + b_1 . \quad (75)$$

It can be easily shown that the n th correction is a polynomial of degree n ,

$$F_n = \alpha_0(b)s^n + \alpha_1(b)s^{n-1} + \dots + b_n .$$

IV. THE ACCURACY OF THE PERTURBATIVE AND THE SEMICLASSICAL LOOP EXPANSIONS

In this section we address the issues of convergence and accuracy of the expressions derived above, comparing the terms of the expansion to each other and to the wave functions obtained numerically. In particular, this comparison will quantify the meaning of “large classical coordinate” $y = g x$ and “small coupling” g . For definiteness, we discuss those issues for the case of AHO system.

Let us first address the issue of convergence of the perturbative series in g at weak coupling. In Fig.5 we plot four subsequent term of the expansion of the wave function phase and their sum, for “small” $g^2 = 1/3$ coupling. and “large” $g^2 = 2$. In all cases there is clear dominance of the classical $O(g^{-2})$ term at large values of the coordinates, $x \gg 1$. This happens because only the classical term grows with x . But if one excludes the leading term and compares the subsequent loop corrections themselves, the series seem to be convergent at *all* x rather well.

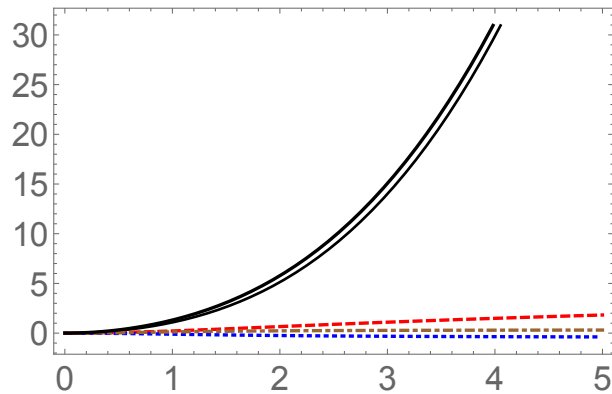


FIG. 5: (color online) The (double) phase of the wave function 2ϕ for the anharmonic oscillator versus the coordinate x . for the coupling $g^2 = 1/3$. The (lower) thin black solid line is the leading term, corresponding to the classical flucton action. The red dashed, blue dotted and brown dot-dashed lines show the magnitude of the one, two and three loop corrections. Their sum is shown by the (upper) thick black solid line.

We now proceed to the case of strong coupling, $g > 1$, and ask whether the semiclassical theory is still applicable in its domain, at large values of the coordinate. In Fig.6 we compare four terms of the loop expansion with the extremely accurate variational wave function derived previously by one of us [16], for rather strong coupling $g^2 = 2$.

Two observations come from this plot. The first is that at strong coupling the convergence at small $x < 1$ is gone. Yet the second is that in the semiclassical domain $x > 1$ one can see that higher loop corrections do in fact improve

the classical result. In fact, for $x > 1.5$ the difference between the flucton series (up to 4th term) and the variational curve is smaller than the width of the line!

The last issue we discuss in this section is that of the overall normalization constant. The flucton method, by construction, is designed to give the *relative* probability to find a particle at different locations. Rather arbitrarily, we have selected in all the discussion above the “normalization point” to be located at the potential minimum. Indeed, our flucton and its action are both zero, for a particle located there. So the same value – taken to be one – is used for all wave functions at the maximum $x = 0$.

The upper Fig.7 shows a comparison of the semiclassical density matrix (the sum of 3 terms) with the exact (numerically calculated at energy $E_0 = 0.69617575$) wave function squared, for $g = 1$. While such normalization is natural for the semiclassical approach used, it is in fact inadequate, in the following sense. As it is clear from the upper plot of Fig.7, this normalization does not provide good description at large x , which is the semiclassical domain.

Such outcome is of course not unexpected. Our derivation from the Schrödinger equation in section III B is based on the *logarithmic derivative* of the wave function (44), which does not depend on the normalization constant. Therefore, a more meaningful comparison between the semiclassical expansion and the exact wave function can be provided by the plot of the corresponding logarithmic derivatives. Such comparison is shown in the lower plot of Fig.7: now the agreement between the two curves is observed for $x > 1$, in the semiclassical domain. Outside it, at $x < 1$, the agreement is not expected, but it is not too bad either. (Note that this figure corresponds to the coupling which is not small, $g^2 = 1$).

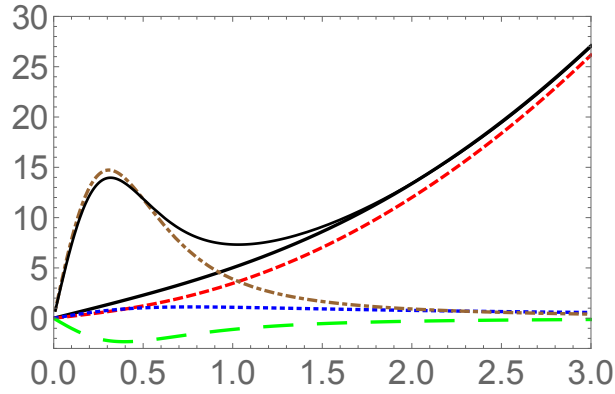


FIG. 6: (color online) The comparison between the variational wave function squared [16] (thick black solid line) with the flucton loop expansion, zero to three loops, at the coupling $g^2 = 2$. The red dashed, blue dotted, green long-dashed, brown dot-dashed and thin solid black lines are for classical action, 1-, 2-, 3-loop contributions and their sum, respectively.

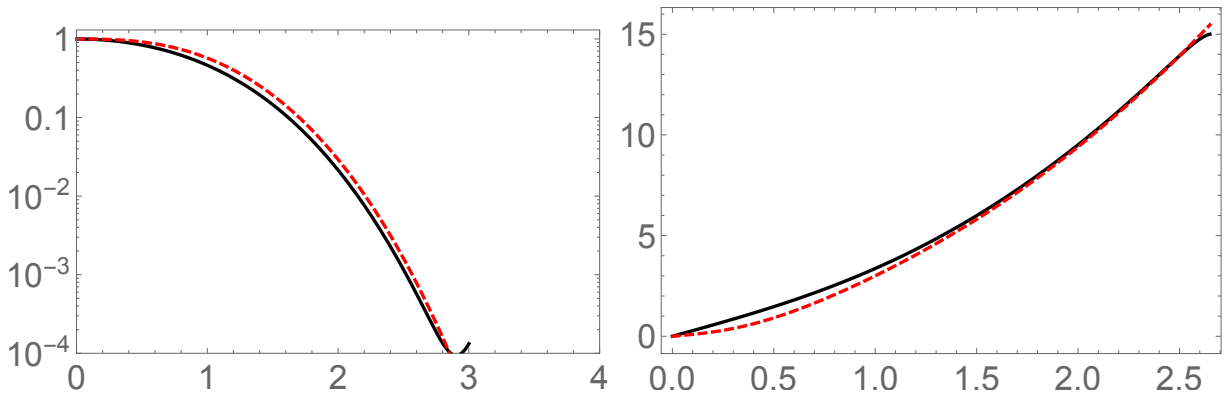


FIG. 7: (color online) The comparisons for the anharmonic oscillator with the coupling $g^2 = 1$ of the numerically calculated wave function (black solid lines) with the flucton expansion (0-1-2-loops summed) shown by the red dashed lines. The upper and lower plots show the density in the ground state and its logarithmic derivative versus the coordinate x , respectively.

V. THE SEMICLASSICAL EXPANSION FOR POTENTIALS WITH MULTIPLE MINIMA

In general, one may think also of the potentials with N minima, and ask how the flucton-based approximation for the path integral we develop should be applied in this case.

The case we start with has all minima to be *degenerate*, corresponding to the same energy (which then can always be put to zero). Since in this case all of the minima can be used for “long-time relaxation” of the flucton paths, one can think of $N \times N$ matrix of fluctons $x_{ij}^f(\tau)$, starting at $\tau \rightarrow -\infty$ in the i -th vacuum and ending at $\tau \rightarrow \infty$ in the j -th one. Of course, for a given “observation point” x_0 one only needs to consider those paths which pass through it. The DWP is the example of such degenerate situation, to be discussed in the subsection V A.

However the problems with the *non – degenerate* minima, such as in the ADWP case, obviously cannot be treated in this way. There are no non-diagonal paths $i \neq j$ between different maxima which can “relax” at both ends, as those have different energy. Long-time “relaxation” is obviously only possible at the global minima. This situation, to be discussed in subsection V B, require complexified classical paths.

A. The density between the minima, for the symmetric double well

So far we only discussed the “outer region” $|x_0| > x_{min} = 1/g$ outside its two minima. Now let us discuss the intermediate region, around the middle point $x \sim 0$. For sufficiently small g – and thus well separated minima – it should also be amenable to a semiclassical treatment.

Following the discussion above, the double well problem should have 4 flucton paths. The $x_{11}^f(\tau)$ and $x_{22}^f(\tau)$ are fluctons we discussed, associated with say the left and the right potential minima. Their contributions generate two familiar maxima in the ground state wave function.

In the outer region $|x_0| > x_{min}$ the $x_{12}^f(\tau)$ and $x_{21}^f(\tau)$ fluctons are a combination of the $x_{11}^f(\tau)$ and $x_{22}^f(\tau)$ fluctons, *plus* the *instanton* or *antiinstanton* paths. So the actions are just

$$S_{12}^f = S_{11}^f(x_0) + S_{instanton} . \quad (76)$$

This means that the density matrix due to the (11) flucton, is just corrected by an exponentially small and x_0 -independent term

$$\psi_0^2(x_0) \sim \exp(-S_{11}^f(x_0)) (1 + O(e^{-S_{instanton}})) . \quad (77)$$

If $|x_0| < x_{min}$, in the inner region, the $x_{12}^f(\tau)$ and $x_{21}^f(\tau)$ fluctons are nothing else as the *instanton* and *antiinstanton* paths. Their timing can be selected so that at $\tau = 0$ their value, as for all other fluctons, should be x_0 . Furthermore, their classical actions

$$\begin{aligned} S_{12}^f = S_{21}^f &= \int_{-x_{min}}^{x_0} p(x') dx' + \int_{x_0}^{x_{min}} p(x') dx' \\ &= \int_{-x_{min}}^{x_{min}} p(x') dx' , \end{aligned} \quad (78)$$

do not depend on the x_0 . Therefore, their contribution to the density matrix in the inner region is – somewhat surprisingly – independent on the observation point x_0 .

It may appear strange that the flucton theory has such unusual contributions in the inner region. since in the familiar WKB-like semiclassical theory one does not have those. Note however, that the WKB is applied to the wave function, while the flucton theory is applied to the density matrix, or its *square* ψ_0^2 : the instanton terms thus come from the product of the two semiclassical contributions in the WKB-like approaches.

To demonstrate its validity, we will use Turbiner trial function $\cosh(A)$, see [16],

$$\cosh^2(A) = (1/2) (\cosh(2A) + 2) \quad (79)$$

B. The density for the asymmetric double well: complex fluctons

As we already noted at the beginning of this section, when the minima of the potential are non-degenerate there are no classical solutions going from one maximum to the other of the potential in Euclidian time and “relaxing” at both

ends, simply because for that one needs two conflicting values of the energy. In particular, there are no “instanton” and “anti-instanton” solutions available.

It was argued in [17, 18] that by complexification of the coordinate, $x(t) \rightarrow z(t) = x(t) + i \cdot y(t)$ and thus by generalizing equations of motion to the so-called *holomorphic Newton's equation* (still for inverted potential)

$$\frac{d^2 z}{dt^2} = + \frac{\partial V}{\partial z} , \quad (80)$$

one can find complex generalization of those. Specifically, in these works it is discussed the contribution of a periodic path with a finite action, called the “complexified bion” (CB), which is an extension to the instanton-antiinstanton pair solutions for the symmetric potential.

For continuity of the notations, let us use the following (Euclidean time) Lagrangian

$$\mathcal{L}_{ADWP} = \frac{1}{2} \dot{x}(\tau)^2 + \frac{1}{2} (x(\tau)^2 - 1)^2 + p g x(\tau) , \quad (81)$$

with the asymmetry parameter p . If p is nonzero but small, the left and right maxima located at x_+ and x_- are of different height, $E_+ = V(x_+) \neq E_- = V(x_-)$ [20]

Like for symmetric case discussed before, for a generic point x_0 in between the two maxima $x_+ < x_0 < x_-$ there are two flucton solutions, also denoted by \pm , which start at x_0 and “relax” for an infinitely long time near either x_+ or x_- . But now there is no symmetry $x \rightarrow -x$, these two fluctons have different energy and the issue of relative normalization of their contributions is rather nontrivial.

The flucton path $f_+(\tau)$, starting from x_+ , can reach any point we discuss. But, an additional problem indicating troubles with such an approach, is that the flucton path f_- cannot reach *all* points x_0 in the interval $x_+ < x_0 < x_-$ since it has the energy E_- insufficient to “climb” all the way to x_+ . This path can only reach to the turning point and get reflected back. A periodic path starting and ending at x_- is known as “bounce” solution.

A complexification of the paths opens many new options. Let us start with the generalizations of the flucton path $f_+(\tau)$, starting from x_+ with the energy E_+ . The initial velocity at the top is zero, but, as for a skier at the mountain top, there is a freedom to slide in any direction.

Where one would like to go? Along the real axes no other point has the sufficient height of the inverted potential, so the real flucton path f_+ can reach any x_0 , inside or outside of the interval indicated. The classical action $S_+ = \int_{x_+}^{x_0} dx p(x)$ is a monotonous function since $p(x) > 0$: therefore the corresponding amplitude $\exp(-S_+(x_0))$ decreases monotonously in both directions, from x_+ . This contribution must be included into the density $P(x_0)$, but it cannot be the only one.

Going into arbitrary direction from the global maximum leads to the family of paths, two of which is shown in Fig. 8. They get reflected at (or a vicinity) of two turning points back to the maximum x_+ . Since at its top “relaxation” takes long time, such paths produce an option of being periodic, with an infinite period but finite action.

Let us find the two turning points. In the particular case of ADWP the potential is the 4th order polynomial in coordinate, and thus it must have 4 roots. Hence, it can be re-written in a form convenient for motion with the maximal energy $E_+ = V(x_+)$ as

$$V - E_+ \sim (z - x_+)^2 (z - z_1)(z - z_2) . \quad (82)$$

Note that x_+ must be a double zero, and two others should be the complex conjugated $z_1^* = z_2$ pair of two *turning points*. (In our concrete example their location is at $z_{1,2} = 1.02412 \pm 0.312482i$.) At these turning points the velocity on the path vanishes, but since it is a not a maximum – a double zero – there is no long-time “relaxation” possible, the “complexified bion” path bounces back.

One of such periodic paths with finite action has been pointed out in Refs. [17, 18] and named “complex bion”. The complex action of these paths contributing to the amplitude the factors $\exp(-\text{Re}S(z_{CB}))\cos(\text{Im}S(z_{CB}))$ produces cosine of certain non-trivial phases.

These phases violate the positivity of the amplitudes present for any real paths, and produce interesting oscillations/cancellations. Some known puzzles associated with the energy spectra of quasi-exactly-solvable (QES) [19] and/or supersymmetric (SUSY) examples have been explained in these works.

Our aim is to find “complex fluctons”, classical paths connecting a generic point x_0 in the real axes with the global maximum at x_+ . However, the paths belonging to the family just described cross the real axes *only* at x_+ .

In general, starting with the real axes, one see that the kinetic energy $K = E - V(x_0)$ is real. Therefore $\text{Im}(K) = \dot{x}\dot{y} = 0$, which means that one of the factors must vanish. Thus there are two sets of paths: they either go along the real axes, or normal to it. The answer obviously is defined by the sign of the kinetic energy $2K = \dot{x}^2 - \dot{y}^2$. Taking generic initial point on the real axes as a starting point $x(t=0) = x_0$ and various initial values of $\dot{y}(t=0)$, one can obtain families of solutions to holomorphic EOM.

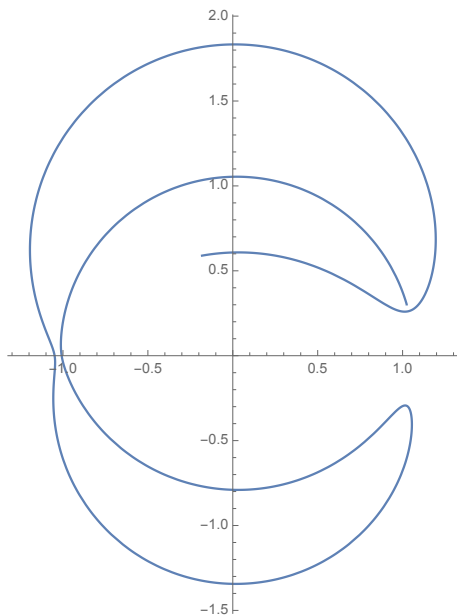


FIG. 8: Two examples of solutions to holomorphic Newton's equation (80). Both start the zero velocity and slightly displaced from the maximum of $-V$ location x_+ . The one going upward has a phase of the displacement tuned so that it goes to the turning point z_1 and is reflected back on the same path, so that one see a single curve. This is the “complex bion” of Ref.[17]. The one going downward, from slightly different location, starts an infinite path going around both turning paths $z_{1,2}$.

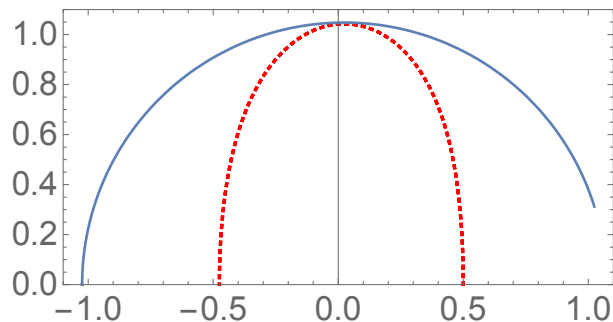


FIG. 9: Two solutions to the holomorphic Newton's equation (80). The blue solid line is the same “complex bion” of Ref.[17] as in the previous figure. The red dashed line starts at some generic point on the real axes, in this case $x_0 = 0.5$, with a velocity tuned so that it touches the “complex bion”.

In Fig. 9 we show only one of them (red dotted line, for $x_0 = 0.5$) tuned to be touching the “complex bion” path of Unsal et al (blue solid line). We propose to use a combination of two segments of those two curves, before and after the touching point z_{cross} , as a “*complex flucton*”, leading from a generic point x_0 to the global maximum x_+ . Note however, that while both curves at the touching point z_{cross} have the same direction of the velocity, its magnitude needs to jump, as the two curves correspond to two different energies.

For a generic paths one can think of them as classical ones, solving EOM with appropriate external force term $z(\tau)f(\tau)$, added into the Lagrangian and to the EOM. The advantage of the complex flucton path just introduced is that in this case the force should only be applied at the crossing time moment only $f(\tau) \sim \delta(\tau - \tau_{cross})$, as an instantaneous kick adjusting the total energy. There is a finite, although small in the example considered, contribution of this kick to the action, which should not be omitted.

Summarizing a construction: there are two paths both leading from x_0 to the maximum x_+ : the original real flucton and the complex one. The corresponding contributions to the density at x_0 are now both normalized in the

same way, and their sum has the form

$$P(x_0) \sim 2\exp(-S_{real\ fl}) + \exp[-\text{Re}(S_{complex\ fl})]2\cos(\text{Im}(S_{complex\ fl})) \quad (83)$$

where we have added the complex conjugate part of the path in the lower hemisphere $\text{Im}(z) < 0$. Like for the complex bion contribution to the ground state energy, the contribution of the complex flucton may be positive or negative, depending on the particular value of the imaginary part of its action.

VI. CONCLUSIONS

In our previous paper I we have outlined a new *flucton – based* semiclassical theory, based on path integral representation of the density matrix. Corrections to leading semiclassical results take the form of Feynman diagrams, well defined to any order by standard Feynman rules. As examples of its applications we calculated one and two-loop corrections for the ground state density (square of the wave function) of the anharmonic oscillator (AHO).

In this second paper we describe the foundations of the method in more details, and also presented a number of new results. At the start of the paper, we summarize the completed one- and two-loop calculations based on Feynman diagrams, for all three physically important examples: AHO, the double well (DWP) and sin-Gordon (SGP), see (41).

We showed that in the case of polynomial potentials the perturbation corrections to the imaginary phase of wavefunction $\phi(x)$ are finite-degree polynomials. We demonstrated that generating functions of their leading degrees coincide to a corresponding terms in a loop expansion. Eventually, we found the Taylor expansion at small distances,

$$\phi(x) = A_0x^2 + A_1x^3 + \dots$$

while the loop expansion is nothing but the expansion at large x . It is sufficiently straightforward to attempt to interpolate between these two regimes. For AHO the following interpolating trial function

$$\phi_{int}(x) = \frac{A + Bx^2 + Cg^2x^4}{\sqrt{D + g^2x^2}} + \frac{1}{2} \log \text{Det} (O_{\text{flucton}}) ,$$

with parameters A, B, C, D gives extremely high local accuracy for all x , in practice solving the problem, c.f. [16]. Similar approximants, interpolating between the small- x series and our loop expansion, can be done for other quantum-mechanical problems.

We also were able to relate these results to the iterative solution of certain equation (46) for the reduced logarithmic derivative of the wave function. As we show explicitly, all our results from Feynman diagrams are reproduced exactly! Among other insights, this way of deriving it explains why all irrational functions – such as logs and polylogs – which appeared in expressions for individual Feynman diagram, are always canceled in their sum. This way of calculation allows us to go to higher orders: in particular, we calculated one more term of the expansion, corresponding to the sum of the 15 three-loop Feynman diagrams, none of which was analytically evaluated so far. (Needless to say, this way of calculation, starting from the Schrödinger equation, is not generalizable to QFT applications at least at present.)

We have studied the issues of convergence and the accuracy of this version of the semiclassical theory. It is shown that in the case of weak coupling, the series for the density are well convergent. Even if the coupling is not small, in the semiclassical domain it seems to be convergent to the exact answer (provided one uses the logarithmic derivative of the wave function, removing sensitivity to the normalization constant.)

Finally, we discussed the generalization of the flucton theory to the case of more than one minimum of the potential using the example of the DWP problem. In the case when two minima are degenerate, the density in between them is defined via a sum of 4 contributions: left and right-side fluctons, the instanton and the anti-instanton solutions. Interestingly, the latter produce constant (x -independent) contribution to the density matrix in this region.

The case of (slightly) non-degenerate minima is much more involved. The authors of [17, 18], in which the ground state energy of the asymmetric double well potential was studied, propose to complexify the coordinate $x(\tau) \rightarrow z(\tau) = x(\tau) + iy(\tau)$ and include contributions of certain finite action solution to the holomorphic EOM (80). We studied such paths and found large families of such solutions, starting from the maximum of the inverted potential. We also show how one can construct “complexified fluctons”, leading from a generic point to the global maximum of the (inverted) potential, can be constructed using (segments of) *two* such paths. While quantitative studies of those “complex fluctons” are deferred to future works, the main qualitative point is made here: since one of them has a complex action, its contribution to the density matrix has a nontrivial phase. It can be either positive or negative, depending on the magnitude of the imaginary part of their action, depending in turn on the asymmetry parameter.

Acknowledgments

This work in its early stage was supported in part by CONACYT grant **166189** (Mexico) for M.A.E.R. and A.V.T., and also by DGAPA grant **IN108815** (Mexico) for A.V.T. M.A.E.R. is grateful to ICN-UNAM, Mexico as well as the Stony Brook University for the kind hospitality during his visits, where a part of the research was done, he was supported in part by DGAPA grant **IN108815** (Mexico) and, in general, by CONACyT grant **250881** (Mexico) for postdoctoral research. The work of E.S. is supported in part by the U.S. Department of Energy under Contract No. **DE-FG-88ER40388**. The authors are grateful to participants of the seminar at Center for Nuclear Theory, Stony Brook University (Dec.21, 2016) for useful discussions.

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 - [20] For illustration below we will use the case $pg = 0.1$, for which these locations are $x_+ \approx -1.02412$, $x_- \approx 0.973994$. Note that plus and minus in our notations do not correspond to the sign of the coordinates, but to the height of the inverted potential, $V(x_+) = 0.10122 > V(x_-) = -0.0987171$.