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Lisa L. Everett and Alexander J. Stuart

Phys. Rev. D **96**, 035030 — Published 30 August 2017

DOI: [10.1103/PhysRevD.96.035030](https://doi.org/10.1103/PhysRevD.96.035030)

Lepton Sector Phases and Their Roles in Flavor and Generalized CP Symmetries

Lisa L. Everett^{1*} and Alexander J. Stuart^{2,3,4†}

¹*Department of Physics, University of Wisconsin,
Madison, WI 53706, USA*

²*SISSA/INFN, Via Bonomea 265, 34136 Trieste, Italy*

³*Facultad de Ciencias - CUICBAS, Universidad de Colima,
C.P. 28045, Colima, México*

⁴*Dual CP Institute of High Energy Physics, C.P. 28045, Colima, México*

Abstract

We study the effects of considering nontrivial unphysical lepton sector phases on the group theoretical properties of the flavor and generalized CP symmetry elements in the case where there are three light, distinct Majorana neutrino species. We highlight the similarities and differences between the charged lepton and neutrino sectors and further elucidate the group properties of the flavor and generalized CP symmetry elements. We show how the inclusion of these leptonic phases affects the bottom-up constructions of these symmetry elements and discuss the implications for top-down model building based on discrete symmetry groups.

*E-mail: leverett@wisc.edu

†E-mail: astuart@ucol.mx

1 Introduction

With the recent measurements of the reactor mixing angle [1–3], the mixing angle pattern of the Maki-Nakagawa-Sakata-Pontecorvo (MNSP) lepton mixing matrix U_{MNSP} [4] is now on solid experimental ground, opening the door to a new era of precision lepton mixing measurements. As the reactor angle is relatively large (roughly Cabibbo-sized), there is an exciting experimental opportunity for a future direct observation of “Dirac-type” leptonic CP violation. Even in the absence of such a direct measurement, hints of a nonzero value of the leptonic Dirac CP-violating phase δ are already originating in global fit analyses of lepton mixing data [5]. Thus in preparation for a nonzero measurement of this phase, it is useful to explore whether the underlying theory violates CP explicitly, or if CP is a spontaneously broken symmetry.

In the case in which CP is conserved in the underlying theory and is broken spontaneously, it is standard to explore this question in the context of theories with spontaneously broken *discrete* lepton family symmetries (see [6] for a review). For the case of Majorana neutrinos, one compelling idea within this framework is to assume the breaking of a high energy discrete flavor symmetry to a residual Klein symmetry in the neutrino sector. The residual Klein symmetry completely fixes the form of U_{MNSP} at leading order, in the diagonal charged lepton basis up to charged lepton rephasing (although it fails to predict values for the Majorana phases [7,8]). To make predictions for the Majorana phases, a CP symmetry can be imposed and then spontaneously broken, resulting in concrete phase predictions. The CP symmetry should of course be defined consistently together with the discrete flavor symmetry [9]. Many such models of CP and flavor symmetries have been proposed and investigated, including models based on A_4 [10,11]/ $\Delta(3n^2)$ [12–15], A_5 [16–19], S_4 [21–26]/ $\Delta(6n^2)$ [14,27–30], $\Sigma(n\phi)$ [31,32], T' [33], and $D_{9n,3n}^{(1)}$ [34].¹

With these studies in mind, it is also worthwhile to take a bottom-up perspective, in which the goal is not to construct a specific top-down model, but instead to elucidate how and when preserving different residual CP and flavor symmetry elements affects the predictions for the lepton mixing parameters. To this end, in previous work [40] we constructed the general residual Klein and generalized CP symmetries in the neutrino sector as a function of the measured lepton mixing parameters. There the simplifying assumption was made that the charged lepton sector is diagonal, such that the neutrino sector mixing parameters are directly related to the experimentally measured (measurable) lepton mixing parameters.

In this paper, we consider the role of general leptonic sector phases. These phases include charged lepton sector phases and overall shifts to Majorana phases, which are typically ignored since by definition they cannot have any effect on physical observables. However, their inclusion clarifies the group theoretical properties at low energies of several quantities of interest for theoretical model-building, which can be obscured when these phases are set to zero. In particular, these phases have relevance for making connections of the family and generalized CP symmetry elements to explicit representations of specific discrete symmetry groups, and hence their inclusion provides a natural generalization of the bottom-up approach given in [40]. Our results provide a set of group theoretical relations that must be satisfied at low energies within any top-down flavor model-building scenario for three light Majorana neutrinos that leaves a residual Klein symmetry in the neutrino sector. The goal is to provide guidelines for developing a better understanding of the generalized CP and flavor symmetries when constructing top-down models, which can in principle lead to new model-building directions within this general framework.

¹We note that CP has been studied for the case of a single preserved Z_2 as the residual neutrino flavor symmetry [26, 35–39].

This paper is structured in the following way. In Section 2, we examine the effects that phase redefinitions in lepton mixing can have on the group structures of the underlying flavor symmetries. We will discuss the effects of elevating the status of certain subsets of these phases to that of flavor symmetries can have on the form of U_{MNSP} , and analyze the group structures of such choices of rephasings/symmetries with a special focus on discrete symmetry groups. In Section 3, we expand this discussion to include the ways in which such phases affect aspects of generalized CP symmetries. Section 4 provides a detailed exposition of the connections of this work to the bottom-up approach given in [40], including several examples. We present our conclusions and outlook in Section 5.

2 Phases and Flavor Symmetries

The starting point of this analysis is the Majorana neutrino mass matrix for the three light neutrino species, M_ν . It is diagonalized by the matrix U_ν , as follows:

$$U_\nu^T M_\nu U_\nu = M_\nu^{\text{Diag}} = \text{Diag}(m_1, m_2, m_3) = \text{Diag}(|m_1|e^{-i\alpha_1}, |m_2|e^{-i\alpha_2}, |m_3|e^{-i\alpha_3}), \quad (1)$$

in which $|m_{1,2,3}|$ are presumed to be nondegenerate and nonzero, i.e., $|m_1| \neq |m_2| \neq |m_3| \neq 0$. The transformation

$$U_\nu \rightarrow U_\nu Q_\nu, \text{ with } Q_\nu = \text{Diag}((-1)^{p_1}, (-1)^{p_2}, (-1)^{p_3}) \text{ where } p_{1,2,3} = 0, 1 \quad (2)$$

also diagonalizes M_ν and leaves M_ν^{Diag} invariant.² There are thus eight possible symmetries contained in Q_ν , corresponding to the eight possible assignments of $p_{1,2,3}$ as given above.

In the charged lepton sector, the mass matrix $M_e = m_e m_e^\dagger$, which connects left-handed states, is diagonalized by U_e , as follows:

$$U_e^\dagger M_e U_e = M_e^{\text{Diag}} = \text{Diag}(|m_e|^2, |m_\mu|^2, |m_\tau|^2), \quad (3)$$

in which again $|m_e| \neq |m_\mu| \neq |m_\tau| \neq 0$. The diagonalization of M_e thus can easily be seen to be left invariant by the transformation

$$U_e \rightarrow U_e Q_e, \text{ where } Q_e = \text{Diag}(e^{i\beta_1}, e^{i\beta_2}, e^{i\beta_3}), \text{ where } \beta_{1,2,3} \in [0, 2\pi). \quad (4)$$

Eqs. (2) and (4) represent the set of transformations which leave the (diagonal) mass matrices of Eqs. (1) and (3) invariant. As these transformations play no role in the diagonalization of M_ν (M_e), they cannot enter any physical predictions that arise from U_ν (U_e). More precisely, $U_\nu \rightarrow U_\nu Q_\nu$ and $U_e \rightarrow U_e Q_e$ imply that the MNSP matrix correspondingly transforms as

$$U_{\text{MNSP}} = U_e^\dagger U_\nu \rightarrow Q_e^\dagger U_{\text{MNSP}} Q_\nu = U'_{\text{MNSP}}, \quad (5)$$

and that U_{MNSP} and (the infinitely many possible) U'_{MNSP} must all yield the same physics predictions (see [41] for a similar discussion in terms of rephasing invariants). The utility of including unphysical phases in Eq. (5) can be seen by observing that Q_e and Q_ν can be related to their nondiagonal forms T_e and S_ν via the unitary transformations [40]

$$S_\nu = U_\nu Q_\nu U_\nu^\dagger, \quad T_e = U_e Q_e U_e^\dagger, \quad (6)$$

²It is possible to put an overall phase in U_ν , i.e., $U_\nu \rightarrow U_\nu e^{i\theta_\nu/2}$, without changing any physical predictions. This is a point that we will revisit later.

These relationships can be derived by using Eq. (1) and Eq. (3) as well as the condition for a flavor symmetry in its nondiagonal basis, i.e.,

$$S_\nu^T M_\nu S_\nu = M_\nu, \quad T_e^\dagger M_e T_e = M_e. \quad (7)$$

Up to this point, we have not yet specified the explicit forms of the mixing matrices U_e and U_ν . Within the top-down approach, one constructs a concrete model of M_e and M_ν in a specific flavor basis (for example, it is often taken to be the basis in which M_e is diagonal). U_e and U_ν are then found through explicit diagonalization (up to rephasings). By contrast, from a bottom-up perspective, the forms of U_e and U_ν can be fixed or arbitrary depending on the choice of and/or number of symmetry elements that are to be imposed from Eq. (2) and Eq. (4). For example, in [40] we explicitly constructed the forms of the S_ν in the case in which the four positive determinant choices of Q_ν are preserved, while the charged leptons are taken to be diagonal. In this case, the mixing parameters of U_ν have a direct connection to the measured mixing parameters of U_{MNSP} , and thus the flavor symmetry elements can be given explicitly in terms of measurable quantities [40]. Here we will explore more general situations in which we allow for different possibilities for the choice and number of conserved symmetry elements. The upshot of this discussion is that although the unphysical phases contained in Q_e and Q_ν do not enter any physical observables by construction, they can clearly play a critical role in model building when it pertains to fixing/predicting lepton mixing patterns.

To see the way in which imposing specific symmetry elements can fix the mixing parameters, it is worthwhile first to consider the case in which there is a two-fold degeneracy in the entries of Q_ν and Q_e . Applying Eq. (6) to Eq. (7) shows that if $p_i = p_j$ for some $i, j = 1, 2, 3$ or $\beta_k = \beta_l$ for some $k, l = 1, 2, 3$, cf. Eq. (2) or (4) respectively, then there will exist an additional unitary rotation U_{ij}^ν or U_{kl}^e that allows for the mixing of the degenerate states, such that

$$Q_\nu^T (U_{ij}^\nu)^T M_\nu^{\text{Diag}} U_{ij}^\nu Q_\nu = (U_{ij}^\nu)^T M_\nu^{\text{Diag}} U_{ij}^\nu, \quad Q_e^\dagger (U_{kl}^e)^\dagger M_e^{\text{Diag}} U_{kl}^e Q_e = (U_{kl}^e)^\dagger M_e^{\text{Diag}} U_{kl}^e, \quad (8)$$

in which $(Q_\nu)_{ii} = (Q_\nu)_{jj}$ and $(Q_e)_{kk} = (Q_e)_{ll}$. This shows that when there are two degenerate phases in either Q_e or Q_ν , this is not enough to fix M_e or M_ν to be of diagonal form. Said again, a two-fold degeneracy in either Q_e or Q_ν is a symmetry not only of M_e^{Diag} or M_ν^{Diag} , but more importantly, $(U_{kl}^e)^\dagger M_e^{\text{Diag}} U_{kl}^e$ and $(U_{ij}^\nu)^T M_\nu^{\text{Diag}} U_{ij}^\nu$, cf. Eq. (8). Thus, to diagonalize these mass matrices completely, it is necessary to map

$$U_\nu \rightarrow U_\nu U_{ij}^\nu Q_\nu \text{ and } U_e \rightarrow U_e U_{kl}^e Q_e, \quad (9)$$

which implies that the mixing matrices U_ν and U_e are only fixed up to these unitary rotations in the degenerate sub-block of eigenvalues.³ The explicit forms of these extra unitary rotations are of course specified by diagonalizing M_e and M_ν .

The preceding discussion assumed a single choice of phases for Q_e and Q_ν . However, if we impose additional choices for the phases, then this can in principle fix the mixing. For example, let us consider the neutrino sector, for which there are eight possible symmetries contained in/denoted by Q_ν . Further imposing $Q_\nu = \pm 1$ does not change the result; neither does $Q'_\nu = -Q_\nu$, i.e., the other phase assignment with the same degenerate sub-block. What clearly affects the result is to impose an additional phase choice such that $(Q_\nu)_{kk} = (Q_\nu)_{ll}$ for some k, l yet to be determined. By demanding that this phase choice also holds, we obtain

$$(U_{ij}^\nu)^T M_\nu^{\text{Diag}} U_{ij}^\nu = (U_{kl}^\nu)^T M_\nu^{\text{Diag}} U_{kl}^\nu \implies M_\nu^{\text{Diag}} = (U_{ij}^\nu (U_{kl}^\nu)^\dagger)^T M_\nu^{\text{Diag}} U_{ij}^\nu (U_{kl}^\nu)^\dagger. \quad (10)$$

³A similar situation occurs if all phases are equal in either Q_e or Q_ν , i.e., $p_i = p_j = p_k$ and $\beta_i = \beta_j = \beta_k$ in Eqs. (2) and (4) respectively. In this case, the mixing is fixed up to an arbitrary unitary 3×3 rotation.

If $kl = ij$ no additional constraints arise, and the mixing still contains the same number of free parameters (after angle and phase redefinitions). However, if $ij \neq kl$, a comparison of Eq. (10) with Eqs. (1)-(2) demonstrates that the arbitrary rotation angles and phases in U_{ij}^ν and U_{kl}^ν are now related to each other because in this case

$$U_{ij}^\nu (U_{kl}^\nu)^\dagger = \text{Diag}(\pm 1, \pm 1, \pm 1) = Q_\nu. \quad (11)$$

Hence, the transformation $U_\nu \rightarrow U_\nu U_{ij}^\nu Q_\nu$ in Eq. (9) is then reduced to $U_\nu \rightarrow U_\nu Q_\nu$, eliminating the additional mixing parameters that entered when only preserving a single set of phases containing a two-fold degeneracy.

This discussion can be generalized to the charged lepton sector, with one notable difference that stems from the fact that the Q_ν contain only ± 1 , while the Q_e depend on the arbitrary β_i phases. We have seen that for the neutrino sector, two such nontrivial rephasings were required to be chosen subject to Eqs. (10)-(11), in order to guarantee the mixing matrix U_ν has no additional free parameters that result from a degenerate sub-block. However, there is clearly no such constraint for Q_e because the phases contained in Q_e can all be chosen to be distinct, as seen in Eq. (4). This then fixes the mixing matrix U_e , and thus it fixes the mass matrix M_e , up to rephasings by Q_e .

Let us now consider the case in which such rephasings are obtained from a flavor symmetry group, for which these arguments can be described within a group theoretical framework. To this end, we note that in the preceding discussion, the forms of U_e and U_ν were completely fixed up to rephasing by Q_e and Q_ν (with the possibility of an additional rotation when there is a degeneracy of phases in Q_e or Q_ν , cf. Eq. (9)). Perhaps just as importantly, Eq. (6) demonstrates that U_ν and U_e are the unitary transformations that relate Q_ν to S_ν and Q_e to T_e . As such, Q_e and Q_ν can be interpreted as representations of the elements of a flavor symmetry in their corresponding diagonal bases. Therefore to understand *all* possible residual flavor symmetries, we need only to understand the group properties of Q_e and Q_ν .

We start by noting that the unphysical phases of Eqs. (2) and (4) generally take multiple values, i.e., Q_ν represents a collection of eight symmetry transformations and Q_e represents an infinite set of symmetry transformations parameterized by the continuous parameters $\beta_{1,2,3}$. Therefore, the full residual neutrino flavor symmetry group G_ν and the full residual charged lepton flavor symmetry group G_e are expressible as⁴

$$G_\nu \cong Z_2^{p_1} \times Z_2^{p_2} \times Z_2^{p_3}, \quad G_e \cong U(1)_{\beta_1} \times U(1)_{\beta_2} \times U(1)_{\beta_3}, \quad (12)$$

(recall Eqs. (2) and (4)). However, the previous discussion shows that it is not necessary to implement the totality of these symmetries to generate a specific mixing pattern (although doing so certainly will). Thus with an eye toward minimality, we observe that it is possible to rewrite Eq. (3) as

$$M_e^{\text{Diag}} = U_e^\dagger M_e U_e = U_e^\dagger (P_e^\dagger P_e) M_e (P_e^\dagger P_e) U_e = U_e^\dagger P_e^\dagger M_e P_e U_e, \quad (13)$$

in which $P_e = e^{i\theta_e}$. Clearly, $U_e \rightarrow P_e U_e Q_e$ still diagonalizes M_e . The phase in P_e can always be chosen to fix $\text{Det}(Q_e) = +1$, for example by choosing $\theta_e = -(\beta_1 + \beta_2 + \beta_3)/3$, cf. Eq. (4).

The same result can be obtained for the neutrino sector, i.e., $\text{Det}(Q_\nu) = +1$, but for different reasons. More precisely, since we have assumed neutrinos are Majorana fermions, the freedom does not exist to rephase the whole mass matrix M_ν by arbitrary phases without affecting the (complex) neutrino mass eigenvalues. However, global phases on Majorana mass

⁴We emphasize again that these symmetry groups are contingent upon having nonzero, nondegenerate lepton masses.

matrices are phenomenologically irrelevant because they contribute to an overall shift of each of the individual phases (which is unmeasurable), i.e.,

$$M_\nu^{\text{Diag}} = U_\nu^T M_\nu U_\nu \rightarrow (P'_\nu)^T M_\nu^{\text{Diag}} P'_\nu = (P'_\nu)^T U_\nu^T M_\nu U_\nu P'_\nu = U_\nu^T (P'_\nu)^T M_\nu P'_\nu U_\nu, \quad (14)$$

where $P'_\nu = e^{-i\theta'_\nu/2}$. Then, letting $\theta'_\nu = \theta_\nu + p\pi$ (p an integer) implies that

$$P'_\nu = (-1)^p P_\nu, \quad (15)$$

in which $P_\nu = e^{-i\theta_\nu/2}$. This allows for the shift $\alpha_{1,2,3} \rightarrow \alpha'_{1,2,3} = \alpha_{1,2,3} + \theta_\nu$, cf. Eq. (1), and the determination of $\text{Det}(Q_\nu) = +1$ by utilizing the $(-1)^p$ factor in Eq. (15), with $p = p_1 + p_2 + p_3$, cf. Eq. (2).⁵ To summarize, it is possible to restrict $\text{Det}(Q_\nu) = \text{Det}(Q_e) = +1$ to remove physically redundant symmetries so that the elements of the minimal, residual leptonic symmetries G_ν and G_e can be expressed as⁶

$$G_\nu = \text{Diag}((-1)^{p_2+p_3}, (-1)^{p_2}, (-1)^{p_3}), \quad G_e = \text{Diag}(e^{-i(\beta_2+\beta_3)}, e^{i\beta_2}, e^{i\beta_3}), \quad (16)$$

in which $p_{2,3} = 0, 1$ and $\beta_{2,3} \in [0, 2\pi)$. This restriction is equivalent to “removing” the four Q_ν with $\text{Det}(Q_\nu) = -1$ and the infinitely many Q_e with $\text{Det}(Q_e) = e^{i\theta_e}$, where $\theta_e \in (0, 2\pi)$.

Motivated by these “new” general forms for G_ν and G_e , we next define

$$\begin{aligned} \tilde{G}_{p_2} &= \text{Diag}((-1)^{p_2}, (-1)^{p_2}, 1), \quad \tilde{G}_{p_3} = \text{Diag}((-1)^{p_3}, 1, (-1)^{p_3}), \\ \tilde{T}_{\beta_2} &= \text{Diag}(e^{-i\beta_2}, e^{i\beta_2}, 1), \quad \tilde{T}_{\beta_3} = \text{Diag}(e^{-i\beta_3}, 1, e^{i\beta_3}), \end{aligned} \quad (17)$$

for all $p_{2,3} = 0, 1$ and for all $\beta_{2,3} \in [0, 2\pi)$. From these definitions, we see that the mappings

$$\tilde{G}_{p_2} \rightarrow (-1)^{p_2} \text{ and } \tilde{G}_{p_3} \rightarrow (-1)^{p_3} \quad (18)$$

for all $p_{2,3} = 0, 1$, imply $\tilde{G}_{p_2} \cong Z_2^{p_2}$ and $\tilde{G}_{p_3} \cong Z_2^{p_3}$. Furthermore, it is clear that $[\tilde{G}_{p_2}, \tilde{G}_{p_3}] = 0$ for every $p_{2,3} = 0, 1$. Hence, $G_\nu \cong Z_2^{p_2} \times Z_2^{p_3}$.

A similar situation occurs in the charged lepton sector, in which it is possible to uniquely map

$$\tilde{T}_{\beta_2} \rightarrow e^{i\beta_2} \text{ and } \tilde{T}_{\beta_3} \rightarrow e^{i\beta_3} \quad (19)$$

for all $\beta_{2,3} \in [0, 2\pi)$, implying $\tilde{T}_{\beta_2} \cong U(1)_{\beta_2}$ and $\tilde{T}_{\beta_3} \cong U(1)_{\beta_3}$. The $\tilde{T}_{\beta_{2,3}}$ also commute ($[\tilde{T}_{\beta_2}, \tilde{T}_{\beta_3}] = 0$ for all $\beta_{2,3} \in [0, 2\pi)$). Therefore, $G_e \cong U(1)_{\beta_2} \times U(1)_{\beta_3}$. Hence, the most general, non-redundant set of residual lepton symmetries is

$$G_\nu \cong Z_2^{p_2} \times Z_2^{p_3}, \quad G_e \cong U(1)_{\beta_2} \times U(1)_{\beta_3}, \quad (20)$$

in which we remind the reader that the superscripts (subscripts) parameterize all elements of the discrete (continuous) symmetry group G_ν (G_e).

Within the context of discrete lepton family symmetries, it is clear that to obtain both G_ν and G_e from the spontaneous breaking of a specific discrete family symmetry group G_f , the infinite parameter space of the continuous group G_e must be restricted. To this end, let us consider the situation in which $\beta_{2,3} = 2\pi k_{2,3}/n_{2,3}$, where $n_{2,3}$ are integers that divide the

⁵Compare to $\theta_e = -(\beta_1 + \beta_2 + \beta_3)/3$.

⁶This is a slight abuse of notation for $p_{1,2,3}$ and $\beta_{1,2,3}$, as they *actually* should be $p'_{1,2,3}$ and $\beta'_{1,2,3}$, where $p'_1 = 2p_1 + p_2 + p_3$, $p'_2 = 2p_2 + p_1 + p_3$, $p'_3 = 2p_3 + p_1 + p_2$, $3\beta'_1 = 2\beta_1 - \beta_2 - \beta_3$, $3\beta'_2 = 2\beta_2 - \beta_1 - \beta_3$, $3\beta'_3 = 2\beta_3 - \beta_1 - \beta_2$.

order of the discrete group G_f and $k_{2,3} = 0, \dots, n_{2,3} - 1$. In this case, the set of residual charged lepton symmetries is expressible as

$$T_{k_2/n_2, k_3/n_3}^{\text{Diag}} = \text{Diag}(e^{-2\pi i(n_3 k_2 + n_2 k_3)/n_2 n_3}, e^{2\pi i k_2/n_2}, e^{2\pi i k_3/n_3}), \quad (21)$$

in which $k_{2,3} = 0, \dots, n_{2,3} - 1$. This clearly represents the elements of $G_e \cong Z_{n_2} \times Z_{n_3}$. Furthermore, if the mixing from the charged lepton sector is to be completely determined, this fixes $n_{2,3} \geq 2$ from the previous logic when discussing two-fold rephasing degeneracies.

Eq. (21) demonstrates that it is never possible to have a diagonal entry of $T_{k_2/n_2, k_3/n_3}^{\text{Diag}}$ that is always real (i.e., it is unity), because $k_{2,3}/n_{2,3} = 0$ for only $k_{2,3} = 0$ and not the remaining $k_{2,3} = 1, \dots, n_{2,3} - 1$. To impose this criteria, one of the phases, for example $\beta_{1,2,3}$ in Eq. (16), must be set to zero. Without loss of generality, let us take $\beta_2 = -\beta_3 = \beta$ ($k_2/n_2 = -k_3/n_3 = k/n$), so that $\beta_1 = 0$. Equation (21) then becomes

$$T_{k/n}^{\text{Diag}} = \text{Diag}(1, e^{-2\pi i k/n}, e^{2\pi i k/n}), \quad (22)$$

in which $k = 0, \dots, n - 1$. Physically, this case corresponds to the complete breaking of one of the original $U(1)$ symmetries, because $Z_n \cong Z_1 \times Z_n$, in which Z_1 is the (trivial) symmetry of a completely broken $U(1)$. Then, as previously discussed, to completely fix charged lepton mixing in this case, it is necessary to impose $n \geq 3$. Therefore, for the remainder of this work, we will assume that if $G_e \cong Z_{n_2} \times Z_{n_3}$, then $n_{2,3} \geq 2$, or if $G_e \cong Z_n$, then $n \geq 3$.

Turning to the neutrino sector, we see that removing the redundant rephasing choices from Eq. (2) with $\text{Det}(Q_\nu) = -1$ yields the well-known result that in the diagonal basis, the elements $(G_\nu)_{i=0,1,2,3} \equiv G_i^{\text{Diag}}$ are given by

$$\begin{aligned} G_0^{\text{Diag}} &= \text{Diag}(1, 1, 1), \quad G_1^{\text{Diag}} = \text{Diag}(1, -1, -1), \\ G_2^{\text{Diag}} &= \text{Diag}(-1, 1, -1), \quad G_3^{\text{Diag}} = \text{Diag}(-1, -1, 1). \end{aligned} \quad (23)$$

This is of course the aforementioned group $G_\nu \cong Z_2^{p_2} \times Z_2^{p_3}$ (cf. Eq. (20)) that is isomorphic to the Klein symmetry group. The previous logic then dictates that we must preserve/impose two⁷ of these nontrivial elements to completely fix U_ν because preserving only one nontrivial element yields a “free” unitary rotation (see Eqs. (8)-(9)).

An inspection of Eq. (23) demonstrates another striking (but perhaps obvious) difference between the largest possible sets of non-redundant neutrino symmetries and charged lepton symmetries. The issue is that while the order of all of the nontrivial neutrino symmetry elements is two, the order of the nontrivial charged lepton symmetries fluctuates depending on the ratio of $\beta_{2,3} = k_{2,3}/n_{2,3}$ or $\beta = k/n$. More precisely, Eqs. (21)-(22) show that the order of the elements of G_e can oscillate between multiple values depending on the value of $k_{2,3} = 0, 1, \dots, n_{2,3} - 1$ or $k = 0, \dots, n - 1$. Hence, we see that

$$(T_{k_2/n_2, k_3/n_3}^{\text{Diag}})^{n_2 n_3} = 1, \quad (T_{k/n}^{\text{Diag}})^n = 1. \quad (24)$$

The orders of $T_{k_2/n_2, k_3/n_3}^{\text{Diag}}$ and $T_{k/n}^{\text{Diag}}$ are thus given by $n_2 n_3$ and n , respectively, unless special conditions are met. For example, if there exist integers $l_{2,3}$ and l such that $n_{2,3} = l_{2,3} k_{2,3}$ and $n = lk$,⁸ then

$$(T_{k_2/n_2, k_3/n_3}^{\text{Diag}})^{l_2 l_3} = (T_{1/l_2, 1/l_3}^{\text{Diag}})^{l_2 l_3} = 1, \quad (T_{k/n}^{\text{Diag}})^l = (T_{1/l}^{\text{Diag}})^l = 1, \quad (25)$$

⁷Actually, when preserving two nontrivial elements of such a symmetry, the third comes for free because $G_i^{\text{Diag}} G_j^{\text{Diag}} = G_k^{\text{Diag}}$ for $i \neq j \neq k$.

⁸Note that $l_{2,3}, l \neq 1$ because by definition $k = 0, \dots, n - 1$ and $k_{2,3} = 0, \dots, n_{2,3} - 1$.

which reduces the order of $T_{k_2/n_2, k_3/n_3}^{\text{Diag}}$ from $n_2 n_3$ to $l_2 l_3$ and the order of $T_{k/n}^{\text{Diag}}$ from n to l . A further reduction occurs in $T_{1/l}^{\text{Diag}}$ if $l = 0$ ($k = 0$), as the order of T_l^{Diag} then is 1. Similarly, if k_2 (k_3) is 0, the order of $T_{k_2/n_2, k_3/n_3}^{\text{Diag}}$ is n_3 (n_2), and if $k_2 = k_3 = 0$, the order of $T_{0,0}^{\text{Diag}}$ is 1.

Hence, an investigation of the orders of all possible elements that can be contained in the residual symmetries G_e and G_ν (assuming these symmetries originate from a discrete symmetry group G_f), shows different results for the two sectors. More precisely, G_ν contains only (3) order 2, nontrivial elements, whereas the orders of the nontrivial elements of G_e can have a wide range of values. These values depend on the ratios $\beta_{2,3} = k_{2,3}/n_{2,3}$ ($\beta = k/n$), in which $k_{2,3} = 0, \dots, n_{2,3} - 1$, and $n_{2,3} \geq 2$ to completely fix charged lepton mixing.

3 Phases and Generalized CP

Having analyzed the relevance of unphysical phases in fixing lepton mixing predictions and explored their possible group theoretical origin in terms of flavor symmetries, we next expand the logic as set forth in Section 2 to include that of generalized CP symmetries. These conditions take the form [42]

$$X_\nu^T M_\nu X_\nu = M_\nu^* \quad (26)$$

and

$$Y_e^\dagger M_e Y_e = M_e^*. \quad (27)$$

In analogy to the the case of the flavor symmetries T_e and S_ν as defined in Eq. (7), Y_e and X_ν represent (possibly infinite) sets of transformations.

To understand the similarities and differences between Eqs. (26)–(27), we will elaborate on the discussion in [40], so that it may be easily extended/connected to the charged lepton sector. We begin with the diagonal neutrino basis. From Eq. (26), we have

$$(X_\nu^{\text{Diag}})^T M_\nu^{\text{Diag}} X_\nu^{\text{Diag}} = (M_\nu^{\text{Diag}})^* \quad (28)$$

with

$$X_\nu^{\text{Diag}} = \text{Diag}(\pm e^{i\alpha'_1}, \pm e^{i\alpha'_2}, \pm e^{i\alpha'_3}). \quad (29)$$

In the above, α'_i is given by $\alpha'_i = \alpha_i + \theta_\nu$, in which θ_ν keeps track of an arbitrary global phase redefinition of M_ν . It is next useful to define

$$X_0^{\text{Diag}} = \text{Diag}(e^{i\alpha'_1}, e^{i\alpha'_2}, e^{i\alpha'_3}) \quad (30)$$

to separate the transformations contained in Q_ν from the X_ν^{Diag} in Eq. (29). From these relations, it is straightforward to obtain [40]

$$X_\nu^{\text{Diag}} = Q_\nu \times X_0^{\text{Diag}}. \quad (31)$$

In analogy with Eq. (6), we then have [40]

$$X_\nu = U_\nu X_\nu^{\text{Diag}} U_\nu^T, \quad (32)$$

which relates X_ν^{Diag} to its generally non-diagonal form X_ν .

For the charged lepton sector, we proceed as before from Eq. (27). We now have

$$(Y_e^{\text{Diag}})^\dagger M_e^{\text{Diag}} Y_e^{\text{Diag}} = (M_e^{\text{Diag}})^* = M_e^{\text{Diag}}, \quad (33)$$

in which we recall that M_e is a Hermitian matrix. From the form of Eq. (3), we see that Y_e^{Diag} takes the form

$$Y_e^{\text{Diag}} = \text{Diag}(e^{i\gamma_1}, e^{i\gamma_2}, e^{i\gamma_3}), \quad (34)$$

where $\gamma_i \in [0, 2\pi)$, cf. Eq. (29). Hence, it should be clear that if we wish to separate the infinitely many nontrivial elements of Q_e from Y_e^{Diag} , it is useful to define

$$Y_0^{\text{Diag}} = \text{Diag}(e^{i\beta'_1}, e^{i\beta'_2}, e^{i\beta'_3}), \quad (35)$$

in which $\beta'_i = \gamma_i - \beta_i$ and β_i is as defined in Eq. (4), so that in analogy with Eq. (31),

$$Y_e^{\text{Diag}} = Q_e \times Y_0^{\text{Diag}}. \quad (36)$$

It is interesting to note [43] that Y_e^{Diag} can be related to its non-diagonal form with a relationship similar to that of Eq. (32), as follows:

$$Y_e = U_e Y_e^{\text{Diag}} U_e^T. \quad (37)$$

We note that even though both Eq. (32) and Eq. (37) preserve the relationships in Eq. (26) and Eq. (27) when $U_\nu \rightarrow U_\nu Q_\nu$ and $U_e \rightarrow U_e Q_e$, they are not generally preserved individually. To see this explicitly, let Q'_ν and Q'_e represent a different choice of unphysical rephasings. Applying $U_\nu \rightarrow U_\nu Q'_\nu$ and $U_e \rightarrow U_e Q'_e$ to Eq. (32) and Eq. (37) yields

$$X_\nu \rightarrow X'_\nu = U_\nu Q_\nu Q'^2_\nu U_\nu^T \text{ and } Y_e \rightarrow Y'_e = U_e Q_e Q'^2_e U_e^T. \quad (38)$$

Since $Q'^2_\nu = 1$ for any choice of phases in Q'_ν , cf. Eq. (2), clearly $X'_\nu = X_\nu$ always. On the other hand, $Q'^2_e \neq 1$ for arbitrary choices of phases in Q'_e , cf. Eq. (4). Therefore, we generally have that $Y_e \neq Y'_e$.

This may naively seem problematic, but in actuality these results are consistent. The consistency of the two mappings in Eq. (38) can be seen by observing that both $Q_\nu Q'^2_\nu$ and $Q_e Q'^2_e$ are actually elements of the original set of unphysical phase choices. In other words, $Q'^2_\nu = 1$ is one of the possible choices of unphysical phase choices (albeit trivial) in Eq. (2), and Q'^2_e is one of the possible phase choices in Eq. (4). Therefore, both mappings in Eq. (38) reduce to those given in Eq. (32) and Eq. (37) as long as we demand all such phase choices to exist. However, it is important to note that it even though the mappings in Eq. (38) can be defined consistently, they can in some cases have important implications for the preserved underlying symmetries.

For example, such mappings may affect the explicit forms of the symmetry elements. In the charged lepton sector, the mapping $U_e \rightarrow U_e Q_e$ can fundamentally alter Eq. (37) because it (potentially) alters the trace of Y_e^{Diag} , as follows:

$$\text{Tr}(Y_e^{\text{Diag}}) \rightarrow \text{Tr}(Q_e Y_e^{\text{Diag}} Q_e^T) = \text{Tr}(Y_e'^{\text{Diag}}), \quad (39)$$

and thus it can change the trace of Y_e^{Diag} to that of $Y_e'^{\text{Diag}}$ if $Q_e^2 \neq 1$. As one may guess, the analogous mapping in the neutrino sector, i.e., $U_\nu \rightarrow U_\nu Q_\nu$, does not have the same effect because it leaves Eq. (38) invariant, since $Q_\nu^2 = 1$. Such a *trace-changing* result for the X_ν^{Diag} can be produced through slightly different means, namely that of a global phase redefinition, cf. Eq. (14). Such a global phase redefinition changes the traces of the X_ν^{Diag} thereby affecting their group character when $\theta_\nu \neq 0$, cf. Eq. (29). Thus, here we see another difference between the two sectors. Note also that Eq. (39) represents a transformation between the diagonal elements and will not transform to an object outside of the set Q_e possibly *unless* the original infinite set of symmetries has been reduced to a smaller more discrete set. However, globally

rephasing M_ν represents an unphysical phase shift to the complex neutrino masses in Eq. (1), and even by including all such symmetries/phase choices, may lead to something that can no longer be realized in a specific family symmetry group.

To this end, let us now turn to the consideration of generalized CP within the context of discrete lepton family symmetries. It is very well known (see e.g. [12, 21, 30, 40, 44]) that a flavor symmetry element can be formed from the action of two generalized CP transformations. For the neutrino sector, we see from Eqs. (1) and (26) that for the generalized CP elements $(X_\nu)_{i=0,1,2,3} \equiv X_i$, we have

$$X_j^\dagger X_i^T M_\nu X_i X_j^* = M_\nu, \quad (40)$$

for $i, j = 0, 1, 2, 3$. By comparing Eq. (40) to Eqs. (29)-(31), it should be clear that the indices in the above are fixed by the Klein symmetry group as given in Eq. (23). Furthermore, we recall that for the phenomenologically desirable case of nondegenerate neutrino masses, $X_i X_j^* \in Z_2 \times Z_2$. Therefore, a flavor symmetry element $(G_\nu)_k \equiv G_k$ can be expressed as

$$G_k = X_i X_j^* \quad (41)$$

for $i, j, k = 0, 1, 2, 3$. For the charged leptons, in analogy, we find from Eqs. (3) and (27) that the generalized CP elements $(Y_e)_{k,l} \equiv Y_{k,l}$

$$Y_l^T Y_k^\dagger M_e Y_k Y_l^* = M_e, \quad (42)$$

which in turn implies that the flavor symmetry element $(T_e)_m \equiv T_m$ is given by

$$T_m = Y_k Y_l^*, \quad (43)$$

for some $m, k, l = 0, \dots, n-1$ if $G_e \cong Z_n$ or $m, k, l = 0, \dots, n_2 + n_3 - 1$ if $G_e \cong Z_{n_2} \times Z_{n_3}$.⁹ Here we note that it is important to realize that the phases α'_i contained in X_0^{Diag} , cf. Eq. (30), are analogous to the β'_i phases contained in Y_0^{Diag} , cf. Eq. (35). Thus, Eq. (40) and Eq. (43) demonstrate that it is possible to relate two successive generalized CP transformations to a flavor symmetry transformation. Note that the complex conjugation in Eq. (41) and Eq. (43) plays a crucial role in that it allows the use of unitarity to reduce the products of flavor symmetry elements when expressed in terms of generalized CP symmetry transformations.

The fact that family symmetry transformations can be expressed in terms of generalized CP transformations has further implications. As seen in Eq. (31) and Eq. (36), it is also possible to express the generalized CP transformations in terms of flavor symmetry transformations. By further assuming Y_e and X_ν are elements of a discrete family symmetry group so that the set of possible phases Q_e/Y_e can possess is reduced, there will exist integers p and q such that $(X_\nu^{\text{Diag}})^p = (Y_e^{\text{Diag}})^q = 1$. From Eq. (31) and Eq. (36), we then have

$$(X_\nu^{\text{Diag}})^p = Q_\nu^p \times \text{Diag}(e^{ip\alpha'_1}, e^{ip\alpha'_2}, e^{ip\alpha'_3}) = 1 \quad (44)$$

and

$$(Y_e^{\text{Diag}})^q = Q_e^q \times \text{Diag}(e^{iq\beta'_1}, e^{iq\beta'_2}, e^{iq\beta'_3}) = 1. \quad (45)$$

For the case of the charged lepton sector, we see that if Eq. (45) is ever to be satisfied without tuning β_i contained in Q_e and β'_i against each other so that they cancel, q must be a multiple of the order of Q_e (such that $Q_e^q = 1$) and $\beta'_i = 2\pi k'_i/q$ for some $k'_i = 0, 1, \dots, q-1$ (such

⁹Note that we could not just apply one general CP transformation to M_e to get a relationship between T_e and Y_e because the resulting space-time indices of the fields would be inconsistent.

that $(Y_0^{\text{Diag}})^q = 1$.¹⁰ Following the same logic for the neutrino sector, it is straightforward to deduce that $\alpha'_i = 2\pi a_i/p$ for some $a_i = 0, 1, \dots, p-1$, in which p must be an even integer ($Q_\nu^2 = 1$) to satisfy Eq. (44). As a result, X_ν^{Diag} must be of *even* order, confirming the results of [12]. Additionally, X_0^{Diag} from Eq. (31) must be the same even order (even though G_0 is order one) because $\alpha'_i = 2\pi a_i/p$ for some $a_i = 0, 1, \dots, p-1$, which it has inherited from the nontrivial Klein group elements. A similar situation occurs for Y_0^{Diag} in Eq. (36); i.e., if $Q_e = 1$, the order of the corresponding Y_e^{Diag} need not be unity.

We note that the previous result that p must be even and q must be a multiple of the order of Q_e may not hold in the nondiagonal basis, i.e., for X_ν and Y_e . This certainly is the case if U_ν and U_e are real so that $U_\nu = U_\nu^*$ and $U_e = U_e^*$, i.e., $U_e U_e^T = 1$ and $U_\nu U_\nu^T = 1$. However, this may not always be true. Thus with an eye towards deriving the most general conditions which X_ν and Y_e must satisfy so that their orders are p and q respectively, we proceed by inverting the transformations in Eq. (32) and Eq. (37):

$$(X_\nu^{\text{Diag}})^p = (U_\nu^\dagger X_\nu U_\nu^*)^p = 1 \text{ and } (Y_e^{\text{Diag}})^q = (U_e^\dagger Y_e U_e^*)^q = 1. \quad (46)$$

Taking the determinant of both of the above relations leads to

$$\text{Det}(U_\nu^*)^{2p} \text{Det}(X_\nu)^p = 1 \text{ and } \text{Det}(U_e^*)^{2q} \text{Det}(Y_e)^q = 1. \quad (47)$$

We also note that the above conditions relating the determinants of X_ν and Y_e to U_ν^* and U_e^* are invariant under the transformation $U_\nu \rightarrow U_\nu Q_\nu$ and $U_e \rightarrow U_e Q_e$ because $Q_e^{2q} = Q_\nu^{2p} = 1$.

4 Generalization of Bottom-Up Constructions

We now discuss how these considerations allow for a generalization of the bottom-up approach given in [40]. This approach, which was based on the hypothesis that the full Klein symmetry is preserved in the neutrino sector such that lepton mixing is fully determined (up to charged lepton rephasings), is summarized as follows. The Klein generators and the generalized CP symmetry elements of the neutrino sector can be expressed as a function of the lepton mixing parameters in the basis in which the charged lepton sector is diagonal. Up to leptonic rephasings, in this basis U_ν can be written as

$$U_\nu = U_e^{\text{Diag}} U_{\text{MNSP}}. \quad (48)$$

Hence, for U_e^{Diag} as the identity, U_ν can be parametrized in terms of the MNSP mixing parameters. With this form of U_ν , the Klein symmetry elements $(G_\nu)_i \equiv G_i$ and the generalized CP elements $(X_\nu)_i \equiv X_i$ can then be constructed explicitly as a function of the lepton mixing parameters, as follows¹¹ (see Eq. (32)):

$$G_i = U_\nu G_i^{\text{Diag}} U_\nu^\dagger, \quad X_i = U_\nu X_i^{\text{Diag}} U_\nu^T. \quad (49)$$

This analysis was then carried out for several popular model scenarios for the MNSP mixing parameters [40]. We note that in addition to working in the diagonal charged lepton sector basis, we also made simplifying assumptions in [40] about the leptonic sector phases. More precisely, we parametrized U_ν as follows:

$$U_\nu = P R_1(\theta_{23}) R'_2(\theta_{13}, \delta) R_3(\theta_{12}), \quad (50)$$

¹⁰In the unlikely case that $\beta_i + \beta'_i = 0$ for every i , then the orders of Y_e and T_e are identical.

¹¹Recall that the G_i are what we previously called S_ν in Eq. (6) with positive determinant.

in which the R_i (R'_i) are the usual (complexified) rotation matrices involving the leptonic mixing angles $(\theta_{12}, \theta_{23}, \theta_{13})$ and the leptonic Dirac phase δ , while P is a diagonal matrix of the form $P = \text{Diag}(1, 1, -1)$. This choice is thus very similar to the standard PDG form, except for the inclusion of P and the neglect of the Majorana phases, which are included in the complex neutrino mass eigenvalues, as given in Eq. (1).

We will now revisit this analysis in the context of this work. We will primarily focus once again on the case of a diagonal charged lepton sector, but now allowing for general leptonic phases. As we will see, in this case useful relations can be obtained that can be of utility in top-down model building scenarios based on discrete groups. We will also comment briefly on the case of a non-diagonal charged lepton sector, in which case the unphysical phases can play a role in the connection of the Klein and generalized CP symmetry elements in both sectors, depending on how U_e and U_ν are reconstructed from the lepton data.

Starting with the case of the diagonal charged lepton basis, it is straightforward to see that the inclusion of nontrivial leptonic phases can affect the reconstruction of U_ν as given in Eq. (48). More precisely, since we have set U_e to be the identity, the inclusion of a general phase matrix Q_e means that Eq. (48) is shifted to

$$U_\nu = U_e^{\text{Diag}} Q_e U_{\text{MNSP}}. \quad (51)$$

As a result, the phases in Q_e can explicitly enter the Klein and generalized CP symmetry elements. Indeed, from Eq. (50), the matrix P itself can be interpreted as a specific choice of Q_e (one with a negative determinant). As discussed in [40], this choice was made for convenience in making an identification between the Klein and generalized CP symmetry elements with standard representations of elements of specific discrete groups. However, another interpretation of P can be understood from considering general leptonic phases, as follows. Let us now take the case in which U_ν is instead parametrized by

$$U_\nu = Q_e P R_1(\theta_{23}) R'_2(\theta_{13}, \delta) R_3(\theta_{12}) \equiv P' R_1(\theta_{23}) R'_2(\theta_{13}, \delta) R_3(\theta_{12}), \quad (52)$$

where Q_e are the charged lepton phases as given in Eq. (4), and we define the matrix P' as

$$P' = \text{Diag}(e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3}) \equiv Q_e P = \text{Diag}(e^{i\beta_1}, e^{i\beta_2}, -e^{i\beta_3}). \quad (53)$$

Therefore, in the basis in which the charged leptons are diagonal, the charged lepton rephasing degrees of freedom as discussed in this paper can be interpreted, in this context, as the following transformation of U_ν :

$$U_\nu \rightarrow Q_e U_\nu. \quad (54)$$

In direct contrast to the Klein symmetry transformations that we previously discussed, for which $U_\nu \rightarrow U_\nu Q_\nu$, it is clear that Eq. (54) is not a symmetry of Eq. (1), but rather changes the specific M_ν that results in a given M_ν^{Diag} . As a result, the Klein and generalized CP elements are necessarily modified, according to Eq. (49). As we will now discuss, these modifications can elucidate certain aspects of connecting these symmetry elements to specific elements of an assumed discrete symmetry group.

For the Klein generators, since the G_i are related to their diagonal forms via a unitary transformation (see Eq. (49)), it is straightforward to see that the elements of G_i are then modified from their forms as given in [40], as follows:

$$(G_i)_{rs} \rightarrow (G_i)_{rs} e^{i(\beta_r - \beta_s)}, \quad (r, s = 1, 2, 3), \quad (55)$$

and we recall that $\beta_{1,2} = \phi_{1,2}$ and $\beta_3 = \phi_3 \pm \pi$ (see Eq. (53)). We see that the diagonal entries of the G_i (and hence the trace) are unaffected by this rephasing, but the off-diagonal

entries are changed. In addition, the modified G_i clearly satisfy the standard Klein relations

$$G_i^2 = 1, \quad G_0 G_{i=1,2,3} = G_{i=1,2,3}, \quad G_i G_j = G_k (i \neq j \neq k \neq 0). \quad (56)$$

Such rephasings, while unphysical, can be helpful in the context of top-down model building based on discrete symmetry groups. As an example, there is a known connection in this context between the eigenvector of each of the $G_{i=1,2,3}$ with a positive +1 eigenvalue and the corresponding (i th) column of the MNSP matrix (up to permutations). This one-to-one correspondence holds irrespective of whether nontrivial charged lepton rephasings are included in the parameterization of U_ν or not. To see the ways in which including the phases can be informative, for concreteness let us express the Klein element G_3 as a function of the mixing parameters and the lepton rephasings. Using Eq. (52) and Eq. (53), G_3 then takes the following form (see also [40]):

$$G_3 = \begin{pmatrix} -c'_{13} & e^{-i(\delta-\phi_1+\phi_2)} s'_{13} s_{23} & e^{-i(\delta-\phi_1+\phi_3)} s'_{13} c_{23} \\ e^{i(\delta-\phi_1+\phi_2)} s'_{13} s_{23} & -c_{13}^2 c_{23}' - s_{13}^2 & e^{i(\phi_2-\phi_3)} c_{13}^2 s'_{23} \\ e^{i(\delta-\phi_1+\phi_3)} c_{23} s'_{13} & e^{-i(\phi_2-\phi_3)} c_{13}^2 s'_{23} & c_{13}^2 c_{23}' - s_{13}^2 \end{pmatrix}, \quad (57)$$

in which (as in [40]), $s_{ij} = \sin \theta_{ij}$, $c_{ij} = \cos \theta_{ij}$, $s'_{ij} = \sin 2\theta_{ij}$, and $c'_{ij} = \cos 2\theta_{ij}$.

Let us now consider the class of models in which $\theta_{23} = \pi/4$ and $\theta_{13} = 0$. This includes the well-known tribimaximal (TBM) mixing scenario [45], for which the solar mixing angle is given by

$$\theta_{12}^{\text{TBM}} = \tan^{-1} \left(\frac{1}{\sqrt{2}} \right). \quad (58)$$

It also includes other situations, such as golden ratio (GR1) mixing [46–49] for which

$$\theta_{12}^{\text{GR1}} = \tan^{-1} \left(\frac{1}{\phi} \right), \quad (59)$$

which depends on the golden ratio $\phi = (1 + \sqrt{5})/2$, as well as other scenarios (see e.g. [6] for a detailed review). In this class of models, it is straightforward to see from Eq. (57) that the Klein element G_3 takes the form

$$G_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & e^{i(\phi_2-\phi_3)} \\ 0 & e^{-i(\phi_2-\phi_3)} & 0 \end{pmatrix}. \quad (60)$$

In the case of tribimaximal mixing, this can be identified with the canonical U generator of the discrete group S_4 , which is known to be the minimal group that contains the three Klein elements SU , S , and U that generate tribimaximal mixing when they are preserved [20]. As is well known, in the group representation typically used in the literature (see e.g. [22]), the S_4 U generator is given by

$$U = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}. \quad (61)$$

Hence, for the choice of phases with $\phi_2 - \phi_3 = \pm\pi$, we see from Eq. (60) that we can make the identification that $G_3 = U$. Clearly, in addition, to make the connection with discrete groups, the phases ϕ_i need to be consistent with a specific subgroup of the discrete symmetry. A minimal implementation of this condition is simply to set $\phi_2 = 0$ and $\phi_3 = \pm\pi$, which results in $P' = P$, as used in [40]. Similar statements can be made for the identification

of G_3 as an element of A_5 , the minimal group that results in GR1 mixing [49]. The same considerations can be explored for the $G_{1,2}$ Klein elements, which also depend on θ_{12} . It can easily be shown that the same phase choice results in the identification of $G_1 = SU$ and $G_2 = S$ of S_4 for the case of tribimaximal mixing, and that a similar identification holds for the elements $G_{1,2}$ in A_5 for GR1 mixing.

An example in which $\theta_{13} \neq 0$ is the case of bitrimaximal (BTM) mixing [50,51], for which the neutrino sector mixing matrix is given by

$$U_\nu^{\text{BTM}} = \begin{pmatrix} \frac{1}{6}(3 + \sqrt{3}) & \frac{1}{\sqrt{3}} & \frac{1}{6}(3 - \sqrt{3}) \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{6}(-3 + \sqrt{3}) & \frac{1}{\sqrt{3}} & \frac{1}{6}(-3 - \sqrt{3}) \end{pmatrix}. \quad (62)$$

For a diagonal charged lepton sector, the lepton mixing angles and Dirac CP-violating phase in this scheme are given by

$$\theta_{12}^{\text{BTM}} = \theta_{23}^{\text{BTM}} = \tan^{-1}(\sqrt{3} - 1), \quad \theta_{13}^{\text{BTM}} = \sin^{-1}\left(\frac{1}{6}(3 - \sqrt{3})\right), \quad \delta^{\text{BTM}} = 0. \quad (63)$$

The BTM mixing pattern as outlined above can naturally arise from the spontaneous breaking of a $\Delta(96)$ flavor symmetry [50,51], which is in fact the smallest group for realizing BTM mixing. Extending the discussion of [40] shows that with the inclusion of general charged lepton rephasings, we see for example that G_3 takes the form

$$G_3 = \begin{pmatrix} -\frac{1}{3} - \frac{1}{\sqrt{3}} & \left(-\frac{1}{3} + \frac{1}{\sqrt{3}}\right) e^{i(\phi_1 - \phi_2)} & \frac{1}{3} e^{i(\phi_1 - \phi_3)} \\ \left(-\frac{1}{3} + \frac{1}{\sqrt{3}}\right) e^{-i(\phi_1 - \phi_2)} & -\frac{1}{3} & \left(\frac{1}{3} + \frac{1}{\sqrt{3}}\right) e^{i(\phi_2 - \phi_3)} \\ \frac{1}{3} e^{-i(\phi_1 - \phi_3)} & \left(\frac{1}{3} + \frac{1}{\sqrt{3}}\right) e^{-i(\phi_2 - \phi_3)} & -\frac{1}{3} + \frac{1}{\sqrt{3}} \end{pmatrix}, \quad (64)$$

which again reduces to the canonical form for G_3 as given in the literature for $\Delta(96)$ models in the case that $\phi_1 - \phi_2 = 0$, $\phi_1 - \phi_3 = \pm\pi$, $\phi_2 - \phi_3 = \pm\pi$. Again, a consistent and minimal implementation of this requirement is the choice of P , as before.

For the generalized CP symmetry elements, the situation is different since the X_i are not related to their diagonal counterparts by a standard similarity transformation, but instead by $X_i = U_\nu X_i^{\text{Diag}} U_\nu^T$, as given in Eq. (32) and Eq. (49). We see that both the Q_e phases and the overall Majorana phase shift θ_ν (see the discussion just after Eq. (29)) affect both the diagonal and the off-diagonal entries of the X_i , and thus the traces are also affected. More explicitly, using Eq. (52), the generalized CP symmetry elements given in [40] take the form

$$(X_i)_{rs} \rightarrow (X_i)_{rs} e^{i(\beta_r + \beta_s)}, \quad (r, s = 1, 2, 3), \quad (65)$$

in which once again the α_i in the expressions for X_i in [40] are to be replaced by $\alpha'_i = \alpha_i + \theta_\nu$ (see Eq. (29)), and we recall the relation between the β_i and ϕ_i as given in Eq. (53). As discussed in [40], if the Majorana phases α'_i are trivial, then for the choice of $\phi_1 - \phi_2 = 0$, $\phi_1 - \phi_3 = \pm\pi$, $\phi_2 - \phi_3 = \pm\pi$, for the case of tribimaximal mixing, the X_i are identical to the Klein elements of the identity, S , SU , and U of S_4 , which are elements of the automorphism group of S_4 .¹² Analogous statements can be made for the cases of GR1 mixing and bitrimaximal mixing in the case of trivial Majorana phases [40].

¹²Recall that a nontrivial prediction for the Dirac phase δ was obtained in S_4 for the case in which a single Z_2 flavor symmetry element was preserved [22].

However, in the case of bitrimaximal mixing and the connection to its minimal discrete group, $\Delta(96)$, there can also be nontrivial rephasings that yield new possible candidates for the generalized CP symmetry elements. In such situations, we can be guided by the general relations given in Eq. (47). For the case in which the phases of Q_e are set to zero, we have

$$\text{Det}(X_i^{\text{BTM}})^p = 1, \quad (66)$$

and hence the order of the product of the eigenvalues of X_i^{BTM} must be an even integer p (note that $\text{Det}(X_i)$ need not be 1, as this is not a necessary condition for Eq. (47)). This can also be seen that by noting that for $Q_e = 1$, $U_\nu^{\text{BTM}} = (U_\nu^{\text{BTM}})^*$, and hence Eq. (32) is a similarity transformation that preserves the eigenvalues of $(X_i^{\text{BTM}})^{\text{Diag}}$. However, if nontrivial phases are included in U_ν as given in Eq. (52), Eq. (66) instead is modified to

$$\text{Det}(X_i^{\text{BTM}})^p = e^{2i(\phi_1 + \phi_2 + \phi_3)}, \quad (67)$$

and recall that when we consider discrete symmetry groups, the ϕ_i by necessity do not take on continuous values, but instead discrete values consistent with group transformations.

With these results in mind, we now consider the group theory of $\Delta(96)$, which is arguably all derivable from the character table of $\Delta(96)$, as given in Table 1. The character table can

$\Delta(96)$	1	1'	2	3	$\tilde{3}$	$\bar{3}$	3'	$\tilde{3}'$	$\bar{3}'$	6
\mathcal{I}	1	1	2	3	3	3	3	3	3	6
$3C_4$	1	1	2	$-1 + 2i$	-1	$-1 - 2i$	$-1 + 2i$	-1	$-1 - 2i$	2
$3C_2$	1	1	2	-1	3	-1	-1	3	-1	-2
$3C_4'$	1	1	2	$-1 - 2i$	-1	$-1 + 2i$	$-1 - 2i$	-1	$-1 + 2i$	2
$6C_4''$	1	1	2	1	-1	1	1	-1	1	-2
$32C_3$	1	1	-1	0	0	0	0	0	0	0
$12C_2'$	1	-1	0	-1	-1	-1	1	1	1	0
$12C_8$	1	-1	0	i	1	$-i$	$-i$	-1	i	0
$12C_4'''$	1	-1	0	1	-1	1	-1	1	-1	0
$12C_8'$	1	-1	0	$-i$	1	i	i	-1	$-i$	0

Table 1: The Character Table of $\Delta(96)$, where kC_n denotes a conjugacy class of k elements all of order n .

be used to deduce the sum of the eigenvalues of X_i^{BTM} , as this is just the trace/character of this element. Thus, we can immediately restrict ourselves to considering the 3-dimensional irreducible representations of $\Delta(96)$, i.e., the **3**, **3'**, **$\tilde{3}$** , **$\tilde{3}'$** , **$\bar{3}$** , and **$\bar{3}'$** . We further restrict ourselves to the four faithful 3-dimensional irreducible representations, i.e., the **3**, **3'**, **$\tilde{3}$** , and **$\tilde{3}'$** , as the **$\bar{3}$** and **$\bar{3}'$** furnish unfaithful representations of $\Delta(96)$ that are isomorphic to S_4 .

For concreteness, let us first set the phases of Q_e to zero, then consider the implications of a nonzero Q_e . From Eq. (32), we thus obtain for a trivial Q_e that

$$\text{Tr}(X_i^{\text{BTM}}) = (-1)^{p_1} e^{i\alpha'_1} + (-1)^{p_2} e^{i\alpha'_2} + (-1)^{p_3} e^{i\alpha'_3}, \quad (68)$$

while allowing for nontrivial Q_e yields

$$\begin{aligned} \text{Tr}(X_i^{\text{BTM}}) &= \frac{1}{3} \left[(-1)^{p_1} e^{i\alpha'_1} (\rho_{11} e^{2i\phi_1} + e^{2i\phi_2} + \rho_{13} e^{2i\phi_3}) \right. \\ &\quad \left. + (-1)^{p_2} e^{i\alpha'_2} (e^{2i\phi_1} + e^{2i\phi_2} + e^{2i\phi_3}) + (-1)^{p_3} e^{i\alpha'_3} (\rho_{31} e^{2i\phi_1} + e^{2i\phi_2} + \rho_{33} e^{2i\phi_3}) \right], \end{aligned} \quad (69)$$

in which

$$\rho_{11} = \rho_{33} = 1 + \sqrt{3}/2, \quad \rho_{13} = \rho_{31} = 1 - \sqrt{3}/2. \quad (70)$$

We now recall the discussion of the BTM example in [40], in which it was posited that for this case, $\alpha_1 = \alpha_3 = \frac{\pi}{6}$ and $\alpha_2 = -\frac{\pi}{3}$, so that $\alpha_{31} = \alpha_3 - \alpha_1 = 0$ and $\alpha_{21} = \alpha_2 - \alpha_1 = -\pi/2$, to match known results in the literature [28]. Let us first consider the case in which the phases of Q_e are trivial. With these assumptions, we can see that for X_0^{BTM} and X_2^{BTM} , their traces satisfy

$$\text{Tr}(X_0^{\text{BTM}}) = e^{2\pi i/3 + i\theta_\nu}(-1 - 2i), \quad (71)$$

and

$$\text{Tr}(X_2^{\text{BTM}}) = e^{2\pi i/3 + i\theta_\nu}(-1 + 2i). \quad (72)$$

We thus see that if we take the overall unphysical phase shift θ_ν to be $\theta_\nu = 4\pi/3 + 2n\pi$, then Eqs. (71) and (72) are in agreement with the entries for the $3C_4$ and $3C'_4$ in Table 1. This phase shift must also be made so that the order of the X_i^{BTM} is changed from 12 to 4 as $\Delta(96)$ does not have a C_{12} conjugacy class, as seen from Table 1. Similar statements can be made for the X_1^{BTM} and X_3^{BTM} elements, with the identification of either the $6C'_4$ or the $12C'''_4$ conjugacy classes. We also note that in each case, the order of X_i is a multiple of 2, as previously discussed. Therefore, the order of the nontrivial $\Delta(96)$ Klein symmetry elements is consistent with this implementation of BTM mixing.

For nontrivial Q_e , there is more freedom to match to specific discrete groups. Here it is not just the trace of X_i that must be fixed, as it is also important to ensure that X_i respects the group multiplication laws. In certain cases, these conditions require that the ϕ_i take trivial values (0 or $\pm\pi$), i.e., the rephasing symmetry associated with the ϕ_i must be restricted to a $Z_2 \times Z_2 \times Z_2$ subgroup along with an overall phase that must itself be restricted by the group multiplication laws. This can easily be seen from the form of Eq. (69), which does not distinguish between $\phi_i = 0$ and $\phi_i = \pm\pi$. In the case previously discussed of connecting the BTM symmetry elements with group elements of $\Delta(96)$ with the nontrivial choice of α'_i as given above, it is straightforward to show that the ϕ_i must indeed be constrained in this way to satisfy the group multiplication laws (with an overall phase of $e^{\pm(i\pi/2 + 2n\pi)}$, which we note is consistent with order 4 symmetry elements). With other choices of α'_i and ϕ_i , there may in principle be other connections of interest that can be made to different discrete groups for a given mixing angle pattern.

Up to this point, we have taken U_e to be the identity, and in so doing, absorbed the effects of Q_e in the parametrization of U_ν (see Eq. (52)). It is of course also possible, and indeed must be physically equivalent, instead to keep Q_e in the charged lepton sector. In either case, the diagonal form of U_e then also usually implies the existence of a residual, diagonal charged lepton flavor symmetry [6] that contains elements that are a subset of the possibilities represented by Q_e of Eqs. (21)-(22). As previously discussed, the unphysical phases contained in the residual charged lepton symmetry should all be distinct, or additional free parameters will in general arise, “forcing” the charged lepton mixing away from the identity unless these parameters are tuned accordingly. Since we are working in the diagonal charged lepton basis, M_e is real and diagonal. The invariance condition given in Eq. (47) with $U_e = 1$ then implies that for the case at hand (neglecting Q_e), the generalized CP elements Y_e obey the relation

$$\text{Det}(Y_e^{\text{BTM}})^q = 1, \quad (73)$$

in which q is a multiple of the order of the corresponding residual charged lepton flavor symmetry. We also note that all possible generalized CP symmetry elements that are consistent with $U_e = 1$ can be found from Eq. (36).

Finally, it is worthwhile to comment on the situation for which a general basis is chosen such that neither the charged leptons or the neutrinos are diagonal. The reason is that even though a basis change can always been made to diagonalize either sector, a general basis may

facilitate the connection between the flavor and generalized CP symmetry elements and the representations of a specific discrete group. In this case, within the bottom-up construction there is then a question of how the observed lepton mixing parameters are split between the two sectors, i.e. the choice of U_e and U_ν such that $U_e^\dagger U_\nu = U_{\text{MNSP}}$ (up to lepton rephasings). When U_e is not the identity, clearly the rephasing matrix Q_e is not so easily translated into U_ν , as in Eq. (51). However, U_e and U_ν can be transformed more generally, as follows:

$$U_e \rightarrow \tilde{Q} U_e Q_e, \quad U_\nu \rightarrow \tilde{Q} U_\nu Q_\nu, \quad (74)$$

in which the transformation given by \tilde{Q} clearly leaves U_{MNSP} invariant. The matrix \tilde{Q} can be a full $U(3)$ transformation, or it can be a subset of this full set of transformations, such as a $U(1)^3$ -preserving transformation. The specific choice of \tilde{Q} thus also has obvious implications for the bottom-up construction of the symmetry elements, in analogy to the effects discussed here in the diagonal case. Once again, it may be that such lines of reasoning open up new model-building directions in the context of family and generalized CP symmetry groups.

In summary, given that there can be a mismatch in the way in which the mixing angles of U_ν and/or U_e are parametrized and the ways in which group representations of discrete groups are given in the literature, the unphysical leptonic phases can be of utility in connecting the bottom-up construction of symmetry elements to specific discrete group representations. The case in which the charged leptons are taken to be diagonal is just one simple example. That being said, since including Q_e into U_ν is *a priori* not necessary since by definition it can be removed by rephasing the charged lepton fields, an equivalent alternative is that these group elements can be shifted via a unitary transformation such that they align with a trivial reconstruction of the Klein generators in this context, based on the standard parametrization of U_{MNSP} and setting P' as the identity. A further (equivalent) alternative is to carry out the bottom-up construction with a different parametrization of U_{MNSP} , as clearly there is nothing sacred from the model-building point of view about the PDG parametrization. However, selecting a specific parametrization and including these phases in the construction of the Klein generators and the generalized CP symmetry elements allows for this freedom to be taken into account in a straightforward way that can facilitate the identification of viable discrete groups for top-down flavor model building.

5 Conclusions

If experiments reveal that neutrinos are Majorana particles, the possibility exists that there is a residual symmetry in the neutrino sector that completely fixes U_{MNSP} in the diagonal charged lepton basis, up to rephasing by unphysical charged lepton phases. However, such a symmetry cannot make predictions for Majorana phases. In order to produce such predictions, a popular and well-motivated approach is to impose a generalized CP symmetry (consistently) alongside of the flavor symmetry and spontaneously break both symmetries (presumably at a high scale, such as the unification scale) to generate mixing angle and phase predictions, accordingly. In such a top-down approach, the angle and phase predictions that arise from this breaking become subject to model-dependent corrections such as renormalization group evolution, canonical normalization and corrections from subleading contributions to either the charged lepton or neutrino sectors.¹³ Alternatively, we can start from a bottom-up approach, in which the flavor and generalized CP symmetry elements can

¹³One may anticipate such corrections to be subleading because renormalization group and canonical normalization effects are expected to be small in realistic models with hierarchical neutrino masses, and the charged lepton corrections are typically at most Cabibbo-sized [52–54]. It is possible to have large charged

be constructed explicitly based on specific mixing angle scenarios, and thus can be used as a roadmap for top-down model building. In this work, we have investigated the effects of considering nontrivial unphysical lepton sector phases in this context, focusing on their group-theoretical properties. We also have discussed how such lepton sector rephasings extend the results of [40], in order to further elucidate the interplay between generalized CP and flavor symmetries in the charged lepton and neutrino sectors.

By extending the results of [40], we have put forth a more complete bottom-up approach that incorporates nontrivial, unphysical charged lepton phases as well as unphysical shifts to Majorana phases. Our analysis further identifies the similarities and differences between generalized CP symmetries in the charged lepton and neutrino sectors while further elucidating the group properties of the generalized CP symmetry elements. The results provide a set of group theoretical relations that must be satisfied at low energies for all models within this general framework. To this end, the methods outlined here can serve as guidance for future model-building by further clarifying the effects that preserving various residual generalized CP and flavor symmetry elements can have on models of lepton masses and mixing angles.

Acknowledgements

A.S. acknowledges support from the research grant 2012CPPYP7 (Theoretical Astroparticle Physics) under the program PRIN 2012 funded by the Italian Ministry of Education, University and Research (MIUR) and support from the ERC Advanced Grant No. 267985 Electroweak Symmetry Breaking, Flavour and Dark Matter: One Solution for Three Mysteries (DaMeSyFla), as well as partial support from the European Union FP7 ITN INVISIBLES (Marie Curie Actions, PITN-GA-2011-289442-INVISIBLES). L.E. is supported by the U.S. Department of Energy under the contract DE-FG-02-95ER40896. L.E. also thanks the Enrico Fermi Institute for its hospitality during the early stages of this work.

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