Quasi-parton distribution functions, momentum distributions, and pseudo-parton distribution functions

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Quasi-PDFs, momentum distributions and pseudo-PDFs

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We show that quasi-PDFs may be treated as hybrids of PDFs and primordial rest-frame momentum distributions of partons. This results in a complicated convolution nature of quasi-PDFs that necessitates using large $p_3 \gtrsim 3$ GeV momenta to get reasonably close to the PDF limit. As an alternative approach, we propose to use pseudo-PDFs $\mathcal{P}(x, z_3^2)$ that generalize the light-front PDFs onto spacelike intervals and are related to Ioffe-time distributions $\mathcal{M}(\nu, z_3^2)$, the functions of the Ioffe time $\nu = p_3 z_3$ and the distance parameter z_3^2 with respect to which it displays perturbative evolution for small z_3. In this form, one may divide out the z_3^2 dependence coming from the primordial rest-frame distribution and from the problematic factor due to lattice renormalization of the gauge link. The ν-dependence remains intact and determines the shape of PDFs.

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I. INTRODUCTION

The parton distribution functions (PDFs) $f(x)$ [1] are related to matrix elements of bilocal operators on the light cone $z^2 = 0$, which prevents a straightforward calculation of these functions in the lattice gauge theory formulated in Euclidean space. The usual way out is to calculate their moments. However, recently, X. Ji [2] suggested a method allowing to calculate PDFs as functions of x. To this end, he proposes to use purely space-like separations $z = (0, 0, 0, z_3)$. Then one deals with quasi-PDFs $Q(y, p_3)$ describing sharing of the p_3 hadron momentum component, and tending to PDFs $f(y)$ in the $p_3 \to \infty$ limit. The same method can be applied to distribution amplitudes (DAs). The results of lattice calculations of quasi-PDFs were reported in Refs. [3–5] and of the pion quasi-DA in Ref. [6].

In our recent papers [7, 8], we have studied nonperturbative p_3-evolution of quasi-PDFs and quasi-DAs using the formalism of virtuality distribution functions [9, 10]. We found that quasi-PDFs can be obtained from the transverse momentum dependent distributions (TMDs) $\mathcal{F}(x, k_\perp^2)$. We built models for the nonperturbative evolution of quasi-PDFs using simple models for TMDs. Our results are in qualitative agreement with the p_3-evolution patterns obtained in lattice calculations.

In the present paper, our first goal is to develop a picture for quasi-PDFs as hybrids of PDFs and primordial momentum distributions of partons in a hadron at rest. As an intermediate step, we demonstrate that the connection between TMDs and quasi-PDFs [7] is a mere consequence of Lorentz invariance. Then we show that, when a hadron is moving, the parton k_3 momentum comes from two sources. The motion of the hadron as a whole gives the xp_3 part, governed by the dependence of the TMD $\mathcal{F}(x, \kappa^2)$ on its x argument. The remaining part $k_3 - xp_3$ is governed by the dependence of the TMD on its second argument, κ^2, dictating the primordial rest-frame momentum distribution. The convolution nature of quasi-PDFs results in a rather complicated pattern of their p_3-evolution, necessitating rather large values $p_3 \sim 3$ GeV for getting close to the PDF limit.

Thus, our second goal is to propose an alternative approach for lattice PDF extraction. To this end, we introduce *pseudo-PDFs* $\mathcal{P}(x, z_3^2)$ that generalize the light-cone PDFs $f(x)$ onto spacelike intervals like $z = (0, 0, 0, z_3)$. The pseudo-PDFs are Fourier transforms of the *Ioffe-time* [11] *distributions* [12] $\mathcal{M}(\nu, z_3^2)$ that are basically given by generic matrix elements like $\langle p|\phi(0)\phi(z)|p\rangle$ written as functions of $\nu = p_3 z_3$ and z_3^2. Unlike quasi-PDFs, the pseudo-PDFs have the "canonical" $-1 \le x \le 1$ support for all z_3^2. They tend to PDFs when $z_3 \to 0$, showing in this limit a usual perturbative evolution with $1/z_3$ serving as an evolution parameter. Finally, we discuss how these properties of pseudo-PDFs may be used for extraction of PDFs on the lattice.

II. PARTON DISTRIBUTIONS

A. Generic matrix element and Lorentz invariance

Historically [1], PDFs were introduced to describe spin-1/2 quarks. Since complications related to spin do not affect the very concept of parton distributions, we start with a simple example of a scalar theory. In that case, information about the target is accumulated in the generic matrix element $\langle p|\phi(0)\phi(z)|p\rangle$. By Lorentz invariance, it is a function of two scalars, $(pz) \equiv -\nu$ and z^2 (or $-z^2$ if we want a positive value for spacelike z):

$$\langle p|\phi(0)\phi(z)|p\rangle = \mathcal{M}(-(pz), -z^2)\,. \quad (1)$$

It can be shown [7, 13] that, for all contributing Feynman diagrams, its Fourier transform $\mathcal{P}(x, -z^2)$ with respect to (pz) has the $-1 \le x \le 1$ support, i.e.,

$$\mathcal{M}(-(pz), -z^2) = \int_{-1}^{1} dx\, e^{-ix(pz)}\, \mathcal{P}(x, -z^2)\ . \quad (2)$$

Note that Eq. (2) gives a covariant definition of x. There is no need to assume that $p^2 = 0$ or $z^2 = 0$ to define x.

B. Collinear PDFs

Choosing some special cases of p and z, one can get expressions for various parton distributions, all in terms of the same function $\mathcal{M}(-(pz), -z^2)$. In particular, taking a light-like z, e.g., that having the light-front minus component z_- only, we parameterize the matrix element by the twist-2 parton distribution $f(x)$

$$\mathcal{M}(-p_+z_-, 0) = \int_{-1}^{1} dx\, f(x)\, e^{-ixp_+z_-} \quad , \tag{3}$$

with $f(x)$ having the usual interpretation of probability that the parton carries the fraction x of the target momentum component p_+. The inverse relation is given by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\nu\, e^{-ix\nu}\, \mathcal{M}(\nu, 0) = \mathcal{P}(x, 0) \ . \tag{4}$$

Since $f(x) = \mathcal{P}(x, 0)$, the function $\mathcal{P}(x, -z^2)$ generalizes PDFs onto non-lightlike intervals z^2, and we will call it *pseudo-PDF*. The variable (pz) is called the *Ioffe time* [11], and $\mathcal{M}(\nu, -z^2)$ is the *Ioffe-time distribution* [12].

Note that the definition of $\mathcal{P}(x, -z^2)$ is simpler than that of $f(x)$ because it does not require taking a subtle $z^2 \to 0$ limit. In renormalizable theories, the function $\mathcal{M}(\nu, z^2)$ has $\sim \ln z^2$ singularities generating perturbative evolution of parton densities. Within the operator product expansion (OPE) approach, the $\ln z^2$ singularities are subtracted using some prescription, say, dimensional renormalization, and the resulting PDFs depend on the renormalization scale μ, i.e., $f(x) \to f(x, \mu^2)$.

C. Transverse momentum dependent distributions

When z^2 is spacelike, one can treat $-z^2$ as the magnitude squared of a two-dimensional vector $\{z_1, z_2\}$, and introduce a two-dimensional Fourier transform with respect to its components, i.e., to write

$$\mathcal{P}(x, z_1^2 + z_2^2) = \int_{-\infty}^{\infty} dk_1 e^{ik_1 z_1} \times \int_{-\infty}^{\infty} dk_2\, e^{ik_2 z_2} \mathcal{F}(x, k_1^2 + k_2^2) \ . \tag{5}$$

Due to rotational invariance of $\mathcal{P}(x, z_1^2 + z_2^2)$ in $\{z_1, z_2\}$ plane, the function $\mathcal{F}(x, k_1^2 + k_2^2)$ depends on k_1, k_2 through $k_1^2 + k_2^2$, the fact already reflected in the notation. Combining this representation with Eq. (2), one has

$$\mathcal{M}(\nu, z_1^2 + z_2^2) = \int_{-1}^{1} dx\, e^{ix\nu} \int_{-\infty}^{\infty} dk_1 e^{ik_1 z_1} \times \int_{-\infty}^{\infty} dk_2\, e^{ik_2 z_2} \mathcal{F}(x, k_1^2 + k_2^2) \ . \tag{6}$$

A physical interpretation of $\mathcal{F}(x, k_1^2 + k_2^2)$ may be given in the frame where the target momentum p is longitudinal, $p = (E, \mathbf{0}_\perp, P)$, while the vector $\{z_1, z_2\}$ is in the transverse plane. Taking z that has z_- and $z_\perp$ components only, one can identify $\mathcal{F}(x, k_\perp^2)$ with the *TMD* and write

$$\mathcal{P}(x, z_\perp^2) = \int d^2\mathbf{k}_\perp e^{i(\mathbf{k}_\perp \mathbf{z}_\perp)} \mathcal{F}(x, k_\perp^2) \ . \tag{7}$$

In this case, the pseudo-PDFs $\mathcal{P}(x, z_\perp^2)$ coincide with the *impact parameter distributions*, a well-known concept actively used in TMD studies.

The $\sim \ln z_\perp^2$ terms in $\mathcal{M}(\nu, z_\perp^2)$ are produced by the $\sim 1/k_\perp^2$ hard tail of $\mathcal{F}(x, k_\perp^2)$. Thus, it makes sense to visualize $\mathcal{M}(\nu, z_\perp^2)$ as a sum of a soft part $\mathcal{M}^{\text{soft}}(\nu, z_\perp^2)$, that has a finite $z_\perp^2 \to 0$ limit and a hard part reflecting the evolution. For TMDs, soft part decreases faster than $1/k_\perp^2$, say, like a Gaussian $e^{-k_\perp^2/\Lambda^2}$. In the $z_\perp$ space, the distributions are then concentrated in $z_\perp \sim 1/\Lambda$ region.

III. QUASI-DISTRIBUTIONS

A. Definition and relation to TMDs

Since one cannot have light-like separations on the lattice, it was proposed [2] to consider spacelike separations $z = (0, 0, 0, z_3)$ [or, for brevity, $z = z_3$]. Then, in the $p = (E, 0_\perp, P)$ frame, one introduces the quasi-PDF $Q(y, P)$ through a parametrization

$$\langle p|\phi(0)\phi(z_3)|p\rangle = \int_{-\infty}^{\infty} dy\, Q(y, P)\, e^{iyPz_3} \quad . \tag{8}$$

According to this definition, the function $Q(y, p)$ characterizes the probability that the parton carries fraction y of hadron's third momentum component P. Viewing the matrix element as a function of the ν and $-z^2$ variables (they are given by Pz_3 and z_3^2 in this case), we have

$$\mathcal{M}(\nu, z_3^2) = \int_{-\infty}^{\infty} dy\, Q(y, P)\, e^{iy\nu} \quad . \tag{9}$$

Noticing that $z_3^2 = \nu^2/P^2$, we get the inverse Fourier transformation in the form

$$Q(y, P) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\nu\ e^{-iy\nu}\, \mathcal{M}(\nu, \nu^2/P^2) \ . \tag{10}$$

It indicates that $Q(y, P)$ tends to $f(y)$ in the $P \to \infty$ limit, as far as $\mathcal{M}(\nu, \nu^2/P^2) \to \mathcal{M}(\nu, 0)$.

Thus, the deviation of the quasi-PDF $Q(y, P)$ from the PDF $f(y)$ is determined by the dependence of $\mathcal{M}(\nu, z_3^2)$ with respect to its second argument. By Eq. (6), this dependence is related to the dependence of the TMD $\mathcal{F}(x, \kappa^2)$ on its second argument κ^2. Hence, the difference between $Q(y, P)$ and $f(y)$ may be described in terms of TMDs.

To this end, we incorporate the fact that Eq. (6) is a *mathematical* relation between the function $\mathcal{M}(\nu, z_1^2 + z_2^2)$ and the function $\mathcal{F}(x, k_1^2 + k_2^2)$, no matter what is a *physical* meaning of the variables z_1, z_2 and

k_1, k_2. Thus, we substitute Eq. (6) with $z_1 = 0$ and $z_2 = \nu/P$ into Eq. (10) to convert it into the expression for quasi-PDFs in terms of TMDs

$$Q(y,P)/P = \int_{-\infty}^{\infty} dk_1 \int_{-1}^{1} dx\, \mathcal{F}(x, k_1^2 + (y-x)^2 P^2) \ . \tag{11}$$

Originally, this relation was derived in Ref. [7] using a Nakanishi-type representation of Refs. [9, 10]. Now, we see that it is a mere consequence of Lorentz invariance.

B. Quantum chromodynamics (QCD) case

The formulas derived above are directly applicable for non-singlet parton densities in QCD. In that case, one deals with matrix elements of

$$\mathcal{M}^{\alpha}(z,p) \equiv \langle p|\bar{\psi}(0)\,\gamma^{\alpha}\,\hat{E}(0,z;A)\psi(z)|p\rangle \tag{12}$$

type, where $\hat{E}(0,z;A)$ is the standard $0 \to z$ straight-line gauge link in the quark (adjoint) representation. These matrix elements may be decomposed into p^{α} and z^{α} parts:

$$\begin{aligned}\mathcal{M}^{\alpha}(z,p) =& 2p^{\alpha}\mathcal{M}_p(-(zp), -z^2) \\ &+ z^{\alpha}\mathcal{M}_z(-(zp), -z^2) \ .\end{aligned} \tag{13}$$

The $\mathcal{M}_p(-(zp), -z^2)$ part gives the twist-2 distribution when $z^2 \to 0$, while $\mathcal{M}_z((zp), -z^2)$ is a purely higher-twist contamination, and it is better to get rid of it.

If one takes $z = (z_-, z_\perp)$ in the $\alpha = +$ component of $\mathcal{M}^{\alpha}$, the z^{α}-part drops out, and one can introduce a TMD $\mathcal{F}(x, k_\perp^2)$ that is related to $\mathcal{M}_p(\nu, z_\perp^2)$ by the scalar formula (6). For quasi-distributions, the easiest way to remove the z^{α} contamination is to take the time component of $\mathcal{M}^{\alpha}(z = z_3, p)$ and define

$$\mathcal{M}^0(z_3, p) = 2p^0 \int_{-1}^{1} dy\, Q(y,P)\; e^{iyPz_3} \ . \tag{14}$$

Then the connection between $Q(y,P)$ and $\mathcal{F}(x, k_\perp^2)$ is given by the scalar formula (11).

One may notice that the operator defining $\mathcal{M}^{\alpha}(z,p)$ involves a straight-line link from 0 to z rather than a stapled link usually used in the definitions of TMDs appearing in the description of Drell-Yan and semi-inclusive DIS processes. As is well-known, the stapled links reflect initial or final state interactions inherent in these processes. The "straight-link" TMDs, in this sense, describe the structure of a hadron when it is in its non-disturbed or "primordial" state. While it is unlikely that such a TMD can be measured in a scattering experiment, it is a well-defined QFT object, and one may hope that it can be measured on the lattice.

C. Momentum distributions

The quasi-PDFs describe the distribution in the fraction $y \equiv k_3/P$ of the third component k_3 of the parton momentum to that of the hadron. One can introduce distributions in k_3 itself: $R(k_3, P) \equiv Q(k_3/P, P)/P$. Then we can rewrite Eq. (11) as

$$R(k_3, P) = \int_{-\infty}^{\infty} dk_1 \int_{-1}^{1} dx\, \mathcal{F}(x, k_1^2 + (k_3 - xP)^2) \tag{15}$$

or, switching to the linear argument $k_3 - xP$,

$$R(k_3, P) = \int_{-1}^{1} dx\, \mathcal{R}(x, k_3 - xP) \ , \tag{16}$$

where

$$\mathcal{R}(x, k_3) \equiv \int_{-\infty}^{\infty} dk_1 \mathcal{F}(x, k_1^2 + k_3^2) \tag{17}$$

is the TMD $\mathcal{F}(x, \kappa^2)$ integrated over the k_1 component of the two-dimensional vector $\kappa = \{k_1, k_3\}$. According to (17), $\mathcal{R}(x, k_3)$ depends on k_3 through k_3^2.

For a hadron at rest, we have

$$R(k_3, P=0) \equiv r(k_3) = \int_{-1}^{1} dx\, \mathcal{R}(x, k_3) \ . \tag{18}$$

This one-dimensional distribution may be directly obtained through a parameterization of the density

$$\rho(z_3^2) \equiv \mathcal{M}(0, z_3^2) = \int_{-\infty}^{\infty} dk_3\, r(k_3)\, e^{ik_3 z_3} \tag{19}$$

given by $\langle p|\phi(0)\phi(z_3)|p\rangle|_{\mathbf{p}=\mathbf{0}}$. Thus, $r(k_3)$ describes a primordial distribution of k_3 (or any other component of $\mathbf{k}$) in a rest-frame hadron.

The formula (16) has a straightforward interpretation. According to it, when the hadron is moving, the parton's k_3 momentum has two sources.

The first part, xP comes from the motion of the hadron as a whole, and the probability to get xP is governed by the dependence of the TMD $\mathcal{F}(x, \kappa^2)$ on its first argument, x.

On the other hand, the probability to get the remaining part $k_3 - xP$ is governed by the dependence of the TMD on its second argument, κ^2, describing the primordial rest-frame momentum distribution.

The parameter x appears in both arguments of $\mathcal{R}(x, k_3 - xP)$ in Eq. (16), i.e., $R(k,P)$ is given by a convolution. In this sense, the momentum distributions $R(k,P)$ and, hence, the quasi-PDFs have a hybrid structure influenced by the shape both of PDFs and rest-frame distributions.

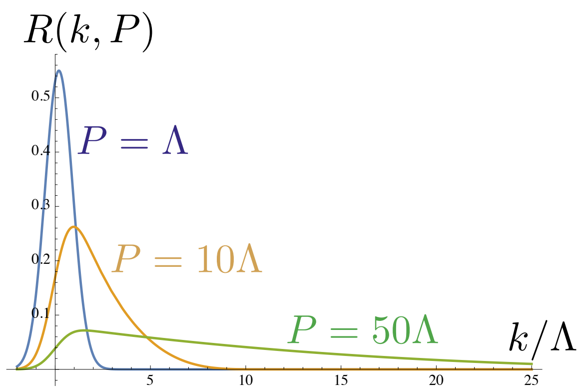

FIG. 1. Momentum distributions $R(k,P)$ in the factorized Gaussian model for $P/\Lambda = 1, 10, 50$.

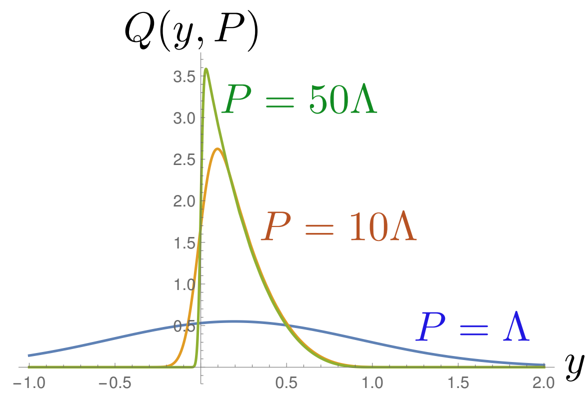

FIG. 2. Evolution of quasi-PDF $Q(y,P)$ in the factorized Gaussian model for $P/\Lambda = 1, 10, 50$.

D. Factorized models.

Since the two sources of k_3 look like independent, it is natural to demonstrate the hybrid nature of momentum distributions and quasi-PDFs using a factorized model $\mathcal{R}(x, k_3 - xP) = f(x) r(k_3 - xP)$ (the x integral of $f(x)$ is normalized to 1). For original $\mathcal{M}(\nu, -z^2)$ function, this Ansatz corresponds to the factorization assumption $\mathcal{M}(\nu, -z^2) = \mathcal{M}(\nu, 0)\mathcal{M}(0, -z^2)$.

For illustration, we take a Gaussian form $\rho_G(z_3^2) = e^{-z_3^2\Lambda^2/4}$ for the rest-frame density. It corresponds to

$$r_G(k_3) = \frac{1}{\sqrt{\pi}\Lambda} e^{-k_3^2/\Lambda^2} \ . \tag{20}$$

For $f(x)$, we take a simple PDF resembling nucleon valence densities $f(x) = 4(1-x)^3\theta(0 \le x \le 1)$. As one can see from Fig. 1, the curve for $R(k,P)$ changes from a Gaussian shape for small P to a shape resembling stretched PDF for large P.

This result is in perfect compliance with a known fact that wave functions of moving hadrons are not given by a mere kinematical "boost" of the rest-frame wave functions. Indeed, with increasing P, the impact of the rest-frame distribution $r(k)$ is less and less visible, and eventually the shape of $R(k,P)$ is determined by a completely different function $f(k/P)$.

Rescaling to the $y = k/P$ variable gives the quasi-PDF $Q(y,P)$ shown in Fig. 2. For large P, it clearly tends to the $f(y)$ PDF form. In particular, using a momentum $P \sim 10\Lambda$ one gets a quasi-PDF that is rather close to the $P \to \infty$ limiting shape. Still, since $\Lambda \sim \langle k_\perp \rangle$, assuming the folklore value $\langle k_\perp \rangle \sim 300$ MeV one translates the $P \sim 10\Lambda$ estimate into $P \sim 3$ GeV, which is uncomfortably large. Thus, a natural question is how to improve the convergence.

E. Pseudo-PDFs

A formal reason for the complicated structure of a quasi-PDF $Q(y,P)$ is the fact that it is obtained by the ν-integral of $\mathcal{M}(\nu, z_3^2)e^{i\nu y}$ along a non-horizontal line $z_3 = \nu/P$ in the (ν, z_3) plane (see Eq. (10)). With increasing P, its slope decreases, the line becomes more horizontal, and quasi-PDFs convert into PDFs.

In contrast, pseudo-PDFs $\mathcal{P}(x, z_3^2)$, by definition, are given by integration of $\mathcal{M}(\nu, z_3^2)e^{i\nu x}$ over horizontal lines $z_3 = \text{const}$. A very attractive feature of the pseudo-PDFs is that they have the $-1 \le x \le 1$ support for all z_3 values. For small z_3, they convert into PDFs.

More precisely, when z_3 is small, $1/z_3$ is analogous to the renormalization parameter μ of scale-dependent PDFs $f(x,\mu^2)$ of the standard OPE approach.

There is a subtlety, however, that while the μ^2-dependence of PDFs $f(x,\mu^2)$ comes solely from the evolution logarithms $\ln(\mu^2/m^2)$, the z_3^2-dependence of quasi-PDFs comes both from the evolution logarithms $\ln(z_3^2 m^2)$ and from the ultraviolet logarithms $\ln(z_3^2\mu_R^2)$, where μ_R is a cut-off parameter for divergences related to the gauge link renormalization (see Ref. [14]). At the leading logarithm level, these divergences do not depend on ν. As a result, the "reduced" Ioffe-time distribution

$$\mathfrak{M}(\nu, z_3^2) \equiv \frac{\mathcal{M}(\nu, z_3^2)}{\mathcal{M}(0, z_3^2)} \tag{21}$$

satisfies, for small z_3, the leading-order evolution equation

$$\frac{d}{d\ln z_3^2}\mathfrak{M}(\nu, z_3^2) = -\frac{\alpha_s}{2\pi}C_F\int_0^1 du\, B(u)\,\mathfrak{M}(u\nu, z_3^2) \tag{22}$$

with respect to $1/z_3$ that coincides with the evolution equation for $f(x,\mu^2)$ with respect to μ. The leading-order evolution kernel $B(u)$ for the non-singlet quark case is given [12] by

$$B(u) = \left[\frac{1+u^2}{1-u}\right]_+ \ , \tag{23}$$

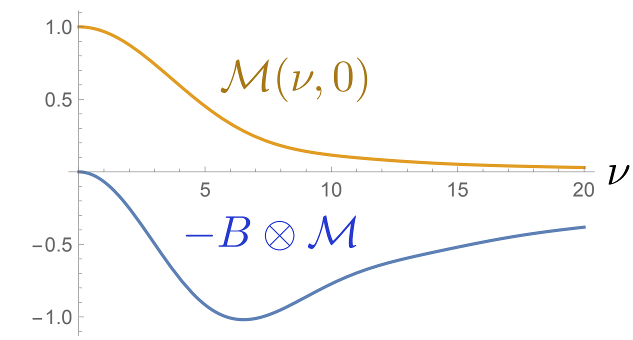

FIG. 3. Model Ioffe-time distribution $\mathcal{M}(\nu, 0)$ and the function $B \otimes \mathcal{M}$ governing its evolution.

with $[\ldots]_+$ denoting the standard "plus" prescription.

For the model used above (and $x \to -x$ symmetrized, as required for non-singlet PDFs), we have $\mathcal{M}(\nu, 0) = 12\left[\nu^2 - 4\sin^2(\nu/2)\right]/\nu^4$. The shape of this function and of the convolution integral $B \otimes \mathcal{M}(\nu)$ are shown in Fig. 3. As one can see, $B \otimes \mathcal{M}(\nu)$ vanishes for $\nu = 0$, which reflects conservation of the vector current. Thus, the rest-frame density $\mathfrak{M}(0, z_3^2)$ is not affected by perturbative evolution.

F. Lattice implementation

A possible way to find the Ioffe-time distributions on the lattice (suggested by K. Orginos) is to calculate $\mathcal{M}(Pz_3, z_3^2)$ for several values of P, and then to fit the results by a function of ν and z_3^2.

Recalling our discussion of two apparently independent sources of obtaining k_3 for a moving hadron, one may hope that $\mathcal{M}(\nu, z_3^2)$ factorizes, i.e., $\mathcal{M}(\nu, z_3^2) = \mathcal{M}(\nu, 0)\mathcal{M}(0, z_3^2)$. Then the reduced function $\mathfrak{M}(\nu, z_3^2)$ defined by Eq. (21) is equal to $\mathcal{M}(\nu, 0)$, and the goal of obtaining $\mathcal{M}(\nu, 0)$ is reached. Formally, what remains is just to take its Fourier transform to get the PDF $f(x)$.

In fact, such a factorization has been already observed several years ago in the pioneering study [15] of the transverse momentum distributions in lattice QCD.

A serious disadvantage of quasi-PDFs is that they have the x-convolution structure (11) even in a favorable situation when the TMD [and $\mathcal{M}(\nu, z_3^2)$] factorizes. On the other hand, using pseudo-PDFs in the form of the ratio $\mathfrak{M}(\nu, z_3^2)$, one divides out the z_3^2-dependence of the primordial distribution without affecting the ν-dependence that dictates the shape of PDF.

A further advantage of using the ratio (pointed out by K. Orginos) is the cancellation of the z_3-dependence generated by the lattice renormalization of the gauge link $\hat{E}(0, z_3; A)$. Such a renormalization is required by linear $|z_3|\delta m$ (where $\delta m \sim 1/a$, and a is the ultraviolet cut-off) and logarithmic $\ln(z_3^2/a^2)$ divergences [16, 17]. Due to their local nature, they are expected to combine into a ν-independent factor $Z(z_3/a)$ that is the same in the numerator and denominator of the ratio $\mathfrak{M}(\nu, z_3^2)$.

The multiplicative renormalizability of the linear divergences of $\mathcal{M}(\nu, z_3^2)$ to all orders was recently argued in Refs. [18, 19]. A general proof for both linear and logarithmic divergences was claimed in Ref. [20] on the basis of a direct analysis of relevant Feynman graphs.

Another approach [21, 22] is to treat $\hat{E}(0, z; A)$ as $h(0)\bar{h}(z)$, where the auxiliary field $h(z)$ is analogous to the infinitely heavy quark field of the heavy quark effective theory (HQET). Since HQET is known to be multiplicatively renormalizable [23] this means that $\bar{\psi}(0)\hat{E}(0, z; A)\psi(z)$ is also multiplicatively renormalizable to all orders in perturbation theory.

In reality, $\mathfrak{M}(\nu, z_3^2)$ will have a residual z_3^2-dependence. It comes both from a possible violation of factorization for the soft part (according to results of Ref. [15], it is expected to be rather mild) and from mandatory perturbative evolution. For a nonzero ν, the latter should be visible as a $\ln(1/z_3^2\Lambda^2)$ spike for small z_3^2.

Hence, a proposed strategy is to extrapolate $\mathfrak{M}(\nu, z_3^2)$ to $z_3^2 = 0$ from not too small values of z_3^2, say, from those above 0.5 fm^2. The resulting function $\mathcal{M}^{\text{soft}}(\nu, 0)$ may be treated as the Ioffe-time distribution producing the PDF $f_0(x)$ "at low normalization point". The remaining $\ln(1/z_3^2\Lambda^2)$ spikes at small z_3 will generate its evolution.

To convert $\mathfrak{M}(\nu, z_3^2)$ into a function of x, one should, in principle, know $\mathfrak{M}(\nu, z_3^2)$ for all ν, which is impossible. The maximal values of ν reached in existing lattice calculations range from 3π [3] to 5π [5] and 6π [24]. Taking a Fourier transform in these limited ranges produces unphysical oscillations in x. Thus, the idea is to avoid the Fourier transform in ν, and just compare the reduced Ioffe-time distributions obtained from the lattice with those derived from experimentally known parton distributions.

Of course, an actual technical implementation of this program should be discussed when the lattice data on $\mathfrak{M}(\nu, z_3^2)$ will become available.

IV. SUMMARY

In this paper, we showed that quasi-PDFs may be seen as hybrids of PDFs and the primordial rest-frame momentum distributions of partons. In this context, the parton's k_3 momentum comes from the motion of the hadron as a whole and from the primordial rest-frame momentum distribution. The complicated convolution nature of quasi-PDFs necessitates using $p_3 \gtrsim 3$ GeV to wipe out the primordial momentum distribution effects and get reasonably close to the PDF limit.

As an alternative approach, we propose to use pseudo-PDFs $\mathcal{P}(x, z_3^2)$ that generalize the light-front PDFs onto spacelike intervals. By a Fourier transform, they are related to the Ioffe-time distributions $\mathcal{M}(\nu, z_3^2)$ given by generic matrix elements written as functions of $\nu = p_3 z_3$ and z_3^2. The advantageous features of pseudo-PDFs are that they, first, have the same $-1 \le x \le 1$ support as PDFs, and second, their z_3^2-dependence for small z_3^2 is governed by a usual evolution equation.

Forming the ratio $\mathcal{M}(\nu, z_3^2)/\mathcal{M}(0, z_3^2)$ of Ioffe-time distributions one divides out the bulk of z_3^2 dependence generated by the primordial rest-frame distribution. Furthermore, taking this ratio one can exclude the z_3^2-dependent factor coming from the lattice renormalization of the $\hat{E}(0, z_3; A)$ link creating difficulties (see, e.g., [18]) for lattice calculations of quasi-PDFs.

Testing the efficiency of using pseudo-PDFs for lattice extractions of PDFs is a challenge for future studies.

In fact, while this paper was in the review process, an actual lattice calculation [24] based on the ideas of the present paper was performed. It has clearly demonstrated the presence of a linear component in the z_3-dependence of the rest-frame function $\mathcal{M}(0, z_3^2)$, that may be attributed to the $Z(z_3^2) \sim e^{-c|z_3|/a}$ behavior generated by the gauge link. It was also observed that the ratio $\mathcal{M}(Pz_3, z_3^2)/\mathcal{M}(0, z_3^2)$ has a Gaussian-type behavior with respect to z_3, which indicates that the $Z(z_3^2/a^2)$ factors entering into the numerator and denominator of the $\mathfrak{M}(Pz_3, z_3^2)$ ratio have been canceled, as we expected.

Furthermore, it was found that when plotted as a function of ν and z_3, the data for the reduced distribution $\mathfrak{M}(\nu, z_3^2)$ have a very mild dependence on z_3^2. This observation indicates that the soft part of the z_3^2-dependence of $\mathcal{M}(\nu, z_3^2)$ has been canceled by the rest-frame density $\mathcal{M}(0, z_3^2)$. This phenomenon corresponds to factorization of the x- and $k_\perp$-dependence for the soft part of the TMD $\mathcal{F}(x, k_\perp^2)$.

It was also demonstrated that the residual z_3-dependence for small $z_3 \le 4a$, may be explained by perturbative evolution, with the α_s value corresponding to $\alpha_s/\pi = 0.1$.

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