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On Large N Expansion of the Sphere Free Energy

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Abstract

We propose formulas for the $1/N$ correction to the sphere free energy of theories with 4-fermion interactions, which are conformal for $d > 2$. We also propose a formula for the scalar $O(N)$ model. Expanding these formulas in small ϵ near various integer dimensions we find a perfect agreement with results obtained using ϵ -expansion technique. In $d = 3$, the large N results with the $1/N$ correction included are in good agreement with the Pade resummed ϵ -expansion.

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1 Introduction and Summary

The free energy of a Quantum Field Theory (QFT) on a d -dimensional sphere $F = -\log Z_{S^d}$ is an extensive quantity, therefore the leading term of it is proportional to the volume of the sphere. The value of this leading term is divergent and depends on the regularization scheme. For 3-d Conformal Field Theory (CFT) the non-divergent constant part of the sphere free energy is universal and does not depend on the sphere radius R . We call this part simply the sphere free energy F . Therefore the sphere free energy F of a CFT is a number. This number "feels" the whole space and is non-local in its nature. The interest in F is due to the F -theorem [1], which suggests that F is a measure of the degrees of freedom of a CFT. More precisely the F -theorem implies that an RG flow from a CFT_1 to a CFT_2 is only possible if

$$F_{\text{CFT}_1} > F_{\text{CFT}_2}. \quad (1.1)$$

Knowledge of F helps to rule out some possible RG flows and support others. The other interesting aspect of the sphere free energy is that it can be formulated in terms of the entanglement entropy across a circle [2,3]. The three-dimensional F -theorem has been proved using properties of the entanglement entropy in relativistic theories [4] (see also [5]).

Unfortunately in the majority of strongly coupled theories it is not possible to compute F exactly. Only in some specific supersymmetric models the sphere free energy can be computed explicitly using localization technique [6–8]. Thus it is important to estimate F in different strongly coupled conformal theories. The sphere free energy for the $O(N)$ scalar, Gross-Neveu, Nambu-Jona-Lasinio and QED models was estimated in series of works [8–11] using the Wilson-Fisher ϵ -expansion. This computation is subtle and involves a careful treatment of curvature terms, which contribute to the sphere free energy at the critical point. The advantage of the ϵ -expansion is that it works quite well for an arbitrary N and gives a rather precise estimate for F at $d = 3$.

Another standard approximation technique is the large N expansion [12, 13]. It was applied to the sphere free energy F in several papers [11, 14–16] but only up to the N^0 order. At this order one has to compute the determinant of a special operator on a sphere.

In the present paper we compute the next order in $1/N$ in four different models, namely in the Gross-Neveu, Nambu-Jona-Lasinio model with $U(1)$ chiral symmetry, Nambu-Jona-Lasinio model with $SU(2)$ chiral symmetry and Wilson-Fisher $O(N)$ model. All these models are CFTs in $2 < d < 4$.

The Gross-Neveu (GN) model [17] in the flat space has the action

$$S_{\text{GN}} = \int d^d x \left(\bar{\psi}_i \not{\partial} \psi^i + \frac{g}{2} (\bar{\psi}_i \psi^i)^2 \right), \quad (1.2)$$

where ψ^i , $i = 1, \dots, N_f$ are the Dirac fermions. This model possesses the discrete chiral symmetry $\psi_i \rightarrow \gamma_5 \psi_i$. In $d = 2$ this theory is asymptotically free for $N > 2$ [17] and it has a unitary UV fixed point in $2 < d < 4$, which describes the second-order phase transition where the discrete chiral symmetry is broken.

The Nambu-Jona-Lasinio models with $U(1)$ and $SU(2)$ chiral symmetries [18] are described by the actions

$$S_{\text{NJL}}^{U(1)} = \int d^d x \left(\bar{\psi}_i \not{\partial} \psi^i + \frac{g}{2} ((\bar{\psi}_i \psi^i)^2 - (\bar{\psi}_i \gamma_5 \psi^i)^2) \right), \quad (1.3)$$

$$S_{\text{NJL}}^{SU(2)} = \int d^d x \left(\bar{\psi}_{i\alpha} \not{\partial} \psi^{i\alpha} + \frac{g}{2} ((\bar{\psi}_{i\alpha} \psi^{i\alpha})^2 - (\bar{\psi}_{i\alpha} \gamma_5 \vec{\tau}_{\alpha\beta} \psi^{i\beta})^2) \right), \quad (1.4)$$

where ψ^i and $\psi^{i\alpha}$, $i = 1, \dots, N_f$, $\alpha = 1, 2$ are the 4-component Dirac fermions, and τ^a , $a = 1, 2, 3$ are the Pauli matrices. For γ^5 we use conventions $\{\gamma^\mu, \gamma^5\} = 0$ and $(\gamma^5)^2 = \mathbf{1}$.

In $d > 2$ the four-fermion interactions (1.2) and (1.3), (1.4) are non-renormalizable, while they are renormalizable in the sense of the $1/N$ expansion [19–22]. At finite N the UV

completion of these theories for $d < 4$ are given by the Gross-Neveu-Yukawa, Nambu-Jona-Lasinio-Yukawa and Gell-Mann-Levy models, considered originally in [23–25].

Using a Hubbard-Stratonovich transformation and introducing auxiliary fields we find that the critical point of the models (1.2), and (1.3), (1.4) on the sphere in the large N approach is described by the actions

$$\begin{aligned} S_{\text{GN}} &= \int d^d x \sqrt{g} \left(\bar{\psi}_i \not{\nabla} \psi^i + \frac{1}{\sqrt{N}} \sigma \bar{\psi}_i \psi^i \right), \\ S_{\text{NJL}}^{U(1)} &= \int d^d x \sqrt{g} \left(\bar{\psi}_i \not{\nabla} \psi^i + \frac{1}{\sqrt{N}} \bar{\psi}_i (\phi_1 + i \phi_2 \gamma^5) \psi^i \right), \\ S_{\text{NJL}}^{SU(2)} &= \int d^d x \sqrt{g} \left(\bar{\psi}_{i\alpha} \not{\nabla} \psi^{i\alpha} + \frac{1}{\sqrt{2N}} \bar{\psi}_{i\alpha} (\pi_0 \delta_{\alpha\beta} + i \tau_{\alpha\beta}^a \pi_a \gamma^5) \psi^{i\beta} \right), \end{aligned} \quad (1.5)$$

where σ , ϕ_1 , ϕ_2 and π_0, \dots, π_3 are the scalar auxiliary fields and $N = N_f \text{Tr} \mathbf{1}$, where $\text{Tr} \mathbf{1} = 4$ is the trace of the identity matrix in the spinor space.

The critical point of the Wilson-Fisher $O(N)$ model at the large N is described by the action

$$S_{O(N)} = \int d^d x \sqrt{g} \left(\frac{1}{2} (\partial_\mu \phi^i)^2 + \frac{d-2}{8(d-1)} \mathcal{R} \phi^i \phi^i + \frac{1}{2\sqrt{N}} \sigma \phi^i \phi^i \right), \quad (1.6)$$

where ϕ^i , $i = 1, \dots, N$ are scalar fields, σ is the scalar auxiliary field and $\mathcal{R} = d(d-1)/R^2$ is the scalar curvature of the sphere. Our main proposals are

$$\begin{aligned} F_{\text{GN}} &= N F_f + \delta F_{\Delta=d-1} + \frac{1}{N} \frac{1}{d} \gamma_{\psi,1}^{\text{GN}} + \mathcal{O}(1/N^2), \\ F_{\text{NJL}}^{U(1)} &= N F_f + 2 \delta F_{\Delta=d-1} + \frac{1}{N} \frac{d-2}{d(d-1)} \gamma_{\psi,1}^{\text{NJL } U(1)} + \mathcal{O}(1/N^2), \\ F_{\text{NJL}}^{SU(2)} &= 2 N F_f + 4 \delta F_{\Delta=d-1} + \frac{1}{N} \frac{d-4}{d(d-1)} \gamma_{\psi,1}^{\text{NJL } SU(2)} + \mathcal{O}(1/N^2) \end{aligned} \quad (1.7)$$

and

$$F_{O(N)} = N F_s + \delta F_{\Delta=d-2} + \frac{1}{N} \left(\frac{1}{d} + \frac{1}{3(d-2)} - \frac{2}{3(d-1)} \right) \gamma_{\phi,1}^{O(N)} + \mathcal{O}(1/N^2), \quad (1.8)$$

where F_s and F_f are sphere free energies for conformally coupled free boson and free massless fermion in arbitrary dimensions [8]

$$F_s = - \frac{1}{\sin(\frac{\pi d}{2}) \Gamma(1+d)} \int_0^1 du u \sin \pi u \Gamma(d/2 + u) \Gamma(d/2 - u), \quad (1.9)$$

$$F_f = -\frac{1}{\sin(\frac{\pi d}{2})\Gamma(1+d)} \int_0^1 du \cos\left(\frac{\pi u}{2}\right) \Gamma\left(\frac{1+d+u}{2}\right) \Gamma\left(\frac{1+d-u}{2}\right) \quad (1.10)$$

and δF_Δ is given by [8, 16]

$$\delta F_\Delta = \frac{1}{2} \log \det s(x, y)^{-2\Delta} = -\frac{1}{\sin(\frac{\pi d}{2})\Gamma(1+d)} \int_0^{\Delta - \frac{d}{2}} du u \sin \pi u \Gamma\left(\frac{d}{2} + u\right) \Gamma\left(\frac{d}{2} - u\right), \quad (1.11)$$

where $s(x, y)$ is the chordal distance on S^d . Finally $\gamma_{\phi,1}^{O(N)}$ and $\gamma_{\psi,1}^{GN}, \gamma_{\psi,1}^{NJL U(1)}, \gamma_{\psi,1}^{NJL SU(2)}$ are $1/N$ corrections to the anomalous dimensions of ϕ and ψ fields [26–28]

$$\gamma_{\phi,1}^{O(N)} = \frac{2 \sin(\pi d/2) \Gamma(d-2)}{\pi \Gamma(d/2-2) \Gamma(d/2+1)}, \quad \gamma_{\psi,1}^{GN} = \frac{(d-2)^2 \Gamma(d-1)}{4 \Gamma(2-d/2) \Gamma(d/2+1) \Gamma(d/2)^2} \quad (1.12)$$

and $\gamma_{\psi,1}^{NJL U(1)} = \gamma_{\psi,1}^{NJL SU(2)} = 2\gamma_{\psi,1}^{GN}$. In $d=3$ using (1.7) and (1.12) we find for the $1/N$ term in the sphere free energy¹

$$F_{1/N}^{O(N)}|_{d=3} = F_{1/N}^{GN}|_{d=3} = F_{1/N}^{NJL U(1)}|_{d=3} = -F_{1/N}^{NJL SU(2)}|_{d=3} = \frac{1}{N} \frac{4}{9\pi^2}. \quad (1.13)$$

We notice that the sphere free energies of the Gross-Neveu and $O(N)$ scalar models are related by $\zeta(3) \rightarrow -\zeta(3)$. One can compare the results (1.7) and (1.8) in $d=3$ with the Pade resummed ϵ -expansion given in [9, 10]. The plots for the GN and NJL $U(1)$ models are depicted in figure 1. The plot for the $O(N)$ scalar model is in figure 2. We see that the large N expansion works quite well even for small N . Below we obtain the results (1.7) and (1.8). Also we compute different ϵ -expansions for the fermionic and bosonic theories, which give a cross-check for the large N results.

The paper is organized as follows: in section 2 we study the fermionic theories. We start with a supersymmetric example, where it is possible to obtain exact result for the sphere free energy. Next we consider the Gross-Neveu-Yukawa model at arbitrary even dimension and calculate the sphere free energy for it using the ϵ -expansion. Finally, we use these ϵ -expansion results to conjecture the $1/N$ correction to the sphere free energy of the Gross-Neveu model as a function of d . Section 3 is devoted to the bosonic theories. Namely, we compute the sphere free energy of the six dimensional cubic and eight dimensional $O(N)$ scalar models using the ϵ -expansion. Then we use these ϵ -expansion results to conjecture the $1/N$ correction to the sphere free energy of the $O(N)$ scalar model as a function of d .

¹In $d=3$ we have $F_f = \frac{\log 2}{8} + \frac{3\zeta(3)}{16\pi^2}$, $F_s = \frac{\log 2}{8} - \frac{3\zeta(3)}{16\pi^2}$ and $\delta F_{\Delta=d-1} = -\delta F_{\Delta=d-2} = \frac{\zeta(3)}{8\pi^2}$.

At the end of the paper there are two Appendices, which discuss some attempts towards the analytic $1/N$ computation.

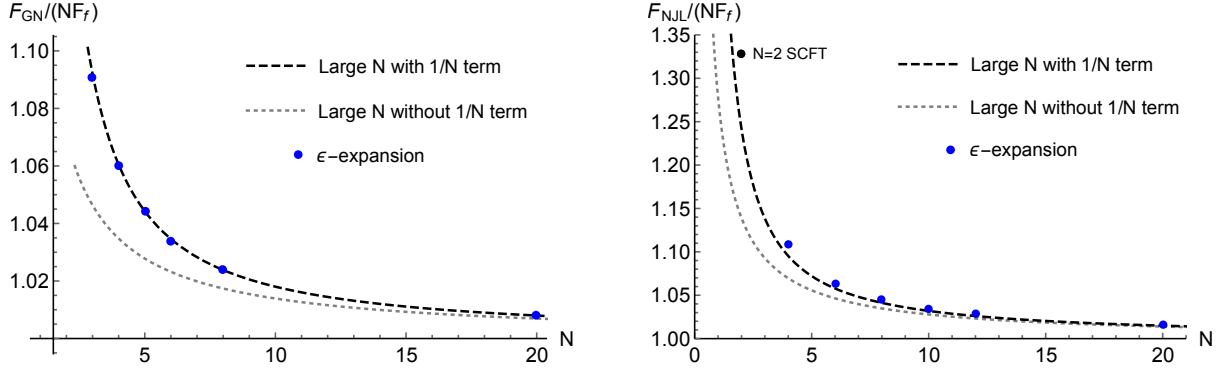


Figure 1: The large N and ϵ -expansion for $F/(NF_f)$ of the GN and NJL $U(1)$ models in $d = 3$. The dashed black line depicts the large N curve including the $1/N$ term, whereas the dotted gray line represents the large N result without the $1/N$ correction. Blue dots stand for the ϵ -expansion obtained in [10]. The black dot is the exact result for the Wess-Zumino model at $d = 3$ obtained via localization technique in [7]: $F_{\text{WZ}}/(2F_f) \approx 1.32806$.

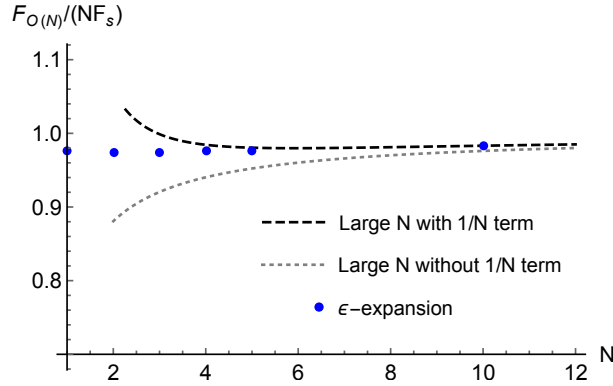


Figure 2: The large N and ϵ -expansion for $F/(NF_s)$ of the $O(N)$ scalar model in $d = 3$. The dashed black line depicts the large N curve including the $1/N$ term, whereas the dotted gray line represents the large N result without the $1/N$ correction. Blue dots stand for the ϵ -expansion obtained in [9].

2 Towards the large N expansion of the sphere free energy in fermionic theory

2.1 Supersymmetric example with $O(N)$ symmetry

An important property of the large N expansion is that all results as functions of dimension d have a specific structure. In this section we show this using as an example a supersymmetric model with $N + 1$ chiral superfields and $O(N)$ symmetric superpotential [8, 29]

$$W = \frac{\lambda}{2} X \sum_{i=1}^N Z^i Z^i, \quad (2.1)$$

where X and Z^i are chiral superfields. As was shown in [8] the exact result for the sphere free energy in arbitrary d is given by

$$F = N\mathcal{F}(\Delta_Z) + \mathcal{F}(\Delta_X), \quad (2.2)$$

where Δ_Z and Δ_X are the anomalous dimensions of the chiral superfields Z and X and

$$\mathcal{F}(\Delta) = 2(F_s + F_f) - \int_{d/2-1}^{\Delta} du \frac{\Gamma(d-1-u)\Gamma(u)\sin(\pi(u-d/2))}{\Gamma(d-1)\sin(\pi d/2)}, \quad (2.3)$$

where F_s and F_f are given in (1.9) and (1.10). At the IR fixed point the conformal dimensions are constrained $2\Delta_Z + \Delta_X = d - 1$ and the value of Δ_Z can be determined by extremizing F . Setting $\Delta_Z = \frac{d}{2} - 1 + \gamma_{Z,1}/N + \mathcal{O}(1/N^2)$ and solving perturbatively in $1/N$ the equation $dF/d\Delta_Z = 0$ one finds [8]

$$\gamma_{Z,1} = -\frac{2\sin(\frac{\pi d}{2})\Gamma(d-2)}{\pi\Gamma(\frac{d}{2}-1)\Gamma(\frac{d}{2})}, \quad \gamma_{Z,2} = 2\gamma_{Z,1}^2 \left(\Psi(d) + \frac{1}{d-2} \right), \quad (2.4)$$

where $\Psi(d) = \psi(d-2) - \psi(\frac{d}{2}-1) + \psi(2-\frac{d}{2}) - \psi(1)$ and $\psi(x) = \Gamma'(x)/\Gamma(x)$. Now substituting (2.4) in (2.2) and expanding in $1/N$ we find for the $1/N$ and $1/N^2$ corrections to the sphere free energy for general d

$$F_{1/N} = \frac{1}{N} \frac{\gamma_{Z,1}}{d-2}, \quad F_{1/N^2} = \frac{1}{N^2} \gamma_{Z,1}^2 \left(\frac{2\Psi(d)}{d-2} + \frac{4}{3(d-2)^2} \right). \quad (2.5)$$

We notice that the $1/N$ terms in (1.7) and (1.8) have the same structure, as in (2.5).

2.2 Gross-Neveu-Yukawa model in general even dimension

In this section we consider the Gross-Neveu-Yukawa model in general even dimension $d = 2k$, where $k = 2, 3, 4, \dots$. The action for this model in flat $d = 2k - \epsilon$ dimensional space reads

$$S_{\text{GN}} = \int d^d x \left(\bar{\psi}_i \not{\partial} \psi^i + (\partial^{k-1} \sigma)^2 + \mu^{\epsilon/2} g_1 \sigma \bar{\psi}_i \psi^i + \dots \right), \quad (2.6)$$

where ψ_i , $i = 1, \dots, N_f$ are Dirac fermions, σ is a higher-derivative scalar field and μ is an auxiliary scale. Dots in (2.6) denote interacting terms made from σ field (and its derivatives), and also all counter-terms. For $k = 2$ the GNY model was proposed in [23, 24] as a UV completion of the GN model. The sphere free energy for the $k = 2$ GNY model was computed recently in [10]. For general k the bare dimensions of ψ and σ fields are $[\psi] = [\mu]^{\frac{d-1}{2}}$ and $[\sigma] = [\mu]^{\frac{d+2-2k}{2}}$. At the leading order in ϵ only g_1 -vertex contributes to the anomalous dimension of ψ and σ fields and to the beta-function β_{g_1} . To compute them, we need to consider three simple one-loop diagrams, depicted in figure 3.

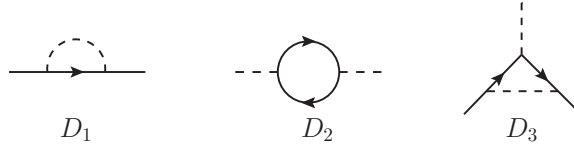


Figure 3: Feynman diagrams contributing to the anomalous dimension of ψ and σ fields and the beta-function β_{g_1} at the leading order in ϵ . The dashed lines represent σ field propagators $\langle \sigma \sigma \rangle = 1/(p^2)^{k-1}$ and the solid lines are for ψ field propagators $\langle \psi \bar{\psi} \rangle = i \not{p}/p^2$.

Working in the $\overline{\text{MS}}$ scheme, it is not difficult to compute the counter-terms δ_ψ , δ_σ and δ_{g_1} , defined as $\psi = Z_\psi^{-1/2} \psi_0$, $\sigma = Z_\sigma^{-1/2} \sigma_0$, where $Z_\psi = 1 + \delta_\psi$ and $Z_\sigma = 1 + \delta_\sigma$ and also $g_{1,0} = Z_g Z_\psi^{-1} Z_\sigma^{-1/2} g$, where $Z_g = 1 + \delta_{g_1}/g_1$. The result is

$$\delta_\psi = -\frac{2^{1-2k}(k-1)g_1^2}{\pi^k \Gamma(k+1)\epsilon}, \quad \delta_\sigma = N \frac{(-1)^{k+1} 2^{3-4k} g_1^2}{\pi^{k-\frac{1}{2}} \Gamma(k-\frac{1}{2})\epsilon}, \quad \delta_{g_1} = \frac{2^{1-2k} g_1^3}{\pi^k \Gamma(k)\epsilon}, \quad (2.7)$$

where $N = N_f \text{Tr} \mathbf{1}$. Therefore for the anomalous dimensions and beta-function we find

$$\begin{aligned} \gamma_\psi &= \frac{k-1}{(4\pi)^k \Gamma(k+1)} g_1^2 + \dots, \quad \gamma_\sigma = -\frac{N(-1)^{k+1}}{(16\pi)^{k-\frac{1}{2}} \Gamma(k-\frac{1}{2})} g_1^2 + \dots, \\ \beta_{g_1} &= -\frac{\epsilon}{2} g_1 + \left(\frac{2\sqrt{\pi}(-1)^k N}{\Gamma(k-\frac{1}{2})} + \frac{4^k(2k-1)}{\Gamma(k+1)} \right) \frac{g_1^3}{2^{4k-1} \pi^k} + \dots, \end{aligned} \quad (2.8)$$

where the full dimensions of ψ and σ fields are $\Delta_\psi = \frac{d-1}{2} + \gamma_\psi$ and $\Delta_\sigma = \frac{d+2-2k}{2} + \gamma_\sigma$. Thus

from (2.8) the fixed point at the leading order in ϵ is

$$g_{1,*}^2 = 4^{2k-1} \pi^k \left(\frac{2\sqrt{\pi}(-1)^k N}{\Gamma(k - \frac{1}{2})} + \frac{4^k(2k-1)}{\Gamma(k+1)} \right)^{-1} \epsilon + \mathcal{O}(\epsilon^2). \quad (2.9)$$

And we find for the anomalous dimensions at the critical point ²

$$\begin{aligned} \gamma_\psi &= \frac{(k-1)^2 k \Gamma(2k-2)}{N(-1)^k \Gamma(k+1)^2 + \Gamma(2k+1)} \epsilon + \mathcal{O}(\epsilon^2), \\ \gamma_\sigma &= \frac{N(-1)^k \sqrt{\pi} \Gamma(k+1)}{2(N(-1)^k \sqrt{\pi} \Gamma(k+1) + 4^k \Gamma(k + \frac{1}{2}))} \epsilon + \mathcal{O}(\epsilon^2). \end{aligned} \quad (2.10)$$

One can compare these results to the anomalous dimensions of ψ and σ fields computed in the Gross-Neveu model in the large N approximation. We have $\Delta_\psi = \frac{d-1}{2} + \gamma_{\psi,1}/N + \gamma_{\psi,2}/N^2 + \mathcal{O}(1/N^3)$ and $\Delta_\sigma = 1 + \gamma_{\sigma,1}/N + \mathcal{O}(1/N^2)$, where the functions $\gamma_{\psi,1}, \gamma_{\psi,2}, \gamma_{\psi,3}$ and $\gamma_{\sigma,1}, \gamma_{\sigma,2}$ are given in [26–28, 30, 31]. Expanding these functions in ϵ at $d = 2k - \epsilon$ one finds perfect agreement with (2.10).

Next we compute the critical free energy on the sphere for the model (2.6) at $d = 2k - \epsilon$ for an arbitrary positive integer k . At the leading order in ϵ only the g_1 -vertex contributes and we have

$$F_{\text{GNY}} = N F_f + F_\sigma - \frac{1}{2!} g_{1,*}^2 \int d^d x d^d y \sqrt{g_x} \sqrt{g_y} \langle \sigma \bar{\psi}_i \psi^i(x) \sigma \bar{\psi}_j \psi^j(y) \rangle + \mathcal{O}(\epsilon^3), \quad (2.11)$$

where F_f is the sphere free energy for the free massless Dirac fermion (1.10) and F_σ is the sphere free energy for the free higher-derivative scalar σ , F_σ is given by (1.11) with $\Delta = \frac{d}{2} + k - 1$. The propagators of ψ and σ fields in flat coordinate space are

$$\langle \psi_i(x) \bar{\psi}^j(0) \rangle = -\delta_i^j \frac{\Gamma(\frac{d}{2})}{2\pi^{d/2}} \frac{\gamma_\mu x_\mu}{(x^2)^{\frac{d}{2}}}, \quad \langle \sigma(x) \sigma(0) \rangle = \frac{\Gamma(\frac{d}{2} - k + 1)}{4^{k-1} \pi^{\frac{d}{2}} \Gamma(k-1)} \frac{1}{(x^2)^{\frac{d}{2}+1-k}}. \quad (2.12)$$

Because the propagators in the flat space are related to the sphere ones by the Weyl transformation, we can compute the correlation function in (2.11) working in the flat space and then project the result on the sphere by making replacement $(x-y)^2 \rightarrow s(x,y)^2$, where $s(x,y)$ is the chordal distance on the sphere. We find

$$F_{\text{GNY}} = N F_f + F_\sigma - \frac{1}{2!} g_{1,*}^2 N \left(\frac{\Gamma(\frac{d}{2})}{2\pi^{d/2}} \right)^2 \frac{\Gamma(\frac{d}{2} - k + 1)}{4^{k-1} \pi^{\frac{d}{2}} \Gamma(k-1)} I_2\left(\frac{3d}{2} - k\right) + \mathcal{O}(\epsilon^3), \quad (2.13)$$

²These results were obtained in collaboration with Lin Fei.

where the integral $I_2(\Delta)$ is defined as [14, 32, 33]

$$I_2(\Delta) = \int \frac{d^d x d^d y \sqrt{g_x} \sqrt{g_y}}{s(x, y)^{2\Delta}} = (2R)^{2(d-\Delta)} \frac{2^{1-d} \pi^{d+\frac{1}{2}} \Gamma(\frac{d}{2} - \Delta)}{\Gamma(\frac{d+1}{2}) \Gamma(d - \Delta)} \quad (2.14)$$

and R is the sphere radius. Now substituting (2.9) in (2.13) and expanding everything in ϵ at $d = 2k - \epsilon$ we find in terms of generalized free energy $\tilde{F} = -\sin(\pi d/2)F$, which is a smooth function of dimension [8]

$$\delta \tilde{F}_{\text{GNY}} = -\frac{N\pi^{3/2}k\Gamma(k)^2\Gamma(2k-1)}{8\Gamma(k-1)\Gamma(2k+1)(N\sqrt{\pi}(-1)^k k\Gamma(k) + 4^k\Gamma(k+\frac{1}{2}))}\epsilon^2 + \mathcal{O}(\epsilon^3), \quad (2.15)$$

where we denoted $\delta \tilde{F}_{\text{GNY}} = \tilde{F}_{\text{GNY}} - N\tilde{F}_f - \tilde{F}_\sigma$. Expanding (2.15) in the large N limit we get

$$\delta \tilde{F}_{\text{GNY}} = \left(-\frac{\pi(-1)^k(k-1)}{16k(2k-1)} + \frac{1}{N} \frac{\sqrt{\pi}2^{2k-5}\Gamma(k-\frac{1}{2})}{k^2\Gamma(k-1)} + \mathcal{O}(1/N^2) \right) \epsilon^2 + \mathcal{O}(\epsilon^3). \quad (2.16)$$

One can check that the ϵ^2/N term obtained by expansion of (1.7) in small ϵ near $d = 2k - \epsilon$ coincides with the same term in (2.16)

2.3 The $1/N$ correction to the GN model

Based on analytical computations described in Appendix A and analogy with the supersymmetric result (2.5) we may guess the expression for the $1/N$ term in the GN sphere free energy

$$\tilde{F}_{1/N}^{\text{GN}} = -\frac{\sin(\pi d/2)}{Nd} \gamma_{\psi,1}^{\text{GN}}, \quad (2.17)$$

where $\gamma_{\psi,1}^{\text{GN}}$ is given in (1.12). One can verify that (2.17) gives the correct ϵ expansion near various integer dimensions. Let us expand it in small ϵ near $d = 2$ and $d = 4$. We find

$$\tilde{F}_{1/N}^{\text{GN}}|_{d=2+\epsilon} = \frac{1}{N} \left(\frac{\pi\epsilon^3}{16} - \frac{\pi\epsilon^4}{16} + \mathcal{O}(\epsilon^5) \right), \quad \tilde{F}_{1/N}^{\text{GN}}|_{d=4-\epsilon} = \frac{1}{N} \left(\frac{\pi\epsilon^2}{16} - \frac{\pi\epsilon^3}{32} + \mathcal{O}(\epsilon^4) \right). \quad (2.18)$$

We have checked that these expansions match with the ϵ -expansions in $d = 2 + \epsilon$ and $d = 4 - \epsilon$ for the sphere free energy of the Gross-Neveu and Gross-Neveu-Yukawa models obtained in [10]. Moreover, as we have already explained, the $1/N$ term in (2.17) matches with the result (2.16) for expansion at $d = 2k - \epsilon$ for all $k = 2, 3, 4, \dots$

The ϵ -expansion for the NJL $U(1)$ model near $d = 2$ and 4 coincides with the result in [10].

3 Towards the large N expansion of sphere free energy in bosonic theory

3.1 Free energy on $S^{6-\epsilon}$ for cubic $O(N)$ scalar theory

In this section we consider the cubic $O(N)$ model on a sphere in $d = 6 - \epsilon$ dimension. This model in flat space was recently proposed in [34] and further studied in [35–37] (non-perturbative studies of this fixed point have been carried out using functional renormalization group [38–41] and conformal bootstrap [42, 43]). The leading term in ϵ -expansion for the sphere free energy was already computed in [8]. Here we calculate the next to leading term. In order to renormalize the theory on a curved manifold, one should add to the action all the relevant curvature couplings that are marginal in $d = 6 - \epsilon$ [44, 45]

$$S_{\text{cubic } O(N)} = \int d^d x \sqrt{g} \left(\frac{1}{2} (\partial_\mu \phi^i)^2 + \frac{1}{2} (\partial_\mu \sigma)^2 + \frac{d-2}{8(d-1)} \mathcal{R} (\sigma^2 + \phi^i \phi^i) + \frac{1}{2} g_{1,0} \sigma \phi^i \phi^i + \frac{1}{6} g_{2,0} \sigma^3 \right) + \frac{\eta_0}{2} \mathcal{R} (\sigma^2 + \phi^i \phi^i) + b_0 \int d^d x \sqrt{g} \mathcal{R}^3, \quad (3.1)$$

where σ and ϕ^i , $i = 1, \dots, N$ are bare scalar fields, $g_{1,0}$ and $g_{2,0}$ are bare couplings and \mathcal{R} is the scalar curvature. The parameters η_0, b_0 are bare curvature couplings whose renormalization is fixed order by order in perturbation theory. Below we will only need the renormalization of the coupling b_0 (the renormalization of conformal coupling η_0 is expected to play a role at higher orders [44–46]). We will find the beta function for b_0 coupling by demanding that the final result for the free energy is free of divergences.

It is convenient to combine the cubic interaction terms in the form

$$\mathcal{L}_{\text{int}} = \frac{1}{3!} g_{abc} \varphi^a \varphi^b \varphi^c, \quad (3.2)$$

with symmetric tensor g_{abc} , where $a, b, c = 1, \dots, N+1$, and we identify $\varphi^0 = \sigma$ and $\varphi^i = \phi^i$ and $g_{000} = g_{2,0}$, $g_{0ii} = g_{i0i} = g_{iio} = g_{1,0}$ and all other components of the tensor g_{abc} are zero. Next we define integrated connected correlation functions on a sphere

$$G_n = \int \prod_{i=1}^n dx_i \sqrt{g_{x_i}} \langle \varphi^3(x_1) \dots \varphi^3(x_n) \rangle_0^{\text{conn}}, \quad (3.3)$$

where $\varphi^3 \equiv g_{abc}\varphi^a\varphi^b\varphi^c$ and the two-point function on the sphere is

$$\langle \varphi^a(x)\varphi^b(y) \rangle = \delta^{ab} \frac{\Gamma(\frac{d}{2}-1)}{4\pi^{d/2}} \frac{1}{s(x,y)^{d-2}} = \delta^{ab} \frac{C_\phi}{s(x,y)^{d-2}}. \quad (3.4)$$

Evidently $G_n = 0$ if n is odd. Therefore for the free energy on the sphere up to the sixth order in couplings we find

$$F_{\text{cubic } O(N)} = (N+1)F_s - \frac{1}{2!} \frac{G_2}{(3!)^2} - \frac{1}{4!} \frac{G_4}{(3!)^4} - \frac{1}{6!} \frac{G_6}{(3!)^6} + b_0 \int d^d x \sqrt{g} \mathcal{R}^3, \quad (3.5)$$

where

$$\begin{aligned} G_2 &= 3!t_2 C_\phi^3 I_2\left(\frac{3}{2}(d-2)\right), \\ G_4 &= 3(3!)^3(3t_{41}G_4^{(1)} + 2t_{42}G_4^{(2)}), \\ G_6 &= 15(3!)^5(18t_{61}G_6^{(1)} + 6t_{62}G_6^{(2)} + 9t_{63}G_6^{(3)} + 36t_{64}G_6^{(4)} + 24t_{65}G_6^{(5)} + 4t_{66}G_6^{(6)}), \end{aligned} \quad (3.6)$$

where the integral $I_2(\Delta)$ is defined in (2.14) and t -coefficients are computed by contracting the tensors g_{abc} for each diagram depicted in figure 4³

$$\begin{aligned} t_2 &= 3Ng_1^2 + g_2^2, \quad t_{41} = (N+4)Ng_1^4 + 2Ng_1^2g_2^2 + g_2^4, \quad t_{42} = 3Ng_1^4 + 4Ng_1^3g_2 + g_2^4, \\ t_{61} &= 4N(N+1)g_1^6 + N(N+4)g_1^4g_2^2 + 2Ng_1^2g_2^4 + g_2^6, \quad t_{62} = 8Ng_1^6 + (Ng_1^2 + g_2^2)^3, \\ t_{63} &= N^2g_1^4(2g_1 + g_2)^2 + 4Ng_1^3g_2^3 + g_2^6, \\ t_{64} &= N(N+4)g_1^6 + 2N(N+2)g_1^5g_2 + Ng_1^4g_2^2 + 2Ng_1^3g_2^3 + Ng_1^2g_2^4 + g_2^6, \\ t_{65} &= N(N+3)g_1^6 + 6Ng_1^5g_2 + 3Ng_1^4g_2^2 + 2Ng_1^3g_2^3 + g_2^6, \quad t_{66} = 6Ng_1^6 + 9Ng_1^4g_2^2 + g_2^6. \end{aligned} \quad (3.7)$$

Here for brevity we suppressed the index ₀ indicating that these couplings are bare.

To compute higher-loop integrals we use the Mellin-Barnes approach, which is described in [9, 11]. We find for the four-point functions

$$G_4^{(k)} = C_\phi^6 (2R)^{2(6-d)} \frac{2^{1-d} \pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})} e^{\frac{3}{2}(d-6)\gamma} \pi^{\frac{3d}{2}} \begin{cases} \frac{1}{18\epsilon} + \frac{43}{162} + \left(\frac{2857}{3888} + \frac{\pi^2}{288}\right)\epsilon + \mathcal{O}(\epsilon^2), \\ -\frac{1}{3\epsilon} - \frac{3}{2} - \left(\frac{191}{48} + \frac{\pi^2}{48}\right)\epsilon + \mathcal{O}(\epsilon^2), \end{cases} \quad (3.8)$$

³These results were obtained in collaboration with Lin Fei.

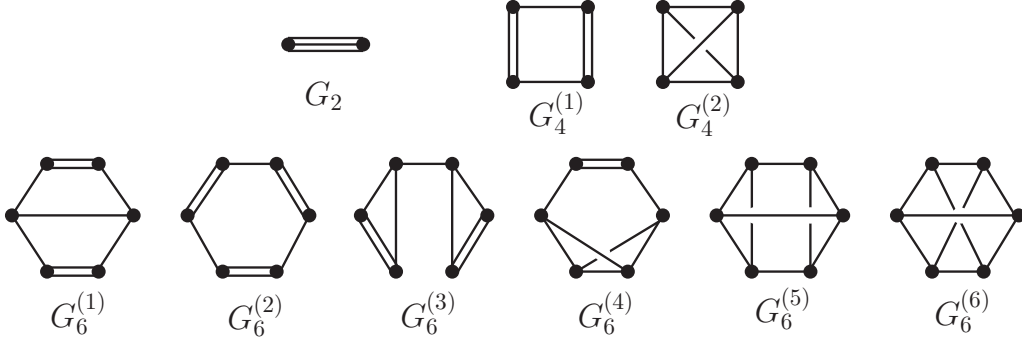


Figure 4: Diagrams contributing to the sphere free energy in the cubic $O(N)$ theory up to the sixth order. Each line represents the propagator $\langle \phi(x)\phi(y) \rangle = C_\phi/s(x,y)^{d-2}$.

where R is the sphere radius and also for the six-point functions we obtain

$$G_6^{(k)} = C_\phi^9 (2R)^{3(6-d)} \frac{2^{1-d} \pi^{\frac{d+1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)} e^{\frac{5}{2}(d-6)\gamma} \pi^{\frac{5d}{2}} \begin{cases} -\frac{1}{54\epsilon^2} - \frac{389}{2592\epsilon} - \left(\frac{87}{128} + \frac{\pi^2}{864}\right) + \mathcal{O}(\epsilon), \\ -\frac{1}{36\epsilon^2} - \frac{139}{648\epsilon} - \left(\frac{3547}{3888} + \frac{\pi^2}{576}\right) + \mathcal{O}(\epsilon), \\ \mathcal{O}(\epsilon), \\ \frac{1}{9\epsilon^2} + \frac{733}{864\epsilon} + \left(\frac{4675}{1296} + \frac{\pi^2}{144}\right) + \mathcal{O}(\epsilon), \\ -\frac{1}{3\epsilon^2} - \frac{703}{288\epsilon} - \left(\frac{34649}{3456} + \frac{\pi^2}{48}\right) + \mathcal{O}(\epsilon), \\ -\frac{5}{24\epsilon} + \left(\frac{\zeta(3)}{8} - \frac{475}{288}\right) + \mathcal{O}(\epsilon). \end{cases} \quad (3.9)$$

The beta-functions for the cubic $O(N)$ model were computed in [34, 35, 37]. For our calculation we need to extract the relation between the bare couplings and the renormalized ones

$$\begin{aligned} g_{0,1} &= \mu^{\epsilon/2} \left(g_1 + \frac{(N-8)g_1^3 - 12g_1^2g_2 + g_1g_2^2}{12(4\pi)^3} \frac{1}{\epsilon} + \dots \right), \\ g_{0,2} &= \mu^{\epsilon/2} \left(g_2 - \frac{3g_2^3 + 4Ng_1^3 - Ng_1^2g_2}{4(4\pi)^3} \frac{1}{\epsilon} + \dots \right), \end{aligned} \quad (3.10)$$

where μ is the auxiliary scale and we have not explicitly written higher order terms. Combining all the results (3.6), (3.8), (3.9) in (3.5) and using (3.10), we find that in order to cancel all poles in ϵ we should renormalize b_0 coupling as

$$b_0 = \mu^{-\epsilon} \left(b + \frac{N+1}{756 \cdot 450(4\pi)^3} \frac{1}{\epsilon} + \frac{b_{61}(g_1, g_2)}{\epsilon} + \dots \right), \quad (3.11)$$

where

$$b_{61}(g_1, g_2) = \frac{1}{2^{12} 3^8 5^3 (4\pi)^{12}} \left(N(2(43N + 268)g_1^6 - 12(11N - 32)g_1^5 g_2 \right. \\ \left. + (11N + 950)g_1^4 g_2^2 + 84g_1^3 g_2^3 - 44g_1^2 g_2^4) + 125g_2^6 \right) \quad (3.12)$$

and the first coupling independent term in (3.11) is due to the trace anomaly of the free boson field. Thus the beta-function for the b coupling reads

$$\beta_b = \epsilon b + \frac{N+1}{756 \cdot 450 (4\pi)^3} + 4b_{61}(g_1, g_2). \quad (3.13)$$

As explained in [9], in order to calculate the free energy at the critical point we should tune all couplings, including b , to their fixed point values. Using (3.13) and the results from [35] for the large N critical point of the $O(N)$ cubic model, we find

$$g_1^* = \sqrt{\frac{6\epsilon(4\pi)^3}{N}} \left(1 + \frac{22}{N} + \frac{726}{N^2} - \left(\frac{155}{6N} + \frac{1705}{N^2} \right) \epsilon + \frac{1777}{144N} \epsilon^2 + \dots \right), \\ g_2^* = 6\sqrt{\frac{6\epsilon(4\pi)^3}{N}} \left(1 + \frac{162}{N} + \frac{68766}{N^2} - \left(\frac{215}{2N} + \frac{86335}{N^2} \right) \epsilon + \frac{2781}{48N} \epsilon^2 + \dots \right), \\ b_* = -\frac{N+1}{756 \cdot 450 (4\pi)^3} \frac{1}{\epsilon} - \frac{4b_{61}(g_1^*, g_2^*)}{\epsilon} + \dots, \quad (3.14)$$

where the dots stand for higher order terms in ϵ and $1/N$. The contribution to the sphere free energy from the curvature coupling reads⁴

$$\delta F_b = b_* \int d^d x \sqrt{g} \mathcal{R}^3 = 450(4\pi)^3 b_* + \mathcal{O}(\epsilon^3), \quad (3.15)$$

where we used that $\text{vol}(S^d) = \int d^d x \sqrt{g} = 2\pi^{(d+1)/2} R^d / \Gamma(\frac{d+1}{2})$ and the scalar curvature on the sphere is $\mathcal{R} = d(d-1)/R^2$. Substituting the values (3.14) in (3.5) we get in terms of $\tilde{F} = -\sin(\pi d/2)F$

$$\tilde{F}_{\text{cubic } O(N)} = (N+1)\tilde{F}_s - \left(\frac{\pi\epsilon^2}{960} + \frac{19\pi\epsilon^3}{43200} \right) - \left(\frac{7\pi\epsilon^2}{120} - \frac{17\pi\epsilon^3}{400} \right) \frac{1}{N} \\ - \left(\frac{91\pi\epsilon^2}{15} - \frac{56843\pi\epsilon^3}{5400} \right) \frac{1}{N^2} + \mathcal{O}\left(\epsilon^4, \frac{1}{N^3}\right). \quad (3.16)$$

⁴Note that we computed the six-point functions in (3.9) essentially only to find the renormalization of b . If one finds the next order term in the renormalization of b , then, using (3.8) and (3.9), it is possible to get the next order term for the sphere free energy.

Expanding the large N result for the $O(N)$ scalar model (1.7) in small ϵ at $d = 6 - \epsilon$ we find perfect agreement with the $1/N$ term in (3.16).

3.2 Free energy on $S^{8-\epsilon}$ for $O(N)$ scalar theory

In this section we compute the leading in ϵ term for the sphere free energy of the $O(N)$ scalar model at $d = 8 - \epsilon$ dimensions. At leading order, one need not to take into account contributions to the free energy coming from the curvature couplings. The curvature couplings contribute only at the next to leading order in ϵ , as we saw in the previous section.

The action for the $O(N)$ model in $d = 8 - \epsilon$ flat space reads [47]

$$S_{8d\ O(N)} = \int d^d x \left(\frac{1}{2} (\partial_\mu \phi^i)^2 + \frac{1}{2} (\Box \sigma)^2 + \frac{1}{2} \mu^{\frac{\epsilon}{2}} g_1 \sigma \phi^i \phi^i + \frac{1}{6} \mu^{\frac{\epsilon}{2}} g_2 \sigma^2 \Box \sigma + \frac{1}{4!} \mu^\epsilon g_3^2 \sigma^4 + \dots \right), \quad (3.17)$$

where σ and ϕ^i , $i = 1, \dots, N$ are scalar fields, g_1, g_2 and g_3 are renormalized couplings and μ is an auxiliary scale. Dots in (3.17) stand for the counter-terms. The beta-functions and anomalous dimensions for this model were computed up to three loops in [47].

At leading order in ϵ only terms with g_1 and g_2 contribute to the sphere free energy. The contribution, which involves g_2 coupling, requires a careful treatment. Let us introduce two operators

$$O_1(x) = \sigma^2 \Box \sigma, \quad O_2(x) = \sigma (\partial \sigma)^2. \quad (3.18)$$

First, an operator $\mathcal{D}(x) = O_1(x) + 2O_2(x)$, being a total derivative, is a descendant operator. Then we can find the primary operator $\mathcal{P}(x)$ as a linear combination of the operators O_1 and O_2 , which is orthogonal to the descendant operator $\langle \mathcal{P}(x) \mathcal{D}(y) \rangle = 0$. One finds that the primary operator is given by the formula

$$\mathcal{P}(x) = \frac{1}{2} (d - 4) O_1(x) + O_2(x). \quad (3.19)$$

Therefore for the operator $O_1(x)$, which appears in the Lagrangian (3.17) we get

$$O_1(x) = \frac{1}{d - 5} (2\mathcal{P}(x) - \mathcal{D}(x)). \quad (3.20)$$

Then at the leading order for the sphere free energy one finds

$$F_{8d\ O(N)} = NF_s + F_\sigma - \frac{1}{2!} \frac{1}{2^2} g_1^2 \int d^d x d^d y \sqrt{g_x} \sqrt{g_y} \langle \sigma \phi^i \phi^i(x) \sigma \phi^j \phi^j(y) \rangle \\ - \frac{1}{2!} \frac{1}{6^2} g_2^2 \left(\frac{2}{d-5} \right)^2 \int d^d x d^d y \sqrt{g_x} \sqrt{g_y} \langle \mathcal{P}(x) \mathcal{P}(y) \rangle + \dots, \quad (3.21)$$

where F_s is the sphere free energy for the free conformally coupled scalar (1.9) and F_σ is the sphere free energy for the free higher-derivative scalar σ , F_σ is given by (1.11) with $\Delta = \frac{d}{2} + 2$. The propagator for the ϕ field is given in (3.4) and the σ field propagator reads

$$\langle \sigma(x) \sigma(y) \rangle = \frac{\Gamma(\frac{d}{2} - 2)}{16\pi^{d/2}} \frac{1}{s(x, y)^{d-4}} = \frac{C_\sigma}{s(x, y)^{d-4}}. \quad (3.22)$$

Using (3.22) we obtain for the two-point function of the primary operators

$$\langle \mathcal{P}(x) \mathcal{P}(y) \rangle = -2d(d-5)(d-4)^2 \frac{C_\sigma^3}{s(x, y)^{3d-8}}. \quad (3.23)$$

Finally computing the correlation functions in (3.21) and tuning the couplings to the critical point we find for the shift of the sphere free energy $\delta \tilde{F} = -\sin(\pi d/2)(F_{8d\ O(N)} - NF_s - F_\sigma)$

$$\delta \tilde{F}_{8d\ O(N)} = \sin(\pi d/2) \left(\frac{1}{4} N g_{1,*}^2 C_\phi^2 C_\sigma - \frac{d(d-4)^2}{9(d-5)} g_{2,*}^2 C_\sigma^3 \right) I_2 \left(\frac{3d}{2} - 4 \right) + \mathcal{O}(\epsilon^3), \quad (3.24)$$

where the integral $I_2(\Delta)$ is defined in (2.14) and the large N critical couplings were computed in [47] and read

$$g_1^* = i \sqrt{\frac{60(4\pi)^4 \epsilon}{N}} \left(1 + (-110 + 126\epsilon) \frac{1}{N} + (18150 - \frac{48475}{2} \epsilon) \frac{1}{N^2} + \mathcal{O}(1/N^3) \right), \quad (3.25) \\ g_2^* = -i \sqrt{\frac{60(4\pi)^4 \epsilon}{N}} \left(15 - (14250 - 10605\epsilon) \frac{1}{N} + (36182250 - \frac{89882625}{2} \epsilon) \frac{1}{N^2} + \mathcal{O}(1/N^3) \right).$$

Expanding the expression (3.24) in small ϵ and $1/N$ we get

$$\delta \tilde{F}_{8d\ O(N)} = \left(\frac{\pi}{1344} - \frac{1}{N} \frac{215\pi}{1008} + \frac{1}{N^2} \frac{32825\pi}{252} + \mathcal{O}(1/N^3) \right) \epsilon^2 + \mathcal{O}(\epsilon^3). \quad (3.26)$$

Expanding the large N result for the $O(N)$ scalar model (1.7) in small ϵ at $d = 8 - \epsilon$ we find perfect agreement with the $1/N$ term in (3.26).

3.3 The $1/N$ correction to the $O(N)$ scalar model

Based on analytical computations described in Appendix B and analogy with the supersymmetric result (2.5) we may guess the expression for the $1/N$ term in the $O(N)$ sphere free energy

$$\tilde{F}_{1/N}^{O(N)} = -\frac{1}{N} \sin\left(\frac{\pi d}{2}\right) \left(\frac{1}{d} + \frac{1}{3(d-2)} - \frac{2}{3(d-1)}\right) \gamma_{\phi,1}^{O(N)}, \quad (3.27)$$

where $\gamma_{\phi,1}^{O(N)}$ is given in (1.12). One can verify that (3.27) gives the correct ϵ -expansion near various integer dimensions. Let us expand it in small ϵ near $d = 4$, $d = 6$ and $d = 8$ dimensions. We find

$$\begin{aligned} \tilde{F}_{1/N}^{O(N)}|_{d=4-\epsilon} &= \frac{1}{N} \left(\frac{7\pi\epsilon^3}{288} + \frac{5\pi\epsilon^4}{1728} - \frac{(239 + 42\pi^2)\pi\epsilon^5}{41472} + \mathcal{O}(\epsilon^6) \right), \\ \tilde{F}_{1/N}^{O(N)}|_{d=6-\epsilon} &= \frac{1}{N} \left(-\frac{7\pi\epsilon^2}{120} + \frac{17\pi\epsilon^3}{400} + \mathcal{O}(\epsilon^4) \right), \\ \tilde{F}_{1/N}^{O(N)}|_{d=8-\epsilon} &= \frac{1}{N} \left(-\frac{215\pi\epsilon^2}{1008} + \mathcal{O}(\epsilon^3) \right). \end{aligned} \quad (3.28)$$

One can check that the $d = 4 - \epsilon$ expansion in (3.28) exactly agrees with the result in [9] and $d = 6 - \epsilon$ and $d = 8 - \epsilon$ expansions agree with formulas (3.16) and (3.26).

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A Large N computation for the GN model

In this appendix we present a direct computation of the $1/N$ term in the sphere free energy of the Gross-Neveu model at general dimension. This computation assumes a special regularization of the Feynman integrals on a sphere. The computation of the integrals is performed with the use of the Mellin-Barnes method [48–50]. Surprisingly it is possible to

find an exact answer for arbitrary d . Unfortunately this answer is off by a factor of $1/3$ from the correct result. The origin of this discrepancy is unclear to us.

We start with the action describing critical fermion theory on a sphere in the large N approach

$$S_{\text{GN}} = \int d^d x \sqrt{g} \left(\bar{\psi}_i \not{\nabla} \psi^i + \frac{1}{\sqrt{N}} \sigma \bar{\psi}_i \psi^i \right), \quad (\text{A.1})$$

where ψ^i , $i = 1, \dots, N_f$ are Dirac fermions, σ is the auxiliary field and $N = N_f \text{Tr} \mathbf{1}$. So we have for the partition function

$$Z_{\text{GN}} = Z_f^N \int D\sigma \frac{\int D\psi e^{-\int d^d x \sqrt{g} \frac{1}{\sqrt{N}} \sigma \bar{\psi}_i \psi^i} \exp(-S_{\text{free ferm}})}{\int D\psi \exp(-S_{\text{free ferm}})}, \quad (\text{A.2})$$

where $Z_f^N = \int D\psi \exp(-S_{\text{free ferm}}) = \exp(-NF_f)$ is the partition function for the N_f free Dirac fermions on the sphere and F_f is given in (1.10). Since the trace of odd number of gamma matrices is zero we have $\langle \bar{\psi}\psi(x_1) \dots \bar{\psi}\psi(x_{2n+1}) \rangle_0 = 0$. Therefore the $1/N$ expansion for the partition function takes the form

$$\begin{aligned} Z_{\text{GN}} = Z_f^N \int D\sigma \exp \left(\frac{1}{2!N} \int d^d x_1 d^d x_2 \sqrt{g_{x_1}} \sqrt{g_{x_2}} \sigma(x_1) \sigma(x_2) \langle \bar{\psi}\psi(x_1) \bar{\psi}\psi(x_2) \rangle_0 \right. \\ \left. + \frac{1}{4!N^2} \int d^d x_1 \sqrt{g_{x_1}} \dots d^d x_4 \sqrt{g_{x_4}} \sigma(x_1) \dots \sigma(x_4) \langle \bar{\psi}\psi(x_1) \dots \bar{\psi}\psi(x_4) \rangle_0^{\text{conn}} + \dots \right). \end{aligned} \quad (\text{A.3})$$

The propagators in flat space are related to those on the sphere by the Weyl transformation, therefore we can compute correlation functions working in the flat space and then project the result on the sphere. The propagators of the ψ and σ fields in flat space are

$$\begin{aligned} \langle \psi^i(x) \bar{\psi}_j(0) \rangle_0 &= -\delta_j^i \frac{\Gamma(\frac{d}{2})}{2\pi^{d/2}} \frac{\gamma^\mu x_\mu}{(x^2)^{d/2}} = \delta_j^i C_\psi \frac{\not{x}}{(x^2)^{d/2}}, \\ \langle \sigma(x) \sigma(0) \rangle_0 &= -\frac{2^d \sin(\frac{\pi d}{2}) \Gamma(\frac{d-1}{2})}{\pi^{3/2} \Gamma(\frac{d}{2} - 1)} \frac{1}{(x^2)^{1-\delta}} = \frac{C_\sigma}{(x^2)^{1-\delta}}, \end{aligned} \quad (\text{A.4})$$

where we have introduced a regulator δ . Next we define $\Gamma_2(x_1, x_2)$ and $\Gamma_4(x_1, \dots, x_4)$ as

$$\begin{aligned} \Gamma_2(x_1, x_2) &\equiv \langle \bar{\psi}\psi(x_1) \bar{\psi}\psi(x_2) \rangle_0 = N_f C_\psi^2 (-1) \frac{\text{Tr}(\not{x}_{12} \not{x}_{21})}{(x_{12}^2)^d}, \\ \Gamma_4(x_1, \dots, x_4) &\equiv \langle \bar{\psi}\psi(x_1) \dots \bar{\psi}\psi(x_4) \rangle_0^{\text{conn}} = 6N_f C_\psi^4 (-1) \frac{\text{Tr}(\not{x}_{12} \not{x}_{23} \not{x}_{34} \not{x}_{41})}{(x_{12}^2 x_{23}^2 x_{34}^2 x_{41}^2)^{d/2}}, \end{aligned} \quad (\text{A.5})$$

where $x_{ij} \equiv x_i - x_j$ and in the second line we assumed that we can symmetrize over the

points x_1, \dots, x_4 under the integral. Taking traces over the gamma matrices in (A.5) we find

$$\Gamma_2(x_1, x_2) = \frac{NC_\psi^2}{(x_{12}^2)^{d-1}}, \quad \Gamma_4(x_1, \dots, x_4) = -3NC_\psi^4 \frac{(2x_{12}^2 x_{34}^2 - x_{13}^2 x_{24}^2)}{(x_{12}^2 x_{23}^2 x_{34}^2 x_{41}^2)^{d/2}}, \quad (\text{A.6})$$

where in the expression for Γ_4 we again used symmetrization over x_1, \dots, x_4 . In order to write these correlation functions on the sphere one simply changes $x_{ij} \rightarrow s(x_i, x_j)$, where $s(x, y)$ is the chordal distance on the sphere. Now the partition function (A.3) can be written in the form

$$Z_{\text{GN}} = Z_f^N \delta Z_{\Delta=d-1} \times \left\langle \exp \left(\frac{1}{4!N^2} \int d^d x_1 \sqrt{g_{x_1}} \dots d^d x_4 \sqrt{g_{x_4}} \sigma(x_1) \dots \sigma(x_4) \Gamma_4(x_1, \dots, x_4) + \dots \right) \right\rangle_\sigma, \quad (\text{A.7})$$

where $\delta Z_{\Delta=d-1} = \exp(-\delta F_{\Delta=d-1}) = \int D\sigma \exp \left(\frac{1}{2N} \int d^d x_1 d^d x_2 \sigma(x_1) \sigma(x_2) \Gamma_2(x_{12}) \right)$ is the partition function on the sphere for the induced σ field and $\delta F_{\Delta=d-1}$ is given by (1.11) with $\Delta = d - 1$. The average in (A.7) is taken over the σ field with propagator (A.4)⁵. Finally for the sphere free energy we obtain

$$F_{\text{GN}} = NF_f + \delta F_{\Delta=d-1} - \frac{1}{4!N^2} \int d^d x_1 \sqrt{g_{x_1}} \dots d^d x_4 \sqrt{g_{x_4}} \langle \sigma(x_1) \dots \sigma(x_4) \rangle_\sigma \Gamma_4(x_1, \dots, x_4) + \mathcal{O}(1/N^2). \quad (\text{A.8})$$

To compute the integral in (A.8) we use the Mellin-Barnes approach, which is described in [9, 11]. Surprisingly it is possible to compute this integral for arbitrary d and the result does not have terms divergent in δ

$$\frac{1}{4!N^2} \int d^d x_1 \sqrt{g_{x_1}} \dots d^d x_4 \sqrt{g_{x_4}} \langle \sigma(x_1) \dots \sigma(x_4) \rangle_\sigma \Gamma_4(x_1, \dots, x_4) = -\frac{1}{N} \frac{3}{d} \gamma_{\psi,1}^{\text{GN}}, \quad (\text{A.9})$$

where $\gamma_{\psi,1}^{\text{GN}}$ is the $1/N$ correction to the anomalous dimension of ψ field

$$\gamma_{\psi,1}^{\text{GN}} = \frac{(d-2)^2 \Gamma(d-1)}{4\Gamma(2-\frac{d}{2}) \Gamma(\frac{d}{2}+1) \Gamma(\frac{d}{2})^2}. \quad (\text{A.10})$$

Nevertheless the result (A.9) is off by a factor $1/3$ from the correct answer

$$F_{\text{GN}} = NF_f + \delta F_{\Delta=d-1} + \frac{1}{N} \frac{1}{d} \gamma_{\psi,1}^{\text{GN}} + \mathcal{O}(1/N^2). \quad (\text{A.11})$$

⁵The propagator of the σ field on the sphere has the form $\langle \sigma(x) \sigma(y) \rangle = C_\sigma / (s(x, y))^{1-\delta}$.

The results for the Nambu-Jona-Lasinio models with $U(1)$ and $SU(2)$ chiral symmetries in (1.7) are obtained using the same method as described above for the Gross-Neveu model. The resulting Feynman integrals are similar to the GN integrals and again we need to multiply the final answer by the factor $1/3$.

B Large N computation for the $O(N)$ model

In this appendix we present a direct calculation of the $1/N$ term in the sphere free energy of the $O(N)$ scalar model. The method of the computation is the same as in the Gross-Neveu model, discussed in previous section. It is also possible to find an exact answer for arbitrary d . Unfortunately this answer is off by some rational factors from the correct result as in the Gross-Neveu case.

We start with the action describing critical boson theory on the sphere in the large N approach

$$S_{O(N)} = \int d^d x \sqrt{g} \left(\frac{1}{2} (\partial_\mu \phi^i)^2 + \frac{d-2}{8(d-1)} \mathcal{R} \phi^i \phi^i + \frac{1}{2\sqrt{N}} \sigma \phi^i \phi^i \right), \quad (\text{B.1})$$

where ϕ^i , $i = 1, \dots, N$ are scalar fields, σ is the auxiliary scalar field and \mathcal{R} is the scalar curvature. For the partition function we get

$$Z_{O(N)} = Z_s^N \int D\sigma \frac{\int D\phi e^{-\int d^d x \sqrt{g} \frac{1}{2\sqrt{N}} \sigma \phi^i \phi^i} \exp(-S_{\text{free bos}})}{\int D\phi \exp(-S_{\text{free bos}})}, \quad (\text{B.2})$$

where $Z_s^N = \int D\phi \exp(-S_{\text{free bos}}) = \exp(-NF_s)$ is the partition function for the N conformally coupled free bosons on the sphere and F_s is given in (1.9). The propagators of the ϕ and σ fields on the sphere are

$$\begin{aligned} \langle \phi^i(x) \phi^j(y) \rangle_0 &= \delta^{ij} \frac{\Gamma(\frac{d}{2} - 1)}{4\pi^{d/2}} \frac{1}{s(x, y)^{d-2}} = \delta^{ij} \frac{C_\phi}{s(x, y)^{d-2}}, \\ \langle \sigma(x) \sigma(y) \rangle &= \frac{2^{d+2} \Gamma(\frac{d-1}{2}) \sin(\frac{\pi d}{2})}{\pi^{\frac{3}{2}} \Gamma(\frac{d}{2} - 2)} \frac{1}{s(x, y)^{2(2-\delta)}} = \frac{C_\sigma}{s(x, y)^{2(2-\delta)}}, \end{aligned} \quad (\text{B.3})$$

where we have introduced a regulator δ . Expanding the interaction term in (B.2) and defining

correlation functions

$$\begin{aligned}
\Gamma_2(x_1, x_2) &= \langle \phi^2(x_1) \phi^2(x_2) \rangle_0 = 2NC_\phi^2(s(x_1, x_2))^{-2(d-2)}, \\
\Gamma_3(x_1, x_2, x_3) &= \langle \phi^2(x_1) \phi^2(x_2) \phi^2(x_3) \rangle_0^{\text{conn}} = 8NC_\phi^3(s(x_{12})s(x_{13})s(x_{23}))^{-(d-2)}, \\
\Gamma_4(x_1, \dots, x_4) &= \langle \phi^2(x_1) \dots \phi^2(x_4) \rangle_0^{\text{conn}} = 48NC_\phi^4(s(x_{12})s(x_{23})s(x_{34})s(x_{41}))^{-(d-2)}.
\end{aligned} \tag{B.4}$$

where $\phi^2 \equiv \phi^i \phi^i$ and $s(x_{ij}) \equiv s(x_i, x_j)$, we obtain

$$\begin{aligned}
Z_{O(N)} &= Z_s^N \int D\sigma \exp \left(\frac{1}{8N} \int d^d x_1 d^d x_2 \sqrt{g_{x_1}} \sqrt{g_{x_2}} \sigma(x_1) \sigma(x_2) \Gamma_2(x_1, x_2) \right. \\
&\quad - \frac{1}{48N^{3/2}} \int d^d x_1 \sqrt{g_{x_1}} \dots d^d x_3 \sqrt{g_{x_3}} \sigma(x_1) \sigma(x_2) \sigma(x_3) \Gamma_3(x_1, x_2, x_3) \\
&\quad \left. + \frac{1}{3 \cdot 2^7 N^2} \int d^d x_1 \sqrt{g_{x_1}} \dots d^d x_4 \sqrt{g_{x_4}} \sigma(x_1) \sigma(x_2) \sigma(x_3) \sigma(x_4) \Gamma_4(x_1, \dots, x_4) + \dots \right). \tag{B.5}
\end{aligned}$$

Finally taking average over the σ field with the propagator (B.3) we find for the free energy

$$F_{O(N)} = NF_s + \delta F_{\Delta=d-2} - \frac{1}{N} \left(\frac{1}{4} I_1 + \frac{1}{8} I_2 + \frac{1}{8} J_1 + \frac{1}{12} J_2 \right) + \mathcal{O}(1/N^2), \tag{B.6}$$

where $\delta F_{\Delta=d-2}$ is the sphere free energy of the induced σ field and is given by (1.11) with $\Delta = d - 2$, and the integrals I_1, I_2 and J_1, J_2 read

$$\begin{aligned}
I_1 &= \int d^d x_1 \sqrt{g_{x_1}} \dots d^d x_4 \sqrt{g_{x_4}} \frac{C_\sigma^2 C_\phi^4}{(s(x_{12})s(x_{34}))^{2(2-\delta)} (s(x_{12})s(x_{23})s(x_{34})s(x_{41}))^{d-2}}, \\
I_2 &= \int d^d x_1 \sqrt{g_{x_1}} \dots d^d x_4 \sqrt{g_{x_4}} \frac{C_\sigma^2 C_\phi^4}{(s(x_{13})s(x_{24}))^{2(2-\delta)} (s(x_{12})s(x_{23})s(x_{34})s(x_{41}))^{d-2}} \tag{B.7}
\end{aligned}$$

and also

$$\begin{aligned}
J_1 &= \int d^d x_i \sqrt{g_{x_i}} \frac{C_\sigma^3 C_\phi^6}{(s(x_{12})s(x_{45}))^{2(d/2+1-\delta)} s(x_{36})^{2(2-\delta)} (s(x_{23})s(x_{31})s(x_{56})s(x_{64}))^{d-2}}, \\
J_2 &= \int d^d x_i \sqrt{g_{x_i}} \frac{C_\sigma^3 C_\phi^6}{(s(x_{14})s(x_{25})s(x_{36}))^{2(2-\delta)} (s(x_{12})s(x_{23})s(x_{31})s(x_{45})s(x_{56})s(x_{64}))^{d-2}}, \tag{B.8}
\end{aligned}$$

where the diagrams for these integrals are represented in figure 5. To compute the integrals in (B.7) and (B.8) we use the Mellin-Barnes approach. As in the Gross-Neveu case it is also possible to compute these integrals for arbitrary d , and the result does not have terms

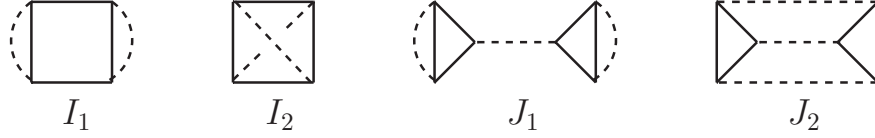


Figure 5: Diagrams contributing to the sphere free energy at the $1/N$ order of the critical $O(N)$ scalar model. The $\langle\sigma\sigma\rangle$ and $\langle\phi\phi\rangle$ propagators are represented by the dashed and solid lines.

divergent in δ . We obtain

$$\begin{aligned} I_1 &= \left(\frac{18}{d-1} - \frac{12}{d} - \frac{6}{d-2} \right) \gamma_{\phi,1}^{\text{O(N)}}, & I_2 &= \left(\frac{12}{d-1} - \frac{12}{d-2} \right) \gamma_{\phi,1}^{\text{O(N)}}, \\ J_1 &= 0, & J_2 &= \left(\frac{20}{d-2} - \frac{40}{d-1} \right) \gamma_{\phi,1}^{\text{O(N)}}, \end{aligned} \quad (\text{B.9})$$

where $\gamma_{\phi,1}^{\text{O(N)}}$ is the $1/N$ correction to the anomalous dimension of the field ϕ

$$\gamma_{\phi,1}^{\text{O(N)}} = \frac{2 \sin(\frac{\pi d}{2}) \Gamma(d-2)}{\pi \Gamma(\frac{d}{2}-2) \Gamma(\frac{d}{2}+1)}. \quad (\text{B.10})$$

To get the correct result for the sphere free energy we again, as in the Gross-Neveu case, we need to multiply the integrals in (B.6) by some rational numbers

$$F_{O(N)} = NF_s + \delta F_{\Delta=d-2} - \frac{1}{N} \left(\frac{1}{3} \left(\frac{1}{4} I_1 + \frac{1}{8} I_2 \right) + \frac{2}{5} \cdot \frac{1}{12} J_2 \right) + \mathcal{O}(1/N^2). \quad (\text{B.11})$$

Then using (B.9) we find the final result

$$F_{O(N)} = NF_s + \delta F_{\Delta=d-2} + \frac{1}{N} \left(\frac{1}{d} + \frac{1}{3(d-2)} - \frac{2}{3(d-1)} \right) \gamma_{\phi,1}^{\text{O(N)}} + \mathcal{O}(1/N^2). \quad (\text{B.12})$$

References

- [1] D. L. Jafferis, I. R. Klebanov, S. S. Pufu, and B. R. Safdi, “Towards the F -Theorem: $\mathcal{N} = 2$ Field Theories on the Three- Sphere,” *JHEP* **06** (2011) 102, 1103.1181.
- [2] R. C. Myers and A. Sinha, “Seeing a C-Theorem with Holography,” *Phys. Rev.* **D82** (2010) 046006, 1006.1263.
- [3] H. Casini, M. Huerta, and R. C. Myers, “Towards a Derivation of Holographic Entanglement Entropy,” *JHEP* **05** (2011) 036, 1102.0440.

- [4] H. Casini and M. Huerta, “On the RG running of the entanglement entropy of a circle,” *Phys.Rev.* **D85** (2012) 125016, 1202.5650.
- [5] H. Liu and M. Mezei, “A Refinement of entanglement entropy and the number of degrees of freedom,” *JHEP* **1304** (2013) 162, 1202.2070.
- [6] A. Kapustin, B. Willett, and I. Yaakov, “Exact Results for Wilson Loops in Superconformal Chern- Simons Theories with Matter,” *JHEP* **03** (2010) 089, 0909.4559.
- [7] D. L. Jafferis, “The Exact Superconformal R-Symmetry Extremizes Z ,” *JHEP* **05** (2012) 159, 1012.3210.
- [8] S. Giombi and I. R. Klebanov, “Interpolating between a and F ,” *JHEP* **1503** (2015) 117, 1409.1937.
- [9] L. Fei, S. Giombi, I. R. Klebanov, and G. Tarnopolsky, “Generalized F -Theorem and the ϵ Expansion,” *JHEP* **12** (2015) 155, 1507.01960.
- [10] L. Fei, S. Giombi, I. R. Klebanov, and G. Tarnopolsky, “Yukawa CFTs and Emergent Supersymmetry,” 1607.05316.
- [11] S. Giombi, I. R. Klebanov, and G. Tarnopolsky, “Conformal QED $_d$, F -Theorem and the ϵ Expansion,” *J. Phys.* **A49** (2016), no. 13 135403, 1508.06354.
- [12] K. G. Wilson, “Field theory in less than four dimensions - the renormalization group,” *eConf* **C720906V2** (1972) 169–173.
- [13] E. Brezin and W. D. J., “CRITICAL BEHAVIOR OF A CLASSICAL HEISENBERG FERROMAGNET WITH MANY DEGREES OF FREEDOM,” *Phys. Rev.* **B7** (1973), no. 5 1967.
- [14] I. R. Klebanov, S. S. Pufu, and B. R. Safdi, “ F -Theorem without Supersymmetry,” *JHEP* **1110** (2011) 038, 1105.4598.
- [15] I. R. Klebanov, S. S. Pufu, S. Sachdev, and B. R. Safdi, “Entanglement Entropy of 3-d Conformal Gauge Theories with Many Flavors,” *JHEP* **1205** (2012) 036, 1112.5342.
- [16] D. E. Diaz and H. Dorn, “Partition functions and double-trace deformations in AdS/CFT,” *JHEP* **0705** (2007) 046, hep-th/0702163.

- [17] D. J. Gross and A. Neveu, “Dynamical Symmetry Breaking in Asymptotically Free Field Theories,” *Phys.Rev.* **D10** (1974) 3235.
- [18] Y. Nambu and G. Jona-Lasinio, “Dynamical Model of Elementary Particles Based on an Analogy with Superconductivity. 1.,” *Phys. Rev.* **122** (1961) 345–358.
- [19] D. J. Gross, “Applications of the Renormalization Group to High-Energy Physics,” *Conf. Proc.* **C7507281** (1975) 141–250.
- [20] G. Parisi, “The Theory of Nonrenormalizable Interactions. 1. The Large N Expansion,” *Nucl.Phys.* **B100** (1975) 368.
- [21] B. Rosenstein, B. J. Warr, and S. H. Park, “The Four Fermi Theory Is Renormalizable in (2+1)-Dimensions,” *Phys. Rev. Lett.* **62** (1989) 1433–1436.
- [22] G. Gat, A. Kovner, B. Rosenstein, and B. J. Warr, “Four Fermi Interaction in (2+1)-dimensions Beyond Leading Order in $1/N$,” *Phys. Lett.* **B240** (1990) 158–162.
- [23] A. Hasenfratz, P. Hasenfratz, K. Jansen, J. Kuti, and Y. Shen, “The Equivalence of the top quark condensate and the elementary Higgs field,” *Nucl.Phys.* **B365** (1991) 79–97.
- [24] J. Zinn-Justin, “Four fermion interaction near four-dimensions,” *Nucl.Phys.* **B367** (1991) 105–122.
- [25] M. Gell-Mann and M. Levy, “The axial vector current in beta decay,” *Nuovo Cim.* **16** (1960) 705.
- [26] J. A. Gracey, “Calculation of exponent eta to $O(1/N^2)$ in the $O(N)$ Gross-Neveu model,” *Int. J. Mod. Phys.* **A6** (1991) 395–408. [Erratum: *Int. J. Mod. Phys.* **A6**,2755(1991)].
- [27] J. Gracey, “Anomalous mass dimension at $O(1/N^2)$ in the $O(N)$ Gross-Neveu model,” *Phys.Lett.* **B297** (1992) 293–297.
- [28] J. A. Gracey, “Computation of Beta-prime ($g(c)$) at $O(1/N^2)$ in the $O(N)$ Gross-Neveu model in arbitrary dimensions,” *Int. J. Mod. Phys.* **A9** (1994) 567–590, [hep-th/9306106](#).
- [29] T. Nishioka and K. Yonekura, “On RG Flow of τ_{RR} for Supersymmetric Field Theories in Three-Dimensions,” *JHEP* **1305** (2013) 165, [1303.1522](#).

- [30] J. A. Gracey, “Computation of critical exponent η at $O(1/N^{**3})$ in the four Fermi model in arbitrary dimensions,” *Int. J. Mod. Phys. A* **9** (1994) 727–744, [hep-th/9306107](#).
- [31] A. N. Vasiliev, S. E. Derkachov, N. A. Kivel, and A. S. Stepanenko, “The $1/n$ expansion in the Gross-Neveu model: Conformal bootstrap calculation of the index η in order $1/n^{**3}$,” *Theor. Math. Phys.* **94** (1993) 127–136. [Teor. Mat. Fiz.94,179(1993)].
- [32] I. Drummond, “Conformally Invariant Amplitudes and Field Theory in a Space-Time of Constant Curvature,” *Phys.Rev.* **D19** (1979) 1123.
- [33] J. L. Cardy, “Is There a c Theorem in Four-Dimensions?,” *Phys.Lett.* **B215** (1988) 749–752.
- [34] L. Fei, S. Giombi, and I. R. Klebanov, “Critical $O(N)$ Models in $6 - \epsilon$ Dimensions,” *Phys.Rev.* **D90** (2014) 025018, [1404.1094](#).
- [35] L. Fei, S. Giombi, I. R. Klebanov, and G. Tarnopolsky, “Three loop analysis of the critical $O(N)$ models in 6- dimensions,” *Phys.Rev.* **D91** (2015), no. 4 045011, [1411.1099](#).
- [36] L. Fei, S. Giombi, I. R. Klebanov, and G. Tarnopolsky, “Critical $Sp(N)$ models in 6 dimensions and higher spin dS/CFT,” *JHEP* **09** (2015) 076, [1502.07271](#).
- [37] J. Gracey, “Four loop renormalization of ϕ^3 theory in six dimensions,” [1506.03357](#).
- [38] P. Mati, “Vanishing beta function curves from the functional renormalization group,” *Phys. Rev.* **D91** (2015), no. 12 125038, [1501.00211](#).
- [39] P. Mati, “Critical scaling in the large- N $O(N)$ model in higher dimensions and its possible connection to quantum gravity,” *Phys. Rev.* **D94** (2016), no. 6 065025, [1601.00450](#).
- [40] A. Eichhorn, L. Janssen, and M. M. Scherer, “Critical $O(N)$ models above four dimensions: Small- N solutions and stability,” *Phys. Rev.* **D93** (2016), no. 12 125021, [1604.03561](#).
- [41] K. Kamikado and T. Kanazawa, “Nonperturbative RG analysis of five-dimensional $O(N)$ models with cubic interactions,” [1604.04830](#).

- [42] S. M. Chester, S. S. Pufu, and R. Yacoby, “Bootstrapping $O(N)$ vector models in $4 < d < 6$,” *Phys. Rev.* **D91** (2015), no. 8 086014, 1412.7746.
- [43] Z. Li and N. Su, “Bootstrapping Mixed Correlators in the Five Dimensional Critical $O(N)$ Models,” 1607.07077.
- [44] L. S. Brown and J. C. Collins, “Dimensional Renormalization of Scalar Field Theory in Curved Space-time,” *Annals Phys.* **130** (1980) 215.
- [45] S. Hathrell, “Trace Anomalies and $\lambda\phi^4$ Theory in Curved Space,” *Annals Phys.* **139** (1982) 136.
- [46] I. Jack and H. Osborn, “Analogues for the c Theorem for Four-dimensional Renormalizable Field Theories,” *Nucl.Phys.* **B343** (1990) 647–688.
- [47] J. A. Gracey, “Six dimensional QCD at two loops,” *Phys. Rev.* **D93** (2016), no. 2 025025, 1512.04443.
- [48] M. Czakon, “Automatized analytic continuation of Mellin-Barnes integrals,” *Comput.Phys.Commun.* **175** (2006) 559–571, hep-ph/0511200.
- [49] A. Smirnov and V. Smirnov, “On the Resolution of Singularities of Multiple Mellin-Barnes Integrals,” *Eur.Phys.J.* **C62** (2009) 445–449, 0901.0386.
- [50] V. A. Smirnov, “Analytic tools for Feynman integrals,” *Springer Tracts Mod.Phys.* **250** (2012) 1–296.