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Non-Abelian strings in $\mathcal{N} = 1$ supersymmetric QCD

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Abstract

Non-Abelian flux tubes (strings) are well studied in $\mathcal{N} = 2$ supersymmetric QCD in (3+1) dimensions. In addition to translational zero modes they have also orientational moduli associated with rotations of their fluxes inside a non-Abelian group. The dynamics of the orientational moduli is described by the two dimensional $\text{CP}(N-1)$ model living on the world sheet of the non-Abelian string. In this paper we consider a deformation of $\mathcal{N} = 2$ supersymmetric QCD with the $\text{U}(N)$ gauge group and $N_f = N$ quark flavors with a mass term μ of the adjoint matter. In the limit of large μ the theory flows to an $\mathcal{N} = 1$ supersymmetric QCD. We study the solution for the non-Abelian string in this limit and derive an effective theory on the string world sheet. The bosonic sector of this theory is still given by the $\text{CP}(N-1)$ model but its scale is exponentially small as compared to the scale of the four dimensional bulk theory in contrast to the $\mathcal{N} = 2$ case where these scales are equal. We study also the fermionic sector of the world sheet theory. Upon the deformation the non-Abelian string is no longer BPS and we show that the fermionic superorientational zero modes are all lifted. This leaves us with the pure bosonic $\text{CP}(N-1)$ model on the string world sheet in the limit of $\mathcal{N} = 1$ QCD. We also discuss what happens to confined monopoles at large μ .

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Introduction

The mechanism of confinement based on a monopole condensation [1] was shown to work [2, 3] in the monopole vacua of $\mathcal{N} = 2$ supersymmetric QCD. This confinement is essentially Abelian [4, 5, 6, 7]. Non-Abelian gauge group is broken down to an Abelian subgroup by condensation of the adjoint scalars at a high scale, while the condensation of monopoles occurs at a much lower scale, in a low-energy Abelian theory. Simultaneously, there occurs the formation of confining flux tubes (strings) of the Abelian Abrikosov-Nielsen-Olesen (ANO) type [8].

On the other hand it is believed that confinement in the real world QCD is essentially non-Abelian. This motivates studies of possible non-Abelian generalizations of the confinement mechanism. One important ingredient of such mechanism, namely the non-Abelian confining strings, was found in $\mathcal{N} = 2$ supersymmetric QCD [9, 10, 11, 12], see also [13, 14, 15, 16] for a review. These strings are formed in the quark vacua of $\mathcal{N} = 2$ QCD with the $U(N)$ gauge group, and they give rise to the confinement of monopoles and the so called "instead-of-confinement" phase for quarks, see [17] for a review.

In much the same way as the real world QCD, $\mathcal{N} = 1$ supersymmetric QCD does not have adjoint scalars. Therefore it is believed to have an essentially non-Abelian dynamics. On the other hand, due to supersymmetry it is more tractable than non-supersymmetric QCD. One may hope that, starting from $\mathcal{N} = 2$ QCD and decoupling the adjoint scalars, one can arrive at a non-Abelian regime. In particular, it was shown that the non-Abelian "instead-of-confinement" phase survives in the limit where the adjoint matter (present in $\mathcal{N} = 2$ QCD) decouples, see review [17] and references therein.

In this paper we make the next step and study what happens to the non-Abelian confining strings upon decoupling of the adjoint matter. Namely, we consider a deformation of $\mathcal{N} = 2$ supersymmetric QCD with the $U(N)$ gauge group and $N_f = N$ quark flavors by a mass term μ of the adjoint matter. The μ -deformation breaks the $\mathcal{N} = 2$ supersymmetry and in the limit of large μ the theory flows to $\mathcal{N} = 1$ supersymmetric QCD.

In addition to the translational zero modes typical for ANO strings, non-Abelian strings have orientational moduli associated with rotations of their fluxes inside the non-Abelian $SU(N)$ group. The dynamics of the orientational moduli in $\mathcal{N} = 2$ QCD is described by the two dimensional $CP(N-1)$ model living on the world sheet of the non-Abelian string. In this paper we study the solution for the non-Abelian string and derive an effective theory on the string world sheet in the limit of large μ .

Similar problem was addressed in [18, 19, 20, 21, 22, 23] where the μ -deformation was considered in $\mathcal{N} = 2$ supersymmetric QCD with the $U(N)$ gauge group and

$N_f = N$ flavors of massless quarks supplemented by the Fayet-Iliopoulos (FI) D -term. In the limit of large μ this theory flows to a theory which differs from $\mathcal{N} = 1$ QCD by the presence of the FI term. In particular, in this theory a scalar quark (squark) condensation is triggered by the FI D -term.

It was shown in the aforementioned papers that bosonic profile functions of the non-Abelian string stay intact upon the μ -deformation, while the fermionic zero modes are changed as compared to the ones in the $\mathcal{N} = 2$ limit. The string remains BPS saturated and the world sheet theory becomes the heterotic $\text{CP}(N - 1)$ model with $\mathcal{N} = (0, 2)$ supersymmetry [19, 20, 21, 23]. In this model, the supertranslational fermionic moduli interact with the superiorientational ones. Large N solution of the world sheet model shows that $\mathcal{N} = (0, 2)$ supersymmetry is spontaneously broken [22]. The model has N vacua corresponding to N different non-Abelian strings and the discrete Z_{2N} symmetry is spontaneously broken.

In this paper we consider the μ -deformation of $\mathcal{N} = 2$ QCD without a FI term in a quark vacuum. Squark condensate is determined by $\sqrt{\mu m}$, where m is a quark mass. In the large μ limit the theory flows to $\mathcal{N} = 1$ QCD in the quark vacuum. Non-Abelian strings cease to be BPS saturated and both bosonic and fermionic profile of the string are modified.

We study solutions for the non-Abelian string profile functions in the large μ limit and derive the effective theory on the string world sheet. The bosonic sector of this theory is still given by the $\text{CP}(N - 1)$ model. The $\text{CP}(N - 1)$ model is asymptotically free, and it is determined by its scale Λ_{CP} (position of the infra-red pole of the coupling constant). At small μ $\Lambda_{CP} = \Lambda_{\mathcal{N}=2}$, where $\Lambda_{\mathcal{N}=2}$ is the scale of four dimensional $\mathcal{N} = 2$ QCD, see, for example, review [15]. We show that in the large μ limit Λ_{CP} is exponentially small. We also derive a potential in two dimensional world sheet theory induced by quark mass differences.

Next we study the fermionic sector of the world sheet theory. Upon the μ -deformation the fermionic superiorientational zero modes are all lifted. This leaves us with the pure bosonic $\text{CP}(N - 1)$ model on the string world sheet in the limit when the bulk theory becomes $\mathcal{N} = 1$ QCD. This ensures that the world sheet theory is in the Coulomb/confinement phase, at least at large N , see [24].

We also address a question of what happens to the confined 't Hooft-Polyakov monopoles present in the $\mathcal{N} = 2$ limit, when we go to the large μ . Studying the world sheet potential we show that confined monopoles seen in the world sheet theory as kinks [11, 12] become unstable at large μ if quark masses are not equal. However, if quarks have equal masses the confined monopoles survive in the limit of $\mathcal{N} = 1$ QCD.

The paper is organized as follows. In Sec. 1 we review our bulk theory, namely μ -deformed $\mathcal{N} = 2$ supersymmetric QCD in the quark vacuum and calculate its

perturbative mass spectrum in the large μ limit. Sec. 2 presents the non-Abelian bosonic string solution at large μ for equal as well as unequal quark masses. The bosonic part of the effective world sheet theory is derived in Sec. 3. First we consider the kinetic term and then discuss the world sheet potential which arises when quark masses are not equal. Moving on, in Sec. 4 we study the fermionic sector and derive superorientational zero modes, starting from the $\mathcal{N} = 2$ limit and then moving to the μ -deformed case. We show that all the superorientational zero modes are lifted upon the μ -deformation. In Sec 5 we review the physics of the world sheet theory on the non-Abelian string and discuss what happen to confined monopoles at large μ . Sec. 6 is devoted to our Conclusions. Details of the derivation of the fermion zero modes are presented in Appendices.

1 Bulk theory

In this section we review our four dimensional bulk theory, see review [15] for more details. The bulk theory is μ -deformed $\mathcal{N} = 2$ supersymmetric QCD (SQCD) with the gauge group $U(N) = SU(N) \times U(1)$. The field content of the theory is as follows. The $\mathcal{N} = 2$ vector multiplet consists of the $U(1)$ gauge field A_μ and $SU(N)$ gauge field A_μ^a , complex scalar fields $a^{U(1)}$ and a^a in the adjoint representation, and their fermion superpartners $(\lambda_\alpha^1, \lambda_\alpha^2)$ and $(\lambda_\alpha^{1a}, \lambda_\alpha^{2a})$. The adjoint index a runs from 1 to $N^2 - 1$, while the spinorial index $\alpha = 1, 2$. The adjoint scalars and fermions λ^2 can be combined into the $\mathcal{N} = 1$ adjoint matter chiral multiplets $\mathcal{A}^{U(1)}$ and $\mathcal{A}^{SU(N)} = \mathcal{A}^a T^a$, where T^a are generators of the $SU(N)$ gauge group normalized as $\text{Tr}(T^a T^b) = (1/2) \delta^{ab}$.

The matter sector consists of $N_f = N$ flavors of quark hypermultiplets in the fundamental representation and scalar components (squarks) q^{kA} and \tilde{q}_{Ak} , while the fermions are represented by ψ^{kA} and $\tilde{\psi}_{Ak}$. Here $A = 1, \dots, N$ is a flavor index and $k = 1, \dots, N$ is a color index.

The superpotential of $\mathcal{N} = 2$ supersymmetric QCD reads

$$\mathcal{W}_{\mathcal{N}=2} = \sqrt{2} \left\{ \frac{1}{2} \tilde{q}_A \mathcal{A}^{U(1)} q^A + \tilde{q}_A \mathcal{A}^a T^a q^A \right\} + m_A \tilde{q}_A q^A, \quad (1.1)$$

where we use the same notations for quark multiplets q^A and \tilde{q}_A and their scalar components, while m_A are quark masses.

The μ -deformation is the mass term for the adjoint matter

$$\mathcal{W}_{\mathcal{N}=1} = \sqrt{\frac{N}{2}} \frac{\mu_1}{2} (\mathcal{A}^{U(1)})^2 + \frac{\mu_2}{2} (\mathcal{A}^a)^2, \quad (1.2)$$

which breaks $\mathcal{N} = 2$ supersymmetry down to $\mathcal{N} = 1$.

In the special case when

$$\mu \equiv \mu_2 = \mu_1 \sqrt{\frac{2}{N}}, \quad (1.3)$$

superpotential (1.2) becomes a single trace operator

$$\mathcal{W}_{\mathcal{N}=1} = \mu \text{Tr}(\Phi^2) \quad (1.4)$$

where we defined a scalar adjoint matrix as

$$\Phi = \frac{1}{2} a^{U(1)} + T^a a^a. \quad (1.5)$$

We will consider bulk QCD in the limit of large μ_1 and μ_2 , when the adjoint matter decouples and the theory becomes $\mathcal{N} = 1$ QCD. Integrating out the adjoint matter in a sum of superpotentials (1.1) and (1.2) we get a quark superpotential of our μ -deformed bulk theory

$$\mathcal{W}(q, \tilde{q}) = -\frac{1}{2\mu_2} \left[(\tilde{q}_A q^B)(\tilde{q}_B q^A) - \frac{\alpha}{N} (\tilde{q}_A q^A)^2 \right] + m_A (\tilde{q}_A q^A), \quad (1.6)$$

where

$$\alpha = 1 - \sqrt{\frac{N}{2}} \frac{\mu_2}{\mu_1}. \quad (1.7)$$

In the case of single trace deformation (1.3) $\alpha = 0$.

The bosonic action of the theory is ¹

$$S_{\text{bos}} = \int d^4x \left(\frac{1}{2g_2^2} \text{Tr} (F_{\mu\nu}^{\text{SU}(N)})^2 + \frac{1}{4g_1^2} (F_{\mu\nu}^{\text{U}(1)})^2 + \right. \\ \left. |\nabla_\mu q^A|^2 + |\nabla_\mu \tilde{q}^A|^2 + V(q^A, \tilde{q}_A) \right). \quad (1.8)$$

Here ∇_μ is a covariant derivative

$$\nabla_\mu = \partial_\mu - \frac{i}{2} A_\mu^{\text{U}(1)} - i A_\mu^a T^a, \quad (1.9)$$

while the scalar potential $V(q^A, \tilde{q}_A)$ is a sum of the D -term and F -term potentials,

$$V(q^A, \tilde{q}_A) = V_D(q^A, \tilde{q}_A) + V_F(q^A, \tilde{q}_A). \quad (1.10)$$

¹From here further on we use a Euclidean notation, that is $F_{\mu\nu}^2 = 2F_{0i}^2 + F_{ij}^2$, $(\partial_\mu a)^2 = (\partial_0 a)^2 + (\partial_i a)^2$, etc. Furthermore, the sigma-matrices are defined as $\sigma^{\alpha\dot{\alpha}} = (1, -i\vec{\tau})$, $\bar{\sigma}_{\dot{\alpha}\alpha} = (1, i\vec{\tau})$. Lowering and raising of spinor indices are performed by virtue of an anti-symmetric tensor defined as $\varepsilon_{12} = \varepsilon_{\dot{1}\dot{2}} = 1$, $\varepsilon^{12} = \varepsilon^{\dot{1}\dot{2}} = -1$. The same raising and lowering convention applies to the flavor $\text{SU}(N)$ indices f, g , etc.

The D -term potential reads

$$V_D = \frac{g_2^2}{2} (\bar{q}_A T^a q^A - \tilde{q}_A T^a \tilde{q}^A)^2 + \frac{g_1^2}{8} (|q^A|^2 - |\tilde{q}^A|^2)^2, \quad (1.11)$$

while the F term potential is determined by superpotential (1.6). It has the form

$$\begin{aligned} V_F = \frac{1}{|\mu_2|^2} & \left\{ (\bar{q}_A q^B) \left[(\bar{q}_C \tilde{q}^A) - \frac{\bar{\alpha}}{N} \delta_C^A (\bar{q}_F \tilde{q}^F) - \bar{\mu}_2 \bar{m}_A \delta_C^A \right] \right. \\ & \times \left[(\tilde{q}_B q^C) - \frac{\alpha}{N} \delta_B^C (\tilde{q}_F q^F) - \mu_2 m_B \delta_B^C \right] \\ & + (\tilde{q}_A \tilde{q}^B) \left[(\bar{q}_B \tilde{q}^C) - \frac{\bar{\alpha}}{N} \delta_B^C (\bar{q}_F \tilde{q}^F) - \bar{\mu}_2 \bar{m}_B \delta_B^C \right] \\ & \left. \times \left[(\tilde{q}_C q^A) - \frac{\alpha}{N} \delta_C^A (\tilde{q}_F q^F) - \mu_2 m_A \delta_C^A \right] \right\}. \end{aligned} \quad (1.12)$$

In this paper we will consider the vacuum (zero of the potential (1.10)) where the maximal possible number of quark flavors equal to N condense (the so called $r = N$ vacuum, where r is the number of condensed squark flavors at weak coupling, see [17] for a review). In this vacuum squark VEVs are given by

$$\langle q^{kA} \rangle = \langle \tilde{q}^{kA} \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\xi_1} & 0 & \dots \\ \dots & \dots & \dots \\ \dots & 0 & \sqrt{\xi_N} \end{pmatrix}, \quad (1.13)$$

where we wrote down the squark field as an $N \times N$ matrix in color and flavor indices, and the parameters ξ_A are defined as

$$\xi_A = 2 \left(\sqrt{\frac{2}{N}} \mu_1 \hat{m} + \mu_2 (m_A - \hat{m}) \right), \quad (1.14)$$

while

$$\hat{m} = \frac{1}{N} \sum_{A=1}^N m_A. \quad (1.15)$$

For single trace deformation (1.3) expressions for the parameters ξ_A simplify:

$$\xi_A = 2 \mu_2 m_A \quad (1.16)$$

In this paper we will mostly consider the non-Abelian limit when all quark masses are equal,

$$m_1 = m_2 = \dots = m_N \equiv m, \quad (1.17)$$

so that the parameters ξ_A degenerate, $\xi_A \equiv \xi$, and the squark VEVs become

$$\langle q^{kA} \rangle = \langle \bar{q}^{kA} \rangle = \sqrt{\frac{\xi}{2}} \begin{pmatrix} 1 & 0 & \dots \\ \dots & \dots & \dots \\ \dots & 0 & 1 \end{pmatrix} \quad (1.18)$$

Note that if we take the limit $\mu \rightarrow \infty$ (keeping the quark masses fixed) the parameters $\xi \sim \mu m$ also go to infinity, and our quark vacuum becomes a run-away vacuum (all the r vacua with the non-zero r become run-away vacua). In this case $\mathcal{N} = 1$ QCD is a theory with only N vacua which originate from N monopole vacua ($r = 0$ vacua) of $\mathcal{N} = 2$ QCD.

Here we define $\mathcal{N} = 1$ QCD in a different way. By taking the limit of large μ we make the quark masses small so that the product μm (and the quark VEVs) are fixed,

$$\mu \rightarrow \infty, \quad m \rightarrow 0, \quad \mu m = \text{fixed}. \quad (1.19)$$

This way we keep track of all the r vacua present in $\mathcal{N} = 2$ QCD. In this paper we will study non-Abelian strings particularly in the $r = N$ quark vacuum (1.18) assuming the limit of large μ when the bulk theory flows to the generalized $\mathcal{N} = 1$ QCD defined above.

In order to keep our bulk theory at weak coupling we assume that the squark VEVs are large as compared with the scale $\Lambda_{\mathcal{N}=1}$ of the $SU(N)$ sector of $\mathcal{N} = 1$ QCD. Namely, we assume that

$$\sqrt{\mu m} \gg \Lambda_{\mathcal{N}=1}. \quad (1.20)$$

Squark VEVs (1.18) result in a spontaneous breaking of both gauge and flavor $SU(N)$'s. The diagonal global $SU(N)$ survives, however,

$$U(N)_{\text{gauge}} \times SU(N)_{\text{flavor}} \rightarrow SU(N)_{C+F}. \quad (1.21)$$

A color-flavor locking takes place in the vacuum. This fact leads to an emergence of non-Abelian strings, see [15] for a review.

Let us briefly summarize a perturbative spectrum of our bulk theory in the large μ limit, cf. [15]. Consider for simplicity the case of equal quark masses. The $U(N)$ gauge group is completely Higgsed and the masses of the gauge bosons are

$$m_G^{SU(N)} = g_2 |\sqrt{\xi}| \quad (1.22)$$

for the $SU(N)$ gauge bosons and

$$m_G^{U(1)} = g_1 \sqrt{\frac{N}{2}} |\sqrt{\xi}| \quad (1.23)$$

for the U(1) one. Below we also assume that the gauge boson masses are of the same order,

$$m_G^{U(1)} \sim m_G^{SU(N)} \equiv m_G \quad (1.24)$$

Extracting a quark mass matrix from potentials (1.11), (1.12) we find that out of $4N^2$ real degrees of freedom of the q^{kA} and \bar{q}^{kA} squarks N^2 phases are eaten by the Higgs mechanism, $(N^2 - 1)$ real squarks have mass (1.22), while one real squark has mass (1.23). These squarks are scalar superpartners of the SU(N) and U(1) gauge bosons in massive vector $\mathcal{N} = 1$ supermultiplets, respectively.

Other $2N^2$ squarks become much lighter in the large μ limit. The masses of $2(N^2 - 1)$ of them forming the adjoint representation of the global color-flavor SU $_{C+F}(N)$ (1.21) are given by

$$m_L^{SU(N)} = \left| \frac{\xi}{\mu_2} \right|, \quad (1.25)$$

while two real SU $_{C+F}(N)$ color-flavor singlets have mass

$$m_L^{U(1)} = \sqrt{\frac{N}{2}} \left| \frac{\xi}{\mu_1} \right|, \quad (1.26)$$

If μ_2 and μ_1 are of the same order (more exactly, we assume below that $\alpha = \text{const}$, see (1.7)), then

$$m_L^{U(1)} \sim m_L^{SU(N)} \equiv m_L \sim m \ll m_G. \quad (1.27)$$

Below we will heavily use this mass hierarchy of the perturbative spectrum.

In particular, in the limit (1.19) $m_L \rightarrow 0$, and $2N^2$ squarks become massless. This reflects the presence of the Higgs branch which develops in this limit. The presence of massless scalars developing VEVs makes the string solution ill-defined [25, 26], see also next section. Below we use the μ -deformed $\mathcal{N} = 2$ QCD at large μ as an infra-red (IR) regularization of $\mathcal{N} = 1$ QCD. At large but finite μ the Higgs branch present in $\mathcal{N} = 1$ QCD is lifted and the IR divergences are regularized, cf. [27].

2 Non-Abelian strings

In this section we derive a vortex solution, assuming the equal quark mass limit (1.17). First we review a general *ansatz* for the non-Abelian string and present equations for string profile functions. Then we solve these equations assuming the mass hierarchy (1.27) in the large μ limit.

2.1 Equations of motion

We consider a static string stretched along the x_3 axis so that the corresponding profile functions depend only on coordinates in the (x_1, x_2) plane. Closely following the strategy developed for $\mathcal{N} = 2$ supersymmetric QCD (see review [15]) we first assume that only those squark fields which develop VEVs have non-trivial profile functions in a string solution. Therefore we set

$$q^{kA} = \bar{q}^{kA} = \frac{1}{\sqrt{2}} \varphi^{kA}. \quad (2.1)$$

and look for the string solutions using the following *ansatz* [10, 11, 15]:

$$\begin{aligned} \varphi &= \phi_2 + n\bar{n}(\phi_1 - \phi_2) \\ &= \frac{1}{N}(\phi_1 + (N-1)\phi_2) + (\phi_1 - \phi_2) \left(n\bar{n} - 1/N \right), \\ A_i^{\text{SU}(N)} &= \varepsilon_{ij} \frac{x^j}{r^2} f_W(r) \left(n\bar{n} - 1/N \right), \\ A_i^{\text{U}(1)} &= \frac{2}{N} \varepsilon_{ij} \frac{x^j}{r^2} f(r), \end{aligned} \quad (2.2)$$

where the index i runs over 1, 2. The profile functions $\phi_1(r)$ and $\phi_2(r)$ determine the profiles of the squarks in the plane orthogonal to the string at rest, while $f(r)$ and $f_W(r)$ are the profiles of the gauge fields. The profile functions depend on the distance r from a given point x^i to the center of the string x_0^i in the (x_1, x_2) plane.

Here we have also introduced the orientational complex vector n^l , $l = 1, \dots, N$, subject to the condition

$$\bar{n}_l \cdot n^l = 1. \quad (2.3)$$

Vector n^l parametrizes the orientational modes of the non-Abelian vortex string. It arises due to a possibility to rotate a given particular string solution with respect to the unbroken color-flavor global group $\text{SU}(N)_{C+F}$, see (1.21).

Boundary conditions for the gauge and scalar profile functions are

$$\begin{aligned} \phi_1(0) &= 0, & \phi_2(0) &\neq 0, & \phi_1(\infty) &= \sqrt{\xi}, & \phi_2(\infty) &= \sqrt{\xi}, \\ f_W(0) &= 1, & f(0) &= 1, & f_W(\infty) &= 0, & f(\infty) &= 0. \end{aligned} \quad (2.4)$$

Substituting *ansatz* (2.2) into action (1.8) we get an energy functional (tension

of the string):

$$T = 2\pi \int r dr \left(\frac{2}{g_1^2 N^2} \frac{f'^2}{r^2} + \frac{N-1}{N} \frac{1}{g_2^2} \frac{f_W'^2}{r^2} + \phi_1'^2 + (N-1)\phi_2'^2 \right. \\ \left. + \frac{1}{N^2} \frac{[f + (N-1)f_W]^2}{r^2} \phi_1^2 + \frac{N-1}{N^2} \frac{[f - f_W]^2}{r^2} \phi_2^2 + V(\phi_1, \phi_2), \right) \quad (2.5)$$

where the potential $V(\phi_1, \phi_2)$ is

$$V(\phi_1, \phi_2) = \frac{1}{4|\mu_2|^2} \left(\phi_1^2 \left[\phi_1^2 + \frac{\alpha}{N} (\phi_1^2 + (N-1)\phi_2^2) - 2\mu_2 m \right]^2 \right. \\ \left. + (N-1)\phi_2^2 \left[\phi_2^2 + \frac{\alpha}{N} (\phi_1^2 + (N-1)\phi_2^2) - 2\mu_2 m \right]^2 \right) \quad (2.6)$$

and we assume that $\mu_2 m$ is real ².

String tension functional (2.5)) gives equations for the profile functions. We get

$$\begin{aligned} f'' - \frac{f'}{r} - \frac{g_1^2}{2} (f + (N-1)f_W) \phi_1^2 - (N-1) \frac{g_1^2}{2} (f - f_W) \phi_2^2 &= 0 \\ f_W'' - \frac{f_W'}{r} - \frac{g_2^2}{N} (f + (N-1)f_W) \phi_1^2 + \frac{g_2^2}{N} (f - f_W) \phi_2^2 &= 0 \\ \phi_1'' + \frac{\phi_1'}{r} - \frac{1}{N^2} \frac{(f + (N-1)f_W)^2}{r^2} \phi_1 - \frac{1}{2} \frac{\partial V}{\partial \phi_1} &= 0 \\ \phi_2'' + \frac{\phi_2'}{r} - \frac{1}{N^2} \frac{(f - f_W)^2}{r^2} \phi_2 - \frac{1}{2(N-1)} \frac{\partial V}{\partial \phi_2} &= 0 \end{aligned} \quad (2.7)$$

These equations are of the second order rather than the first order. This is because our string is not BPS saturated. Note, that for a BPS string the masses of the scalars forming the string are equal to masses of the gauge bosons (1.22) and (1.23), see [15]. For our μ -deformed theory this is not the case. Masses of singlet and adjoint scalars in the scalar matrix φ^{kA} in (2.1) are given by (1.25) and (1.26), and in the large μ limit they are much smaller than the masses of gauge bosons. In particular, as we mentioned already, in the limit (1.19) $m_L \rightarrow 0$, and our μ -deformed theory develops a Higgs branch.

2.2 String profile functions

It is quite often that supersymmetric gauge theories have Higgs branches. These are flat directions of the scalar potential on which charged scalar fields can develop VEVs

²If it is in fact a complex quantity, we should modify relation (2.1) inserting there the phase of $\mu_2 m$.

breaking the gauge symmetry. In many instances this breaking provides topological reasons behind formation of vortex strings. A dynamical side of the problem of the vortex string formation in theories with Higgs branches was addressed in [25, 26, 27]. A priori it is not clear at all whether or not stable string solutions exist in this class of theories. The fact is that a theory with a Higgs branch represents a limiting case of type I superconductor with vanishing Higgs mass. In particular, it was shown in [25] that infinitely long strings cannot be formed in this case due to infrared divergences.

Later this problem was studied in [26, 27]. It was shown that vortices on Higgs branches become logarithmically "thick" due to the presence of massless scalars in the bulk. Still, they are well defined if IR divergences are regularized. One way of regularization is to consider a vortex string of the finite length L [26]. This setup is typical for the confinement problem. It was shown in [26] that confining potential between heavy trial charges becomes nonlinear,

$$V(L) \sim \frac{L}{\log L}, \quad (2.8)$$

in theories with Higgs branches.

Another way of IR regularization is to lift the Higgs branch so that scalar fields forming the string have small but non-zero masses m_L , cf. [27]. We use this approach here, see Eqs. (1.25) and (1.26), assuming that μ is large but finite.

To the leading order in $\log m_G/m_L$ the vortex solution has the following structure in the (x_1, x_2) plane [26]. The gauge fields are localized inside the core region of the radius R_g and almost zero outside this region³. In contrast, scalar profiles are almost constant inside the core. In particular, the ϕ_1 profile function associated with winding of the vortex is almost zero inside the core (see (2.4)),

$$\begin{aligned} \phi_1 &\approx 0 \\ \phi_2 &\approx (1-c)\sqrt{\xi}, \end{aligned} \quad (2.9)$$

where c is a constant to be determined.

Then the two first equations for gauge profile functions in (2.7) have solutions

$$f = f_W \approx 1 - \frac{r^2}{R_g^2} \quad (2.10)$$

inside the core.

Outside the core in a logarithmically wide region

$$1/m_G \lesssim r \lesssim 1/m_L \quad (2.11)$$

³We will determine R_g shortly.

gauge fields are almost zero and two last equations in (2.7) reduce to the equations for free massless scalars. Their solutions have a logarithmic form

$$\begin{aligned}\phi_1 &\approx \sqrt{\xi} \left(1 - \frac{\ln \frac{1}{rm_L}}{\ln \frac{1}{m_L R_g}} \right), \\ \phi_2 &\approx \sqrt{\xi} \left(1 - c \cdot \frac{\ln \frac{1}{rm_L}}{\ln \frac{1}{m_L R_g}} \right),\end{aligned}\tag{2.12}$$

where the normalization is fixed by matching with the behavior inside the core (2.9) and with the boundary conditions at infinity (2.4).

In the region of very large r , $r \gg 1/m_L$, the scalar fields exponentially approach their VEVs ($\sim \exp\{-m_L r\}$), see (2.4)). This region gives a negligible contribution to the string tension and a particular form of the scalar potential (2.6) is not important.

Upon a substitution of the above solution into tension functional (2.5) one arrives at

$$T \approx \frac{\text{const}}{R_g^2} \left(\frac{2}{g_1^2 N^2} + \frac{N-1}{g_2^2 N} \right) + \frac{2\pi|\xi|}{\ln \frac{1}{R_g m_L}} [1 + (N-1)c^2],\tag{2.13}$$

where the first term comes from the gauge fields inside the core while the second term is produced by the logarithmic integral over the region (2.11) coming from the kinetic terms of scalars.

Minimization of this expression with respect to the constant c yields

$$c = 0,\tag{2.14}$$

so that the profile function ϕ_2 does not depend on r and is given by its VEV $\sqrt{\xi}$.

Minimizing (2.13) with respect to R_g we find

$$R_g \sim \frac{\text{const}}{m_G} \ln \frac{m_G}{m_L}.\tag{2.15}$$

The solutions for string profile functions in the intermediate region (2.11) becomes

$$\begin{aligned}\phi_1 &\approx \sqrt{\xi} \left(1 - \frac{\ln \frac{1}{rm_L}}{\ln \frac{m_G}{m_L}} \right), \\ \phi_2 &\approx \sqrt{\xi}, \\ f &\approx f_W \approx 0,\end{aligned}\tag{2.16}$$

while the final result for the tension of a non-Abelian string takes the form

$$T = \frac{2\pi|\xi|}{\ln \frac{m_G}{m_L}} + \dots, \quad (2.17)$$

where corrections are suppressed by powers of large logarithm $\log m_G/m_L$. The leading term here comes from quark kinetic energy $((\phi'_1)^2)$ integrated over intermediate region (2.11), see (2.5). Note, that the logarithmic suppression of the string tension is not specific for non-Abelian strings. Similar expression was found for the ANO string on a Higgs branch [26, 27].

2.3 Non-equal quark masses

In this section we relax condition (1.17) and consider a string solution assuming that quark mass differences are small,

$$\Delta m_{AB} = m_A - m_B \ll \hat{m}, \quad (2.18)$$

where \hat{m} is the average quark mass, (1.15).

Non-equal quark masses break color-flavor symmetry (1.21) down to $U(1)^N$, so the orientational modes of the non-Abelian string are no longer zero modes. They become quasizero modes in the approximation of small quark mass differences (2.18), cf. [15]. In fact, in Sect. 3.2 we will derive a shallow world sheet potential with N extreme points associated with Z_N strings.

Now we generalize the *ansatz* for the string solution (2.2) as follows. First we set an orientational vector

$$n^l = \delta^{lA_0}, \quad A_0 = 1, \dots, N \quad (2.19)$$

separating the A_0 -th Z_N string (the string associated with the winding of A_0 quark flavor, see [15]).

We expect that, much in the same way as for the equal quark masses case, the main contribution to the string tension comes from logarithmically wide intermediate region (2.11), while the string core does not contribute to the leading order. Then taking into account (2.18) we can neglect mass differences of different gauge bosons setting

$$m_G \approx g_2 \sqrt{|\hat{\xi}|}, \quad (2.20)$$

where

$$\hat{\xi} = \frac{1}{N} \sum_{A=1}^N \xi_A. \quad (2.21)$$

In this approximation we can use the same *ansatz* for the gauge fields as for the case of equal quark masses, see two last equations in (2.2) with n^l from (2.19). Gauge fields are still parametrized by only two gauge profile functions $f(r)$ and f_W , which are non-zero inside the string core determined by m_G (2.20).

The *ansatz* for the squark fields in (2.2) is generalized as follows

$$\varphi = \begin{pmatrix} \phi_1(r) & 0 & \dots & 0 & \dots & 0 \\ 0 & \phi_2(r) & \dots & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \phi_{A_0}(r) & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \dots & \phi_N(r) \end{pmatrix}, \quad (2.22)$$

where we introduce the profile functions ϕ_1, \dots, ϕ_N for non-winding flavors $A \neq A_0$ while the profile function ϕ_{A_0} is associated with A_0 -th winding flavor.

Boundary conditions for gauge profile functions are the same as in (2.4) while for quarks we require

$$\phi_{A_0}(0) = 0 \quad (2.23)$$

$$\phi_A(\infty) = \sqrt{\xi_A}, \quad A = 1, \dots, N, \quad (2.24)$$

where ξ_A are given by (1.14).

Equations for the profile functions now read

$$f'' - \frac{f'}{r} - \frac{g_1^2}{2}(f + (N-1)f_W)\phi_{A_0}^2 - \frac{g_1^2}{2}(f - f_W) \sum_{A \neq A_0} \phi_A^2 = 0 \quad (2.25)$$

$$\begin{aligned} f_W'' - \frac{f_W'}{r} - \frac{g_2^2}{N}(f + (N-1)f_W)\phi_{A_0}^2 + \frac{g_2^2}{N(N-1)}(f - f_W) \sum_{A \neq A_0} \phi_A^2 &= 0 \\ \phi_{A_0}'' + \frac{\phi_{A_0}'}{r} - \frac{1}{N^2} \frac{(f + (N-1)f_W)^2}{r^2} \phi_{A_0} - \frac{1}{2} \frac{\partial V}{\partial \phi_{A_0}} &= 0 \\ \phi_A'' + \frac{\phi_A'}{r} - \frac{1}{N^2} \frac{(f - f_W)^2}{r^2} \phi_A - \frac{1}{2} \frac{\partial V}{\partial \phi_A} &= 0, \quad A \neq A_0. \end{aligned}$$

Solving these equations in much the same way as we did in the previous subsection we get

$$\phi_A \approx \sqrt{\xi_A}, \quad A \neq A_0. \quad (2.26)$$

Moreover, gauge profile functions are determined by (2.10) inside the core, while they are zero outside it. Here the size of the core is still given by (2.15).

In much the same way as for the case of equal quark masses the profile function ϕ_1 is almost zero inside the core and is given by

$$\phi_{A_0} \approx \sqrt{\xi_{A_0}} \left(1 - \frac{\ln \frac{1}{rm_L}}{\ln \frac{m_G}{m_L}} \right) \quad (2.27)$$

in the region (2.11) of intermediate r .

The results for the string tensions of Z_N strings have the form

$$T_{A_0} = \frac{2\pi|\xi_{A_0}|}{\ln \frac{m_G}{m_L}} + \dots, \quad A_0 = 1, \dots, N. \quad (2.28)$$

We see that now string tensions of N Z_N strings are split.

3 World sheet effective theory

A non-Abelian string has both translational and orientational zero modes. If we allow a slow dependence of the associated moduli on the world sheet coordinates $z = x_3$ and t they become fields in the effective two dimensional low energy theory on the string world sheet [10, 11], see [15] for a review. Namely, we will have translational moduli $x_0^i(t, z)$ (position of the string in the (x_1, x_2) plane, $i = 1, 2$) and orientational moduli $n^l(t, z)$, $l = 1, \dots, N$. Translational sector is free and decouples therefore we will focus on the orientational sector.

In this section we will derive the bosonic part of the effective world sheet theory on the string.

3.1 $\text{CP}(N-1)$ model on the string world sheet

First we consider the limit of equal quark masses (1.17). In this limit color-flavor symmetry (1.21) is unbroken and the orientational moduli n^l describe the zero modes of the non-Abelian string. Namely, consider a particular Z_N solution (2.2) with $n^l = \delta^{lA_0}$, $A_0 = 1, \dots, N$. It breaks the $\text{SU}(N)_{C+F}$ group down to $\text{SU}(N-1) \times \text{U}(1)$. Therefore the $\text{SU}(N)_{C+F}$ rotation of the Z_N string solution generates the whole family of solutions (non-Abelian string) parametrized by the vector n^l from the moduli space

$$\frac{\text{SU}(N)_{C+F}}{\text{SU}(N-1) \times \text{U}(1)} = \text{CP}(N-1). \quad (3.1)$$

Since for equal quark masses the $\text{SU}(N)_{C+F}$ group is unbroken there is no world sheet potential for orientational moduli n^l . To derive the kinetic term we closely follow

the general procedure developed in [10, 11] (see [15] for a review) for the $\mathcal{N} = 2$ case. We substitute solution (2.2) into four-dimensional action (1.8) assuming slow t and z dependence of the orientational moduli n^l .

Once the moduli n^l cease to be constant, the gauge field components $A_0^{\text{SU}(N)}$ and $A_3^{\text{SU}(N)}$ also become non-vanishing. We use the *ansatz* [10, 11, 28]

$$A_k^{\text{SU}(N)} = -i \left[\partial_k n \cdot \bar{n} - n \cdot \partial_k \bar{n} - 2n \cdot \bar{n} (\bar{n} \partial_k n) \right] \rho(r), \quad k = 0, 3, \quad (3.2)$$

for these components, where we assume a contraction of the color indices inside the parentheses in the third term. We also introduced a new profile function $\rho(r)$ which will be determined through a minimization procedure.

Substituting (2.2) and (3.2) into (1.8) we get $\text{CP}(N-1)$ model

$$S^{(1+1)} = 2\beta \int dt dz \left\{ (\partial_k \bar{n} \partial_k n) + (\bar{n} \partial_k n)^2 \right\}, \quad (3.3)$$

with the coupling constant β given by

$$\beta = \frac{2\pi}{g_2^2} I, \quad (3.4)$$

where I is a normalization integral determined by the string profile functions integrated over (x_1, x_2) plane.

$$\begin{aligned} I = & \int_0^\infty r dr \left\{ \left(\frac{d}{dr} \rho(r) \right)^2 + \frac{1}{r^2} f_W^2 (1 - \rho)^2 \right. \\ & \left. + g_2^2 \left[\frac{\rho^2}{2} (\phi_1^2 + \phi_2^2) + (1 - \rho) (\phi_2 - \phi_1)^2 \right] \right\}. \end{aligned} \quad (3.5)$$

The above functional determines an equation of motion for ρ ,

$$-\frac{d^2}{dr^2} \rho - \frac{1}{r} \frac{d}{dr} \rho - \frac{1}{r^2} f_W^2 (1 - \rho) + \frac{g_2^2}{2} (\phi_1^2 + \phi_2^2) \rho - \frac{g_2^2}{2} (\phi_1 - \phi_2)^2 = 0, \quad (3.6)$$

while boundary conditions for ρ read

$$\rho(\infty) = 0, \quad \rho(0) = 1, \quad (3.7)$$

see [15] for the details.

The formulas above are valid for a non-Abelian string in both $\mathcal{N} = 2$ QCD and μ -deformed QCD. For our large μ limit we use the string profile functions found in Sec. 2.2. To find the solution for ρ we first note that inside the core $\rho \approx 1$. In the intermediate region (2.11) $f_W \approx 0$. The terms of equation (3.6) which involve

derivatives of ρ are negligible compared to the others (we will check that afterwards), and so an approximate solution can be easily found:

$$\rho \approx \frac{(\phi_1 - \phi_2)^2}{(\phi_1^2 + \phi_2^2)} \approx \frac{\left(\frac{\ln \frac{1}{rm_L}}{\ln \frac{1}{Rg m_L}} \right)^2}{2 - 2 \frac{\ln \frac{1}{rm_L}}{\ln \frac{1}{Rg m_L}} + \left(\frac{\ln \frac{1}{rm_L}}{\ln \frac{1}{Rg m_L}} \right)^2}, \quad (3.8)$$

where we used solutions (2.16) for quark profile functions.

It is easy to check that $\rho' \sim m_L \rho$ and $\rho'' \sim m_L^2 \rho$, so it was indeed consistent to drop the derivatives out of the equation (3.6).

Our next step is to substitute this solution into the (3.5) and calculate I . As we will see, only the region $r \lesssim 1/m_L$ gives a significant contribution to this integral. We have:

$$\begin{aligned} I &\approx g_2^2 \int r dr \left(\frac{1}{2} \frac{(\phi_1 - \phi_2)^4}{(\phi_1^2 + \phi_2^2)} + \frac{2\phi_1\phi_2(\phi_1 - \phi_2)^2}{(\phi_1^2 + \phi_2^2)} \right) \\ &= \frac{g_2^2}{2} \int r dr \frac{(\phi_1^2 - \phi_2^2)^2}{(\phi_1^2 + \phi_2^2)} \end{aligned} \quad (3.9)$$

Calculation yields

$$I \approx c \frac{m_G^2}{m_L^2} \frac{1}{\ln^2 \frac{m_G^2}{m_L^2}} \sim \frac{g^2 |\mu|}{|m|} \frac{1}{\ln^2 \frac{g^2 |\mu|}{|m|}}, \quad (3.10)$$

where we used Eqs. (1.22) and (1.25) while the constant c is associated with the ambiguity of the upper limit ($\sim 1/m_L$) of the integral above. As for the region of large r , $r \gg 1/m_L$, the function ρ falls off exponentially, and a contribution from this region is therefore negligible.

Substituting (3.10) into (3.4) we get the final result for the coupling β of the world sheet $\text{CP}(N-1)$ model (3.3)

$$\beta \approx c \frac{2\pi}{g_2^2} \frac{m_G^2}{m_L^2} \frac{1}{\ln^2 \frac{m_G^2}{m_L^2}} \sim \frac{|\mu|}{|m|} \frac{1}{\ln^2 \frac{g^2 |\mu|}{|m|}}. \quad (3.11)$$

$\text{CP}(N-1)$ model (3.3) is a low energy effective theory on the string world sheet. It describes the dynamics of massless orientational moduli at energies much below the inverse thickness of the string proportional to m_L . If we go to higher energies we have to take into account higher derivative corrections to (3.3).

Relation (3.11) is derived at the classical level. In quantum theory the coupling constant β runs. Relation (3.11) defines the $\text{CP}(N-1)$ model coupling at a scale of

the ultra-violet (UV) cutoff of the world sheet theory equal to m_G . In fact, $\text{CP}(N-1)$ model is an asymptotically free theory. Its coupling at the UV scale $m_G \sim \sqrt{\xi}$ at one loop is given by

$$4\pi\beta(\sqrt{\xi}) = N \ln \frac{\sqrt{\xi}}{\Lambda_{CP}}, \quad (3.12)$$

where Λ_{CP} is the scale of the $\text{CP}(N-1)$ model. This gives for the scale Λ_{CP}

$$\Lambda_{CP} \approx \sqrt{\xi} \exp \left\{ -\text{const} \frac{|\mu|}{|m|} \frac{1}{\ln^2 \frac{g^2 |\mu|}{|m|}} \right\}. \quad (3.13)$$

We see that the scale of $\text{CP}(N-1)$ model Λ_{CP} is exponentially small, so the world sheet theory is weakly coupled in a wide region of energies $\gg \Lambda_{CP}$. This should be contrasted to non-Abelian string in $\mathcal{N} = 2$ QCD where world sheet theory has a scale Λ_{CP} equal to scale $\Lambda_{\mathcal{N}=2}$ of the bulk QCD [15].

3.2 World-sheet potential at large μ

In this subsection we relax the condition of equal quark masses (1.17) and consider the effect of quark mass differences to the leading order in Δm_{AB} , see (2.18). Non-equal quark masses break color-flavor symmetry (1.21) down to $\text{U}(1)^N$ so as we already mentioned above the orientational modes of the non-Abelian string are no longer zero modes. They become quasizero modes in the approximation of small quark mass differences (2.18). We still can introduce the orientational quasimoduli n^l , $l = 1, \dots, N$ and consider a shallow potential in the $\text{CP}(N-1)$ world sheet theory (3.3) generated by the mass differences. We neglect effects of small mass differences in the kinetic term assuming that it is still given by Eq. (3.3).

Our general strategy is to take string solution (2.2) with the unperturbed string profile functions of Sec. 2.2 and substitute it into potential (1.12) taking into account explicit m_A dependence of this potential to the leading order in Δm_{AB} . After a rather involved calculation we arrive at the potential of the world sheet theory

$$\delta V_{1+1} = \chi \sum_{A=1}^N \frac{\text{Re} \left[(\xi_A - \hat{\xi}) \bar{\xi} \right]}{|\hat{\xi}|} |n^A|^2, \quad (3.14)$$

where δV_{1+1} is the potential up to a constant, ξ_P are given by (1.14), while the factor χ is determined by the string profile functions integrated over (x_1, x_2) plane,

$$\chi = \frac{\pi}{|\mu_2|^2} \int_0^\infty r dr (\phi_2^2 - \phi_1^2) \left[\phi_1^2 - \frac{\alpha}{N} (\phi_1^2 - \phi_2^2) \right]. \quad (3.15)$$

Now we use our solutions for ϕ_1 and ϕ_2 (see Sect. 2.2) and integrate here over the region $r \lesssim 1/m_L$. We also assume that μ_1 and μ_2 scale in such a way that the parameter α in (1.7) is fixed. More explicitly, we assume that

$$\mu \equiv \mu_2 = \text{const} \cdot \mu_1 \sqrt{\frac{2}{N}}, \quad (3.16)$$

This gives for χ

$$\chi \approx \text{const} \cdot \frac{2\pi}{\ln \frac{m_G}{m_L}}. \quad (3.17)$$

Moreover, the region of integration $r \gg 1/m_L$ in (3.15) does not contribute to the leading order. The unknown constant above appears due to an ambiguity of the upper limit of the integral over r , $r \sim 1/m_L$.

Substituting this into (3.14) we get

$$\delta V_{1+1} \approx \text{const} \cdot 2\pi \sum_{A=1}^N \frac{\text{Re} \left[(\xi_A - \hat{\xi}) \bar{\hat{\xi}} \right]}{|\hat{\xi}| \ln \frac{m_G}{m_L}} |n^A|^2. \quad (3.18)$$

Now let us fix the unknown constant in the equation above comparing it with expressions (2.28) for the string tensions of Z_N strings. Z_N strings are extreme points of the world sheet potential V_{1+1} , therefore a value of V_{1+1} at the extreme point $n^l = \delta^{lA_0}$ corresponding to A_0 -th Z_N string should be equal to string tension (2.28). This gives $\text{const} = 1$ in (3.18) and leads us to the final expression for the potential in the world sheet $\text{CP}(N-1)$ theory

$$V_{1+1} \approx \frac{4\pi}{\ln \frac{m_G}{m_L}} \left| \sqrt{\frac{2}{N}} \mu_1 \hat{m} + \mu_2 \left(\sum_{A=1}^N m_A |n^A|^2 - \hat{m} \right) \right|. \quad (3.19)$$

This potential integrated over world sheet coordinates t and z should be added to the kinetic term in (3.3). Note, that this potential is a generalization of our result in (3.18) since it includes all terms in the expansion in powers of $(m_A - \hat{m})/\hat{m}$. For the single trace μ -deformation (1.3) the world sheet potential takes a particularly simple form

$$V_{1+1} \approx \frac{4\pi}{\ln \frac{m_G}{m_L}} |\mu_2| \left| \sum_{A=1}^N m_A |n^A|^2 \right|. \quad (3.20)$$

The potential (3.19) has only one minimum and one maximum at generic Δm_{AB} . Other $(N-2)$ extreme points are saddle points. All these extreme points are located at

$$n^A = \delta^{AA_0}, \quad A_0 = 1, \dots, N, \quad (3.21)$$

and associated with the Z_N strings. A value of the potential at a given extreme point coincides with the tension of the A_0 -th Z_N string,

$$V_{1+1}(n^A = \delta^{AA_0}) = T_{A_0}, \quad A_0 = 1, \dots, N. \quad (3.22)$$

Absolute minimum (the unique vacuum) of (3.19) corresponds to the Z_N string associated with winding of a squark with the smallest mass.

Note that our derivation of Eq. (3.18) reproduced the logarithmic suppression typical for string tensions in extreme type I superconductors (with small Higgs mass m_L), see (2.28) and [26].

Potential (3.19) is similar to the potential in the world sheet theory on the non-Abelian string derived in [29] for μ -deformed $\mathcal{N} = 2$ QCD in the limit of small μ . In this case the world sheet theory is heterotic $CP(N-1)$ model with $\mathcal{N} = (0, 2)$ supersymmetry. For small μ the world sheet potential obtained in [29] can be written in the form

$$V_{1+1}^{\mu \rightarrow 0} = 4\pi \left| \sqrt{\frac{2}{N}} \mu_1 \hat{n} + \mu_2 \left(\sum_{A=1}^N m_A |n^A|^2 - \hat{n} \right) \right|. \quad (3.23)$$

It differs from the one in (3.19) by the absence of the logarithmic suppression. This has a natural explanation. At small μ in much the same way as in our case the saddle points of the world sheet potential correspond to the Z_N strings and relation (3.22) is still valid. On the other hand in the limit of small μ the Z_N strings are BPS saturated and their tensions are given by $T_{A_0}^{\mu \rightarrow 0} = 2\pi |\xi_{A_0}|$, see [29]. This explains the absence of the logarithmic suppression in the potential (3.23).

3.3 Mass spectrum on the string

Let us assume that m_1 is the smallest quark mass. Then the vacuum of the world sheet potential (3.19) is located at

$$n^A = \delta^{A1} \quad (3.24)$$

and the minimum value $V_{1+1}^{\min} = T_{A=1}$. Let us calculate a perturbative mass spectrum of the world sheet theory in this vacuum. Expanding

$$|n^1|^2 = 1 - \sum_{A \neq 1} |n^A|^2 \quad (3.25)$$

and extracting the quadratic in fluctuations n^A terms from potential (3.19), we get masses of world sheet excitations n^A , $A \neq 1$

$$m_{A \neq 1}^2 = \frac{\pi}{\beta \ln \frac{m_G}{m_L}} \frac{\text{Re}[(\xi_A - \xi_1)\bar{\xi}_1]}{|\xi_1|}. \quad (3.26)$$

Note the factor $1/(2\beta)$ here that comes from the kinetic term in (3.3). Substituting here the coupling β from (3.11) we see that the masses of the perturbative world sheet excitations behave as

$$m_{A \neq 1}^2 \sim m(m_A - m_1) \ln \frac{m_G}{m_L} \quad (3.27)$$

The coupling constant of $\text{CP}(N-1)$ model grows at low energies and gets frozen at the scale of the masses calculated above. If these masses are much larger than Λ_{CP} (3.13) then the world sheet theory is at weak coupling. Since Λ_{CP} is exponentially small we see that world sheet theory is at weak coupling even at rather small mass differences Δm_{AB} . However in the equal quark mass limit (1.17) when $\Delta m_{AB} = 0$ the world sheet $\text{CP}(N-1)$ model becomes strongly coupled.

Our result (3.19) for the world sheet potential on the non-Abelian string in μ -deformed theory can be compared with the world sheet potential for the non-Abelian string in $\mathcal{N} = 2$ supersymmetric QCD with FI D -term generated by quark mass differences, see [15] for a review. In the $\mathcal{N} = 2$ case all the Z_N strings are degenerate, with tensions given by the FI parameter. The world sheet potential in this case has N minima located at (3.21) separated by shallow barriers quadratic in Δm_{AB} . The world sheet theory has $\mathcal{N} = (2, 2)$ supersymmetry and the presence of N vacua is ensured by the Witten index for $\text{CP}(N-1)$ supersymmetric model. There are kinks interpolating between these vacua which are interpreted as confined monopoles of bulk QCD [11, 12], see Sec. 5 and [15] for a review.

In the limit of large μ potential (3.19) dominates over the quadratic in Δm_{AB} potential, and one can neglect the latter one. We see that most of the vacua present in the $\mathcal{N} = 2$ case are lifted and the world sheet theory has a single vacuum at non-zero Δm_{AB} . Moreover, the lifted vacua are saddle points rather than local minima and therefore classically they are unstable. This means that there are no kinks in the world sheet theory.

Thus we come to the conclusion that confined monopoles present in $\mathcal{N} = 2$ QCD with FI term do not survive large μ limit when μ -deformed theory flows to $\mathcal{N} = 1$ QCD, provided that Δm_{AB} are non-zero. Only when $\Delta m_{AB} = 0$, the potential (3.19) vanishes (and the world sheet theory enters into the strong coupling), and we can consider kinks/confined monopoles. We will discuss this case below in Sec. 5

4 Fermion zero modes

In this section we consider the fermion zero modes of the non-Abelian string. First we briefly review the limit of small μ , see [15] for a more detailed review. In this limit deformation superpotential (1.2) reduces to the FI F -term and does not break

$\mathcal{N} = 2$ supersymmetry [5, 7]. In the $\mathcal{N} = 2$ limit both superorientational and supertranslational fermion zero modes of the non-Abelian string can be obtained by a supersymmetry transformation of the bosonic string solution [11, 21, 23]. Next we gradually increase μ and study perturbations of superorientational zero modes at small μ . We show that all the superorientational fermion zero modes are lifted by the μ -deformation. As a result fermionic moduli which become fermion fields in the two dimensional low energy $CP(N-1)$ model on the string acquire masses. Eventually they disappear from the world sheet theory in the large μ limit. Finally we comment on supertranslational fermion zero modes which in much the same way as in $\mathcal{N} = 2$ theory can be obtained by supersymmetry transformations from the bosonic string solution.

4.1 Superorientational modes in $\mathcal{N} = 2$ limit

The fermionic part of the $\mathcal{N} = 2$ QCD defined by superpotentials (1.1) and (1.2) (before integrating out adjoint fields) is as follows:

$$\begin{aligned}
\mathcal{L}_{4d} = & \frac{2i}{g_2^2} \text{Tr} \overline{\lambda_f^{\text{SU}(N)}} \not{D} \lambda^{f\text{SU}(N)} + \frac{i}{g_1^2} \overline{\lambda_f^{\text{U}(1)}} \not{D} \lambda^{f\text{U}(1)} + \text{Tr} i \overline{\psi} \not{\nabla} \psi + \text{Tr} i \tilde{\psi} \not{\nabla} \tilde{\psi} \\
& + i\sqrt{2} \text{Tr} \left(\overline{q}_f \lambda^{f\text{U}(1)} \psi + \tilde{\psi} \lambda_f^{\text{U}(1)} q^f + \overline{\psi} \lambda_f^{\text{U}(1)} q^f + \overline{q^f \lambda_f^{\text{U}(1)} \tilde{\psi}} \right) \\
& + i\sqrt{2} \text{Tr} \left(\overline{q}_f \lambda^{f\text{SU}(N)} \psi + \tilde{\psi} \lambda_f^{\text{SU}(N)} q^f + \overline{\psi} \lambda_f^{\text{SU}(N)} q^f + \overline{q^f \lambda_f^{\text{SU}(N)} \tilde{\psi}} \right) \quad (4.1) \\
& + i\sqrt{2} \text{Tr} \tilde{\psi} \left(\frac{1}{2} a^{\text{U}(1)} + \frac{m_A}{\sqrt{2}} + a^{\text{SU}(N)} \right) \psi + i\sqrt{2} \text{Tr} \overline{\psi} \left(\frac{1}{2} \overline{a}^{\text{U}(1)} + \frac{m_A}{\sqrt{2}} + \overline{a}^{\text{SU}(N)} \right) \tilde{\psi} \\
& - 2\sqrt{\frac{N}{2}} \mu_1 \left[(\lambda^{2\text{U}(1)})^2 + (\overline{\lambda}_2^{\text{U}(1)})^2 \right] - \mu_2 \text{Tr} \left[(\lambda^{2\text{SU}(N)})^2 + (\overline{\lambda}_2^{\text{SU}(N)})^2 \right],
\end{aligned}$$

where derivatives acting on fermion fields are defined by the σ -matrices, for example $\not{\nabla} = \nabla_\mu \overline{\sigma}_{\dot{\alpha}\alpha}^\mu$, and a color-flavor matrix notation is used for the quark fermions ψ_α^{kA} , $\tilde{\psi}_{A\dot{\alpha}}^\alpha$. Index f is $\text{SU}(2)_R$ index of the $\mathcal{N} = 2$ theory, $q^f = (q, \tilde{q})$, $\lambda_\alpha^f = (\lambda_\alpha^1, \lambda_\alpha^2)$. Note the μ -deformation mass terms for the $f = 2$ gauginos in (4.1). In the $\mathcal{N} = 2$ limit these terms vanish.

A string solution in the $\mathcal{N} = 2$ limit at small μ is 1/2 BPS, which means that half of the supercharges of the $\mathcal{N} = 2$ theory act trivially on solution (2.2), provided the orientational vector n^l is a constant vector. Namely, the four supercharges (out of eight supercharges $Q^{\alpha f}$) that satisfy the constraints

$$Q^{21} = Q^{22}, \quad Q^{11} = -Q^{12}. \quad (4.2)$$

act trivially on the BPS string in the $\mathcal{N} = 2$ theory with the FI F -term [7, 11, 15].

The other four supercharges generate four supertranslational modes which are superpartners of the two translational modes.

However once the orientational vector n^l acquires a slow t and z dependence, the supercharges selected by (4.2) become supersymmetry generators acting in the $\mathcal{N} = (2, 2)$ supersymmetric $CP(N - 1)$ model on the string world sheet [11]. This allows one to obtain the orientational fermionic zero modes from a bosonic solution using supersymmetry transformations selected by (4.2) [11, 15]. The result is

$$\begin{aligned}
\bar{\psi}_2 &= \frac{\phi_1^2 - \phi_2^2}{\phi_2} \cdot n_{\bar{\xi}_L} , \\
\bar{\psi}_1 &= - \frac{\phi_1^2 - \phi_2^2}{\phi_2} \cdot \xi_R \bar{n} , \\
\lambda^{11 \text{ SU}(N)} &= i \frac{\phi_1}{\phi_2} f_W \frac{x^1 - i x^2}{r^2} \cdot n_{\bar{\xi}_L} \\
\lambda^{22 \text{ SU}(N)} &= - i \frac{\phi_1}{\phi_2} f_W \frac{x^1 + i x^2}{r^2} \cdot \xi_R \bar{n} \\
\lambda^{12 \text{ SU}(N)} &= \lambda^{11 \text{ SU}(N)} , \quad \lambda^{21 \text{ SU}(N)} = - \lambda^{22 \text{ SU}(N)} ,
\end{aligned} \tag{4.3}$$

where we suppress color and flavor indices while superscripts of adjoint fermions mean $\lambda^{\alpha f}$.

Note that the bosonic profile functions of the string $\phi_{1,2}(r)$, $f(r)$ and f_W in this section are the profile functions of the BPS string in the $\mathcal{N} = 2$ limit of small μ rather than the string profile functions of Sec. 2.2, which corresponds to the large μ limit. Former are solutions of first order equations rather than second order equations (2.7). They satisfy boundary conditions (2.4) and were found numerically in [10], see [15] for a review.

The Grassmann variables $\xi_{R,L}^l$, $l = 1, \dots, N$ in (4.3) are proportional to the parameters of the supersymmetry transformations $\epsilon^{\alpha f}$ selected by (4.2), namely

$$\xi_L^l \sim \epsilon^{21} + \epsilon^{22}, \quad \xi_R^l \sim \epsilon^{12} - \epsilon^{11}. \tag{4.4}$$

These parameters become fermion fields (superpartners of n^l) in the effective world sheet $CP(N - 1)$ model once we allow their slow dependence on the world sheet coordinates t and z [11, 15]. They are subject to the conditions

$$\bar{n}_l \xi_{L,R}^l = 0 , \tag{4.5}$$

which are a supersymmetric generalization of the $CP(N - 1)$ condition $|n|^2 = 1$.

4.2 Small μ expansion for fermion orientational zero modes

As we switch on the mass terms for the $f = 2$ gauginos (see the last line in (4.1)), the theory becomes $\mathcal{N} = 1$ supersymmetric and half of the supercharges $Q^{\alpha f=2}$ are lost. There are no SUSY transformations which act trivially on the string with constant n^l (they were used to generate superorientational modes in $\mathcal{N} = 2$ limit), and the string is no longer BPS. Therefore, to calculate zero modes one has to solve the Dirac equations.

Note that for the case of the massless μ -deformed theory with FI D -term considered in [23] the supercharges that act trivially on the string with constant n^l in the $\mathcal{N} = 2$ limit are Q^{12} and Q^{21} instead of the linear combinations selected by (4.2). Therefore as we switch on the μ -deformation only one (two real) of the above supercharges is lost, namely Q^{12} . The other one (two real), Q^{21} , still acts trivially and ensures that the string is still BPS-saturated. In our case all four supercharges of $\mathcal{N} = 1$ theory $Q^{\alpha 1}$ act non-trivially on the string. This is the reason why the string ceases to be a BPS one as we switch on μ .

Dirac equations which follow from action (4.1) read

$$\begin{aligned}
& \frac{i}{g_1^2} \left(\bar{\phi} \lambda^{fU(1)} \right) + i\sqrt{2} \text{Tr} \left(\bar{\psi} q^f + \bar{q}^f \tilde{\psi} \right) - 4\delta_2^f \sqrt{\frac{N}{2}} \mu_1 \bar{\lambda}_2^{U(1)} = 0, \\
& \frac{i}{g_2^2} \left(\bar{\phi} \lambda^{fSU(N)} \right)^a + i\sqrt{2} \text{Tr} \left(\bar{\psi} T^a q^f + \bar{q}^f T^a \tilde{\psi} \right) - \delta_2^f \mu_2 \bar{\lambda}_2^{aSU(N)} = 0, \\
& -i\bar{\psi} \overleftarrow{\nabla} + i\sqrt{2} \left[\bar{q}_f \left\{ \lambda^{fU(1)} + \lambda^{fSU(N)} \right\} + \tilde{\psi} \left\{ \frac{1}{2} \bar{a}^{U(1)} + \frac{m_A}{\sqrt{2}} + \bar{a}^{SU(N)} \right\} \right] = 0, \\
& i\nabla \tilde{\psi} + i\sqrt{2} \left[\left\{ \lambda_f^{U(1)} + \lambda_f^{SU(N)} \right\} q^f + \left\{ \frac{1}{2} \bar{a}^{U(1)} + \frac{m_A}{\sqrt{2}} + \bar{a}^{SU(N)} \right\} \psi \right] = 0, \\
& i\bar{\nabla} \psi + i\sqrt{2} \left[\left\{ \bar{\lambda}_f^{U(1)} + \bar{\lambda}_f^{SU(N)} \right\} q^f + \left\{ \frac{1}{2} a^{U(1)} + \frac{m_A}{\sqrt{2}} + a^{SU(N)} \right\} \tilde{\psi} \right] = 0, \\
& -i\tilde{\psi} \overleftarrow{\nabla} + i\sqrt{2} \left[\bar{q}^f \left\{ \bar{\lambda}_f^{U(1)} + \bar{\lambda}_f^{SU(N)} \right\} + \bar{\psi} \left\{ \frac{1}{2} a^{U(1)} + \frac{m_A}{\sqrt{2}} + a^{SU(N)} \right\} \right] = 0.
\end{aligned} \tag{4.6}$$

To simplify the problem we will use below the following strategy. We consider the region of small μ and look for solutions of the Dirac equations above perturbatively in μ . Of course, zero modes (4.3) satisfy Dirac equations (4.6) [21, 23] at $\mu = 0$. We take these modes as a zero order solutions and solve for perturbations proportional to μ .

Similar method was used in [21, 23] for a massless μ -deformed theory with a

FI D -term. In that case it was shown that the orientational fermion zero modes survive the μ -deformation, however, their profile functions become deformed. Below we will show that in our case of μ -deformed theory with massive quarks without the FI D -term the answer is different: orientational fermion zero modes do not survive the μ -deformation.

In analogy with the method of [23] we will use an *ansatz* for the superorientational modes:

$$\lambda^{1f \text{ SU}(N)} = 2 \frac{x^1 - ix^2}{r} \lambda_+^{1f}(r) n_{\xi_L} \bar{\xi}_L + 2 \lambda_-^{1f}(r) \xi_L \bar{n}, \quad (4.7)$$

$$\begin{aligned} \lambda^{2f \text{ SU}(N)} &= 2 \frac{x^1 + ix^2}{r} \lambda_+^{2f}(r) \xi_R \bar{n} + 2 \lambda_-^{2f}(r) n_{\xi_R} \bar{\xi}_R, \\ \bar{\psi}_{\dot{1}} &= 2 \bar{\psi}_{\dot{1}+}(r) \xi_R \bar{n} + 2 \frac{x^1 - ix^2}{r} \bar{\psi}_{\dot{1}-}(r) n_{\xi_R} \bar{\xi}_R, \\ \bar{\psi}_{\dot{2}} &= 2 \bar{\psi}_{\dot{2}+}(r) n_{\xi_L} \bar{\xi}_L + 2 \frac{x^1 + ix^2}{r} \bar{\psi}_{\dot{2}-}(r) \xi_L \bar{n}. \end{aligned} \quad (4.8)$$

$$\begin{aligned} \bar{\psi}_{\dot{1}} &= 2 \bar{\psi}_{\dot{1}+}(r) \xi_R \bar{n} + 2 \frac{x^1 - ix^2}{r} \bar{\psi}_{\dot{1}-}(r) n_{\xi_R} \bar{\xi}_R, \\ \bar{\psi}_{\dot{2}} &= 2 \bar{\psi}_{\dot{2}+}(r) n_{\xi_L} \bar{\xi}_L + 2 \frac{x^1 + ix^2}{r} \bar{\psi}_{\dot{2}-}(r) \xi_L \bar{n}. \end{aligned}$$

Here $\lambda_+(r)$ and $\psi_+(r)$ represent "undeformed" profile functions present in the $\mathcal{N} = 2$ case, while $\lambda_-(r)$ and $\psi_-(r)$ are the "perturbations" due to μ -deformation. Of course this terminology makes sense only in the small μ limit, then "-"-components will be of order μ . More generally, the "+" profile functions are expanded in even powers of μ , while "-" components are expanded in odd powers of μ .

Let us consider the equations for the perturbative "-"-components (solutions for the "+"-components are given by (4.3) up to the $O(\mu^2)$ terms). Half of them are very similar to those solved in [23]. If we denote

$$\lambda_-^{22} - \lambda_-^{21} \equiv \lambda_-,$$

then two of these equations for λ_- and $\bar{\psi}_{\dot{1}-}$ take the form

$$\begin{aligned} \partial_r \bar{\psi}_{\dot{1}-}(r) + \frac{1}{r} \bar{\psi}_{\dot{1}-}(r) - \frac{1}{Nr} (f + f_W(N-1)) \bar{\psi}_{\dot{1}-}(r) + i\phi_2 \lambda_- &= 0 \\ -\partial_r \lambda_- - \frac{f_W}{r} \lambda_- + i g_2^2 \phi_2 \bar{\psi}_{\dot{1}-}(r) - \mu_2 g_2^2 \frac{i}{2} \frac{f_W}{r} \frac{\phi_1}{\phi_2} &= 0 \end{aligned} \quad (4.9)$$

These equations can be solved in much the same way as in [23]. The solutions are

$$\begin{aligned}\overline{\psi}_{1-} &= -\mu_2 g_2^2 \frac{r}{8\phi_1} (\phi_1^2 - \phi_2^2) + O(\mu^3), \\ \lambda_- &= \lambda_-^{22} - \lambda_-^{21} = -\mu_2 g_2^2 \frac{i}{4} \left((f_W - 1) \frac{\phi_2}{\phi_1} + \frac{\phi_1}{\phi_2} \right) + O(\mu^3).\end{aligned}\quad (4.10)$$

Another pair of the profile functions $\overline{\psi}_{2-}$ and $(\lambda_-^{12} + \lambda_-^{11})$ satisfies the same equations (4.9). Hence the solution reads

$$\begin{aligned}\overline{\psi}_{2-} &= \overline{\psi}_{1-}, \\ \lambda_-^{12} + \lambda_-^{11} &= \lambda_-.\end{aligned}\quad (4.11)$$

Let us study behavior of these solutions in the limits $r \rightarrow \infty$ and $r \rightarrow 0$. The bosonic profile functions fall off exponentially at infinity

$$f_W(r) \sim \exp\{-m_G r\}, \quad \phi_{1,2} - \sqrt{\xi} \sim \exp\{-m_G r\}, \quad (4.12)$$

while their behavior at $r \rightarrow 0$ is as follows:

$$f_W(r) - 1 \sim r^2, \quad \phi_1 \sim r, \quad \phi_2 \sim \text{const}, \quad (4.13)$$

see (2.4).

From this behavior we see that fermion zero modes (4.10), (4.11) are normalizable. They fall off exponentially at $r \rightarrow \infty$ and are regular at $r \rightarrow 0$.

Now consider solutions for the other components. They turn out to be more complicated. Denoting

$$\lambda_-^{22} + \lambda_-^{21} \equiv \lambda_{(1)},$$

one gets:

$$\begin{aligned}\partial_r \overline{\psi}_{1-}(r) + \frac{1}{r} \overline{\psi}_{1-}(r) + \frac{1}{Nr} (f - f_W) \overline{\psi}_{1-}(r) + i\phi_1 \lambda_{(1)} &= 0, \\ -\partial_r \lambda_{(1)} - \frac{f_W}{r} \lambda_{(1)} + i g_2^2 \phi_1 \overline{\psi}_{1-}(r) - \mu_2 g_2^2 \frac{i}{2} \frac{f_W}{r} \frac{\phi_1}{\phi_2} &= 0.\end{aligned}\quad (4.14)$$

So far solutions for our equations were given by certain algebraic combinations of the bosonic profile functions. However, for the functions $\overline{\psi}_{1-}$ and $\lambda_{(1)}$ it is not the case. The above equations are solved in Appendix A. The solutions are given by Eqs. (A.6) and (A.7).

Two remaining modes $\overline{\psi}_{2-}$ and $(\lambda_-^{12} - \lambda_-^{11})$ satisfy the same equations (4.14). Therefore these modes are given by the same expressions,

$$\overline{\psi}_{2-}(r) = \overline{\psi}_{1-}(r), \quad \lambda_-^{12} - \lambda_-^{11} = \lambda_{(1)}(r). \quad (4.15)$$

Solutions (A.6) and (A.7) fall off exponentially at infinity, however, the behavior of the field λ in (A.7) is singular at $r \rightarrow 0$, namely it is proportional to $1/r$. This means that these modes are non-renormalizable. Our perturbative approach does not work: the corrections to (4.3) proportional to μ turn out to be non-normalizable. We will show in the next subsection that the resolution of this puzzle is that the fermion orientational modes get lifted by the μ -deformation.

4.3 Lifted fermion orientational modes

Let us consider instead of Dirac equations (4.6) equations with a non-zero eigenvalue for quark fermions, namely

$$\begin{aligned} -i \overline{\psi} \overleftarrow{\nabla} + i\sqrt{2} \left(\overline{q}_f \{ \lambda^{f\text{U}(1)} + \lambda^{f\text{SU}(N)} \} \right. \\ \left. + \tilde{\psi} \left\{ \frac{1}{2} \overline{a}^{\text{U}(1)} + \frac{m_A}{\sqrt{2}} + \overline{a}^{\text{SU}(N)} \right\} \right) = -m_{or} \tilde{\psi}, \end{aligned} \quad (4.16)$$

$$\begin{aligned} i \nabla \tilde{\psi} + i\sqrt{2} \left(\{ \lambda_f^{\text{U}(1)} + \lambda_f^{\text{SU}(N)} \} q^f \right. \\ \left. + \left\{ \frac{1}{2} \overline{a}^{\text{U}(1)} + \frac{m_A}{\sqrt{2}} + \overline{a}^{\text{SU}(N)} \right\} \psi \right) = -m_{or} \psi. \end{aligned} \quad (4.17)$$

with the mass m_{or} to be determined from the condition of normalizability of super-orientational modes.

Proceeding exactly as it was done in the previous subsection, instead of Eqs. (4.14) we arrive at

$$\begin{aligned} \partial_r \overline{\psi}_{1-}(r) + \frac{1}{r} \overline{\psi}_{1-}(r) + \frac{1}{Nr} (f - f_N) \overline{\psi}_{1-}(r) + i\phi_1 \lambda_{(1)} &= m_{or} \frac{\phi_1^2 - \phi_2^2}{2\phi_2}, \\ -\partial_r \lambda_{(1)} - \frac{f_N}{r} \lambda_{(1)} + i g_2^2 \phi_1 \overline{\psi}_{1-}(r) - \mu_2 g_2^2 \frac{i}{2} \frac{f_N}{r} \frac{\phi_1}{\phi_2} &= 0. \end{aligned} \quad (4.18)$$

We consider these equations in the Appendix A. The solutions are given by (A.8) and

(A.9). The condition of regularity of these solutions at $r \rightarrow 0$ gives the eigenvalue

$$m_{or} = - \frac{\mu_2 g_2^2 \int_0^\infty dy \frac{f_N^2(y) \phi_1^2(y)}{y \phi_2^2(y)}}{1 - 2 \int_0^\infty dy \frac{f_N^2(y) \phi_1^2(y)}{y \phi_2^2(y)}}. \quad (4.19)$$

Solutions for $\widetilde{\psi}_{2-}$ and the combination $(\lambda_-^{12} - \lambda_-^{11})$ satisfy the same equations (4.18) and are related to solutions (A.8) and (A.9) via (4.15).

4.4 Effective action in the orientational sector

Now to see the effect of lifting of the orientational fermion zero modes let us derive a fermionic part of the two-dimensional effective action on the string world sheet with the $O(\mu)$ accuracy. In order to do so, we assume a slow t and z dependence of the fermionic moduli $\xi_{L,R}^l$, substitute our *ansatz* (4.7), (4.8) into the four dimensional fermion action (4.1) and integrate over x_1, x_2 . Kinetic terms for bulk fermions (containing derivatives ∂_0 and ∂_3) produce corresponding kinetic terms for two-dimensional fermions, and mass terms are generated because fermionic modes are now lifted. The result for the quadratic terms in the two dimensional fermionic action is

$$\mathcal{S}_{2d} = \int dt dz \left\{ \frac{4\pi}{g_2^2} (\bar{\xi}_L i \partial_R \xi_L + \bar{\xi}_R i \partial_L \xi_R) + m_{or} \gamma (\bar{\xi}_R \xi_L + \bar{\xi}_L \xi_R) + \dots \right\}, \quad (4.20)$$

where dots stand for higher order terms in fields and

$$\gamma = -4 \int dx_1 dx_2 \widetilde{\psi}_{1+} \overline{\psi}_{2+} = 4 \int dx_1 dx_2 |\overline{\psi}_{2+}|^2, \quad (4.21)$$

while

$$\partial_R = \partial_0 + i \partial_3, \quad \partial_L = \partial_0 - i \partial_3.$$

We see that all two-dimensional fermionic fields $\xi_{L,R}^l$ become massive with mass m_{or} proportional to μ . We expect that in the limit of large μ these fermions decouple from bosonic $CP(N-1)$ model (3.3).

4.5 Supertranslational zero modes

As we already mentioned, supertranslational modes can be obtained via a supersymmetry transformation from bosonic string solution even in the μ -deformed theory.

String solution ceases to be a BPS one, and all of the four remaining supercharges $Q^{\alpha 1}$ of the $\mathcal{N} = 1$ theory act non-trivially on the string solution. Much in the same way as the bosonic translational modes, the supertranslational ones decouple from orientational $CP(N-1)$ model and are described by free fermions on the string world sheet. This can be anticipated on general grounds. To see this note that the orientational fermion fields $\xi_{L,R}^l$ become heavy at large μ and without them we cannot construct interaction terms of n^l and supertranslational moduli $\zeta_{L,R}$ compatible with symmetries of the theory (if we do not consider higher derivative corrections).

For the sake of completeness we construct explicitly supertranslational zero modes in the large μ limit in Appendix B acting by $\mathcal{N} = 1$ supersymmetry transformations on the string solution of Sec. 2.

5 Physics of the world sheet theory and confined monopoles

As we have seen above the fermionic fields ξ^l of the effective world sheet theory become heavy in the large μ limit and decouple. Moreover, the translational sector is free and does not interact with the orientational sector. Thus, our effective world sheet theory on the non-Abelian string is given by bosonic $CP(N-1)$ model (3.3) without fermions in the large μ limit. If quark masses are small but not equal, the orientational moduli n^l are lifted by shallow potential (3.19).

As we already mentioned, our four dimensional bulk theory is in the Higgs phase where squarks develop condensate (1.18). Therefore 't Hooft-Polyakov monopoles present in the theory in the $\mathcal{N} = 2$ limit of small μ are confined by non-Abelian strings. In fact in $U(N)$ gauge theories confined monopoles are junctions of two distinct strings [30, 11, 12]. In the effective world sheet theory on the non-Abelian string they are seen as kinks interpolating between different vacua of $CP(N-1)$ model, see [15] for a review.

The question of the crucial physical importance is whether monopoles survive the limit of large μ when the bulk theory flows to $\mathcal{N} = 1$ QCD. Quasiclassically we do not expect this to happen. From a quasiclassical point of view, the very existence of 't Hooft-Polyakov monopoles relies on the presence of adjoint scalars which develop VEVs. At large μ adjoint fields become heavy and decouple in our bulk theory, so quasiclassically we do not expect monopoles to survive.

We will see now that in quantum theory the story becomes more interesting. Confined monopoles are represented by kinks of $CP(N-1)$ model on the non-Abelian string. Therefore to address the above problem we have to study kinks in the world

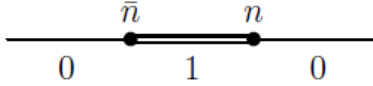


Figure 1: Configuration of a string with kink and anti-kink on it. Zero and one represent the true vacuum and the first quasi-vacuum respectively.

sheet theory. Certain results in this direction were already obtained. As we mentioned before, in the framework of massless μ -deformed $\mathcal{N} = 2$ QCD with FI D -term it was shown that the effective theory on the string world sheet is heterotic $\mathcal{N} = (0, 2)$ supersymmetric $CP(N-1)$ model [19, 20, 21, 23]. This model has N degenerative vacua and kinks interpolating between them. This means that kinks/confined monopoles do survive the large μ limit in the above mentioned theory.

In this paper we study a more "realistic" version of μ -deformed theory without the FI D -term. This theory flows to $\mathcal{N} = 1$ QCD in the large μ -limit. As we have shown the world sheet theory on the non-Abelian string reduces to non-supersymmetric $CP(N-1)$ model without fermions in the large μ limit. If quark mass differences are non-zero, a potential (3.19) is generated. It does not have multiple local minima, therefore kinks (confined monopoles of the bulk theory) become unstable and disappear.

Consider the case when quark masses are equal. Then $CP(N-1)$ model is at strong coupling. This model was solved by Witten [24] in the large N approximation. It was shown that kinks in this model are in a confinement phase. In terms more suitable for application to monopole physics of the bulk theory this can be understood as follows, see also [15] for a more detail review.

The vacuum structure of the $CP(N-1)$ model was studied in [31]. It was shown that the genuine vacuum is unique. There are, however, of order N quasi-vacua, which become stable in the limit $N \rightarrow \infty$, since an energy split between the neighboring quasi-vacua is $O(1/N)$. Thus, one can imagine a kink interpolating between the true vacuum and the first quasi-vacuum and the anti-kink returning to the true vacuum as in Fig.(1). Linear confining potential between kink and anti-kink is associated with excited quasi-vacuum.

This two dimensional confinement of kinks was interpreted in terms of strings and monopoles of the bulk theory in [28]. The fine structure of vacua in $CP(N-1)$ model on the non-Abelian string means that N elementary strings are split by quantum effects and have slightly different tensions. The difference between the tensions of "neighboring" strings is proportional to Λ_{CP}^2 , see (3.13). Therefore monopoles, in

addition to the four dimensional confinement (which ensures that they are attached to the string), acquire two-dimensional confinement along the string. Monopole and antimonopole connected by a string with larger tension form a mesonic bound state.

Consider a monopole-antimonopole pair interpolating between strings 0 and 1, see Fig. 1. Energy of the excited part of the string (labeled as 1) is proportional to the distance R between the kink and anti-kink as

$$V(R) \sim \Lambda_{CP}^2 R. \quad (5.1)$$

When it exceeds the mass of two monopoles which is of order of Λ_{CP} then the second monopole-antimonopole pair emerges breaking the excited part of the string. This gives an estimate for the typical length of the excited part of the string, $R \sim N/\Lambda_{CP}$. Since this length grows in the large N limit, kinks are metastable with an exponentially small decay rate $\exp\{-N\}$.

6 Conclusions

In this paper we considered non-Abelian strings in $\mathcal{N} = 2$ supersymmetric QCD deformed by a large mass term for adjoint matter. In the limit of large μ this theory flows to $\mathcal{N} = 1$ SQCD. We found a solution for the non-Abelian string and derived a two dimensional effective theory on the string world sheet which describes the dynamic of its orientational zero modes. This theory turns out to be bosonic $CP(N-1)$ model (3.3) with shallow potential (3.19) generated by small quark mass differences. The fermionic superpartners ξ^l of the bosonic orientational moduli n^l present in the $\mathcal{N} = 2$ limit become heavy at large μ and decouple.

We addressed the question of what happen to confined 't Hooft-Polyakov monopoles at large μ . We showed that, if the quark mass differences are larger than (exponentially small) Λ_{CP} , the confined monopoles become unstable at large μ . However, if the quarks have equal masses, the confined monopoles survive in the $\mathcal{N} = 1$ QCD limit. This result is quite remarkable since $\mathcal{N} = 1$ QCD is in the non-Abelian regime and quasiclassically we do not expect monopoles in this theory. It also supports the picture of "instead-of-confinement" phase for $\mathcal{N} = 1$ QCD at strong coupling [17].

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Appendix A:

In this Appendix we solve Dirac equations (4.14). After a substitution

$$\begin{aligned}\bar{\psi}_{1-}(r) &= \frac{1}{r\phi_2(r)}\Psi(r), \\ \lambda_{(1)}(r) &= ig_2^2\Lambda(r)\end{aligned}$$

equations (4.14) reduce to

$$\begin{aligned}\frac{1}{rg_2^2\phi_1\phi_2}\partial_r\Psi &= \Lambda, \\ r\partial_r\Lambda + f_N\Lambda - \frac{\phi_1}{\phi_2}\Psi &= -\frac{\mu_2 f_N}{2}\frac{\phi_1}{\phi_2},\end{aligned}\tag{A.1}$$

which in turn gives an equation of second order for Ψ :

$$\partial_r^2\Psi - \frac{1}{r}\left(1 + \frac{2}{N}(f - f_N)\right)\partial_r\Psi - g_2^2\phi_1^2\Psi = -\frac{\mu_2 f_N}{2}g_2^2\phi_1^2.\tag{A.2}$$

First let us solve the homogeneous version of (4.14), i. e. put $\mu_2 = 0$. The solutions are

$$\begin{aligned}\bar{\psi}_{1-} &= c\frac{f_N}{r\phi_2}, \\ \lambda_{(1)} &= c\frac{ig_2^2}{2}\left(\frac{\phi_1}{\phi_2} - \frac{\phi_2}{\phi_1}\right).\end{aligned}$$

with some constant c . They correspond to $\Psi = f_N$; indeed, this is a solution for homogeneous version of (A.2). With the help of it we can reduce the order of this equation. Let us take

$$\Psi(r) = \mu_2 f_N(r) \left(\int_0^r dx \chi(x) + c_1 \right),$$

with some constant c_1 , then from (A.2) it follows that

$$\partial_r\chi + \frac{1}{r}\left(\frac{1}{f_N}r^2g_2^2(\phi_1^2 - \phi_2^2) - 1 - \frac{2}{N}(f - f_N)\right)\chi = -\frac{1}{2}g_2^2\phi_1^2.\tag{A.3}$$

This is just an equation of the first order; its solution can be found very easily as

$$\chi = -\frac{g_2^2 r \phi_2^2}{2f_N^2} \left(\int_0^r \frac{dy}{y} \frac{\phi_1^2}{\phi_2^2} f_N^2 + c_2 \right). \quad (\text{A.4})$$

Putting all this together, we obtain:

$$\bar{\psi}_{1-}(r) = -\mu_2 g_2^2 \frac{f_N(r)}{r \phi_2(r)} \left(\int_0^r dx \frac{x \phi_2^2(x)}{2f_N^2(x)} \left(\int_0^x \frac{dy}{y} \frac{\phi_1^2(y)}{\phi_2^2(y)} f_N^2(y) + c_2 \right) + c_1 \right). \quad (\text{A.5})$$

with some new constant c_1 .

For this solution to behave well at the origin we have to put $c_1 = 0$. Considering the infinity, we should also require that

$$c_2 = - \int_0^\infty \frac{dy}{y} \frac{\phi_1^2(y)}{\phi_2^2(y)} f_N^2(y).$$

This gives

$$\bar{\psi}_{1-}(r) = \frac{\mu_2 g_2^2}{2} \frac{f_W(r)}{r \phi_2(r)} \int_0^r dx \frac{x \phi_2^2(x)}{f_W^2(x)} \int_x^\infty \frac{dy}{y} \frac{\phi_1^2(y)}{\phi_2^2(y)} f_W^2(y). \quad (\text{A.6})$$

for $\bar{\psi}_{1-}$ and

$$\begin{aligned} \lambda_{(1)}(r) &\equiv \lambda_-^{22} + \lambda_-^{21} = \\ &= \frac{i \mu_2 g_2^2}{2} \left(\frac{g_2^2}{2} \left(\frac{\phi_1}{\phi_2} - \frac{\phi_2}{\phi_1} \right) \int_0^r dx \frac{x \phi_2^2(x)}{f_W^2(x)} \int_x^\infty \frac{dy}{y} \frac{\phi_1^2(y)}{\phi_2^2(y)} f_W^2(y) \right. \\ &\quad \left. + \frac{\phi_2}{\phi_1 f_W} \int_r^\infty \frac{dy}{y} \frac{\phi_1^2(y)}{\phi_2^2(y)} f_W^2(y) \right). \end{aligned} \quad (\text{A.7})$$

for $\lambda_{(1)}$. By direct substitution we verified that these modes indeed satisfy the Dirac equations.

Now let us consider Dirac equations (4.18) with the non-zero eigenvalue m_{or} . Applying the method developed above we find the solutions

$$\bar{\psi}_{1-}(r) = - \left(m_{or} - \mu_2 \frac{g_2^2}{2} \right) \frac{f_N(r)}{r \phi_2(r)} \int_0^r dx \frac{x \phi_2^2(x)}{f_N^2(x)} \int_x^\infty dy \frac{f_N^2(y) \phi_1^2(y)}{y \phi_2^2(y)} \quad (\text{A.8})$$

for $\bar{\psi}_{i-}$ and

$$\begin{aligned} \lambda_{(1)}(r) = & -i \frac{m_{or}}{2} \left(\frac{\phi_1}{\phi_2} - \frac{\phi_2}{\phi_1} \right) - i g_2^2 \left(m_{or} - \mu_2 \frac{g_2^2}{2} \right) \frac{1}{2} \left(\frac{\phi_1}{\phi_2} - \frac{\phi_2}{\phi_1} \right) \times \\ & \int_0^r dx \frac{x \phi_2^2(x)}{f_N^2(x)} \int_x^\infty dy \frac{f_N^2(y) \phi_1^2(y)}{y \phi_2^2(y)} - i \left(m_{or} - \mu_2 \frac{g_2^2}{2} \right) \frac{\phi_2}{f_N \phi_1} \int_r^\infty dy \frac{f_N^2(y) \phi_1^2(y)}{y \phi_2^2(y)} \end{aligned} \quad (\text{A.9})$$

for $\lambda_{(1)}$.

One can see, that the first and the last terms in the last expression behave at the origin as $1/r$. We can choose eigenvalue m_{or} to insure that $1/r$ terms cancel out. This gives the expression (4.19) for the mass m_{or} .

Appendix B:

In this Appendix we generate supertranslational fermion zero modes in the large μ limit acting by $\mathcal{N} = 1$ supersymmetry transformations on the string solution of Sec. 2. The $\mathcal{N} = 1$ supersymmetry transformations have the form

$$\begin{aligned} \delta \bar{\psi}_{\dot{\alpha}}^{kA} &= i\sqrt{2} \bar{\nabla}_{\dot{\alpha}\alpha} q^{kA} \epsilon^\alpha + \sqrt{2} \bar{\epsilon}^\alpha \bar{F}^{kA} , \\ \delta \bar{\psi}_{\dot{\alpha}Ak} &= i\sqrt{2} \bar{\nabla}_{\dot{\alpha}\alpha} \bar{q}_{Ak} \epsilon^\alpha + \sqrt{2} \bar{\epsilon}^\alpha \bar{F}_{kA} , \end{aligned} \quad (\text{B.1})$$

where the F -terms are given by derivatives of the superpotential (1.6),

$$\begin{aligned} \bar{F}_{Ak} &= -\frac{\partial \mathcal{W}}{\partial q^{kA}} = \frac{i}{\mu_2} \left(\tilde{q}_{Ck} (\tilde{q}_A q^C) - \frac{\alpha}{N} (\tilde{q}_C q^C) \tilde{q}_{Ak} \right) + m \tilde{q}_{Ak} \\ &= \frac{i}{\mu_2} \left(\varphi_{Ck} (\varphi_A \varphi^C) - \frac{\alpha}{N} (\varphi_C \varphi^C) \varphi_{Ak} \right) + m \varphi_{Ak} , \end{aligned} \quad (\text{B.2})$$

$$\begin{aligned} \bar{\tilde{F}}^{kA} &= -\frac{\partial \mathcal{W}}{\partial \tilde{q}_{Ak}} = \frac{i}{\mu_2} \left(q^{kC} (\tilde{q}_C q^A) - \frac{\alpha}{N} (\tilde{q}_C q^C) q^{kA} \right) + m q^{kA} \\ &= \frac{i}{\mu_2} \left(\varphi^{kC} (\varphi_C \varphi^A) - \frac{\alpha}{N} (\varphi_C \varphi^C) \varphi^{kA} \right) + m \varphi^{kA} , \end{aligned} \quad (\text{B.3})$$

where we also used (2.1).

Consider first the region of intermediate r , in the range $1/m_G \lesssim r \lesssim 1/m_L$. As we will see, the fermion zero modes behave as $1/r$. This will give us leading logarithmic contributions to the kinetic terms for fermions of two dimensional effective theory on the string world sheet.

To calculate the fermionic modes one should substitute bosonic solutions (2.16) into the transformations (B.1). In (B.1), the first terms in the first and second lines

give $1/r$ contributions, whereas the F -terms adds constant and logarithmic terms, which does not produce leading logarithmic terms in the effective action. We neglect these last terms, and get non-zero fermionic profiles

$$\begin{aligned}
\bar{\psi}_1 &\approx (n\bar{n}) \frac{x^1 - i x^2}{r} \frac{1}{r} \frac{\sqrt{\xi}}{\ln \frac{g_2 \sqrt{\xi}}{m_L}} \zeta_R, \\
\bar{\psi}_2 &\approx (n\bar{n}) \frac{x^1 + i x^2}{r} \frac{1}{r} \frac{\sqrt{\xi}}{\ln \frac{g_2 \sqrt{\xi}}{m_L}} \zeta_L, \\
\tilde{\bar{\psi}}_1 &\approx (n\bar{n}) \frac{x^1 - i x^2}{r} \frac{1}{r} \frac{\sqrt{\xi}}{\ln \frac{g_2 \sqrt{\xi}}{m_L}} \zeta_R, \\
\tilde{\bar{\psi}}_2 &\approx (n\bar{n}) \frac{x^1 + i x^2}{r} \frac{1}{r} \frac{\sqrt{\xi}}{\ln \frac{g_2 \sqrt{\xi}}{m_L}} \zeta_L.
\end{aligned} \tag{B.4}$$

One can see that these modes are indeed proportional to $1/r$. Here $\zeta_{L,R}$ are the Grassmann parameters generated by supersymmetry transformations, $\zeta_L = \frac{1}{\sqrt{2}}\epsilon^1$, $\zeta_R = -\frac{1}{\sqrt{2}}\epsilon^2$. These parameters become fermionic fields in the two dimensional effective theory on the string world sheet.

The region of small r , $r \ll 1/m_G$ does not contribute because quark fields vanish in this limit.

To find the effective world sheet action, one should substitute solutions (B.4) into four-dimensional fermionic action (4.1). For the kinetic term, we obtain:

$$\mathcal{L}_{2d} = 2\pi\xi I_\xi (\bar{\zeta}_L i \partial_R \zeta_L + \bar{\zeta}_R i \partial_L \zeta_R), \tag{B.5}$$

where the normalization constant is

$$I_\xi = \frac{2N}{\ln \frac{m_W}{m_L}}. \tag{B.6}$$

As we already mentioned, deriving this effective action we integrated over the transversal coordinates in the range $1/m_G \lesssim r \lesssim 1/m_L$. The integral over r is logarithmically enhanced. The contributions of other regions do not have logarithmic enhancement and can be neglected.

We see that (B.5) is the action for free fermions which decouples from the orientational sector given by $CP(N-1)$ model (3.3).

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