Quasidecoupled state for dark matter in nonstandard thermal histories

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A Quasi-Decoupled State for Dark Matter In Non-Standard Thermal Histories

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Dark matter drops out of kinetic equilibrium with Standard Model particles when the momentum-transfer rate equals the expansion rate. In a radiation-dominated universe, this occurs at essentially the same time as dark matter kinetically decouples from the plasma. Dark matter may also fall out of kinetic equilibrium with Standard Model particles during an early matter-dominated era (EMDE), which occurs when the energy content of the Universe is dominated by either a decaying oscillating scalar field or a semi-stable massive particle before Big Bang Nucleosynthesis. Until now, it has been assumed that kinetic decoupling during an EMDE happens similarly to the way it does in a radiation-dominated era. We show that this is not the case. By studying the evolution of the dark matter temperature, we establish a quasi-decoupled state for dark matter in an EMDE, during which the dark matter temperature cools faster than the plasma temperature but slower than it would cool if the dark matter were fully decoupled. The dark matter does not fully decouple until the EMDE ends and the Universe becomes radiation dominated. We also extend the criteria for quasi-decoupling to other non-standard thermal histories and consider how quasi-decoupling affects the free-streaming length of dark matter.

I. INTRODUCTION

Weakly interacting massive particles (WIMPs) are prime candidates for cold dark matter. WIMPs interact with other Standard Model (SM) particles solely via the electroweak force, which allows them to fall out of kinetic equilibrium with the relativistic plasma as early as one second after the Big Bang. This departure from equilibrium occurs when the momentum-transfer rate, γ, between WIMPs and SM particles equals the Hubble expansion rate, H:

$$\gamma(T_{\mathrm{neq}}) \equiv H(T_{\mathrm{neq}}), \tag{1}$$

which defines the non-equilibrium temperature T_{neq} [1].[1] After dark matter (DM) kinetically decouples from the plasma, its temperature T_χ begins to scale as a^{-2}, where a is the scale factor. It has been customary to define the kinetic decoupling temperature for DM based on the late-time behavior of T_χ [2, 3]:

$$T_{\mathrm{kd}} \equiv T_\chi|_{T \to 0} \left(\frac{a}{a_{\mathrm{kd}}}\right)^2, \tag{2}$$

where T is the plasma temperature and a_{kd} is the value of the scale factor when $T = T_{\mathrm{kd}}$.

In a radiation-dominated (RD) universe, Eqs. (1) and (2) imply that $T_{\mathrm{kd}} = T_{\mathrm{neq}}/K_n$, where K_n is a numerical factor of order unity that depends on how the velocity-averaged DM scattering cross section scales with temperature ($\langle \sigma v \rangle \propto T^n$): $K_2 \simeq 1.03$ for p-wave

scattering [4, 5]. Therefore, in a RD universe, the temperature at which DM falls out of equilibrium with the plasma is essentially the same as the kinetic decoupling temperature. The timing of kinetic decoupling sets the cutoff scale in the matter power spectrum [1–3, 6–10]. The small-scale cutoff, in turn, fixes the mass of the smallest protohalos that can form at high redshift [1–3, 6–11].

Several investigations have explored the possibility that the Universe was not radiation dominated when DM kinetically decoupled from the plasma [4, 12–19]. Big Bang Nucleosynthesis (BBN) requires the Universe to be radiation dominated at a plasma temperature of $T \simeq 3$ MeV [20–23], but the evolution of the Universe at higher temperatures is unknown. Both delayed inflationary reheating and the presence of gravitationally coupled scalar fields support the possibility that the Universe could have been dominated by an oscillating scalar field prior to BBN. Since scalar fields that oscillate around the minimum of a quadratic potential behave like a pressureless fluid, these scenarios include an early matter-dominated era (EMDE) prior to BBN [24–31]. A recent review [32] concluded that EMDEs are a generic consequence of gravitationally coupled scalars in string theories. EMDEs have also been explored in the context of models where the inflaton decays to a hidden sector [33] and where a long-lived light mediator is responsible for interactions between DM and SM particles [34].

If DM is produced thermally from interactions with SM particles in the plasma, then an EMDE raises the value of T_{neq} relative to its value in a RD era, which leads to a smaller free-streaming length [12]. However, if DM is produced non-thermally through energy injection from a decaying scalar field, then an EMDE can lower the value of T_{neq} and make the transition from fully coupled to fully decoupled less sharp [13]. Most recently, Ref. [4] derived analytic expressions for how the DM temperature evolves during an EMDE and other non-standard

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[1] The EMDE literature typically uses Eq. (1) to define T_{kd} in a generic non-standard thermal history, but we are using Eq. (1) to define a new temperature T_{neq} in order to distinguish it from T_{kd} defined in Eq. (2).

thermal histories.

DM kinetic decoupling in an EMDE has received a great deal of recent attention because it has been shown that an EMDE enhances the small-scale matter power spectrum and boosts the abundance of microhalos if DM stops interacting with the plasma before the onset of radiation domination [15–18].[2] Matter perturbations that enter the horizon during radiation domination grow logarithmically with the scale factor, but they grow linearly during an EMDE.[3] The resulting enhancement in the small-scale matter power spectrum significantly increases the abundance of microhalos at high redshift ($z \gtrsim 100$) [15, 18], which can boost the DM annihilation rate by many orders of magnitude [18]. Such a boost to the annihilation rate can bring formerly untestable DM candidates within reach of current and future observations [19].

In this short paper, we reconsider the differential equation governing the evolution of the DM temperature, T_χ. All prior analyses of DM kinetic decoupling in an EMDE assume that $T_\chi \propto a^{-2}$ when $\gamma < H$, just as it does in a RD universe. We show that this is not the case: at plasma temperatures $T < T_{\mathrm{neq}}$ in an EMDE, T_χ decays faster than T, but slower than a^{-2}, which implies that DM does not fully decouple in an EMDE. This *quasi-decoupled* state for DM implies that the value of T_{neq} is much greater than the value of T_{kd} in an EMDE and that T_{kd} is *not* the temperature at which DM starts cooling as a^{-2}. We will show that T_{kd}, as defined by Eq. (2), does not correspond to any physical transition in the evolution of T_χ during an EMDE. Rather, T_{neq} marks the moment that DM drops out of equilibrium with the plasma, and the reheat temperature T_{RH} marks the end of the EMDE and the onset of full kinetic decoupling: $T_\chi \propto a^{-2}$. Finally, we establish the criteria for quasi-decoupling in other non-standard cosmologies.

A quasi-decoupled state for DM is likely to have profound implications for our understanding of DM behavior in EMDE scenarios, including the growth of density perturbations, the free-streaming length, the abundance of microhalos, and the observational signatures of an EMDE. As a first pass at understanding these ramifications, we calculate the DM free-streaming length with quasi-decoupling in an EMDE. While the DM free-streaming length is still smaller than it would be in the absence of an EMDE, it is an order of magnitude greater than it would be if DM fully decoupled during the EMDE.

We begin in Sec. II by establishing the cosmologi-

cal framework that we use throughout this paper. In Sec. III, we solve the evolution equation for the DM temperature in a generic cosmology in the post-equilibrium limit and show how a quasi-decoupled state for DM occurs during an EMDE. We then explore the conditions for quasi-decoupling in other thermal histories, and we confirm that the general solution to the differential equation for the DM temperature agrees with the solution to the differential equation for the DM temperature in the post-equilibrium limit. We also discuss the difference between the non-equilibrium temperature T_{neq}, as defined by Eq. (1), and the kinetic decoupling temperature T_{kd}, as defined by Eq. (2), in an EMDE. In Sec. IV we use a piecewise model for the DM velocity to study the effect of quasi-decoupling in an EMDE on the free-streaming length of DM. Finally, we summarize our results in Sec. V. The appendix presents a proof of the asymptotic series expansion for the upper incomplete gamma function that enters into the general solution for the DM temperature in a generic cosmology. We use natural units ($\hbar = c = 1$) throughout this work.

II. THE COSMOLOGICAL FRAMEWORK

We employ the same cosmological framework as Ref. [4]: the plasma temperature and scale factor are related by $T \propto a^{-\alpha}$, where α is a positive constant. The Hubble rate H is

$$H = H_i \left(\frac{T}{T_i} \right)^\nu = H_i \left(\frac{a_i}{a} \right)^{\alpha\nu} \propto \rho^{1/2}, \qquad (3)$$

where ρ is the energy density of the Universe, ν is a positive constant, and H_i, T_i, and a_i are the expansion rate, plasma temperature, and scale factor, respectively, at some initial reference point. During a RD era, $\alpha = 1$ and $\nu = 2$ if we neglect changes in the number of relativistic degrees of freedom as the Universe cools. During an EMDE, entropy is generated by a decaying scalar field, so that the energy density of radiation scales as $\rho_R \propto a^{-3/2}$, which implies $\alpha = 3/8$ [37]. Since the scalar field behaves like a pressureless fluid during an EMDE, its energy density scales as $\rho_\phi \propto a^{-3}$, which forces $H \propto a^{-3/2}$ and implies that $\nu = 4$ in an EMDE. In our analysis, we restrict to thermally produced DM and neglect any energy transfer into DM from the decaying scalar field.

An EMDE ends when the scalar field's decay rate Γ_ϕ is approximately equal to the Hubble rate. At that time, the scalar field's energy density begins to decrease exponentially, and the Universe becomes radiation dominated shortly thereafter. It is customary to define the reheat temperature T_{RH} by

$$\Gamma_\phi \equiv \sqrt{\frac{8\pi^3 g_*(T_{\mathrm{RH}})}{90}} \frac{T_{\mathrm{RH}}^2}{m_{\mathrm{pl}}}, \qquad (4)$$

where $m_{\mathrm{pl}} = G^{-1/2}$ is the Planck mass and $g_*(T) = \rho_R(T)/[(\pi^2/30)T^4]$ is the number of relativistic

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[2] If DM remains kinetically coupled during the EMDE, then the evolution of the matter and radiation perturbations generate isocurvature perturbations that enhance the small-scale matter power spectrum [35].

[3] The enhanced growth of density perturbations in an EMDE may enable primordial black holes (PBHs) to form on sub-horizon scales, but PBHs only constrain EMDE scenarios if the primordial power spectrum is blue-tilted [36].

degrees of freedom. It is also convenient to define $a_{\rm RH}$ as the value of the scale factor when $\Gamma_\phi = H_i(a_i/a)^{3/2}$:

$$\frac{a_{\rm RH}}{a_i} \equiv \left(\frac{H_i}{\Gamma_\phi}\right)^{2/3} . \tag{5}$$

Since the transition from scalar domination to radiation domination is not instantaneous, these definitions of $T_{\rm RH}$ and $a_{\rm RH}$ do not imply that $T(a_{\rm RH}) = T_{\rm RH}$. On the contrary, numerically solving the equations that govern the evolution of the scalar and radiation energy densities reveals that $T(a_{\rm RH}) \simeq 0.74 T_{\rm RH}$.

The kinetic decoupling of DM is governed by the elastic collision rate $\Gamma = \langle \sigma v \rangle n_{\rm rel}$ between DM and relativistic particles with number density $n_{\rm rel}$. In most WIMP models, the p-wave ($n = 2$) scattering channel dominates [9], but we maintain arbitrary n for generality. Relative to the elastic collision rate, the momentum-transfer rate, γ, is suppressed by a factor of (T/m_χ), where m_χ is the mass of the DM particle: $\gamma \simeq (T/m_\chi) \Gamma$. The suppression factor (T/m_χ) encodes the fact that it takes $(m_\chi/T) \gg 1$ elastic scatterings to appreciably alter the DM particle's momentum [1]. Thus, the momentum-transfer rate is given by

$$\gamma = \gamma_i \left(\frac{T}{T_i}\right)^{4+n} = \gamma_i \left(\frac{a_i}{a}\right)^{\alpha(4+n)} , \tag{6}$$

where $\gamma_i = \gamma(T_i)$. Lastly, since $T_{\rm neq}$ is defined by Eq. (1), we denote the equilibrium regime by $(\gamma/H) \gg 1$ and the post-equilibrium regime by $(\gamma/H) \ll 1$. The equilibrium regime describes the period in which $T_\chi(a) \simeq T(a)$ and the post-equilibrium regime includes all phases during which $T_\chi(a)$ diverges from $T(a)$.

III. THE POST-EQUILIBRIUM BEHAVIOR OF THE DM TEMPERATURE

Using a Fokker-Planck equation for the DM particle occupation number f_χ with the approximation $1 \pm f_\chi \approx 1$, it can be shown that T_χ satisfies the differential equation [2, 9]

$$a \frac{dT_\chi}{da} + 2 T_\chi(a) \left[1 + \frac{\gamma(a)}{H(a)}\right] = 2 \frac{\gamma(a)}{H(a)} T(a). \tag{7}$$

In the post-equilibrium regime, $(\gamma/H) \ll 1$, so if we drop the (γ/H) term on the l.h.s. of Eq. (7) and assume that $(\gamma/H) T \ll T_\chi$, then $T_\chi \propto a^{-2}$, and DM fully kinetically decouples from the plasma. However, the assumption that $(\gamma/H) T \ll T_\chi$ is not always valid, even in regimes where $(\gamma/H) \ll 1$. To demonstrate this point, we solve Eq. (7) in the post-equilibrium limit

$$a \frac{dT_\chi^{\rm PE}}{da} + 2 T_\chi^{\rm PE}(a) \simeq 2 \frac{\gamma(a)}{H(a)} T(a). \tag{8}$$

We can re-write $T(a)$ and (γ/H), respectively, as $T(a) = T_{\rm neq} \left(a_{\rm neq}/a\right)^\alpha$ and $(\gamma/H) = \left(a_{\rm neq}/a\right)^{\alpha\beta}$, where $a_{\rm neq}$ is the value of the scale factor when $T = T_{\rm neq}$ and $\beta \equiv (4+n-\nu)$. We restrict to the case $\beta > 0$ to guarantee that (γ/H) monotonically decreases with time.

If we perform the variable transformations $g(a) \equiv (a/a_{\rm neq})^2 T_\chi^{\rm PE}(a)$ and $y \equiv (a/a_{\rm neq})$, then we obtain $(dg/dy) = 2 T_{\rm neq} y^{[1-\alpha(\beta+1)]}$, which integrates to

$$T_\chi^{\rm PE} \simeq C \left(\frac{a_{\rm neq}}{a}\right)^2 + \frac{2 T_{\rm neq}}{2 - \alpha(\beta+1)} \left(\frac{a_{\rm neq}}{a}\right)^{\alpha(\beta+1)} , \tag{9}$$

where C is a constant of integration. If $\alpha(\beta+1) > 2$, then Eq. (9) shows that the $C \left(a_{\rm neq}/a\right)^2$ term dominates over the second term in the post-equilibrium regime, such that $T_\chi^{\rm PE} \simeq C(a_{\rm neq}/a)^2$. In other words, DM fully kinetically decouples from the plasma, which also follows from Eq. (8) if we neglect the $(\gamma/H) T$ term.

The second term in Eq. (9) is proportional to $(\gamma/H) T$. If $\alpha(\beta+1) < 2$, then the second term in Eq. (9) dominates in the post-equilibrium regime, such that $T_\chi^{\rm PE} \sim (\gamma/H) T$, and $(\gamma/H) T$ in Eq. (8) cannot be approximated as zero. Under these conditions, $T_\chi^{\rm PE}$ falls off *faster* than the plasma temperature, but *slower* than a^{-2}, which implies that DM never fully kinetically decouples from the plasma. Instead, DM enters a *quasi-decoupled state*, which represents a surprising new behavior, because all prior analyses of kinetic decoupling in non-standard cosmologies assume that DM is fully decoupled from the plasma when $T < T_{\rm neq}$.

Table I shows how the DM temperature evolves in several cosmological scenarios. An EMDE is the only cosmology listed that permits quasi-decoupling. In an EMDE, $\alpha(\beta+1) = (3/8)(1+n)$ is less than 2 for $n \leq 4$. Equation (9) then demands $T_\chi^{\rm PE} \propto a^{-(3/8)(1+n)}$, which forces $T_\chi^{\rm PE}$ to decrease *faster than* the plasma temperature ($T \propto a^{-3/8}$) but *slower than* $T_\chi \propto a^{-2}$. Figure 1 illustrates the quasi-decoupled behavior of DM in an EMDE for p-wave scattering; this plot of T_χ is obtained from the numerical solution to Eq. (7). The moment when $\gamma = H$ and the onset of radiation domination (reheating) are labeled in Fig. 1: we see that the equilibrium regime ($\gamma \gg H$) describes the era during which $T_\chi(a) \simeq T(a) \propto a^{-3/8}$, while the post-equilibrium regime includes the quasi-decoupled phase, where $T_\chi(a) \propto a^{-9/8}$, and the fully decoupled phase, where $T_\chi(a) \propto a^{-2}$.

While Eq. (9) highlights the sensitivity of a quasi-decoupled state to the size of $\alpha(\beta+1)$ relative to 2, Table I suggests a deeper explanation for quasi-decoupling. In all thermal histories listed where entropy is conserved, such that $T \propto a^{-1}$, quasi-decoupling is not permitted for $n > 0$. Is quasi-decoupling possible in thermal histories for which entropy is conserved and is an EMDE the only entropy-producing scenario that allows quasi-decoupling? To answer these questions, we recast our treatment in terms of the parameter $w \equiv P/\rho$, which we assume to be constant. First, we fix $T \propto a^{-1}$

Thermal History	$\alpha(\beta+1)$	$(\gamma/H)\,T$	$T_\chi^{\mathrm{PE}} \sim$	Quasi-Decoupling?
Λ Dom. ($T \propto a^{-1}, H \propto T^0, w=-1$)	$5+n$	$T_{\mathrm{neq}}\left(a_{\mathrm{neq}}/a\right)^{(5+n)}$	$C\left(a_{\mathrm{neq}}/a\right)^2$	No
Matter Dom. ($T \propto a^{-1}, H \propto T^{3/2}, w=0$)	$(7/2)+n$	$T_{\mathrm{neq}}\left(a_{\mathrm{neq}}/a\right)^{[(7/2)+n]}$	$C\left(a_{\mathrm{neq}}/a\right)^2$	No
RD ($T \propto a^{-1}, H \propto T^2, w=1/3$)	$3+n$	$T_{\mathrm{neq}}\left(a_{\mathrm{neq}}/a\right)^{(3+n)}$	$C\left(a_{\mathrm{neq}}/a\right)^2$	No
Kination ($T \propto a^{-1}, H \propto T^3, w=1$)	$2+n$	$T_{\mathrm{neq}}\left(a_{\mathrm{neq}}/a\right)^{(2+n)}$	$C\left(a_{\mathrm{neq}}/a\right)^2$	No
EMDE ($T \propto a^{-3/8}, H \propto T^4, w=0$)	$(3/8)(1+n)$	$T_{\mathrm{neq}}\left(a_{\mathrm{neq}}/a\right)^{[(3/8)(1+n)]}$	$T_{\mathrm{neq}}\left(a_{\mathrm{neq}}/a\right)^{[(3/8)(1+n)]}$	Yes

TABLE I: The impact of cosmology and scattering process on the post-equilibrium behavior of DM temperature T_χ^{PE} in various thermal histories. The equation of state parameter $w \equiv \left(P/\rho\right)$ is defined in terms of the pressure, P, and energy density ρ of the dominant energy component. The abbreviations Λ Dom., and Matter Dom. denote cosmological constant-dominated and matter-dominated scenarios, respectively.
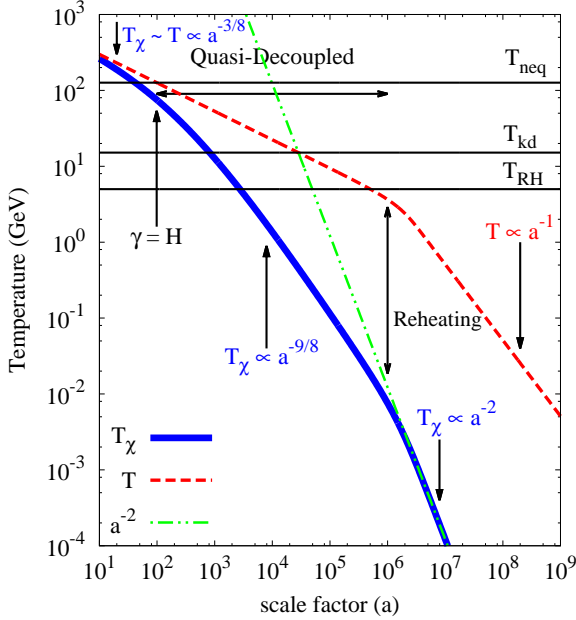

FIG. 1: The evolution of the DM temperature T_χ (solid curve) and plasma temperature T (dashed curve) vs. a for p-wave ($n = 2$) scattering in an EMDE with $T_{\mathrm{RH}} = 5$ GeV. The EMDE ends and the Universe becomes radiation-dominated when $a \simeq a_{\mathrm{RH}}$: in this figure, $a_{\mathrm{RH}} = 10^6$. The momentum-transfer rate γ equals H when $T = T_{\mathrm{neq}} = 126.5$ GeV, which corresponds to a DM particle that would have $T_{\mathrm{neq}} = 20$ GeV in a radiation-dominated universe (i.e. $T_{\mathrm{kdS}} = 20$ GeV). Equation (2) gives $T_{\mathrm{kd}} = 15.1$ GeV. In the quasi-decoupled regime, T_χ decays faster than the plasma temperature, but slower than a^{-2}.

and use $\rho \propto a^{-3(1+w)}$ to find $w = (2/3)\nu - 1$. Our quasi-decoupling condition, $\alpha(\beta+1) < 2$, then implies that $w > 1 + (2/3)n$. Thus, quasi-decoupling is not possible if entropy is conserved and $n \geq 0$ unless $w > 1$ (which does not necessarily violate causality [38–40]).

In cases of entropy production, we look beyond an EMDE by searching for a condition on w that permits quasi-decoupling. If the Universe is dominated by a scalar field with energy density ρ_ϕ that decays into relativistic particles with energy density ρ_R, then

$$a\,H\,\frac{d}{da}\,\rho_R + 4\,H\,\rho_R = \Gamma_\phi\,\rho_\phi. \qquad (10)$$

Equation (10) implies that $\rho_R \propto \sqrt{\rho_\phi}$ while $\rho_\phi \gg \rho_R$, which fixes $\nu = 4$ and $\alpha = (3/8)(1+w)$. Applying $\alpha(\beta+1) < 2$ yields $w < 16/[3(1+n)] - 1$. Therefore, any $w < 5/3$ allows quasi-decoupling for $n = 1$ (s-wave) scattering. For p-wave scattering, quasi-decoupling is permitted for all $w < 7/9$. Table 1 shows that a standard kination scenario does not permit quasi-decoupling, but a *decaying* kination scenario ($\nu = 4$, $w = 1$) would support quasi-decoupling if the s-wave channel dominates the elastic scattering cross section.

We now verify that the general solution to Eq. (7) is consistent with T_χ^{PE} in Eq. (9). The general solution to Eq. (7) with $T_\chi(a_i) = T(a_i)$ is [4]

$$T_\chi(a) = T(a)\,s^\lambda\,e^s\,\Gamma(1-\lambda, s), \qquad (11)$$

where $s(a) = [2/(\alpha\beta)](a_{\mathrm{neq}}/a)^{\alpha\beta}$ is the decoupling parameter for $\beta > 0$, $\lambda \equiv (2-\alpha)/(\alpha\beta)$, and $\Gamma(p,s)$ is the upper incomplete gamma function defined by $\Gamma(p,s) = \int_s^\infty dt\,t^{p-1}\,e^{-t}$ for all p and all $s \geq 0$. Ref. [4] obtained Eq. (11), but failed to notice that T_χ does *not* scale as a^{-2}, i.e. that DM does *not* fully decouple, when $(\gamma/H) \ll 1$ in an EMDE. However, we can coax the quasi-decoupled behavior from Eq. (11) by noting that $\Gamma(1-\lambda, s)$ obeys the asymptotic series expansion (see Appendix)

$$\Gamma(1-\lambda, s) = \Gamma(1-\lambda) - \frac{s^{1-\lambda}}{1-\lambda} + \mathcal{O}(s^{2-\lambda}), \qquad (12)$$

where $\Gamma(p) = \Gamma(p,0) = \int_0^\infty dt\,t^{p-1}\,e^{-t}$ for $p > 0$ and $\Gamma(p) = (1/p)\,\Gamma(p+1)$ for $p < 0$. Equation (12) shows that $\Gamma(1-\lambda, s)$ approaches a constant value as $s \to 0$ if $(1-\lambda) > 0$, which is consistent with the RD scenario for all $n > 0$. However, $\Gamma(1-\lambda, s)$ *diverges* as $s \to 0$ if $(1-\lambda) < 0$, which is equivalent to $\alpha(\beta+1) < 2$. Using $e^s \simeq 1$ for small s and substituting Eq. (12) and $s(a) = [2/(\alpha\beta)]\left(a_{\mathrm{neq}}/a\right)^{\alpha\beta}$ into Eq. (11) yields

$$T_\chi \simeq \left[\left(\frac{2}{\alpha\beta}\right)^\lambda T_{\mathrm{neq}}\,\Gamma(1-\lambda)\right]\left(\frac{a_{\mathrm{neq}}}{a}\right)^2 \qquad (13)$$

$$+ \frac{2\,T_{\mathrm{neq}}}{2 - \alpha(\beta+1)}\left(\frac{a_{\mathrm{neq}}}{a}\right)^{\alpha(\beta+1)}\left[1 + \mathcal{O}\left(\left[\frac{a_{\mathrm{neq}}}{a}\right]^{\alpha\beta}\right)\right],$$

which is consistent with Eq. (9) for $T_\chi^{\rm PE}$ and fixes the constant of integration, C.

In a RD universe, the values of $T_{\rm neq}$ and $T_{\rm kd}$ are nearly identical. In an EMDE, the value of $T_{\rm neq}$ is much greater than the value of $T_{\rm kd}$, as demonstrated in Fig. 1. In an EMDE, Eq. (2) implies that $T_{\rm neq} > T_{\rm kd} > T_{\rm RH}$. Moreover, $T_{\rm kd}$ does not mark a transition in the evolution of T_χ. For small values of γ, corresponding to small scattering cross sections, the quasi-decoupled stage is very long ($T_{\rm neq} \gg T_{\rm RH}$), so treating the DM as fully coupled and then fully decoupled with a transition point marked by $T_{\rm kd}$ is inappropriate.

We can use the numerical solutions to Eq. (7) to determine the discrepancy between $T_{\rm neq}$ and $T_{\rm kd}$ in an EMDE. These numerical solutions reveal that $T_\chi|_{T \to 0} = \kappa_1 T_\chi(a_{\rm RH}) (a/a_{\rm RH})^{-2}$, where $\kappa_1 = 1.37$, $a_{\rm RH}$ is defined by Eq. (5), and $T_\chi(a)$ is given by Eq. (11).[4] During an EMDE, $a/a_{\rm RH} = (2/5)^{2/3}(T_{\rm RH}/T)^{8/3}$ [18], which implies that $T_{\rm kd} = (2/5)^{4/13} T_{\rm RH} \left[T_{\rm RH}/[\kappa_1 T_\chi(a_{\rm RH})]\right]^{3/13}$, where $T_{\rm RH}$ is defined by Eq. (4). Since T_χ decreases faster than T while the DM is quasi-decoupled, $\kappa_1 T_\chi(a_{\rm RH}) \ll T_{\rm RH}$, and therefore $T_{\rm kd} \gg T_{\rm RH}$.

If $T_{\rm neq}/T_{\rm RH} \gg 1$, then we can approximate $T_\chi(a_{\rm RH})$ using Eq. (13). This approximation allows us to relate the value of $T_{\rm kd}$ to the value of $T_{\rm neq}$ in a RD universe, which we will refer to as $T_{\rm kdS}$ to maintain consistency with prior investigations of kinetic decoupling in nonstandard thermal histories. If DM decouples during an EMDE, then $T_{\rm neq} = \sqrt{5/2}\,(T_{\rm kdS}^2/T_{\rm RH})$ [12, 18]. During an EMDE, the second term on the r.h.s. of Eq. (13) is dominant for $a \gg a_{\rm neq}$. It follows that

$$\frac{T_{\rm kd}}{T_{\rm neq}} = 0.53 \left(\frac{T_{\rm RH}}{T_{\rm kdS}}\right)^{14/13}. \tag{14}$$

For $(T_{\rm RH}/T_{\rm kdS}) \ll 1$, $(T_{\rm kd}/T_{\rm neq}) \ll 1$, which confirms the generality of the temperature hierarchy $T_{\rm neq} > T_{\rm kd} > T_{\rm RH}$ shown in Fig. 1. The temperatures $T_{\rm neq}$ and $T_{\rm kd}$ are vastly different in an EMDE with a sufficiently long quasi-decoupled phase.

IV. IMPACT ON THE MATTER POWER SPECTRUM

The existence of a quasi-decoupled phase could affect the evolution of DM perturbations in two ways. First, the DM perturbations may remain at least partially coupled

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[4] Equation (11) and the following expressions in this section assume that the number of relativistic degrees of freedom, $g_*(T)$, is constant. If $g_*(T)$ changes, then Eq. (11) is no longer an exact solution to Eq. (7). However, using $s = [2/(\alpha\beta)](T/T_{\rm neq})^{4+n}(a/a_{\rm neq})^{3/2}$ in Eq. (11) provides an accurate approximate solution, and a value of κ_1 can be found for each value of $T_{\rm RH}$.

to the perturbations in the relativistic plasma during the EMDE. The radiation density perturbations grow during the EMDE because they are sourced by an increasingly inhomogeneous scalar field, but they do not grow as quickly as the uncoupled DM perturbations [15–17, 35], so any residual interactions between the plasma and the DM particles may suppress the growth of matter perturbations during the EMDE. Second, the fact that the DM temperature does not scale as a^{-2} during the EMDE implies that the velocities of the DM particles, v_χ, do not scale as a^{-1}, which profoundly affects the calculation of the comoving DM free-streaming horizon: $\lambda_{\rm fs} = \int_{t_{\rm neq}}^{t_0} (v_\chi/a)dt$, where t_0 is the present age of the Universe. The evolution of the DM and plasma perturbations lies beyond the scope of this article, but we can estimate the impact of the quasi-decoupled phase on $\lambda_{\rm fs}$ by assuming that $v_\chi \propto \sqrt{T_\chi}$. We restrict our analysis to p-wave scattering in an EMDE. In this case, $T_\chi \propto a^{-9/8}$ during the quasi-decoupled phase.

To isolate the effect of the EMDE on the post-reheating velocities of the DM particles, we consider the velocity at the reheat temperature $T_{\rm RH}$. We also use the fact that $T_{\rm neq} \simeq T_{\rm kdS}^2/T_{\rm RH}$ for $T_{\rm kdS} > T_{\rm RH}$ [12, 18]. Since $T \propto a^{-3/8}$ during the EMDE, $v_\chi(T_{\rm RH}) \simeq \sqrt{(T_{\rm kdS}/m_\chi)}(T_{\rm RH}/T_{\rm kdS})^{5/2}$. Therefore, the velocities of the DM particles are suppressed by a factor of $(T_{\rm RH}/T_{\rm kdS})^{3/2}$ if they quasi-decouple during an EMDE, as opposed to if they fully decouple during radiation domination. While this suppression is not as large as it would have been if the DM fully decoupled during an EMDE (in which case the suppression factor is $(T_{\rm RH}/T_{\rm kdS})^{23/6}$ [18]), it does imply that an EMDE still reduces the DM free-streaming horizon even though the DM particles do not fully decouple from the plasma.

To quantify the impact of quasi-decoupling on the free-streaming horizon, we compute

$$\lambda_{\rm fs} = \int_{a_{\rm neq}}^{a_0} da \frac{v_\chi(a)}{a^2 H(a)} \tag{15}$$

using a piecewise model for the DM velocity: $v_\chi \simeq \sqrt{T_{\rm kd}/m_\chi}(a_{\rm kd}/a)$ for $a > a_{\rm RH}$ and $v_\chi \simeq \sqrt{0.8\,T_\chi(a_{\rm RH})/m_\chi}(a_{\rm RH}/a)^{9/16}$ for $a < a_{\rm RH}$, where $T_\chi(a_{\rm RH})$ is evaluated using Eq. (11). The factor of 0.8 in the latter expression accounts for the fact that the DM temperature obtained by numerically solving Eq. (7) is 0.8 times the value given by Eq. (11) when $a = a_{\rm RH}$. This model overestimates v_χ for $a \simeq a_{\rm neq}$; to compensate, we start the integration in Eq. (15) when $0.8\,T_\chi(a_{\rm RH})(a_{\rm RH}/a)^{9/8} = T(a)$, instead of at $a = a_{\rm neq}$. In contrast, if we had assumed that $T_\chi \propto a^{-2}$ while $\gamma \lesssim H$, then $v_\chi \simeq \sqrt{(T_{\rm neq}/m_\chi)}(a_{\rm neq}/a)$ for all $T < T_{\rm neq}$. Figure 2 shows how these two models predict different free-streaming horizons if DM decouples during an EMDE. In this figure, we plot the ratio $\lambda_{\rm fs}/\lambda_{\rm fs}({\rm RD})$, where $\lambda_{\rm fs}({\rm RD})$ is computed assuming that the Universe is radiation dominated when DM decouples (at $T = T_{\rm kdS}$ and $a = a_{\rm kdS}$) and that
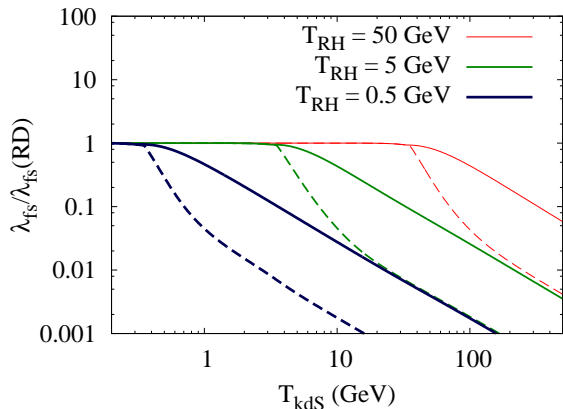
FIG. 2: The comoving free-streaming horizon in an EMDE divided by the comoving free-streaming horizon in a radiation-dominated universe for DM particles that would kinetically decouple at a temperature $T_{\rm kdS}$ in a radiation-dominated universe. If $T_{\rm kdS} > T_{\rm RH}$, then the DM kinetically decouples during an EMDE. The solid curves show how the free-streaming is reduced relative to its value in a radiation-dominated universe given that the DM is quasi-decoupled during the EMDE ($v_\chi \propto \sqrt{T_\chi} \propto a^{-9/16}$). The dashed curves show how the free-streaming horizon would be even smaller if the DM fully decoupled during the EMDE, as assumed in earlier investigations of EMDE scenarios [12–19].

$v_\chi \simeq \sqrt{(T_{\rm kdS}/m_\chi)}(a_{\rm kdS}/a)$ for all $T < T_{\rm kdS}$.

Figure 2 confirms the expectations stated above: it shows that including the effects of the quasi-decoupled period increases $\lambda_{\rm fs}$ by about an order of magnitude compared to what it would be if DM fully decoupled during an EMDE, but $\lambda_{\rm fs}$ is still smaller than it would have been in the absence of an EMDE. Since the mass enclosed within the free-streaming horizon sets a lower bound on the mass of the smallest DM halos [1–3, 6–11], an EMDE will still decrease the minimum halo mass if $T_{\rm kdS} > T_{\rm RH}$, but the minimum halo mass will be ~ 1000 times greater than stated in Ref. [12], which lay the foundations for subsequent treatments of DM free-streaming in EMDE cosmologies [13–19].

V. CONCLUDING REMARKS

We have established the existence of a quasi-decoupled state for DM in thermal histories in which entropy is produced by a decaying scalar field with $w < 16/[3(1+n)] - 1$, where n determines how the DM scattering cross section depends on temperature ($\langle \sigma v \rangle \propto T^n$). The non-equilibrium temperature $T_{\rm neq}$ is still higher than it is in a radiation-dominated era, but it marks a transition from tightly coupled to quasi-decoupled as opposed to a shift from tightly coupled to fully decoupled. The DM will remain quasi-decoupled until the onset of radiation domination; the transition from quasi-decoupled to fully decoupled does not depend on the interactions between DM and the SM. Therefore, the quasi-decoupled phase should be considered as a fun-

damentally new state of DM.

These surprising results force us to reconsider the body of work on kinetic decoupling of DM in non-standard thermal histories, which, until now, has been built on the assumption that DM fully decouples from the plasma when the momentum-transfer rate falls below the expansion rate. Quasi-decoupling calls all previous work on the behavior of DM during an EMDE into question and merits new investigations. We have demonstrated that quasi-decoupling increases $\lambda_{\rm fs}$ by an order of magnitude compared to the fully decoupled calculation, which has profound implications for the small-scale cutoff in the matter power spectrum, the size of the first microhalos, and the timing of their formation [18]. In future work, we will explore these ramifications in more detail [41].

Acknowledgments

Appendix: Derivation of Eq. (12)

The upper incomplete gamma function is given by $\Gamma(p, s) = \int_s^\infty dt\, t^{p-1} e^{-t}$ for all p and all $s \geq 0$. In this appendix, we derive Eq. (12) for the series expansion of $\Gamma(p, s)$ for $s \ll 1$. If $p > 0$, then the complete gamma function, $\Gamma(p)$, is given by

$$\Gamma(p) = \Gamma(p, 0) = \int_0^\infty dt\, t^{p-1} e^{-t}, \qquad \text{(A.1)}$$

which gives

$$\Gamma(p, s) = \left[\int_0^\infty dt\, e^{-t} t^{p-1} - \int_0^s dt\, e^{-t} t^{p-1} \right],$$
$$= \Gamma(p) - \sum_{n=0}^\infty \frac{(-1)^n s^{p+n}}{n!\, (p+n)}, \qquad \text{(A.2)}$$

where we used the series expansion for the e^{-t} in the second integral. The series in Eq. (A.2) converges for all s.

If $p < 0$, then $\Gamma(p)$ is defined by the recursion relation

$$\Gamma(p) = \frac{1}{p}\, \Gamma(p+1). \qquad \text{(A.3)}$$

We now show that it is still possible to apply Eq. (A.2) for $\Gamma(p, s)$ if $p < 0$ and non-integer, provided that $\Gamma(p)$ is defined by Eq. (A.3). We follow the method described in Ref. [42] and use tautologies to decompose $\Gamma(p, s)$ as

follows:

$$\Gamma(p,s) = \int_s^\infty dt\, t^{p-1} \left[\mathcal{S} + e^{-t} - \mathcal{S}\right],$$

$$= \int_s^\infty dt\, t^{p-1}\mathcal{S} - \int_0^s dt\, f(t) + \int_0^\infty dt\, f(t),$$

$$= I_1 - I_2 + I_3, \tag{A.4}$$

where

$$\mathcal{S} \equiv \sum_{n=0}^N \frac{(-1)^n t^n}{n!},$$

$$f(t) \equiv t^{p-1}[e^{-t} - \mathcal{S}],$$

$$I_1 \equiv \int_s^\infty dt\, t^{p-1} \sum_{n=0}^N \frac{(-1)^n t^n}{n!},$$

$$I_2 \equiv \int_0^s dt\, t^{p-1} \left[e^{-t} - \sum_{n=0}^N \frac{(-1)^n t^n}{n!}\right],$$

$$I_3 \equiv \int_0^\infty dt\, t^{p-1} \left[e^{-t} - \sum_{n=0}^N \frac{(-1)^n t^n}{n!}\right],$$

and where N is the largest integer less than $|p|$. Integrals I_1 and I_2 can be evaluated term-by-term. Integral I_1 becomes

$$I_1 = \sum_{n=0}^N \frac{(-1)^n}{n!} \int_s^\infty dt\, t^{p+n-1} = -\sum_{n=0}^N \frac{(-1)^n s^{p+n}}{n!\,(p+n)}. \tag{A.5}$$

The integral in Eq. (A.5) converges because $-1 < (p+N) < 0$ demands $(p+n) < 0$ for $0 \le n \le N$. Using the series expansion for e^{-t} and then combining the sums in I_2, integral I_2 can be written as

$$I_2 = \sum_{n=N+1}^\infty \frac{(-1)^n}{n!} \int_0^s dt\, t^{p+n-1} = \sum_{n=N+1}^\infty \frac{(-1)^n s^{p+n}}{n!\,(p+n)}. \tag{A.6}$$

The integral in Eq. (A.6) converges because $(p+n) > 0$ for $N+1 \le n \le \infty$.

Substituting the results of Eqs. (A.5) and (A.6) into Eq. (A.4) yields

$$\Gamma(p,s) = I_3 - \sum_{n=0}^\infty \frac{(-1)^n s^{p+n}}{n!\,(p+n)}. \tag{A.7}$$

We show that $I_3 = \Gamma(p)$ by integrating I_3 repeatedly by parts. It is simplest to start with the case of $N = 1$, which corresponds to p-wave scattering in an EMDE, where $p = -7/6$. Suppose we integrate I_3 by parts once, then:

$$I_3 = B\Big|_{t=0}^{t=\infty} + \mathcal{I}_3,$$

where the boundary term

$$B = \frac{t^{-7/6}}{(-7/6)} \left[e^{-t} - (1-t)\right], \tag{A.8}$$

$$= \frac{1}{(-7/6)} \sum_{n=2}^\infty \frac{(-1)^n t^{-7/6+n}}{n!}, \tag{A.9}$$

and

$$\mathcal{I}_3 = \frac{1}{(-7/6)} \int_0^\infty dt\, \left(e^{-t} - 1\right) t^{-7/6}. \tag{A.10}$$

The $(1-t)$ term in Eq. (A.8) removes the first two terms from the series expansion of e^{-t}, so we combined it with the series expansion of e^{-t} to produce Eq. (A.9). Equation (A.8) makes it easy to see that B vanishes at the upper limit $t = \infty$: the polynomial terms in Eq. (A.8) vanish at $t = \infty$ because they are both proportional to a negative power of t. Equation (A.9) shows that B will also vanish at $t = 0$ because the sum in Eq. (A.9) starts at $n = 2$, such that the lowest power of t is $(-7/6 + 2)$, which is positive. Thus $B\big|_{t=0}^{t=\infty} = 0$, which implies that $I_3 = \mathcal{I}_3$.

Equation (A.10) for $\mathcal{I}_3$ is easily evaluated by integrating by parts once more: the boundary term vanishes at $t = 0$ and $t = \infty$ leaving a convergent integral

$$I_3 = \frac{1}{(-7/6)(-7/6+1)} \int_0^\infty dt\, t^{(-7/6+2)-1} e^{-t}, \tag{A.11}$$

$$= \frac{\Gamma(-7/6+2)}{(-7/6)(-7/6+1)} = \Gamma\left(-7/6\right). \tag{A.12}$$

Since $(-7/6 + 2) > 0$, the integral in Eq. (A.11) is equal to $\Gamma(-7/6 + 2)$, as defined by Eq. (A.1). The second equality in Eq. (A.12) follows from applying the recursion relation in Eq. (A.3) *two* times to $\Gamma(-7/6+2)$. In summary, to obtain $I_3 = \Gamma(-7/6)$ for $N = 1$, we performed $N+1 = 2$ integrations by parts on I_3 in order to guarantee that I_3 could be expressed in terms of a complete gamma function with positive argument. To this function, we then applied Eq. (A.3) $N+1 = 2$ times to achieve $\Gamma(-7/6)$, which shows that Eq. (A.7) is identical in form to Eq. (A.2) for $p = -7/6$ as long as $\Gamma(-7/6)$ in Eq. (A.7) is given by Eq. (A.3). Therefore, we can apply Eq. (A.2) for $\Gamma(-7/6, s)$ which gives Eq. (12) for $1 - \lambda = -7/6$:

$$\Gamma(-7/6, s) = \Gamma(-7/6) - \frac{s^{-7/6}}{(-7/6)} + \mathcal{O}(s^{-1/6}).$$

The iterative procedure outlined above for $p = -7/6$ and $N = 1$ generalizes to arbitrary non-integer values of $p < 0$. No matter how negative such a value of p is, integrating I_3 by parts $N+1$ times generates a $\Gamma(p+N+1)$, where $(p+N+1) > 0$, and the boundary term always vanishes at $t = \infty$ and $t = 0$ for reasons similar to those stated above: the $t^{-7/6}(1-t)$ term in Eq. (A.8) generalizes to a contribution that is proportional to $\sum_0^N t^{p+n}$,

which vanishes at $t = \infty$ since $(p + n)$ remains negative for all $0 \leq n \leq N$, and the sum in Eq. (A.9) vanishes at $t = 0$ because it attains a new lower limit such that the smallest power of t in Eq. (A.9) becomes $(p + N + 1)$, which is positive. Then, Eq. (A.3) can be applied to $\Gamma(p + N + 1)$ $N + 1$ times to produce $I_3 = \Gamma(p)$. Therefore, Eq. (A.2) provides a series expansion for $\Gamma(p, s)$ if $p < 0$ and non-integer, provided that $\Gamma(p)$ is defined by Eq. (A.3).

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