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Second-order dissipative hydrodynamics for plasma with chiral asymmetry and vorticity

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By making use of the chiral kinetic theory in the relaxation-time approximation, we derive an Israel–Stewart type formulation of the hydrodynamic equations for a chiral relativistic plasma made of neutral particles (e.g., neutrinos). The effects of chiral asymmetry are captured by including an additional continuity equation for the axial charge, as well as the leading-order quantum corrections due to the spin of particles. In a formulation of the chiral kinetic theory used, we introduce a symmetric form of the energy-momentum tensor that is suitable for the description of a weakly nonuniform chiral plasma. By construction, the energy and momentum are conserved to the same leading order in the Planck constant as the kinetic equation itself. By making use of such a chiral kinetic theory and the Chapman-Enskog approach, we obtain a set of second-order dissipative hydrodynamic equations. The effects of the fluid vorticity and velocity fluctuations on the dispersion relations of chiral vortical waves are analyzed.

I. INTRODUCTION

The concept of spin of elementary particles has been known for almost a century now. It has vast practical applications in science and technology. For massive particles, the spin is defined as the intrinsic angular momentum of the particle in its rest frame. For massless particles, such a frame is absent, and the spin can be defined indirectly via the particle's helicity as the spin projection on the particle's momentum. In the case of massless fermions, one can also use the concept of chirality instead of helicity. In fact, chirality and helicity are same for particles (positive-energy states) and opposite of each other for antiparticles (negative-energy states). In a classical theory, the chirality of massless fermions is a conserved charge (quantum number). However, the chiral charge conservation is anomalous [1, 2] and, thus, cannot be enforced in a quantum theory.

In recent years, there was a surge of interest in chiral relativistic plasmas, in which chirality and/or chiral structure play a fundamental role. Theoretical studies of such plasmas revealed a number of unusual phenomena, including the chiral magnetic [3], chiral separation [4] and chiral vortical [5–7] effects among others. Their applications range from a possible generation of primordial magnetic fields in cosmology [8–11] to observable correlations of charged particle created in heavy-ion collisions [12–15], to unusual transport properties of Dirac/Weyl materials in condensed matter physics [16]. For recent reviews, see Refs. [17–21].

In addition to the first-principles quantum-field theoretical methods in studies of chiral relativistic plasmas, several quasiclassical approaches were proposed as well. They include the chiral kinetic theory [22–26] and chiral hydrodynamics [27–29]. In particular, in the kinetic theory, which has an intermediate status between the microscopic approach and hydrodynamics, a chiral plasma is described in terms of a one-particle distribution function $f(x,p)$ in the phase-space spanned by spatial coordinates and momenta. The fact that such a description may be possible for a plasma of massless fermions is interesting by itself. It is even more amazing, however, that the corresponding framework reproduces exactly the quantum chiral anomaly.

The chiral anomaly is also taken into account in chiral hydrodynamics [27–29], which describes local properties of plasma in terms of its conserved charge densities and the energy-momentum tensor. Among the three approaches discussed above, hydrodynamics is the least detailed one. Also, its range of validity is limited to the states of matter in the vicinity of equilibrium. Often, however, this is the most efficient and practical framework for the description of physical properties of matter in the long-wavelength limit.

One of the most difficult tasks in a hydrodynamic description is the inclusion of dissipative effects. This is a particularly sensitive issue in the case of relativistic hydrodynamics, where a naïve use of the gradient expansion is in conflict with the causality of the theory [30–32]. The problem can be resolved by inclusion of higher moments of the distribution function [33, 34] beyond the basic hydrodynamic variables (e.g., the density of matter, energy density, and fluid velocity) [35]. The second-order theory by Israel and Stewart [36, 37], which employs additional purely damped degrees of freedom, solves the acausality problem and is widely used in the analysis of relativistic hydrodynamic systems. In principle, the dissipative form of hydrodynamics for chiral plasma can be derived from the chiral kinetic theory. Several generic algorithms for building such a theory are well known [38]. They utilize the definitions of conserved charges in the kinetic theory together with the gradient expansion for the distribution function in order to derive a consistent set of equations for hydrodynamic quantities. In order to obtain a closed set of equations, however, certain approximations are usually required. For example, one can use the moment expansion [39] or the Chapman-Enskog method [40] and truncate the expansion at a given finite number of moments and gradients.

In this paper, we will derive a closed set of dissipative hydrodynamic equations for a relativistic plasma by using the Chapman-Enskog method and following a truncation method similar to that in Refs. [41–44], but paying a special attention to the effects due to chiral asymmetry and fluid vorticity.[1] In this approach, instead of the 14-moment approximation of the original Israel–Stewart theory, an iterative solution of the Boltzmann equation is employed in order to derive the dissipative evolution equations. In particular, one of the novel features of our analysis will be the inclusion of the effects associated with a fluid vorticity in a chiral plasma.

The paper is organized as follows. The key details of the chiral kinetic theory in the relaxation-time approximation are presented in Sec. II. The dissipative hydrodynamic equations are derived in Sec. III. In Sec. IV, we discuss several types of solutions in the form of attenuated propagating waves that involve the oscillations of chirality. The summary of the main results and general conclusions are given in Sec. V. Some technical details and derivations are presented in Appendix A. In this paper we use units in which the speed of light is $c = 1$.

II. CHIRAL KINETIC THEORY FOR PLASMA WITH NON-UNIFORM FLOW

The starting point in our analysis is the chiral kinetic theory [22–26] in the relaxation-time approximation. The governing equations of such a theory are generalized Boltzmann equations for the distribution functions of chiral (Weyl) fermions. The corresponding particles can be of the left-hand ($\lambda = -1$) or right-hand chirality ($\lambda = +1$). Also, in view of the relativistic nature of the system, the semi-classical framework at hand will not be complete without introducing both particles ($\chi = +1$) and antiparticles ($\chi = -1$) as independent species. In general, therefore, the chiral kinetic theory has four different distribution functions for the description of all four species of particles, $f_{\lambda,\chi}(x, p)$, where $\lambda = \pm 1$ is the chirality and $\chi = \pm 1$ is the sign of energy. In order to simplify the notation, we will suppress the indices λ and χ in most formulas below.

As already stated, we will use the chiral kinetic theory with a relaxation-time collision term as a starting point in the derivation of the second-order dissipative chiral hydrodynamics. The relativistic form of the relaxation-time approximation was developed in Ref. [46]. However, it should be noted that the chiral kinetic theory in the relaxation-time approximation, while providing a great toy model, appears to be in conflict with the Lorentz covariance of the theory. In fact, it is argued in Ref. [47] that a collision term consistent with the Lorentz covariance should necessarily be nonlocal. For the purposes of this study, however, we will ignore this deficiency of the relaxation-time approximation in order to explore the structure of the theory in the simplest possible framework.

For the purposes of this study, we require that the chiral kinetic equation be valid up to the linear order in $\hbar$ (or, equivalently, in spin). While such a form was proposed in Ref. [47], it has to be recast in a format that allows one to describe a plasma with a spatially inhomogeneous flow velocity. By implementing the relaxation-time approximation as in Ref. [46], we write the kinetic equation in the following form:

$$p^\mu \partial_\mu f + (\partial_\mu S^{\mu\nu})\partial_\nu f = -\frac{p\cdot u}{\tau}(f - f_{\rm eq}), \tag{1}$$

where p^μ is the four-momentum of the particle, u^μ is the time-like four-velocity of the local plasma flow (by assumption, $u^\mu u_\mu = 1$), and τ is the relaxation time. The spin tensor $S^{\mu\nu}$ [47] and the equilibrium distribution function $f_{\rm eq}(x, p)$ are defined as follows:

$$S^{\mu\nu} = \lambda\frac{\hbar}{2}\frac{\varepsilon^{\mu\nu\alpha\beta}p_\alpha u_\beta}{p\cdot u}, \tag{2}$$

$$f_{\rm eq} = \frac{1}{1 + e^{\beta(\varepsilon_{p,\rm eq} - \chi\mu_\lambda)}}. \tag{3}$$

As is easy to check, the only nonvanishing components of the spin tensor in the local rest frame of the fluid are the spatial components: $S^{ij} = \lambda\hbar\varepsilon^{ijk}p^k/2|\mathbf{p}|$. Note that, in a general frame determined by the four-velocity u^μ, the antisymmetric spin tensor satisfies the following relations: $u_\mu S^{\mu\nu} = p_\mu S^{\mu\nu} = 0$.

The equilibrium distribution function (3) is defined in terms of the local values of the temperature $T \equiv 1/\beta$ and the chiral chemical potentials μ_λ. (Instead of using the chemical potentials $\mu_\lambda = \mu + \lambda\mu_5$, it may be also convenient

[1] A different approach to study the role of spin polarization in relativistic plasmas was presented in Ref. [45].

to use the number-density and axial-charge density chemical potentials μ and μ_5, respectively.) In this formalism, the dispersion relation for a chiral fermion is given by

$$\varepsilon_{p,\mathrm{eq}} = \chi p^\mu u_\mu + \lambda \frac{\hbar}{2} \frac{p\cdot\omega}{p\cdot u}, \tag{4}$$

where the last term accounts for the spin contribution to the particle's energy connected with a nonzero vorticity of the flow $\omega^\mu \equiv \frac{1}{2}\varepsilon^{\mu\alpha\beta\gamma}u_\alpha\partial_\beta u_\gamma$. Here we use the conventional notation $\varepsilon^{\mu\nu\alpha\beta}$ for the four-dimensional Levi-Civita symbol. In the fluid rest frame, the vorticity takes its usual non-relativistic form: $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\mathbf{u}/2$.

In terms of the distribution function, the fermion-number and the axial-charge current densities are defined by the following expressions [47]:

$$j^\mu = \sum_\lambda \int \left(p^\mu f + S^{\mu\nu}\partial_\nu f\right), \tag{5}$$

$$j_5^\mu = \sum_\lambda \lambda \int \left(p^\mu f + S^{\mu\nu}\partial_\nu f\right), \tag{6}$$

where we introduced the shorthand notation for the Lorentz-invariant momentum integration and the particle-antiparticle summation over χ

$$\int F(p_0,\mathbf{p}) = \sum_\chi \int \frac{d^4p}{(2\pi)^3} 2\delta(p^\mu p_\mu)\theta(\chi p^0)F(p_0,\mathbf{p}) = \sum_\chi \int \frac{d^3\mathbf{p}}{(2\pi)^3|\mathbf{p}|}F(\chi|\mathbf{p}|,\mathbf{p}). \tag{7}$$

It should be noted that, in addition to the usual orbital (or convective) part described by the first term in Eqs. (5) and (6), the definition of currents also contains a magnetization contribution connected with the spin. In the fluid rest frame, the latter for the fermion-number current takes the standard form of the curl of magnetization [24, 25], i.e., $\boldsymbol{\nabla}\times\boldsymbol{\mathcal{M}}$, where $\boldsymbol{\mathcal{M}} \equiv \frac{\hbar}{2}\sum_\lambda \lambda \int \hat{\mathbf{p}} f$ and $\hat{\mathbf{p}} \equiv \mathbf{p}/|\mathbf{p}|$.

In this study, we neglect the correction to the current densities associated with the so-called side jumps during the collisions [47]. There are two reasons for this. Firstly, it is not clear whether such a correction is meaningful and how to account it in the kinetic equation when the relaxation-time approximation is used. Secondly, we assume that the relaxation time τ is rather large and, thus, the current corrections due to the side jumps are small.

In terms of the particle distribution function, the energy-momentum tensor is defined as follows:

$$T^{\mu\nu} = \sum_\lambda \int \left(p^\mu p^\nu f + \frac{1}{2}p^\mu S^{\nu\alpha}\partial_\alpha f + \frac{1}{2}p^\nu S^{\mu\alpha}\partial_\alpha f\right). \tag{8}$$

It is important to note that, unlike the case of the currents, there is no room for an "axial" counterpart of the energy-momentum tensor. In the context of hydrodynamics, as we will see later, this is intimately connected with the fact that the two chiral components of the plasma should have the same temperature, even if they are characterized by different chemical potentials.

The definition in Eq. (8) is a straightforward generalization of the energy-momentum tensor introduced in Ref. [24] to the case of a plasma with non-uniform flow. Just like the charge density, the energy-momentum tensor (8) contains both orbital and spin contributions. By construction, this tensor is manifestly symmetric. Such a symmetric form of the tensor appears natural because the kinetic theory can be viewed as a semiclassical approximation to the microscopic quantum-field theoretical description, in which the corresponding tensor can be always symmetrized. A symmetric form of the energy-momentum tensor is also the most "physical" from the viewpoint of general relativity, see Ref. [48]. It is interesting to mention, however, that hydrodynamics may allow for a nonzero antisymmetric part in the energy-momentum tensor (i.e., the torque tensor) which can be connected, for example, with the spin [49]. While we do not explore such a possibility here, it is intriguing to suggest that a spin-related torque tensor might be induced in a chiral plasma made of Weyl fermions.

III. HYDRODYNAMIC EQUATIONS

In this section, we derive a closed set of hydrodynamic equations for an inhomogeneous chiral plasma slightly out of local equilibrium. The corresponding local state is described in terms of the hydrodynamic variables T, μ_λ, and u^μ. Within the chiral kinetic theory, which is the starting point in our derivation, these variables are sufficient to specify the local equilibrium distribution function (3).

In essence, the hydrodynamic equations are the continuity equations for conserved quantities, such as the energy and momentum, as well as various conserved charges. In order to close the corresponding system of equations, one should also add a number of constitutive relations. One of such relations is the equation of state that relates the energy density ϵ with the pressure P of the fluid. In a relativistic plasma at hand, the latter is given by $P = \epsilon/3$. [Note that the corresponding equation of state also follows from the definition of the energy-momentum tensor in Eq. (8) combined with its representation in terms of the pressure and energy density, see Eq. (11) below.]

The hydrodynamic equations in a chiral plasma are governed by continuity equations for the current densities j^μ and j_5^μ, as well as the energy-momentum tensor $T^{\mu\nu}$. Before deriving the equations for j^μ, j_5^μ, and $T^{\mu\nu}$, however, let us first discuss their vector/tensor structure. As usual, we will decompose these quantities using projections onto the Lorentz subspaces parallel and perpendicular to the four-velocity u^μ. From a physics viewpoint, there are several possibilities for the choice of the four-velocity u^μ connected, e.g., with the energy flow (Landau frame) or the particle flow (Eckart frame). While we will keep u^μ arbitrary for now, later we will see that the consistency of our hydrodynamic equations will single out a modified version of the Landau frame [46]. The latter for particles with spin may differ from the usual Landau frame by corrections of order $\hbar$. In general, the current densities j^μ and j_5^μ, and the energy-momentum tensor $T^{\mu\nu}$ have the following decompositions:

$$j^\mu = n u^\mu + \nu^\mu, \tag{9}$$

$$j_5^\mu = n_5 u^\mu + \nu_5^\mu, \tag{10}$$

$$T^{\mu\nu} = \epsilon u^\mu u^\nu - \Delta^{\mu\nu} P + (h^\mu u^\nu + u^\mu h^\nu) + \pi^{\mu\nu}, \tag{11}$$

where $n \equiv u_\mu j^\mu$ and $n_5 \equiv u_\mu j_5^\mu$ are the fermion-number and axial-charge densities, $\nu^\mu \equiv \Delta^{\mu\nu} j_\nu$ and $\nu_5^\mu \equiv \Delta^{\mu\nu} j_{5,\nu}$ are the corresponding diffusion current densities perpendicular to the fluid four-velocity [39, 43, 44], $\epsilon \equiv u_\mu u_\nu T^{\mu\nu}$ is the energy density, $P \equiv -\Delta_{\mu\nu} T^{\mu\nu}/3$ is the thermodynamic pressure, $h^\mu \equiv \Delta^\mu_\alpha u_\beta T^{\alpha\beta}$ is the momentum density or the energy-flow density, and $\pi^{\mu\nu} \equiv \Delta^{\mu\nu}_{\alpha\beta} T^{\alpha\beta}$ is the shear stress tensor. By definition, the projector onto the subspace perpendicular to the fluid four-velocity is $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$, and the traceless symmetric projector used in the definition of the shear stress tensor is $\Delta^{\mu\nu}_{\alpha\beta} \equiv (1/2)\Delta^\mu_\alpha \Delta^\nu_\beta + (1/2)\Delta^\mu_\beta \Delta^\nu_\alpha - (1/3)\Delta^{\mu\nu}\Delta_{\alpha\beta}$.

It should be noted that, the viscous pressure Π, which would shift the thermodynamic pressure $P \to P + \Pi$ in Eq. (11), is absent. From a physics viewpoint, such a correction would capture the effects of bulk viscosity. In the case of a nearly scale-invariant chiral plasma of (massless) fermions, however, the approximation with $\Pi = 0$ is well justified. Indeed, the bulk viscosity vanishes in scale-invariant theories. Of course, in realistic models such as high-temperature QCD, the corresponding property is not exact because of subtle quantum effects. Nevertheless, as direct calculations in Ref. [50] show, the bulk viscosity is negligible compared to shear viscosity. Therefore, in the following we will completely ignore the effects due to the viscous pressure.

In order to derive a closed system of hydrodynamic equations from chiral kinetic theory, we use the approximation similar to that used in Refs. [37, 39, 43, 44]. The relevant moments are ϵ, n, n_5, P, ν^μ, ν_5^μ, h^μ, and $\pi^{\mu\nu}$. (Because of the spin contributions, strictly speaking, these quantities are not simple moments of the distribution function.) In the kinetic theory, the corresponding quantities can be expressed in terms of the particle distribution function, using the definitions for the current densities and the energy-momentum tensor in Eqs. (5) and (8).

By making use of the general decomposition of the current densities (9) and (10), and the energy-momentum tensor (11) in terms of independent moments, the continuity equations $\partial_\mu j^\mu = 0$, $\partial_\mu j_5^\mu = 0$, and $\partial_\nu T^{\mu\nu} = 0$ take the following form:

$$\dot{n} + n\partial_\mu u^\mu + \partial_\mu \nu^\mu = 0, \tag{12}$$

$$\dot{n}_5 + n_5\partial_\mu u^\mu + \partial_\mu \nu_5^\mu = 0, \tag{13}$$

$$\dot{\epsilon} + (\epsilon + P)\partial_\mu u^\mu + \partial_\mu h^\mu + u_\mu \dot{h}^\mu - \pi^{\mu\nu}\partial_\mu u_\nu = 0, \tag{14}$$

$$(\epsilon + P)\dot{u}^\alpha - \nabla^\alpha P + h^\mu \partial_\mu u^\alpha + h^\alpha(\partial_\mu u^\mu) + \Delta^\alpha_\mu \dot{h}^\mu + \Delta^\alpha_\mu \partial_\nu \pi^{\mu\nu} = 0, \tag{15}$$

where we introduced the comoving and transverse derivatives as $\dot{A} \equiv u^\mu \partial_\mu A$ and $\nabla^\alpha P \equiv \Delta^{\alpha\mu}\partial_\mu P$, respectively. In dissipative regime, Eqs. (12)–(15) are not sufficient to describe unambiguously the hydrodynamic behavior of plasma. These equations should be supplemented by the equations for functions ν^μ, ν_5^μ, h^μ, and $\pi^{\mu\nu}$. As is clear, their derivation would be impossible without additional information about the microscopic processes responsible for dissipative effects. In this study, the corresponding details will be supplied by the chiral kinetic theory in the relaxation-time approximation.

Before attempting to derive the equations that govern the dynamics of dissipative functions, we should first discuss the generic constraints that the chiral kinetic theory imposes on the hydrodynamic variables T, μ_λ, and u^μ. The corresponding variables determine the equilibrium distribution function, see Eq. (3), and thus define the local state of equilibrium in plasma. It remains to determine, however, the connection between these variables and the out-of-equilibrium hydrodynamic functions that satisfy Eqs. (12)–(15). The needed relations can be established by analyzing the continuity equations, i.e., $\partial_\mu j^\mu = 0$, $\partial_\mu j_5^\mu = 0$, and $\partial_\nu T^{\mu\nu} = 0$, for j^μ, j_5^μ, and $T^{\mu\nu}$ given by Eqs. (5), (6), and (8), respectively, in the framework of the chiral kinetic theory where the kinetic equation (1) plays the role of a microscopic equation.

Let us first discuss the implication of the continuity equations for the current densities. By making use of the definitions in Eqs. (5) and (6), it is straightforward to derive the following relation:

$$\partial_\mu j^\mu = -\frac{1}{\tau}(n - n_{\rm eq}), \tag{16}$$

$$\partial_\mu j_5^\mu = -\frac{1}{\tau}(n_5 - n_{5,\rm eq}), \tag{17}$$

where we used the definitions for the densities $n \equiv \sum_\lambda \int (p\cdot u) f$ and $n_5 \equiv \sum_\lambda \lambda \int (p\cdot u) f$. The explicit expressions for the corresponding quantities in equilibrium are obtained by making use of the equilibrium distribution function in Eq. (3), i.e.,

$$n_{\rm eq} \equiv \sum_\lambda \left(\frac{\mu_\lambda^3}{6\pi^2} + \frac{\mu_\lambda T^2}{6} \right) = \frac{\mu\left(\mu^2 + 3\mu_5^2 + \pi^2 T^2\right)}{3\pi^2}, \tag{18}$$

$$n_{5,\rm eq} \equiv \sum_\lambda \lambda \left(\frac{\mu_\lambda^3}{6\pi^2} + \frac{\mu_\lambda T^2}{6} \right) = \frac{\mu_5\left(\mu_5^2 + 3\mu^2 + \pi^2 T^2\right)}{3\pi^2}. \tag{19}$$

As follows from Eqs. (16) and (17), the requirements of the fermion-number and axial-charge conservation, $\partial_\mu j^\mu = 0$ and $\partial_\mu j_5^\mu = 0$, give

$$n = n_{\rm eq}, \qquad\qquad n_5 = n_{5,\rm eq}. \tag{20}$$

These equations can be interpreted as definitions of the out-of-equilibrium charge densities in terms of given local values of the chemical potentials μ_λ, or vice versa, as the equations that define μ_λ in terms of the local charge densities n and n_5.

By applying the same method to the definition in Eq. (8), we can also calculate the divergence of the energy-momentum tensor in the chiral kinetic theory. The corresponding details are presented in Appendix A 1. The final result reads

$$\partial_\nu T^{\mu\nu} = -\frac{u^\mu}{\tau}\left(\epsilon - \epsilon_{\rm eq} + \frac{\hbar}{2}\omega_\alpha(\nu_5^\alpha - \nu_{5,\rm eq}^\alpha)\right) - \frac{1}{\tau}\left(h^\mu - h^\mu_{\rm eq} - \frac{\hbar}{4}\epsilon^{\mu\alpha\beta\gamma}u_\alpha \dot{u}_\beta(\nu_{5,\gamma} - \nu_{5,{\rm eq},\gamma})\right) + O(\hbar^2), \tag{21}$$

where we used Eq. (20). Due to the chiral vortical effect, the equilibrium axial-charge current density is nonzero in the presence of a background vorticity, i.e., $\nu^\mu_{5,\rm eq} \propto \hbar\omega^\mu$. Considering that the corresponding result is already linear in $\hbar$, it contributes to the right-hand side of Eq. (21) only at the quadratic order in $\hbar$. Therefore, while we will formally keep the equilibrium current density in similar expressions below, it would be consistent to set $\nu^\mu_{5,\rm eq} \simeq 0$ there.

By separating the two independent projections with respect to the four-velocity u^μ and enforcing the continuity equation $\partial_\nu T^{\mu\nu} = 0$, we then arrive at the following constraints:

$$\epsilon + \frac{\hbar}{2}\omega_\mu \nu_5^\mu = \epsilon_{\rm eq} + \frac{\hbar}{2}\omega_\mu \nu^\mu_{5,\rm eq}, \tag{22}$$

$$h^\mu - \frac{\hbar}{4}\epsilon^{\mu\alpha\beta\gamma}u_\alpha \dot{u}_\beta \nu_{5,\gamma} = h^\mu_{\rm eq} - \frac{\hbar}{4}\epsilon^{\mu\alpha\beta\gamma}u_\alpha \dot{u}_\beta \nu_{5,{\rm eq},\gamma}, \tag{23}$$

where, by definition, the equilibrium quantities are given by

$$\nu^\mu_{\rm eq} \equiv \hbar\omega^\mu \sum_\lambda \lambda \left(\frac{\mu_\lambda^2}{4\pi^2} + \frac{T^2}{12} \right) = \frac{\mu\mu_5}{\pi^2}\hbar\omega^\mu, \tag{24}$$

$$\nu^\mu_{5,\rm eq} \equiv \hbar\omega^\mu \sum_\lambda \left(\frac{\mu_\lambda^2}{4\pi^2} + \frac{T^2}{12} \right) = \frac{3(\mu^2 + \mu_5^2) + \pi^2 T^2}{6\pi^2}\hbar\omega^\mu, \tag{25}$$

$$\epsilon_{\rm eq} \equiv \sum_\lambda \left(\frac{\mu_\lambda^4}{8\pi^2} + \frac{\mu_\lambda^2 T^2}{4} + \frac{7\pi^2 T^4}{120} \right) = \frac{\mu^4 + 6\mu^2\mu_5^2 + \mu_5^4}{4\pi^2} + \frac{T^2}{2}(\mu^2 + \mu_5^2) + \frac{7\pi^2 T^4}{60}, \tag{26}$$

$$h^\mu_{\rm eq} \equiv \hbar\omega^\mu n_{5,\rm eq} = \frac{\mu_5\left(\mu_5^2 + 3\mu^2 + \pi^2 T^2\right)}{3\pi^2}\hbar\omega^\mu. \tag{27}$$

The constraint in Eq. (22) for the energy density is analogous to the relations in Eq. (20). This is most evident in the limit of vanishing vorticity or in the absence of axial current density, when the second term on each side of Eq. (22)

is trivial. Then, the equation $\epsilon = \epsilon_{\rm eq}$ allows one to define the out-of-equilibrium energy density in terms of a given local value of the temperature T or, alternatively, to determine the value of T in terms of the local energy density ϵ. Because of the sum over λ, Eq. (22) gives a single relation that allows one to determine only one (common) local temperature T for both chiral components of the plasma. This is a natural consequence of the definition for the energy-momentum tensor in Eq. (8), which includes the contributions of both chiralities.

Now let us turn to the second constraint, given by Eq. (23). It can be viewed as the relation that defines the four-velocity u^μ of a generalized Landau frame, in which the hydrodynamic equations derived from the kinetic theory are consistent with the momentum conservation. This agrees with a similar constraint (i.e., $h^\mu = 0$) obtained in a model without chiral asymmetry in Refs. [43, 46]. Indeed, after taking into account that all $\hbar$-terms in Eq. (23) vanish at $n_5 = 0$ and $\nu_5^\mu = 0$, we obtain the standard relation that defines the Landau frame: $h^\mu = 0$. Because of the $\hbar$-corrections, however, our definition of the generalized Landau frame in Eq. (23) is different. It would be interesting to investigate whether the underlying reason for the difference is related to the subtleties of defining a thermodynamically preferred frame in Ref. [51].

It is interesting that constraints (22) and (23) contain corrections of order $\hbar$ when there is a nonzero background vorticity. In essence, the corresponding corrections to the energy and momentum densities are the consequences of chirality, which locks the orbital motion of particles with their spin.

Even after taking into account Eqs. (20), (22), and (23), we still need to determine the equations that govern the evolution of the dissipative functions ν^μ and $\pi^{\mu\nu}$. Here we will follow the approach similar to that in Ref. [43] in order to derive the missing equations. We start by rewriting the kinetic equation in the following form:

$$\dot{f} + \frac{f}{\tau} = \frac{f_{\rm eq}}{\tau} - \frac{1}{p\cdot u} p^\rho \nabla_\rho f - \frac{1}{p\cdot u}(\partial_\sigma S^{\sigma\rho})\partial_\rho f. \tag{28}$$

Using the definitions for the dissipative functions, $\nu^\mu = \Delta^\mu_\nu j^\nu$, $\nu^\mu_5 = \Delta^\mu_\nu j^\nu_5$, and $\pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta} T^{\alpha\beta}$, we can express their comoving derivatives in the following form:

$$\dot{\nu}^{\langle\mu\rangle} = -\dot{u}^\mu n + \Delta^\mu_\nu \sum_\lambda \int (p^\nu + S^{\nu\alpha}\partial_\alpha)\, \dot{f} - \Delta^\mu_\nu \sum_\lambda \int S^{\nu\alpha}(\partial_\alpha u^\beta)\partial_\beta f + \Delta^\mu_\nu \sum_\lambda \int \dot{S}^{\nu\alpha}\partial_\alpha f, \tag{29}$$

$$\dot{\nu}_5^{\langle\mu\rangle} = -\dot{u}^\mu n_5 + \Delta^\mu_\nu \sum_\lambda \lambda \int (p^\nu + S^{\nu\alpha}\partial_\alpha)\, \dot{f} - \Delta^\mu_\nu \sum_\lambda \lambda \int S^{\nu\alpha}(\partial_\alpha u^\beta)\partial_\beta f + \Delta^\mu_\nu \sum_\lambda \lambda \int \dot{S}^{\nu\alpha}\partial_\alpha f, \tag{30}$$

$$\begin{aligned}\dot{\pi}^{\langle\mu\nu\rangle} &= -2\Delta^{\mu\nu}_{\alpha\beta} h^\alpha \dot{u}^\beta + \Delta^{\mu\nu}_{\alpha\beta} \sum_\lambda \int \left(p^\alpha p^\beta + p^\alpha S^{\beta\gamma}\partial_\gamma\right) \dot{f} - \Delta^{\mu\nu}_{\alpha\beta} \sum_\lambda \int p^\alpha S^{\beta\gamma}(\partial_\gamma u^\delta)\partial_\delta f \\ &\quad + \Delta^{\mu\nu}_{\alpha\beta} \sum_\lambda \int p^\alpha \dot{S}^{\beta\gamma}\partial_\gamma f,\end{aligned} \tag{31}$$

where, by definition, the quantities with the Lorentz indices in angle brackets are the projections of the corresponding quantities onto the subspace orthogonal to the four-velocity, i.e., $\dot{\nu}^{\langle\mu\rangle} \equiv \Delta^\mu_\alpha \dot{\nu}^\alpha$ and $\dot{\pi}^{\langle\mu\nu\rangle} \equiv \Delta^{\mu\nu}_{\alpha\beta}\dot{\pi}^{\alpha\beta}$. The use of projectors here is needed in order to force the dissipative current densities and the shear stress tensor to remain consistent with their generic definitions. This can be also viewed as a necessary condition for a self-consistent truncation of the evolution equations.

By making use of the kinetic equation (28), the relations for the comoving derivatives Eqs. (29)–(31) can be equivalently rewritten as follows:

$$\begin{aligned}\dot{\nu}^{\langle\mu\rangle} + \frac{\nu^\mu}{\tau} &= -\dot{u}^\mu n + \sum_\lambda \int (\Delta^\mu_\nu p^\nu + S^{\mu\nu}\partial_\nu) \left(\frac{f_{\rm eq}}{\tau} - \frac{1}{p\cdot u} p^\rho \nabla_\rho f - \frac{1}{p\cdot u}(\partial_\sigma S^{\sigma\rho})\partial_\rho f\right) \\ &\quad - (\partial_\nu u^\rho) \sum_\lambda \int S^{\mu\nu}\partial_\rho f + \Delta^\mu_\rho \sum_\lambda \int \dot{S}^{\rho\nu}\partial_\nu f,\end{aligned} \tag{32}$$

$$\begin{aligned}\dot{\nu}_5^{\langle\mu\rangle} + \frac{\nu_5^\mu}{\tau} &= -\dot{u}^\mu n_5 + \sum_\lambda \lambda \int (\Delta^\mu_\nu p^\nu + S^{\mu\nu}\partial_\nu) \left(\frac{f_{\rm eq}}{\tau} - \frac{1}{p\cdot u} p^\rho \nabla_\rho f - \frac{1}{p\cdot u}(\partial_\sigma S^{\sigma\rho})\partial_\rho f\right) \\ &\quad - (\partial_\nu u^\rho) \sum_\lambda \lambda \int S^{\mu\nu}\partial_\rho f + \Delta^\mu_\rho \sum_\lambda \lambda \int \dot{S}^{\rho\nu}\partial_\nu f,\end{aligned} \tag{33}$$

$$\begin{aligned}\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau} &= -2\Delta^{\mu\nu}_{\alpha\beta} h^\alpha \dot{u}^\beta + \Delta^{\mu\nu}_{\alpha\beta} \sum_\lambda \int \left(p^\alpha p^\beta + p^\alpha S^{\beta\gamma}\partial_\gamma\right) \left(\frac{f_{\rm eq}}{\tau} - \frac{1}{p\cdot u} p^\rho \nabla_\rho f - \frac{1}{p\cdot u}(\partial_\sigma S^{\sigma\rho})\partial_\rho f\right) \\ &\quad - (\partial_\gamma u^\rho) \sum_\lambda \Delta^{\mu\nu}_{\alpha\beta} \int p^\alpha S^{\beta\gamma}\partial_\rho f + \Delta^{\mu\nu}_{\alpha\beta} \sum_\lambda \int p^\alpha \dot{S}^{\beta\gamma}\partial_\gamma f.\end{aligned} \tag{34}$$

These equations for dissipative functions contain the distribution function f. In order to obtain a closed set of equations, the right-hand sides of the equations above should be reexpressed in terms of the hydrodynamic variables and dissipative functions. To achieve this, we replace the distribution function with its iterative solution in the form

$$f \simeq f_{\rm eq} - \frac{\tau}{p\cdot u} p\cdot\partial f_{\rm eq} \tag{35}$$

and further approximate the equilibrium distribution function by its expansion to the linear order in $\hbar$,

$$f_{\rm eq} \simeq f_0 + \lambda\frac{\hbar}{2}\frac{p^\mu\omega_\mu}{p\cdot u} f_0' + \ldots, \tag{36}$$

where f_0 is the equilibrium function at a vanishing vorticity and $f_0' \equiv \partial f_0/\partial\varepsilon_p$. Now, by using the moments of the equilibrium distribution function from Appendix A 2, we rewrite the evolution equations for the dissipative functions in the following form:

$$\begin{aligned}
\dot{\nu}^{\langle\mu\rangle} + \frac{\nu^\mu}{\tau} = &-\dot{u}^\mu n + \sum_\lambda \Bigg[\frac{1}{3}\nabla^\mu I_3 + \lambda\frac{\hbar}{\tau}\omega^\mu I_2 \\
&- \frac{2\tau}{5}\nabla^\mu(\partial\cdot u)I_3 - \frac{7\tau}{15}(\partial\cdot u)\nabla^\mu I_3 + \tau\dot{u}^\mu(\partial\cdot u)I_3 + \frac{4\tau}{5}\dot{u}^\rho(\partial_\rho u^\mu)I_3 - \frac{\tau}{3}\nabla^\mu \dot{I}_3 \\
&- \frac{7\tau}{15}(\nabla_\rho u^\mu)\nabla^\rho I_3 - \frac{2\tau}{5}(\nabla^\mu u_\rho)\dot{u}^\rho I_3 - \frac{2\tau}{15}(\nabla^\mu u^\rho)\partial_\rho I_3 - \frac{\tau}{5}\Delta^\mu_\nu(\partial\cdot\partial u^\nu)I_3 + \frac{\tau}{5}\Delta^\mu_\nu \ddot{u}^\nu I_3 \\
&- \lambda\frac{14\hbar}{15}\omega^\mu(\partial\cdot u)I_2 - \lambda\frac{14\hbar}{15}\omega^\nu(\partial_\nu u^\mu)I_2 + \lambda\frac{\hbar}{15}\omega^\nu(\nabla^\mu u_\nu)I_2 - \lambda\frac{2\hbar}{3}\omega^\mu\dot{I}_2 \\
&+ \lambda\frac{\hbar}{6}\varepsilon^{\mu\nu\alpha\beta}u_\alpha\dot{u}_\beta\partial_\nu I_2 - \lambda\frac{\hbar}{3}\varepsilon^{\mu\nu\alpha\beta}u_\beta(\partial_\nu u^\rho)(\partial_\rho u_\alpha)I_2\Bigg],
\end{aligned} \tag{37}$$

$$\begin{aligned}
\dot{\nu}_5^{\langle\mu\rangle} + \frac{\nu_5^\mu}{\tau} = &-\dot{u}^\mu n_5 + \sum_\lambda \lambda\Bigg[\frac{1}{3}\nabla^\mu I_3 + \lambda\frac{\hbar}{\tau}\omega^\mu I_2 \\
&- \frac{2\tau}{5}\nabla^\mu(\partial\cdot u)I_3 - \frac{7\tau}{15}(\partial\cdot u)\nabla^\mu I_3 + \tau\dot{u}^\mu(\partial\cdot u)I_3 + \frac{4\tau}{5}\dot{u}^\rho(\partial_\rho u^\mu)I_3 - \frac{\tau}{3}\nabla^\mu \dot{I}_3 \\
&- \frac{7\tau}{15}(\nabla_\rho u^\mu)\nabla^\rho I_3 - \frac{2\tau}{5}(\nabla^\mu u_\rho)\dot{u}^\rho I_3 - \frac{2\tau}{15}(\nabla^\mu u^\rho)\partial_\rho I_3 - \frac{\tau}{5}\Delta^\mu_\nu(\partial\cdot\partial u^\nu)I_3 + \frac{\tau}{5}\Delta^\mu_\nu \ddot{u}^\nu I_3 \\
&- \lambda\frac{14\hbar}{15}\omega^\mu(\partial\cdot u)I_2 - \lambda\frac{14\hbar}{15}\omega^\nu(\partial_\nu u^\mu)I_2 + \lambda\frac{\hbar}{15}\omega^\nu(\nabla^\mu u_\nu)I_2 - \lambda\frac{2\hbar}{3}\omega^\mu\dot{I}_2 \\
&+ \lambda\frac{\hbar}{6}\varepsilon^{\mu\nu\alpha\beta}u_\alpha\dot{u}_\beta\partial_\nu I_2 - \lambda\frac{\hbar}{3}\varepsilon^{\mu\nu\alpha\beta}u_\beta(\partial_\nu u^\rho)(\partial_\rho u_\alpha)I_2\Bigg],
\end{aligned} \tag{38}$$

$$\begin{aligned}
\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau} = &-2\Delta^{\mu\nu}_{\alpha\beta}h^\alpha\dot{u}^\beta + \Delta^{\mu\nu}_{\alpha\beta}\sum_\lambda\Bigg[\frac{8}{15}(\partial^\alpha u^\beta)I_4 \\
&- \frac{32\tau}{35}(\partial^\alpha u^\beta)(\partial\cdot u)I_4 - \frac{8\tau}{15}\partial^\alpha(\dot{u}^\beta I_4) - \frac{16\tau}{35}(\nabla_\rho u^\alpha)(\nabla^\rho u^\beta)I_4 - \frac{8\tau}{21}(\partial^\alpha u^\rho)(\partial^\rho u^\beta)I_4 \\
&+ \frac{2\tau}{15}\partial^\alpha\partial^\beta I_4 - \frac{2\tau}{3}(\partial^\alpha u^\beta)\dot{I}_4 + \frac{8\tau}{105}(\partial^\alpha u_\rho)(\partial^\beta u^\rho)I_4 \\
&+ \lambda\frac{\hbar}{5}(\partial^\alpha\omega^\beta)I_3 + \lambda\frac{7\hbar}{15}\omega^\alpha\partial^\beta I_3 + \lambda\frac{\hbar}{5}\dot{u}^\alpha\omega^\beta I_3 + \lambda\frac{\hbar}{10}\varepsilon^{\beta\sigma\rho\delta}u_\delta\partial_\sigma(I_3\nabla^\alpha u_\rho) \\
&+ \lambda\frac{\hbar}{10}\varepsilon^{\beta\sigma\rho\delta}u_\delta(\partial_\sigma u^\alpha)\partial_\rho I_3 + \lambda\frac{\hbar}{5}\dot{u}^\alpha\varepsilon^{\beta\sigma\rho\delta}u_\rho(\partial_\sigma u_\delta)I_3 + \lambda\frac{\hbar}{5}\varepsilon^{\beta\sigma\rho\delta}u_\sigma\dot{u}_\rho(\partial_\delta u^\alpha)I_3\Bigg].
\end{aligned} \tag{39}$$

As is easy to check, these equations for dissipative functions are finally sufficient to close the whole system of equations of the second-order dissipative hydrodynamics. Indeed, we have Eqs. (12)–(15) and (23) for hydrodynamic variables n, n_5, ϵ, u^μ, and h^μ. Note also that the thermodynamic pressure is defined by the corresponding constitutive equation, $P=\epsilon/3$. The corresponding equations are supplemented by Eqs. (37)–(39) for functions ν^μ, ν_5^μ, and $\pi^{\mu\nu}$. According to Eqs. (A15)–(A17) in Appendix A 2, quantities I_2, I_3, and I_4 on the right-hand side of Eqs. (37)-(39) are expressed through the local equilibrium chemical potentials μ, μ_5 and temperature T, which in turn could be expressed through the local values of n, n_5, and ϵ, respectively, see the constraints in Eqs. (20) and (22).

The right-hand side of the equations for dissipative functions can be further simplified by making use of the following first-order relations:

$$\nu^\mu = \sum_\lambda \left[\lambda\hbar\omega^\mu I_2 + \frac{\tau}{3}\nabla^\mu I_3 - \tau\dot{u}^\mu I_3\right] + O(\partial^2), \tag{40}$$

$$\nu_5^\mu = \sum_\lambda \lambda \left[\lambda\hbar\omega^\mu I_2 + \frac{\tau}{3}\nabla^\mu I_3 - \tau\dot{u}^\mu I_3\right] + O(\partial^2), \tag{41}$$

$$\pi^{\mu\nu} = \sum_\lambda \frac{8\tau}{15}\Delta^{\mu\nu}_{\alpha\beta}(\partial^\alpha u^\beta) I_4 + O(\partial^2), \tag{42}$$

which follow from Eqs. (37) – (39). [Let us note in passing that the above first-order relations define the diffusion constant and the shear viscosity in terms of the relaxation time: $D = \tau/3$ and $\zeta = 8\tau\epsilon/15$, respectively.] Indeed, by making use of these equations as well as the continuity equations in the leading order in derivatives, we can re-express most of the terms with an explicit dependence on the relaxation time in Eqs. (37)–(39) in terms of the hydrodynamic functions themselves. After doing this, the final set of equations for dissipative functions takes a simpler form, i.e.,

$$\begin{aligned}\dot{\nu}^{\langle\mu\rangle} + \frac{\nu^\mu - \nu^\mu_{\rm eq}}{\tau} &= -\dot{u}^\mu n + \frac{1}{3}\nabla^\mu n - \frac{n}{\epsilon+P}\Delta^{\mu\nu}\partial^\rho\pi_{\rho\nu} - \nu_\rho\omega^{\rho\mu} - (\partial\cdot u)\nu^\mu - \frac{9}{5}(\partial^{\langle\mu}u^{\rho\rangle})\nu_\rho + \frac{14}{15}(\nabla^{\langle\mu}u^{\rho\rangle})\nu_{\rho,\rm eq}\\ &- \frac{2}{9}(\partial\cdot u)\nu^\mu_{\rm eq} - \frac{2\hbar}{3}\omega^\mu\sum_\lambda \lambda\dot{I}_2 + \sum_\lambda \lambda\frac{\hbar}{6}\varepsilon^{\mu\nu\alpha\beta}\left[u_\alpha\dot{u}_\beta\partial_\nu I_2 - 2u_\beta(\partial_\nu u^\rho)(\partial_\rho u_\alpha)I_2\right],\end{aligned} \tag{43}$$

$$\begin{aligned}\dot{\nu}_5^{\langle\mu\rangle} + \frac{\nu_5^\mu - \nu^\mu_{5,\rm eq}}{\tau} &= -\dot{u}^\mu n_5 + \frac{1}{3}\nabla^\mu n_5 - \frac{n_5}{\epsilon+P}\Delta^{\mu\nu}\partial^\rho\pi_{\rho\nu} - \nu_{5,\rho}\omega^{\rho\mu} - (\partial\cdot u)\nu_5^\mu - \frac{9}{5}(\partial^{\langle\mu}u^{\rho\rangle})\nu_{5,\rho} + \frac{14}{15}(\nabla^{\langle\mu}u^{\rho\rangle})\nu_{5,{\rm eq},\rho}\\ &- \frac{2}{9}(\partial\cdot u)\nu^\mu_{5,\rm eq} - \frac{2\hbar}{3}\omega^\mu\sum_\lambda \dot{I}_2 + \sum_\lambda \frac{\hbar}{6}\varepsilon^{\mu\nu\alpha\beta}\left[u_\alpha\dot{u}_\beta\partial_\nu I_2 - 2u_\beta(\partial_\nu u^\rho)(\partial_\rho u_\alpha)I_2\right],\end{aligned} \tag{44}$$

$$\begin{aligned}\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau} &= -2h^{\langle\mu}\dot{u}^{\nu\rangle} + 2\pi_\rho^{\langle\mu}\omega^{\nu\rangle\rho} - \frac{10}{7}\pi_\rho^{\langle\mu}\sigma^{\nu\rangle\rho} - \frac{4}{3}\pi^{\mu\nu}\partial_\alpha u^\alpha + \frac{8}{15}(\partial^{\langle\mu}u^{\nu\rangle})\epsilon\\ &+ \frac{\hbar}{5}\left((\partial^{\langle\mu}\omega^{\nu\rangle})n_5 + \frac{7}{3}\omega^{\langle\mu}\partial^{\nu\rangle}n_5 - \dot{u}^{\langle\mu}\omega^{\nu\rangle}n_5\right)\\ &+ \frac{\hbar}{5}\Delta^{\mu\nu}_{\alpha\beta}\varepsilon^{\beta\sigma\rho\delta}\left[\frac{1}{2}u_\delta\partial_\sigma(n_5\nabla^\alpha u_\rho) + \frac{1}{2}u_\delta(\partial_\sigma u^\alpha)\partial_\rho n_5 + u_\sigma\dot{u}_\rho(\partial_\delta u^\alpha)n_5\right],\end{aligned} \tag{45}$$

where $\sigma^{\mu\nu} = \partial^{\langle\alpha}u^{\beta\rangle} = \Delta^{\mu\nu}_{\alpha\beta}(\partial^\alpha u^\beta)$, $\omega^{\mu\nu} = (\nabla^\mu u^\nu - \nabla^\nu u^\mu)/2$, and $A^{\langle\mu\nu\rangle} \equiv \Delta^{\mu\nu}_{\alpha\beta}A^{\alpha\beta}$. In the derivation, we used the following relation:

$$\frac{1}{\epsilon+P}\Delta^{\mu\nu}\partial^\rho\pi_{\rho\nu} = \frac{\tau}{5}\Delta^{\mu\nu}(\partial^2 u_\nu) - \frac{\tau}{5}\Delta^{\mu\nu}\ddot{u}_\nu + \frac{\tau}{15}\nabla^\mu(\partial\cdot u) - \frac{3\tau}{5}\dot{u}^\mu(\partial\cdot u) + \frac{3\tau}{5}\dot{u}^\rho(\partial_\rho u^\mu) + \frac{4\tau}{5}\dot{u}^\rho(\nabla^\mu u_\rho) + O(\partial^3), \tag{46}$$

which follows from Eq. (42) as well as the first-order continuity equations.

The set of second-order equations (43), (44), and (45) for dissipative functions in a chiral plasma is our main result. This is a generalization of the previous results of Refs. [41–44, 46], which were obtained for massless plasmas without a chiral asymmetry (i.e., $n_5 = 0$ and $\nu_5^\mu = 0$) and without $\hbar$ corrections due to the spin. In the current study, in contrast, we treated the fermion chiralities as two components of a relativistic fluid. The (approximate) conservation of the axial-charge in the chiral plasma gives rise to an additional continuity equation, see Eq. (44). Moreover, the quantum effects of the chiral plasma are captured by the linear in $\hbar$ corrections in the second-order theory.

IV. CHIRAL VORTICAL WAVE

In order to illustrate how the hydrodynamic equations derived in the previous section could be used in practice, we discuss in this section one of its simplest solutions describing a chiral vortical wave. As we will see, a proper account of the fluid flow affects the properties of such a wave.

We begin our analysis by recalling that the existence of the chiral vortical wave is a direct consequence of the chiral vortical effect. In essence, the later states that a nonzero fluid vorticity in a chiral plasma induces the following fermion-number and axial-charge currents [52–55]:

$$\mathbf{j} = \frac{\mu\mu_5}{\pi^2}\,\boldsymbol{\omega}, \tag{47}$$

$$\mathbf{j}_5 = \left(\frac{T^2}{6} + \frac{\mu^2+\mu_5^2}{2\pi^2}\right)\boldsymbol{\omega}. \tag{48}$$

Note that while the fermion-number current (47) exists only when both μ and μ_5 are nonzero, the chiral current (48) exists even if $\mu = \mu_5 = 0$ due to the T^2 term. The latter is related to the gravitational anomaly [52, 53, 56]. As suggested in Ref. [15], an interplay between the fermion-number and axial-charge fluctuations induced by the chiral vortical effect results in a gapless collective excitation that was called the chiral vortical wave. As we will see below, the inclusion of the hydrodynamic flow profoundly modifies this simple picture.

To start with, let us note that the dissipative equations derived in the previous section reproduce the correct equilibrium expressions for the fermion-number and axial-charge currents given by Eqs. (47) and (48). Indeed, as the system approaches equilibrium, all gradient terms in Eqs. (43) and (44) vanish and their solutions take a particularly simple form: $\nu^\mu = \nu^\mu_{\rm eq}$ and $\nu^\mu_5 = \nu^\mu_{5,\rm eq}$, where the equilibrium currents are defined by Eqs. (24) and (25). As is easy to check, the latter coincide with the results in Eqs. (47) and (48) in the rest frame of the fluid $u^\mu = (1, 0, 0, 0)$. In this connection, it should be noted that the currents ν^μ and ν^μ_5 are generically defined as dissipative quantities. At the same time, the equilibrium chiral vortical effect currents $\nu^\mu_{\rm eq}$ and $\nu^\mu_{5,\rm eq}$ are non-dissipative parts of j^μ and j^μ_5. For simplicity of notations, however, we still include them in the dissipative functions ν^μ and ν^μ_5.

In order to analyze the chiral vortical wave by using the hydrodynamic equations obtained in the previous section, we choose the local background velocity of the fluid in the following form:

$$u^\mu = u^\mu_0 + \epsilon^{\mu\nu\alpha\beta} x_\nu u_{0\alpha}\bar{\omega}_\beta, \tag{49}$$

where the first term describes a uniform motion and the second one describes a rotation. We will assume that relation (49) is valid for a sufficiently slow rotation and sufficiently small distances $L \ll |\bar{\omega}|^{-1}$. The above expression for the four-velocity is normalized in the usual way, $u^\mu u_\mu = 1$. Up to quadratic terms in vorticity, which are negligible in the case of a slow rotation, the normalization condition for u^μ is valid if the four-vectors u^μ_0 and $\bar{\omega}^\mu$ satisfy $u^\mu_0\bar{\omega}_\mu = 0$ and $u^\mu_0 u_{0,\mu} = 1$. One should also note that, to leading order, the four-vector $\bar{\omega}^\mu$ coincides with the definition of vorticity given by $\omega^\mu \equiv \frac{1}{2}\varepsilon^{\mu\alpha\beta\gamma}u_\alpha\partial_\beta u_\gamma$.

Let us search for a solution to hydrodynamic equations in the form of a propagating wave. In the most general case, the chemical potentials, temperature, and fluid velocity will oscillate around their average values, i.e.,

$$\delta\mu(x) = e^{-ikx}\delta\mu_0, \qquad \delta\mu_5(x) = e^{-ikx}\delta\mu_{5,0}, \qquad \delta T(x) = e^{-ikx}\delta T_0, \qquad \delta u^\mu(x) = e^{-ikx}\delta u^\mu_0, \tag{50}$$

where k^μ is the wavevector, and $\delta\mu_0$, $\delta\mu_{5,0}$, δT_0, and δu^μ_0 are the amplitudes of oscillations of the corresponding quantities. The requirement of normalization constrains the oscillations of the fluid velocity to be orthogonal to the background velocity, $u_\mu\delta u^\mu = 0$. This is automatically satisfied for the waves with the fluid velocity oscillations along the direction of the vorticity, i.e., $\delta u^\mu \parallel \bar{\omega}^\mu$.

For the sake of simplicity, let us analyze the dissipative equations in the first-order theory. In this case, we find from Eqs. (43)–(45) that

$$\nu^\mu = \nu^\mu_{\rm eq} - \tau n\dot{u}^\mu + \frac{\tau}{3}\nabla^\mu n, \tag{51}$$

$$\nu^\mu_5 = \nu^\mu_{5,\rm eq} - \tau n_5\dot{u}^\mu + \frac{\tau}{3}\nabla^\mu n_5, \tag{52}$$

$$\pi^{\mu\nu} = \frac{8\tau}{15}\epsilon(\partial^{\langle\mu}u^{\nu\rangle}), \tag{53}$$

where we used the constraints (20), (22), and (23). By substituting these expressions into the continuity equations

(12)–(15) and linearizing them in fluctuations, we derive the following system of coupled equations:

$$\sum_{z_i}\left(\Omega\frac{\partial n_{\rm eq}}{\partial z_i}-i\frac{\tau}{3}k_\perp^2\frac{\partial n_{\rm eq}}{\partial z_i}+k_\mu\frac{\partial \nu^\mu_{\rm eq}}{\partial z_i}\right)\delta z_i+n_{\rm eq}\left(1+i\tau\Omega\right)(k\cdot\delta u)-2\Omega(\nu_{\rm eq}\cdot\delta u)=0, \tag{54}$$

$$\sum_{z_i}\left(\Omega\frac{\partial n_{5,\rm eq}}{\partial z_i}-i\frac{\tau}{3}k_\perp^2\frac{\partial n_{5,\rm eq}}{\partial z_i}+k_\mu\frac{\partial \nu^\mu_{5,\rm eq}}{\partial z_i}\right)\delta z_i+n_{5,\rm eq}\left(1+i\tau\Omega\right)(k\cdot\delta u)-2\Omega(\nu_{5,\rm eq}\cdot\delta u)=0, \tag{55}$$

$$\sum_{z_i}\left(\Omega\frac{\partial \epsilon_{\rm eq}}{\partial z_i}+k_\mu\frac{\partial h^\mu_{\rm eq}}{\partial z_i}\right)\delta z_i+\frac{4}{3}\epsilon_{\rm eq}(k\cdot\delta u)-3\Omega(h_{\rm eq}\cdot\delta u)=0, \tag{56}$$

$$\sum_{z_i}\left(\Omega\frac{\partial h^\mu_{\rm eq}}{\partial z_i}-k_\perp^\mu\frac{1}{3}\frac{\partial \epsilon_{\rm eq}}{\partial z_i}\right)\delta z_i+\left(\frac{4}{3}\epsilon_{\rm eq}\Omega+\frac{3}{4}(k\cdot h_{\rm eq})\right)\delta u^\mu+h^\mu_{\rm eq}(k\cdot\delta u)$$
$$+\frac{1}{4}(h_{\rm eq}\cdot\delta u)k_\perp^\mu-\frac{i}{2}\hbar n_{5,\rm eq}\Omega\varepsilon^{\mu\nu\alpha\beta}u_\nu k_\alpha\delta u_\beta-i\tau\frac{8}{15}\Delta^{\mu\nu}_{\alpha\beta}k_\nu\epsilon_{\rm eq}k^\alpha\delta u^\beta=0. \tag{57}$$

Here we introduced the shorthand notations $\Omega=(k\cdot u)$ and $k_\perp^\mu=k^\mu-u^\mu(k\cdot u)$, and used the summation index $z_i=(\mu,\mu_5,T)$. The analysis of these equations is simplified in the rest frame with $u_0^\mu=(1,0,0,0)$ and $\bar\omega^\mu=(0,0,0,\bar\omega)$. For a wave propagating along the direction of vorticity, the wave vector takes the form $k^\mu=\Omega u_0^\mu+k_z\hat\omega^\mu=(\Omega,0,0,k_z)$.

The obtained system of homogeneous linear equations has nontrivial solutions only when the determinant of the corresponding matrix of coefficients vanishes. Thus, by solving the characteristic equation, we obtain dispersion relations for four different types of waves: two sound waves and two modes that resemble chiral vortical waves. As we will see, the latter differ from the simplified solutions of the chiral vortical waves [15] because their propagation is profoundly affected by the hydrodynamic flow of the fluid itself.

To the linear order in ω and τ, the resulting dispersion relations for the sound waves are given by

$$\Omega=\pm\frac{k_z}{\sqrt{3}}+\frac{3}{8}\hbar\bar\omega\frac{n_{5,\rm eq}}{\epsilon_{\rm eq}}k_z+\frac{2}{15}i\tau k_z^2, \tag{58}$$

where the second term is a vorticity correction to the usual speed of sound and the third term describes the attenuation of the sound wave. The dispersion relations of the chiral vortical waves read

$$\Omega=\hbar\bar\omega v_1k_z-\frac{1}{3}i\tau k_z^2,\qquad \Omega=\hbar\bar\omega v_2k_z-\frac{1}{3}i\tau k_z^2, \tag{59}$$

where $v_{1,2}$ are the roots of a quadratic equation $av^2+bv+c=0$ with the following coefficients:

$$a=\epsilon\Big[45\left(\mu^2-\mu_5^2\right)^2\left(\mu^2+\mu_5^2\right)+7\pi^6T^6+27\pi^4T^4\left(\mu^2+\mu_5^2\right)+3\pi^2T^2\left(11\mu^4+18\mu^2\mu_5^2+11\mu_5^4\right)\Big], \tag{60}$$

$$b=\frac{\mu_5}{10\pi^2}\Big[225(\mu^2-\mu_5^2)\left(2\mu^6+5\mu^4\mu_5^2+8\mu^2\mu_5^4+\mu_5^6\right)-14\pi^8T^8-\pi^6T^6\left(78\mu^2+127\mu_5^2\right)$$
$$-45\pi^4\mu_5^2T^4\left(11\mu^2+9\mu_5^2\right)+15\pi^2T^2\left(30\mu^6+5\mu^4\mu_5^2-72\mu^2\mu_5^4-43\mu_5^6\right)\Big], \tag{61}$$

$$c=\frac{3}{20\pi^2}\Big[75\mu_5^8-4\left(\pi^3\mu T^3+5\pi\mu^3T\right)^2+225\mu_5^6\left(3\mu^2+\pi^2T^2\right)-3\mu_5^2\left(5\mu^2+\pi^2T^2\right)^2\left(5\mu^2+7\pi^2T^2\right)$$
$$+5\mu_5^4\left(-75\mu^4+13\pi^4T^4+30\pi^2\mu^2T^2\right)\Big]. \tag{62}$$

It is worth noting that there are two different modes of the chiral vortical wave. This result seems to qualitatively agree with the dispersion relations obtained in Refs. [60, 61]. From a physics point of view, they correspond to two opposite directions of propagation with respect to the vorticity. In general, the speeds of such waves are different. It is interesting to note that the corresponding waves have nonzero velocities $\hbar\bar\omega v_{1,2}$ even at $\mu=0$, which appears to contradict the prediction of Ref. [62], where similar waves were analyzed. We may suggest that this is the result of using a more general scheme in this study, in which both the fermion-number and axial-charge conservations are enforced (see also Refs. [60, 61]).

It is instructive to consider a special case of a plasma with the vanishing axial-charge chemical potential $\mu_5=0$. In this case, the dispersion relation for the sound waves is similar to that in Eq. (58), but has no correction due to vorticity. This should not be surprising for a plasma without a chiral asymmetry. As for the dispersion relations of the chiral vortical waves, they are given by the following explicit expression:

$$\Omega=\pm\frac{\bar\omega\hbar T\mu(\pi^2T^2+5\mu^2)k_z}{2\pi\sqrt{5\epsilon_{\rm eq}(5\epsilon_{\rm eq}-2T^2\mu^2)(\pi^2T^2+3\mu^2)}}-\frac{1}{3}i\tau k_z^2. \tag{63}$$

As is easy to check from Eq. (54)–(57), the propagation of the chiral vortical waves is characterized by oscillations of all thermodynamic parameters. In fact, in this case, the explicit relations between their oscillation amplitudes are given by

$$\delta\mu_{5,0} = \pm\frac{4\pi}{(\partial\epsilon_{\rm eq}/\partial T)}\sqrt{\frac{\epsilon_{\rm eq}(5\epsilon_{\rm eq}-2T^2\mu^2)}{5(\pi^2T^2+3\mu^2)}}\delta\mu_0, \tag{64}$$

$$\delta T_0 = -\frac{3n_{\rm eq}}{(\partial\epsilon_{\rm eq}/\partial T)}\delta\mu_0, \tag{65}$$

$$\delta u_0^\mu = \mp\frac{\hbar\bar{\omega}^\mu}{\pi(\partial\epsilon_{\rm eq}/\partial T)}\sqrt{\frac{(5\epsilon_{\rm eq}-2T^2\mu^2)(\pi^2T^2+3\mu^2)}{5\epsilon_{\rm eq}}}\delta\mu_0. \tag{66}$$

As we see from the last equation, the chiral vortical wave is accompanied by oscillations of the fluid velocity along the direction of the vorticity. This is in addition to the usual oscillations of the fermion-number and axial-charge densities. With the model assumptions used here, we also see that the chiral vortical waves come with local oscillations of the temperature. At the same time, as is easy to check, the chiral vortical waves do not drive the oscillations of the local energy density, $\delta\epsilon = 0$.

V. CONCLUSION

In this study, we derived a closed system of second-order dissipative hydrodynamic equations that governs the evolution of a chiral plasma made of neutral particles such as neutrinos. The corresponding results can be applied for studies of the early states of protoneutron star evolution, where neutrinos are trapped in dense matter and achieve a hydrodynamic regime. In such a plasma, the effects of chirality could play an important role in driving an inverse cascade that may be relevant for the origin of the supernova explosion [63]. The system of hydrodynamic equations obtained here can be used to numerically simulate the corresponding dynamics. While the corresponding detailed study is beyond the scope of this paper, we see that the presence of a chiral asymmetry modifies the hydrodynamic equations. It also appears that the leading-order quantum corrections due to spin of chiral particles can play profound effects in hydrodynamics, especially when combined with a chiral asymmetry and vorticity of the fluid.

By making use of the hydrodynamic equations, in this paper we also briefly addressed the modification of the chiral vortical waves associated with the fluid flow. In this part, for simplicity, we used the first-order theory. Of course, such an approximation is sufficient for the problem of the propagating modes with long wavelengths when there is no issue with the stability of solutions. We found that the propagation of the chiral vortical wave also induces oscillations of the local fluid velocity. As a result, its dispersion relation differs from that predicted in a simplified model where only the oscillations of fermion-number and axial-charge densities are taken into account. Interestingly, we find that the local energy density does not oscillate during the propagation of the chiral vortical wave.

While the effects of electromagnetism were neglected in this study, there is no conceptual limitation to take them into account. In fact, a generalization of the second-order hydrodynamic equations to the case of a chiral plasma made of charged particles is of great interest. The corresponding plasmas play a profound role in cosmology, heavy-ion collisions, and even Dirac/Weyl materials. Obviously, many interesting phenomena may be expected from a nontrivial interplay of electromagnetic fields and vorticity. In principle, the derivation of the corresponding equations is a straightforward although tedious task that we plan to address in the future.

Another interesting extension of the current study would be the derivation of the third-order dissipative hydrodynamics [43] and, perhaps, even the inclusion of the quantum corrections beyond the leading order in the Planck constant. One should keep in mind, however, that additional corrections of quantum origin could be expected even in the chirally symmetric matter at the second order in $\hbar$ [64]. Concerning the quantum corrections, it should be noted that, in the framework proposed in this study, the conservations of the energy and momentum were enforced only to the linear order in $\hbar$, or equivalently in spin. Of course, before attempting the inclusion of higher-order quantum corrections, this limitation should be lifted first. One of the promising approaches that could help to advance the problem is based on the use of the Wigner function [54, 57].

In view of the obvious importance of preserving the Lorentz covariance in relativistic models, it will be desirable to generalize the current analysis to the case of models with nonlocal collision integrals consistent with the Lorentz covariance. While the problem is expected to be much more challenging technically, it may not be hopeless [65].

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Appendix A: The energy-momentum conservation and other technical details

In this Appendix, we present selected technical details that support the results in the main text of the paper.

1. The energy-momentum conservation

The conservation of the energy and momentum to the leading order in the Planck constant plays a very important role in our analysis. So, here we present the derivation of the general relation for the energy-momentum tensor, i.e., $\partial_\nu T^{\mu\nu} = 0$, in the chiral kinetic theory. As we show, this condition is not automatically satisfied in the relaxation-time approximation. It can be enforced by choosing a special frame of reference.

From the definition of the energy-momentum tensor in Eq. (8), we obtain

$$\begin{aligned}\partial_\nu T^{\mu\nu} &= \frac{1}{2}\int (\delta^\mu_\nu \delta^\kappa_\lambda - \delta^\mu_\lambda \delta^\kappa_\nu)\, p^\lambda (\partial_\kappa S^{\nu\alpha})\partial_\alpha f - \frac{1}{\tau}\int p^\mu (p\cdot u)(f-f_{\rm eq}) + \frac{1}{2}\int S^{\mu\alpha}\partial_\alpha (p\cdot\partial f)\\ &\simeq \frac{1}{4}\int \varepsilon^{\mu\kappa\sigma\rho}\varepsilon_{\nu\lambda\sigma\rho} p^\lambda (\partial_\kappa S^{\nu\alpha})\partial_\alpha f - \frac{1}{\tau}\int p^\mu (p\cdot u)(f-f_{\rm eq}) - \frac{1}{2\tau}\int S^{\mu\alpha}\partial_\alpha \left[(p\cdot u)(f-f_{\rm eq})\right], \end{aligned} \tag{A1}$$

where we temporarily omitted the sum over λ and dropped the terms of order $\hbar^2$. (Note that $S^{\mu\alpha}$ is linear in $\hbar$.) It is convenient to analyze the first integral on the right-hand side separately. By making use of the explicit form of the spin tensor $S^{\nu\alpha}$, we can rewrite the corresponding integrand as follows:

$$\frac{1}{4}\int \varepsilon^{\mu\kappa\sigma\rho}\varepsilon_{\nu\lambda\sigma\rho} p^\lambda (\partial_\kappa S^{\nu\alpha})\partial_\alpha f = \lambda\frac{\hbar}{8}\int \varepsilon^{\mu\kappa\sigma\rho}\delta^{\alpha\beta\gamma}_{\lambda\sigma\rho} p^\lambda p_\beta \partial_\kappa \left(\frac{u_\gamma}{p\cdot u}\right)\partial_\alpha f, \tag{A2}$$

where $\delta^{\alpha\beta\gamma}_{\lambda\sigma\rho} = \epsilon^{\alpha\beta\gamma\delta}\epsilon_{\lambda\sigma\rho\delta}$ is the generalized Kronecker symbol. Due to the Kronecker symbol, the contraction over index λ will lead to one of the following three possibilities: (i) $p^\lambda \to p^\alpha$, (ii) $p^\lambda \to p^\beta$, or (iii) $p^\lambda \to p^\gamma$. The latter two cases give vanishing results either because $p^\beta p_\beta = 0$ (massless particles), or because $\partial_\kappa \left(\frac{p^\gamma u_\gamma}{p\cdot u}\right) = 0$. The only nontrivial contribution comes from the contraction that turns p^λ into p^α, i.e.,

$$\lambda\frac{\hbar}{4}\int \varepsilon^{\mu\kappa\beta\gamma} p^\alpha p_\beta \partial_\kappa \left(\frac{u_\gamma}{p\cdot u}\right)\partial_\alpha f \simeq -\frac{1}{2\tau}\int (\partial_\kappa S^{\mu\kappa})\,(p\cdot u)(f-f_{\rm eq}), \tag{A3}$$

where we used the kinetic equation and dropped the terms of order $\hbar^2$ on the right-hand side. Now, by combining the results in Eqs. (A1) and (A3), we finally obtain

$$\begin{aligned}\partial_\nu T^{\mu\nu} &= -\frac{1}{\tau}\int p^\mu (p\cdot u)(f-f_{\rm eq}) - \lambda\frac{\hbar}{4\tau}\int \varepsilon^{\mu\alpha\nu\beta} p_\nu \partial_\alpha \left[u_\beta (f-f_{\rm eq})\right] + O(\hbar^2)\\ &= -\frac{1}{\tau}\left[u^\mu(\epsilon-\epsilon_{\rm eq}) + (h^\mu - h^\mu_{\rm eq}) - \lambda\frac{\hbar}{2}\omega^\mu (n-n_{\rm eq}) - \lambda\frac{\hbar}{4}\epsilon^{\mu\alpha\beta\gamma}(\partial_\alpha u_\beta)(\nu_\gamma - \nu^{\rm eq}_\gamma)\right] + O(\hbar^2). \end{aligned} \tag{A4}$$

Here we used the chiral kinetic theory definitions for the energy density $\epsilon \equiv u_\mu u_\nu T^{\mu\nu}$, momentum density $h^\mu \equiv \Delta^\mu_\alpha u_\beta T^{\alpha\beta}$, charge density $n \equiv u_\mu j^\mu$, and current density $\nu^\mu \equiv \Delta^{\mu\nu} j_\nu$ that follow directly from the definitions for the number-density current (5) and the energy-momentum tensor (8).

After restoring the sum over λ, the result in Eq. (A4) can be rewritten in the following equivalent form:

$$\partial_\nu T^{\mu\nu} = -\frac{u^\mu}{\tau}\left(\epsilon - \epsilon_{\rm eq} + \frac{\hbar}{2}\omega_\mu(\nu^\mu_5 - \nu^\mu_{5,\rm eq})\right) - \frac{1}{\tau}\left(h^\mu - h^\mu_{\rm eq} - \frac{\hbar}{4}\epsilon^{\mu\alpha\beta\gamma}u_\alpha \dot{u}_\beta (\nu_{5,\gamma} - \nu_{5,{\rm eq},\gamma})\right) + O(\hbar^2), \tag{A5}$$

where we used the constraint in Eq. (20) and separated the components along the four-vector u^μ from the projection perpendicular to u^μ. Note the last term in the perpendicular component can be given in several alternative forms

$$\frac{\hbar}{4}\epsilon^{\mu\alpha\beta\gamma}u_\alpha\dot{u}_\beta(\nu_{5,\gamma}-\nu_{5,\text{eq},\gamma})=\Delta^\mu_\nu\frac{\hbar}{4}\epsilon^{\nu\alpha\beta\gamma}(\partial_\alpha u_\beta)(\nu_{5,\gamma}-\nu_{5,\text{eq},\gamma})=\frac{\hbar}{4}\epsilon^{\mu\alpha\beta\gamma}(\partial_\alpha u_\beta)(\nu_{5,\gamma}-\nu_{5,\text{eq},\gamma})+\frac{\hbar}{2}u^\mu\omega^\gamma(\nu_{5,\gamma}-\nu_{5,\text{eq},\gamma}). \tag{A6}$$

2. Useful integrals

In the calculation of moments of the distribution function, the following integrals are useful:

$$\int(p\cdot u)^n f_0=-\sum_{\chi=\pm1}\frac{\Gamma(n+2)}{2\pi^2}\chi^n T^{n+2}\text{Li}_{n+2}\left(-e^{\frac{\chi\mu_\lambda}{T}}\right)\equiv I_{n+2}, \tag{A7}$$

$$\int(p\cdot u)^n p^\alpha f_0=u^\alpha I_{n+3}, \tag{A8}$$

$$\int(p\cdot u)^n p^\alpha p^\beta f_0=\left(-\frac{1}{3}g^{\alpha\beta}+\frac{4}{3}u^\alpha u^\beta\right)I_{n+4}, \tag{A9}$$

$$\int(p\cdot u)^n p^\alpha p^\beta p^\gamma f_0=\left(-g^{(\alpha\beta}u^{\gamma)}+2u^\alpha u^\beta u^\gamma\right)I_{n+5}, \tag{A10}$$

$$\int(p\cdot u)^n p^\alpha p^\beta p^\gamma p^\delta f_0=\left(\frac{1}{5}g^{(\alpha\beta}g^{\gamma\delta)}-\frac{12}{5}g^{(\alpha\beta}u^\gamma u^{\delta)}+\frac{16}{5}u^\alpha u^\beta u^\gamma u^\delta\right)I_{n+6}, \tag{A11}$$

$$\int(p\cdot u)^n p^{\mu_1}...p^{\mu_5} f_0=\left(g^{(\mu_1}gu^{\mu_5)}-\frac{16}{3}g^{(\mu_1}u..u^{\mu_5)}+\frac{16}{3}u^{\mu_1}..u^{\mu_5}\right)I_{n+7}, \tag{A12}$$

$$\int(p\cdot u)^n p^{\mu_1}...p^{\mu_6} f_0=\left(-\frac{1}{7}g^{(\mu_1}gg^{\mu_6)}+\frac{24}{7}g^{(\mu_1}guu^{\mu_6)}-\frac{80}{7}g^{(\mu_1}u..u^{\mu_6)}+\frac{64}{7}u^{\mu_1}..u^{\mu_6}\right)I_{n+8}, \tag{A13}$$

where f_0 is the equilibrium function at a vanishing vorticity and the round brackets denote a symmetrization over all possible permutations, e.g., $A^{(\alpha}B^\beta C^{\gamma)}\equiv(A^\alpha B^\beta C^\gamma+A^\alpha B^\gamma C^\beta+A^\beta B^\alpha C^\gamma+A^\beta B^\gamma C^\alpha+A^\gamma B^\beta C^\alpha+A^\gamma B^\alpha C^\beta)/3!$.

It is easy to check that lower moments can be obtained from the higher ones multiplying the latter by the four-velocity u^μ. Similar integral chains can be obtained also for derivatives of the distribution function $f_0'=\partial f_0/\partial\varepsilon_p$ if one makes a substitution $I_n\to-(n-1)I_{n-1}$. For f_0'', the substitution is $I_n\to(n-1)(n-2)I_{n-2}$ and so on. As is easy to check, the explicit results for several lowest-order moments read

$$I_1=\frac{\mu_\lambda}{2\pi^2}, \tag{A14}$$

$$I_2=\frac{\mu_\lambda^2}{4\pi^2}+\frac{T^2}{12}, \tag{A15}$$

$$I_3=\frac{\mu_\lambda^3}{6\pi^2}+\frac{\mu_\lambda T^2}{6}, \tag{A16}$$

$$I_4=\frac{\mu_\lambda^4}{8\pi^2}+\frac{\mu_\lambda^2 T^2}{4}+\frac{7\pi^2T^4}{120}. \tag{A17}$$

Note that these moments satisfy the following recurrent relation: $\partial I_{n+1}/\partial\mu_\lambda=nI_n$.

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