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A discrete quantum spin system is presented in which several modern methods of canonical quantum gravity can be tested with promising results. In particular, features of interacting dynamics are analyzed with an emphasis on homogeneous configurations and the dynamical building-up and stability of long-range correlations. Different types of homogeneous minisuperspace models are introduced for the system, including one based on condensate states, and shown to capture different aspects of the discrete system. They are evaluated with effective methods and by means of continuum limits, showing good agreement with operator calculations whenever the latter are available. As a possibly quite general result, it is concluded that an analysis of the building-up of long-range correlations in discrete systems requires non-perturbative solutions of the dynamical equations. Some questions related to stability can be analyzed perturbatively, but suggest that matter couplings may be relevant for this question in the context of quantum cosmology.

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I. INTRODUCTION

Minisuperspace models of quantum-field theories, in particular quantum gravity, are usually constructed by quantizing a set of configurations obtained from the full classical theory by imposing spatial homogeneity. While homogeneous configurations are exact (though special) solutions of the classical theory, for various reasons they are not expected to be exact solutions of the full quantum-field theory. For instance, uncertainty relations would prevent both the amplitude and momentum of an inhomogeneous mode from having zero quantum fluctuations. In an interacting theory, fluctuations couple to expectation values, and non-zero fluctuations usually imply that the mode expectation values cannot remain zero in time. An exactly homogeneous (non-vacuum) solution therefore cannot be realized in a quantum-field theory. The question of what kind of an approximation to the full quantum theory a minisuperspace model may provide has remained open, but recently canonical effective methods have shed some light on this question for scalar quantum-field theories on a flat background space-time [1].

The main application of minisuperspace models is in the context of quantum gravity, where space-time is no longer a background but quantized as well. Several approaches to quantum gravity suggest that space or space-time may no longer be continuous in this setting. (See for instance [2, 3].) Discrete space may present a further obstacle to finding exact or approximate homogeneous solutions of the theory: even if we disregard quantum fluctuations or their back-reaction on expectation values, local moves in a discrete structure do not respect homogeneity. At most, a coarse-grained model which collects the accumulated action of many local moves in a single evolution step could lead to approximate homogeneous solutions. However, coarse-graining remains incompletely understood in discrete approaches to background-independent gravity. (See for instance [4, 5] for recent realizations.)

In order to probe these questions, we introduce here a discrete quantum system which exhibits several interesting aspects regarding minisuperspace models. Starting from the discrete quantum theory rather than a classical continuum theory allows us to analyze how different features of the interacting dynamics can be captured in simpler systems. As is well known, a discrete theory can give rise to different continuum limits. Each of them would then lead to a different minisuperspace model. The same result can be seen directly by minisuperspace constructions performed for the discrete quantum theory.

We will also analyze the discrete quantum theory in qualitative terms. In particular, we are interested in the question of how long-range correlations can build up in a fundamental theory and under which conditions they are stable. If such correlations can be achieved, it is at least possible that nearly homogeneous configurations can be the result of evolution in the theory, rather than just of specific initial choices as implicitly made in minisuperspace constructions. Of course, homogeneous configurations require long-range correlations of a very specific kind which is more difficult to analyze for a generic interacting theory. But the building-up of some kind of long-range correlations is a pre-requisite for near homogeneity, and it can be studied in our model in qualitative terms. The stability question

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will lead us back to the ground-state configurations discussed for the various minisuperspace models introduced here. An interesting interplay between the full discrete theory and the models is important for the physical interpretation of minisuperspace results.

II. THE MODEL

In the absence of a consistent canonical quantum theory of gravity, it is not clear what Hamiltonian one should use to model its discrete dynamics. (See for instance [6–10] for some issues involved in such a construction.) In an attempt to construct tractable models, we focus here on some of the ingredients that seem to be rather general. We do not intend to capture the precise dynamics of quantum gravity but rather plan to explore some properties of possible candidates for fundamental degrees of freedom.

Several proposals of quantum gravity, going back to [11], are based on mathematical versions of angular momentum or spin as a fundamental degrees of freedom. Not only spin eigenvalues but also their proposed arrangement on a graph in space, or a spin network, are discrete. Certain invariant combinations of spin quantum numbers can then be defined as discrete analogs of the common continuum expressions of geometrical measures. Moreover, spin-spin interactions can be used to introduce possible Hamiltonians.

Handling the arrangement of spins on an arbitrary graph in three spatial dimensions can be a difficult combinatorial problem. The first simplification we will use is a dimensional reduction: We will consider only one spatial dimension, which could be thought of as the radial direction measuring the distance from a non-rotating and spherical star or black hole, or more generally a so-called midisuperspace model of general relativity. Aligned in this direction are then several different types of spins, which roughly correspond to independent components of the spatial metric. We arrive at a 1-dimensional graph model as illustrated in Fig. 1. Here, spins on links in the horizontal direction, called "horizontal spins" in what follows, would then have a geometrical interpretation distinct from that of "vertical spins" on upward and open-ended links. However, such a geometrical interpretation will not be relevant for our analysis of the interacting dynamics.

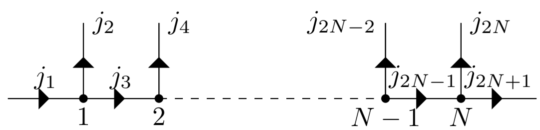

FIG. 1: The inhomogeneous one-dimensional graph Γ with N vertices and $2N+1$ links. The Hamiltonian (1) is invariant under a mapping of spins that corresponds to a reflection of the graph in a horizontal direction. If the orientation of a link, indicated by an arrow, changes under this reflection, the corresponding mapped spin has a negative sign.

Specific versions of such quantum midisuperspace models with explicit Hamiltonians have been constructed for spherically symmetric models [12, 13] and certain types of gravitational waves [14, 15]. In tractable versions, one makes use of a further reduction of the group SU(2) to the Abelian U(1). In order to have interesting spin-spin interactions, we will not make use of this reduction here. However, we will simplify the combinatorics by working with a single spin on each link, instead of distinguishing between left- and right-invariant vector fields on SU(2) as would be done in a full spin network.

The dynamics on a spin system, such as the one illustrated in Fig. 1, is in general spin-changing as well as graph-changing if it comes from a generic proposal of canonical quantum gravity. That is, the Hamiltonian can contain terms that change the irreducible representation of SU(2) on each link of the graph, as well as terms that can create new vertices and corresponding links of the graph. Such a dynamics is hard to control, and therefore we assume a simplified version in which no spin-changing or graph-changing terms occur. Therefore, for given irreducible representations and a fixed graph, only spin-spin interactions are present in the Hamiltonian. We consider only local (next-neighbor) pairwise interactions and require a certain reflection symmetry as indicated in Fig. 1 and spelled out in the explicit construction that follows.

As the discrete theory, we introduce a spin system which for a given integer N has $2N+1$ interacting spins J_i, $i = 1, \ldots, 2N+1$. We define the dynamics in canonical form, generated by the Hamiltonian

$$\hat{\mathrm{H}}_\Gamma = \alpha \sum_{i=1}^{N} \left(-\hat{\mathbf{J}}_{2i-1} \cdot \hat{\mathbf{J}}_{2i} + \hat{\mathbf{J}}_{2i-1} \cdot \hat{\mathbf{J}}_{2i+1} + \hat{\mathbf{J}}_{2i} \cdot \hat{\mathbf{J}}_{2i+1} \right) \tag{1}$$

with a coupling constant α. This operator is invariant under global rotations of the spins: the sum of all horizontal spins,

$$\mathbf{G} := \sum_{i=1}^{N+1} \hat{\mathbf{J}}_{2i-1}\,, \tag{2}$$

commutes with the Hamiltonian. This conserved quantity can be used in some cases to simplify equations of motion, as in Sec. V B. We have chosen the signs of individual coefficients of the spin products so as to make the Hamiltonian reflection symmetric under the operation $\hat{\mathbf{J}}_{2i-1} \mapsto -\hat{\mathbf{J}}_{2N-2i+3}$ while $\hat{\mathbf{J}}_{2i} \mapsto \hat{\mathbf{J}}_{2N-2i+2}$ (or $i \mapsto N-i+1$). These properties can be illustrated by the graph model presented in Fig. 1. The arrows indicate the sign in the reflection symmetry.

In 1-dimensional models of gravity, it is often convenient to impose polarization conditions which eliminate one of the metric components as an independent field. In our discrete model, such a condition would then relate the different types of spins (horizontal and vertical) to each other. Our polarization condition used here is a constraint that corresponds to the classical conditions

$$\mathbf{C}_i = -\mathbf{J}_{2i-1} + \mathbf{J}_{2i} + \mathbf{J}_{2i+1} = 0 \quad , \quad i = 1, \ldots, N\,. \tag{3}$$

They can be used to eliminate the vertical spins. Also this system of constraints has coefficients chosen so as to make it reflection symmetric: $\mathbf{C}_i \mapsto \mathbf{C}_{N-i+1}$.

We have a system of constraints in a non-symplectic Poisson manifold with coordinates given by spin components J_i^a, such that standard classifications of first or second class constraints are not available [16]. It is, however, straightforward to see that the constraints do not all (Poisson) commute with one another, nor with the Hamiltonian. The non-zero Poisson brackets are $\{C_i^a, C_i^b\} = \epsilon^{abc}(J^c_{2i-1} + J^c_{2i} + J^c_{2i+1}) \approx 2\epsilon^{abc} J^c_{2i-1}$ (the weak equality $\approx$ indicating that the constraints have been used) and $\{C_i^a, C_{i+1}^b\} = -\epsilon^{abc} J^c_{2i+1}$ for $a \neq b$, while all other components of the constraints commute. We are only interested in imposing the constraints as a reduction of vertical degrees of freedom. The constraint surface remains well-defined if the reduction constraints are imposed strongly. In particular, we can solve the constraints so as to eliminate all vertical spins $\mathbf{J}_{2i}$ (or $\hat{\mathbf{J}}_{2i}$), and use standard Poisson brackets (or commutators) for the remaining $\mathbf{J}_{2i-1}$ (or $\hat{\mathbf{J}}_{2i-1}$). All our derivations exclusively use Poisson brackets or commutators, and therefore the model is sufficient as a non-symplectic Poisson system.

We use the system of partially non-commuting constraints as an example of reduction, better known from the context of symmetry reduction. In fact, if we combine the constraints $\mathbf{C}_i$ with additional constraints that set all the vertical spins equal to zero, the reduction imposes homogeneity: it requires that the remaining, horizontal spins are all equal, $\mathbf{J}_{2i-1} = \mathbf{J}_{2i+1}$ for all $i = 1, \ldots, N$. If we impose only $\mathbf{C}_i = 0$ without restricting vertical spins, we can solve for the vertical spins and obtain a single 1-dimensional spin chain closely related to the next-neighbor Heisenberg spin chain. The fact that the quantized constraints do not commute with $\hat{H}$ allows us to probe for potential effects of local discrete moves not respecting reduction constraints.

An important question in classical symmetry reduction is whether variation commutes with reduction. It is not always guaranteed that equations of motion of the reduced system (extrema of the reduced action) agree with the field equations of the full theory restricted to fields that obey the reduction condition. Certain general conditions are known that guarantee this commutation property (symmetric criticality), formulated mainly as conditions on properties of the corresponding symmetry group [17, 18]. In our case, we have a reduction constraint which shares with minisuperspace reductions the feature that it is (partially) non-commuting, but it does not directly correspond to a symmetry group. Moreover, we are working exclusively with Hamiltonians rather than action principles, and we do not have a symplectic phase space.

However, instead of using general conditions on symmetry groups, it is not difficult to test the commutation property directly. We have the reduced Hamiltonian

$$\hat{\mathrm{H}}_{\mathrm{red}} = \alpha \sum_{i=1}^{N} \left(-\hat{J}^2_{2i-1} + 3\,\hat{\mathbf{J}}_{2i-1} \cdot \hat{\mathbf{J}}_{2i+1} - \hat{J}^2_{2i+1} \right) \tag{4}$$

$$= \alpha \left(-\hat{J}_1^2 - 2\sum_{i=2}^{N} \hat{J}^2_{2i-1} - \hat{J}^2_{2N+1} + 3\sum_{i=1}^{N} \hat{\mathbf{J}}_{2i-1} \cdot \hat{\mathbf{J}}_{2i+1} \right)\,, \tag{5}$$

which generates Heisenberg equations of motion

$$\frac{\mathrm{d}\hat{J}^a_{2i+1}}{\mathrm{d}t} = 3\,\alpha\,\epsilon^{abc} \left(\hat{J}^b_{2i-1} \hat{J}^c_{2i+1} + \hat{J}^c_{2i+1} \hat{J}^b_{2i+3} \right)\,. \tag{6}$$

The full equations for horizontal spins are

$$\frac{\mathrm{d}\hat{J}^a_{2i+1}}{\mathrm{d}t} = \alpha\,\epsilon^{abc}\left(-\hat{J}^c_{2i+1}\hat{J}^b_{2i+2} + \hat{J}^b_{2i-1}\hat{J}^c_{2i+1} + \hat{J}^c_{2i+1}\hat{J}^b_{2i+3} + \hat{J}^b_{2i}\hat{J}^c_{2i+1}\right), \tag{7}$$

coupled to vertical spins $\hat{\mathbf{J}}_{2i}$. If we use the constraint in order to eliminate the vertical spins in the equation of motion, we obtain

$$\frac{\mathrm{d}\hat{J}^a_{2i+1}}{\mathrm{d}t} = 2\,\alpha\,\epsilon^{abc}\left(\hat{J}^b_{2i-1}\hat{J}^c_{2i+1} + \hat{J}^c_{2i+1}\hat{J}^b_{2i+3}\right) \tag{8}$$

which are not identical with the equations generated by the reduced Hamiltonian. However, the difference is merely a constant numerical factor of the time derivatives. We have interpreted the constraint imposed here as a polarization condition. The preceding calculations have shown that there is a small difference between imposing the polarization condition before or after deriving equations of motion. The coupling of modes is therefore slightly different if it is described by a reduced Hamiltonian, compared with the full Hamiltonian on whose equations of motion the same condition would be imposed. One can account for the difference by a simple rescaling (or a classical renormalization) of the coupling constant, using $\frac{2}{3}\alpha$ instead of α in the reduced Hamiltonian.

In what follows, we will, for simplicity, work mainly with reduced Hamiltonians. (The conserved quantity (2) now commutes strongly with the Hamiltonian.) We will compare different versions of homogeneous minisuperspace models and effective continuum theories.

III. MINISUPERSPACE MODELS

All spins in the model are coupled. It might therefore be possible that long-distance correlations build up over time, which could be of classical or quantum nature. A minisuperspace configuration would be one example of a classically correlated system. There is a difference between such a minisuperspace configuration and a homogeneous configuration as it might be realized as a ground state of the unreduced system because all spins would have to be identical as degrees of freedom, not just equal as values assigned to different links of the graph. In this subsection, we explore several questions related to this conceptual difference. We will also see that there is some freedom in defining different minisuperspace models, and that selecting a model that gives predictions close to the discrete theory requires knowledge of solutions of the discrete theory.

A. A minimal minisuperspace model

The smallest number of minisuperspace degrees of freedom can be realized by identifying all horizontal spins on the links. Setting $\hat{\mathbf{J}}_{2i-1} = \hat{\mathbf{J}}_{2i+1} =: \hat{\mathbf{J}}$ in the reduced Hamiltonian (4) gives us the minisuperspace Hamiltonian

$$\hat{H}^{(1)}_{\mathrm{mini}} = \alpha N \hat{J}^2\,. \tag{9}$$

We obtain the same result if we start with the unreduced Hamiltonian and set vertical spins equal to zero. Clearly, this Hamiltonian commutes with all minisuperspace degrees of freedom, given by the three components of a single $\hat{\mathbf{J}}$. The resulting trivial dynamics is very different from the coupled equations of the inhomogeneous theory.

B. A condensate model

Alternatively, homogeneity can be imposed at the level of states by working with condensate states of the full theory, given by

$$\Psi = \bigotimes_{i=1}^{N+1}\chi \tag{10}$$

where χ is a state in the Hilbert space of a single horizontal spin. The same individual state is therefore assumed for all links, but unlike in the minisuperspace model, the spins remain as independent degrees of freedom. This procedure is well known from the description of Bose–Einstein condensates, where it results in the non-linear Gross-Pitaevsky equation for the analog of χ. In quantum gravity, the procedure has been used in particular in the context of group-field theory [19–26], and it can also be seen in certain approximations that go beyond minisuperspace models by including perturbative inhomogeneity at an effective quantum level [27].

A non-linear equation for the single-spin wave function χ can be derived by first computing an effective Hamiltonian $\langle\Psi|\hat{H}|\Psi\rangle$ for (5) in a state of the form (10):

$$\langle\Psi|\hat{H}|\Psi\rangle = -2\alpha N\langle\chi|\hat{J}^2|\chi\rangle + 3\alpha N\langle\chi|\hat{\mathbf{J}}|\chi\rangle^2 \tag{11}$$

where we have identified $\langle\chi|\hat{\mathbf{J}}_i|\chi\rangle = \langle\chi|\hat{\mathbf{J}}_j|\chi\rangle$ for $i \neq j$. Effective equations of motion or variational ground states can be related to those of a state-dependent single-spin Hamiltonian

$$\hat{H}_{\rm condensate} = -2\alpha N\hat{J}^2 + 3\alpha N\langle\hat{\mathbf{J}}\rangle\cdot\hat{\mathbf{J}}\,. \tag{12}$$

It generates the non-linear equation

$$i\hbar\frac{\mathrm{d}\chi}{\mathrm{d}t} = \hat{H}\chi = -2\alpha N\hat{J}^2\chi + 3\alpha N\langle\chi|\hat{\mathbf{J}}|\chi\rangle\cdot\hat{\mathbf{J}}\chi\,. \tag{13}$$

Unlike in the minimal minisuperspace model, the condensate dynamics is non-trivial.

C. Two interacting minisuperspace models

The inhomogeneous dynamics can be probed more faithfully by using more than one triple of degrees of freedom, but still of small number for a minisuperspace model. Starting with the unreduced Hamiltonian, we split the spins into different subsets by introducing the following notation:

$$\begin{aligned}
\hat{\mathbf{J}}_{\mathrm{h}1,j} &:= \hat{\mathbf{J}}_{4j-3} &(14)\\
\hat{\mathbf{J}}_{\mathrm{v}1,j} &:= \hat{\mathbf{J}}_{4j-2} &(15)\\
\hat{\mathbf{J}}_{\mathrm{h}2,j} &:= \hat{\mathbf{J}}_{4j-1} &(16)\\
\hat{\mathbf{J}}_{\mathrm{v}2,j} &:= \hat{\mathbf{J}}_{4j}\,. &(17)
\end{aligned}$$

No analogous version of such a reduction has been used in quantum cosmology yet, but we will see that, in the present model, it can improve the simpler reduction given by $\hat{H}^{(1)}_{\rm mini}$.

Assuming even N, we can pair up neighboring vertices ($i = 2j-1$ and $i = 2j$ in (1)), and obtain the Hamiltonian

$$\begin{aligned}
\hat{H} &= \alpha\sum_{j=1}^{N/2}\Big(-\hat{\mathbf{J}}_{4j-3}\cdot\hat{\mathbf{J}}_{4j-2} + \hat{\mathbf{J}}_{4j-3}\cdot\hat{\mathbf{J}}_{4j-1} + \hat{\mathbf{J}}_{4j-2}\cdot\hat{\mathbf{J}}_{4j-1} &(18)\\
&\qquad -\hat{\mathbf{J}}_{4j-1}\cdot\hat{\mathbf{J}}_{4j} + \hat{\mathbf{J}}_{4j-1}\cdot\hat{\mathbf{J}}_{4j+1} + \hat{\mathbf{J}}_{4j}\cdot\hat{\mathbf{J}}_{4j+1}\Big) &(19)\\
&= \alpha\sum_{j=1}^{N/2}\Big(-\hat{\mathbf{J}}_{\mathrm{h}1,j}\cdot\hat{\mathbf{J}}_{\mathrm{v}1,j} + \hat{\mathbf{J}}_{\mathrm{h}1,j}\cdot\hat{\mathbf{J}}_{\mathrm{h}2,j} + \hat{\mathbf{J}}_{\mathrm{v}1,j}\cdot\hat{\mathbf{J}}_{\mathrm{h}2,j} &(20)\\
&\qquad -\hat{\mathbf{J}}_{\mathrm{h}2,j}\cdot\hat{\mathbf{J}}_{\mathrm{v}2,j} + \hat{\mathbf{J}}_{\mathrm{h}2,j}\cdot\hat{\mathbf{J}}_{\mathrm{h}1,j+1} + \hat{\mathbf{J}}_{\mathrm{v}2,j}\cdot\hat{\mathbf{J}}_{\mathrm{h}1,j+1}\Big)\,. &(21)
\end{aligned}$$

(We refer to $\hat{\mathbf{J}}_{2N+1}$ as a single spin $\hat{\mathbf{J}}_{\mathrm{h}1,N/2+1}$ without vertical or a second horizontal spins for $j = N/2+1$.) The new configurations and their interactions can be illustrated as in Fig. 2.

We have two constraints for each value of j:

$$\begin{aligned}
\hat{C}_{1,j} &= -\hat{\mathbf{J}}_{4j-3} + \hat{\mathbf{J}}_{4j-2} + \hat{\mathbf{J}}_{4j-1} = -\hat{\mathbf{J}}_{\mathrm{h}1,j} + \hat{\mathbf{J}}_{\mathrm{v}1,j} + \hat{\mathbf{J}}_{\mathrm{h}2,j} &(22)\\
\hat{C}_{2,j} &= -\hat{\mathbf{J}}_{4j-1} + \hat{\mathbf{J}}_{4j} + \hat{\mathbf{J}}_{4j+1} = -\hat{\mathbf{J}}_{\mathrm{h}2,j} + \hat{\mathbf{J}}_{\mathrm{v}2,j} + \hat{\mathbf{J}}_{\mathrm{h}1,j+1}\,. &(23)
\end{aligned}$$

The reduced Hamiltonian is

$$\hat{H}_{\rm red} = \alpha\sum_{j=1}^{N/2}\left(-\hat{J}^2_{\mathrm{h}1,j} - 2\hat{J}^2_{\mathrm{h}2,j} - \hat{J}^2_{\mathrm{h}1,j+1} + 3\,\hat{\mathbf{J}}_{\mathrm{h}1,j}\cdot\hat{\mathbf{J}}_{\mathrm{h}2,j} + 3\,\hat{\mathbf{J}}_{\mathrm{h}2,j}\cdot\hat{\mathbf{J}}_{\mathrm{h}1,j+1}\right)\,. \tag{24}$$

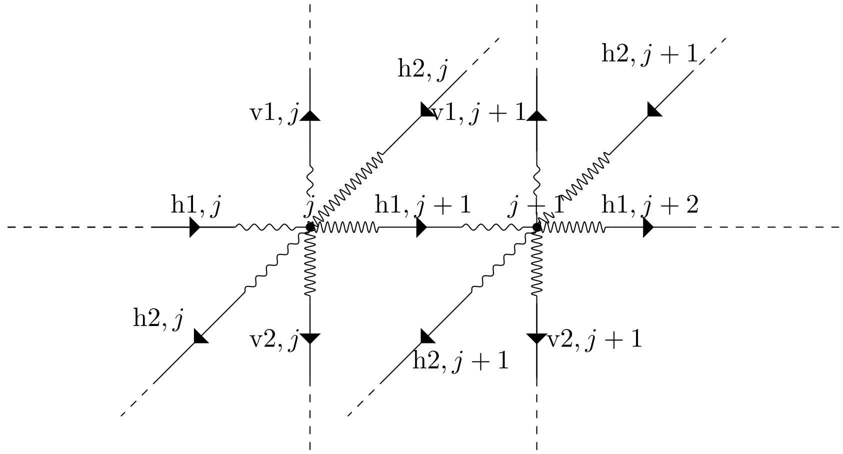

FIG. 2: A re-arranged representation of the discrete spin model. As before, lines with a given orientation stand for the spin operators. There are now five different spins meeting at a given vertex j, among which only the two triplets $(\hat{\mathbf{J}}_{\mathrm{h1},j}, \hat{\mathbf{J}}_{\mathrm{v1},j}, \hat{\mathbf{J}}_{\mathrm{h2},j})$ and $(\hat{\mathbf{J}}_{\mathrm{h2},j}, \hat{\mathbf{J}}_{\mathrm{v2},j}, \hat{\mathbf{J}}_{\mathrm{h1},j+1})$ are interacting as indicated by wavy lines with different wave lengths. The spin $\hat{\mathbf{J}}_{\mathrm{h2},j}$ appears in both triplets, and is therefore doubled in the diagrammatic visualization. Also the constraints relate only the spins that occur together in a triplet.

Setting $\hat{\mathbf{J}}_{\mathrm{h1},j} = \hat{\mathbf{J}}_{\mathrm{h1}}$ and $\hat{\mathbf{J}}_{\mathrm{h2},j} = \hat{\mathbf{J}}_{\mathrm{h2}}$ for all j (but $\hat{\mathbf{J}}_{\mathrm{h1}} \neq \hat{\mathbf{J}}_{\mathrm{h2}}$), we obtain a new minisuperspace Hamiltonian

$$\hat{H}^{(2)}_{\mathrm{mini}} = \alpha N(-\hat{J}^2_{\mathrm{h1}} - \hat{J}^2_{\mathrm{h2}} + 3\,\hat{\mathbf{J}}_{\mathrm{h1}} \cdot \hat{\mathbf{J}}_{\mathrm{h2}}) \tag{25}$$

with non-trivial dynamics. We note that this minisuperspace model is closely related to the mean-field model introduced for finite Heisenberg spin chains in [28]. The main difference is that our minisuperspace treatment identifies spin degrees of freedom on alternating links, while the mean-field treatment couples these spins. This relation, which we do not pursue further in this paper (except for one conclusion drawn in Sec. VI), could be useful in an extension of discrete minisuperspace models to controlled mean-field theories.

Alternatively, we can split up the range of N vertices into two disjoint averaging regions. Unlike $\hat{H}^{(2)}_{\mathrm{mini}}$, such a reduction can be interpreted as an analog of reductions proposed in quantum cosmology. In an attempt to include degrees of freedom relevant for the evolution of inhomogeneous perturbations on an isotropic background cosmology, [29] proposed that independent spatial regions can be pasted together to allow for more general degrees of freedom. (Our model here has only two such regions, but we will comment on effects of subdivisions in our discussion of stability in Sections V B and VI.) Such a quantum cosmology is a version of the classical separate-universe approximation of [30–32].

In the reduced theory, we now assume N odd, such that we have an even number $N+1$ of horizontal spins. We can group them in two sets, one for the spins around vertices $i = 1$ to $i = (N-1)/2$ and one for vertices from $i = (N+3)/2$ to $i = N$. (The central vertex $i_{\mathrm{c}} = (N+1)/2$ is not included in this counting. The spin $\hat{\mathbf{J}}_{2i_{\mathrm{c}}-1} = \hat{\mathbf{J}}_N$ to its left is contained in the first set, while $\hat{\mathbf{J}}_{2i_{\mathrm{c}}+1} = \hat{\mathbf{J}}_{N+2}$ to its right is contained in the second set.) Calling the first $(N+1)/2$ horizontal spins $\hat{\mathbf{J}}_{\mathrm{h1}} := \hat{\mathbf{J}}_1 = \hat{\mathbf{J}}_3 \cdots = \hat{\mathbf{J}}_N$ and the last $(N+1)/2$ spins $\hat{\mathbf{J}}_{\mathrm{h2}} := \hat{\mathbf{J}}_{N+2} = \hat{\mathbf{J}}_{N+4} = \cdots = \hat{\mathbf{J}}_{2N+1}$, we obtain, starting from the reduced Hamiltonian, the minisuperspace Hamiltonian

$$\hat{H}^{(3)}_{\mathrm{mini}} = \alpha \frac{N-3}{2} (\hat{J}^2_{\mathrm{h1}} + \hat{J}^2_{\mathrm{h2}}) + 3\alpha \hat{\mathbf{J}}_{\mathrm{h1}} \cdot \hat{\mathbf{J}}_{\mathrm{h2}}\,. \tag{26}$$

(There are $(N-1)/2$ non-interacting contributions of $\hat{J}^2_{\mathrm{h1}}$ and $\hat{J}^2_{\mathrm{h2}}$ from spins in the interior of the two averaging regions, as well as one contribution of $-\hat{J}^2_{\mathrm{h1}} + 3\hat{\mathbf{J}}_{\mathrm{h1}} \cdot \hat{\mathbf{J}}_{\mathrm{h2}} - \hat{J}^2_{\mathrm{h2}}$ with interactions at the border between the regions, located at the central vertex $i_{\mathrm{c}} = (N+1)/2$.)

The two Hamiltonians are rather different from each other. They are both of the form

$$\hat{H}_{\beta\gamma} := \beta(\hat{J}^2_{\mathrm{h1}} + \hat{J}^2_{\mathrm{h2}}) + \gamma \hat{\mathbf{J}}_{\mathrm{h1}} \cdot \hat{\mathbf{J}}_{\mathrm{h2}}\,, \tag{27}$$

but while $\mathrm{sgn}\gamma = \mathrm{sgn}\alpha$ in both cases, we have $\mathrm{sgn}\beta = -\mathrm{sgn}\alpha$ for $\hat{H}^{(2)}$ and $\mathrm{sgn}\beta = \mathrm{sgn}\alpha$ for $\hat{H}^{(3)}$ (with $N > 3$). We should therefore expect different ground states or effective potentials in the two cases. In particular, if $\langle \hat{J}^2_{\mathrm{h1/2}} \rangle$ is considered a free variable, we can minimize the energy of $\langle\hat{H}^{(3)}\rangle$ by zero spins, while the energy range of $\langle\hat{H}^{(2)}\rangle$ is unbounded from below. If $\langle\hat{J}^2_{\mathrm{h}1/2}\rangle$ is fixed, however, only the interaction term matters for ground states or effective potentials, for which the two models provide the same sign. The ground-state properties are then similar to those of the Heisenberg spin chain related to the reduced Hamiltonian (4): parallel alignment of next-neighbor spins (ferromagnetic) if $\gamma < 0$ and antiparallel alignment (antiferromagnetic) if $\gamma > 0$.

We assume that the individual spin states have the same eigenvalue of $\hat{J}^2_{\mathrm{h}1}$ and $\hat{J}^2_{\mathrm{h}2}$, given by $s(s+1)\hbar^2$ with some half-integer s. We have two cases according to sgnγ: For $\gamma > 0$, the interaction term is minimized by antiparallel spins $\hat{\mathbf{J}}_{\mathrm{h}1/2}$. The two spins then form a combined spin eigenstate $|0,0\rangle$ of the total spin $\hat{\mathbf{J}} := \hat{\mathbf{J}}_{\mathrm{h}1} + \hat{\mathbf{J}}_{\mathrm{h}2}$ in which

$$\langle\hat{\mathbf{J}}_{\mathrm{h}1}\cdot\hat{\mathbf{J}}_{\mathrm{h}2}\rangle = \frac{1}{2}\langle\hat{J}^2 - \hat{J}^2_{\mathrm{h}1} - \hat{J}^2_{\mathrm{h}2}\rangle = -s(s+1)\hbar^2\,, \tag{28}$$

and the energy eigenvalue is given by

$$E_{\gamma>0} = (2\beta - \gamma)s(s+1)\hbar^2\,. \tag{29}$$

For $\gamma < 0$, the parallel configuration minimizes the interaction term, for which we have a whole multiplet of different sates with total spin $2s$. In any such state, $\langle\hat{\mathbf{J}}_{\mathrm{h}1}\cdot\hat{\mathbf{J}}_{\mathrm{h}2}\rangle = s^2\hbar^2$, and the energy eigenvalue is

$$E_{\gamma<0} = (2\beta s(s+1) + \gamma s^2)\hbar^2\,. \tag{30}$$

The two different types of ground states, with antiparallel spins for $\gamma > 0$ and parallel spins for $\gamma < 0$, have interesting implications for the reliability of the two different interacting minisuperspace models. If $\gamma < 0$, the two models predict the same ground-state configuration with all fundamental spins aligned. There is only a quantitative difference between the models in the predicted ground-state energy. For $\gamma > 0$, however, the antiparallel alignment of $\hat{\mathbf{J}}_{\mathrm{h}1}$ and $\hat{\mathbf{J}}_{\mathrm{h}2}$ corresponds to very different fundamental configurations. With $\hat{H}^{(2)}_{\mathrm{mini}}$, the two antiparallel spins are alternating along the full spin chain, which agrees with the ground state of the discrete theory. With $\hat{H}^{(3)}_{\mathrm{mini}}$, however, we have two averaging regions with equal spins in each region, but antiparallel alignment between the two regions. Knowing the fundamental configuration, we can tell that an energy preference of antiparallel alignment at the border between the two regions means that the configuration should be unstable under splitting it up further into smaller and smaller averaging regions with antiparallel alignment at all borders. With complete splitting, each link being an averaging region of its own, a configuration as with $\hat{H}^{(2)}_{\mathrm{mini}}$ or in the fundamental theory is obtained, but one would have left the minisuperspace stage. The second minisuperspace model with Hamiltonian $\hat{H}^{(2)}_{\mathrm{mini}}$, on the other hand, realizes the correct ground state within a minisuperspace model, and without the need for further refinement. (There is, however, a difference between the ground state of $\hat{H}^{(2)}_{\mathrm{mini}}$ and the corresponding discrete theory. The former has an even number of spins, but it is derived from a spin chain with an odd number of spins. These two cases are known to have different behaviors [33–35].)

We conclude that the way degrees of freedom are included in a minisuperspace model can have significant implications for how well fundamental properties are modelled, which however can be evaluated only if one knows a great deal about the fundamental theory. Transferring this lesson to quantum gravity suggests that caution toward minisuperspace results would be advisable. However, there is a difference between stability as discussed so far, where it is implicitly assumed that the spin chain can exchange energy with an environment and settle down to its ground state, and quantum-cosmology models, where there is no environment outside of the system. We will return to this question in Secs. V and VI.

D. Effective equations and potentials

We continue to analyze the dynamics by means of effective equations in canonical form. Following [36–38], we assign infinitely many numbers to a set of quantum spin degrees of freedom with operator $\hat{\mathbf{J}}_i$. In a given state, these numbers correspond to the expectation values $\langle\hat{J}^a_i\rangle$ of spin components and the moments

$$\Delta(J^{a_1}_{i_1}\cdots J^{a_n}_{i_n}) := \langle(\hat{J}^{a_1}_{i_1} - \langle\hat{J}^{a_1}_{i_1}\rangle)\cdots(\hat{J}^{a_n}_{i_n} - \langle\hat{J}^{a_n}_{i_n}\rangle)\rangle_{\mathrm{symm}} \tag{31}$$

in totally symmetric ordering. (For uniform notation of all moments, we write fluctuations as $\Delta(J^aJ^a) = (\Delta J^a)^2$.) In a semiclassical expansion, which we will perform in most of our derivations, it is sufficient to include only moments up to second order, $n = 2$. We then have a finite-dimensional system. For a semiclassical state, defined as a state with moments of the order $\Delta(J^{a_1}_{i_1}\cdots J^{a_n}_{i_n}) = \mathcal{O}(\hbar^{(a_1+\cdots+a_n)/2})$, terms of order $\hbar$ are included in a truncation up to second order in moments.

The quantum degrees of freedom form a phase space with Poisson bracket derived from the commutator via

$$\{\langle\hat{A}\rangle, \langle\hat{B}\rangle\} := \frac{\langle[\hat{A},\hat{B}]\rangle}{i\hbar}\,. \tag{32}$$

For instance,

$$\begin{aligned}
\{\langle\hat{J}^a_i\rangle, \langle\hat{J}^b_j\rangle\} &= \epsilon^{abc}\langle\hat{J}^c_i\rangle\delta_{ij} \\
\{\Delta(J^a_i J^b_j), \langle\hat{J}^c_k\rangle\} &= \delta_{ik}\epsilon^{acd}\Delta(J^d_i J^b_j) + \delta_{jk}\epsilon^{bcd}\Delta(J^a_i J^d_j) \\
\{\Delta(J^a_i J^b_j), \Delta(J^c_k J^d_l)\} &= \delta_{ik}\epsilon^{ace}\langle\hat{J}^e_i\rangle\Delta(J^b_j J^d_l) + \delta_{il}\epsilon^{ade}\langle\hat{J}^e_i\rangle\Delta(J^b_j J^c_k) + \delta_{jk}\epsilon^{bce}\langle\hat{J}^e_j\rangle\Delta(J^a_i J^d_l) \\
&+ \delta_{jl}\epsilon^{bde}\langle\hat{J}^e_j\rangle\Delta(J^a_i J^c_k) + \mathcal{O}(\Delta^3)\,.
\end{aligned} \tag{33}$$

The first two lines are exact, while the third line is valid to second order in moments.

The Poisson bracket allows us to compute equations of motion. For instance, in the minimal minisuperspace model we have the effective Hamiltonian

$$H^{(1)}_{\text{eff}} = \alpha N\left(\langle\hat{J}^x\rangle^2 + \langle\hat{J}^y\rangle^2 + \langle\hat{J}^z\rangle^2 + \Delta(J^xJ^x) + \Delta(J^yJ^y) + \Delta(J^zJ^z)\right)\,. \tag{34}$$

All expectation values and moments then have zero time derivatives

$$\frac{\mathrm{d}O}{\mathrm{d}t} = \{O, H^{(1)}_{\text{eff}}\}\,, \tag{35}$$

as is to be expected from the non-interacting nature of the system.

1. *Minimal minisuperspace model*

The moments in (34) provide a quantum correction to the classical Hamiltonian. For the ground state, we can derive their minimal values by saturating uncertainty relations

$$\Delta(J^xJ^x)\Delta(J^yJ^y) \geq \frac{\hbar^2}{4}|\langle\hat{J}^z\rangle|^2 \tag{36}$$

and cyclic permutations. In order to evaluate these equations, we may assume that the expectation value $\langle\hat{\mathbf{J}}\rangle$ points in the z-direction. Therefore, only $\langle\hat{J}^z\rangle$ is non-zero among the spin components. The saturated uncertainty relations then require that $\Delta(J^zJ^z) = 0$ while

$$\Delta(J^xJ^x)\Delta(J^yJ^y) = \frac{1}{4}\hbar^2|\langle\hat{J}^z\rangle|^2\,. \tag{37}$$

By symmetry, $\Delta(J^xJ^x) = \Delta(J^yJ^y)$. The quantum correction to the classical Hamiltonian, or the effective potential, is then

$$V^{(1)}_{\text{eff}} = \alpha N(\Delta(J^xJ^x) + \Delta(J^yJ^y) + \Delta(J^zJ^z)) = \alpha N\hbar|\langle\hat{J}^z\rangle| = \alpha N\hbar\sqrt{\delta_{ab}\langle\hat{J}^a\rangle\langle\hat{J}^b\rangle}\,. \tag{38}$$

Although the effective potential is non-zero, it does not imply interactions but rather provides a zero-point energy because it only depends on a constant of motion. The effective Hamiltonian is given by

$$H^{(1)}_{\text{eff}} = \alpha N\langle\hat{J}^z\rangle^2 + V^{(1)}_{\text{eff}} = \alpha N\left(\langle\hat{J}^z\rangle^2 + \hbar|\langle\hat{J}^z\rangle|\right) = \alpha N s(s+1)\hbar^2 \tag{39}$$

if $\langle\hat{J}^z\rangle = \pm s\hbar$ for a spin pointing in the z-direction according to our assumptions. This value of the effective Hamiltonian is in agreement with the operator result, where the well-known eigenvalues of $\hat{J}^2$ are $s(s+1)\hbar^2$.

The same effective potential can be obtained from a canonical version of the spin system. Again assuming that the spin vector points in the z-direction, we introduce canonical coordinates

$$\begin{aligned}
q &:= \frac{\delta\langle\hat{J}^x\rangle}{\sqrt{\langle\hat{J}^z\rangle}} = \frac{\delta\langle\hat{J}^x\rangle}{\sqrt[4]{\delta_{ab}\langle\hat{J}^a\rangle\langle\hat{J}^b\rangle - (\delta\langle\hat{J}^x\rangle)^2 - (\delta\langle\hat{J}^y\rangle)^2}} \\
p &:= \frac{\delta\langle\hat{J}^y\rangle}{\sqrt{\langle\hat{J}^z\rangle}} = \frac{\delta\langle\hat{J}^y\rangle}{\sqrt[4]{\delta_{ab}\langle\hat{J}^a\rangle\langle\hat{J}^b\rangle - (\delta\langle\hat{J}^x\rangle)^2 - (\delta\langle\hat{J}^y\rangle)^2}}
\end{aligned} \tag{40}$$

for small values $\delta\langle\hat{J}^x\rangle$ and $\delta\langle\hat{J}^y\rangle$ much less than $\langle\hat{J}^z\rangle$: We have

$$\{q,p\} = \frac{\{\delta\langle\hat{J}^x\rangle, \delta\langle\hat{J}^y\rangle\}}{\langle\hat{J}^z\rangle} + \frac{\delta\langle\hat{J}^x\rangle\{\delta\langle\hat{J}^y\rangle, \langle\hat{J}^z\rangle\} - \delta\langle\hat{J}^y\rangle\{\delta\langle\hat{J}^x\rangle, \langle\hat{J}^z\rangle\}}{2\langle\hat{J}^z\rangle^2} = 1 + \mathcal{O}((\delta\langle\hat{J}^{x/y}\rangle/\langle\hat{J}^z\rangle)^2)\,. \tag{41}$$

These relations imply the quadratic equation

$$\left((\delta\langle\hat{J}^x\rangle)^2 + (\delta\langle\hat{J}^y\rangle)^2\right)^2 + (q^2+p^2)^2\left((\delta\langle\hat{J}^x\rangle)^2 + (\delta\langle\hat{J}^y\rangle)^2\right) - (q^2+p^2)^2\delta_{ab}\langle\hat{J}^a\rangle\langle\hat{J}^b\rangle = 0\,. \tag{42}$$

For $\delta_{ab}\langle\hat{J}^a\rangle\langle\hat{J}^b\rangle$ much greater than $(\delta\langle\hat{J}^x\rangle)^2 + (\delta\langle\hat{J}^y\rangle)^2$, we can solve (42) for $(\delta\langle\hat{J}^x\rangle)^2 + (\delta\langle\hat{J}^y\rangle)^2$ by

$$\begin{aligned}(\delta\langle\hat{J}^x\rangle)^2 + (\delta\langle\hat{J}^y\rangle)^2 &= -\frac{1}{2}(q^2+p^2)^2 \pm \sqrt{\delta_{ab}\langle\hat{J}^a\rangle\langle\hat{J}^b\rangle}(q^2+p^2)\sqrt{1+\frac{(q^2+p^2)^2}{4\delta_{ab}\langle\hat{J}^a\rangle\langle\hat{J}^b\rangle}} \\ &\approx \sqrt{\delta_{ab}\langle\hat{J}^a\rangle\langle\hat{J}^b\rangle}(q^2+p^2) - \frac{1}{2}(q^2+p^2)^2 + \cdots\,.\end{aligned} \tag{43}$$

(Only the plus sign gives a positive solution.) The first term is a harmonic-oscillator Hamiltonian with $m^{-1} = 2\sqrt{\delta_{ab}\langle\hat{J}^a\rangle\langle\hat{J}^b\rangle} = \omega$, which has zero-point energy $\frac{1}{2}\hbar\omega = \hbar\sqrt{\delta_{ab}\langle\hat{J}^a\rangle\langle\hat{J}^b\rangle}$ in agreement with the effective potential (38).

2. *Interacting minisuperspace models*

For the two interacting models, we need the covariances $\Delta(J^a_{\rm h1}J^a_{\rm h2})$ for $a = x, y, z$ (no sum over a). In addition to the fluctuations as in the minimal minisuperspace model with a Hamiltonian of the type (27), we then have the term

$$\Delta V_{\rm interaction} = \gamma(\Delta(J^x_{\rm h1}J^x_{\rm h2}) + \Delta(J^y_{\rm h1}J^y_{\rm h2}) + \Delta(J^z_{\rm h1}J^z_{\rm h2})) \tag{44}$$

in the effective potential.

Covariances can often be ignored in the context of uncertainty relations, but they do contribute in the complete form

$$\Delta(A^2)\Delta(B^2) - \Delta(AB)^2 \geq \frac{1}{4}|\langle[\hat{A},\hat{B}]\rangle|^2\,. \tag{45}$$

Unlike fluctuations, which obtain a lower bound for non-commuting operators, covariances are subject to an upper bound depending on the fluctuations. For two commuting spin components as in (44), the uncertainty relation is

$$\Delta(J^x_{\rm h1}J^x_{\rm h1})\Delta(J^x_{\rm h2}J^x_{\rm h2}) \geq \Delta(J^x_{\rm h1}J^x_{\rm h2})^2\,. \tag{46}$$

If $\gamma > 0$ in (44), we can minimize the effective interaction potential by choosing the value

$$\Delta(J^x_{\rm h1}J^x_{\rm h2}) = -\sqrt{\Delta(J^x_{\rm h1}J^x_{\rm h1})\Delta(J^x_{\rm h2}J^x_{\rm h2})} = -\frac{1}{2}\hbar|\langle\hat{J}^z\rangle|\,. \tag{47}$$

The last equality holds if we also minimize the uncertainty relations for the two individual spins and use the antiparallel alignment for spin expectation values, without loss of generality assumed to point in the z-direction. The same value is obtained for $\Delta(J^y_{\rm h1}J^y_{\rm h2})$, while $\Delta(J^z_{\rm h1}J^z_{\rm h2}) = 0$ because the z-components have zero fluctuations with our choice of directions. Combining all terms in the effective Hamiltonian (27) then yields

$$H_{{\rm eff},\gamma>0} = 2\beta s(s+1)\hbar^2 - \gamma(s^2\hbar^2 + s\hbar^2) = s(s+1)(2\beta-\gamma)\hbar^2\,. \tag{48}$$

The first term is twice the non-interacting contribution from a single spin with the same form as in (39), the second term, $-\gamma s^2\hbar^2$, is $\gamma\langle\hat{J}^z_{\rm h1}\rangle\langle\hat{J}^z_{\rm h2}\rangle$ for antiparallel alignment in the z-direction, and the last term, $-\gamma s\hbar^2$, adds up the two non-zero covariances in (44). This result agrees with the ground-state energy (29). For $\gamma < 0$, one can see in the same way that the operator result (30) is reproduced if the covariances vanish and parallel alignment is used in $\gamma\langle\hat{J}^z_{\rm h1}\rangle\langle\hat{J}^z_{\rm h2}\rangle = \gamma s^2\hbar^2$.

3. Condensate model

The condensate model has provided non-trivial dynamics for a single spin. However, some of the equations for expectation values and moments are trivial. We can derive effective equations from the state-dependent Hamiltonian (12) if we assign to it the effective Hamiltonian

$$H_{\rm eff,\,cond} = -2\alpha N\delta_{ab}\left(\langle\hat{J}^a\rangle\langle\hat{J}^b\rangle + \Delta(J^aJ^b)\right) + 3\alpha N\delta_{ab}j^a\langle\hat{J}^b\rangle \tag{49}$$

where the vector j^a is treated in the following way: It is a constant for purposes of computing Poisson brackets, for instance in equations of motion. After the Poisson brackets have been derived, one sets $j^a = \langle\hat{J}^a\rangle$. In this way, the correct equations follow for expectation values taken in a state that evolves according to the non-linear equation (13).

Following this procedure, we obtain the equation

$$\frac{{\rm d}\langle\hat{J}^c\rangle}{{\rm d}t} = 3\alpha N\epsilon^{abc}\langle\hat{J}^a\rangle\langle\hat{J}^b\rangle = 0\,. \tag{50}$$

For second-order moments, we have the contribution

$$\frac{{\rm d}\langle\hat{J}^c\hat{J}^d\rangle}{{\rm d}t} = 3\alpha N\langle\hat{J}^a\rangle\left\langle\epsilon^{abc}\hat{J}^b\hat{J}^d + \epsilon^{abd}\hat{J}^c\hat{J}^b\right\rangle\,. \tag{51}$$

Since covariances couple to expectation values and fluctuations, non-zero correlations can build up during evolution even if they vanish in an initial state.

IV. CONTINUUM THEORIES

One question about minisuperspace models derived from classical continuum theories, addressed in [1], is about the coordinate volume V_0 that characterizes the size of an averaging region. (See [39] for a review of minisuperspace models and quantum cosmology.) For the simplest minisuperspace models of general relativity, space is flat and infinite, and the canonical form $\int {\rm d}^3x\dot{\phi}p_\phi$ of any local degree of freedom $\phi(x)$ with momentum $p_\phi(x)$ is infinite for homogeneous configurations. In order to obtain a well-defined canonical structure, one can choose a finite region $\mathcal{V}$ of coordinate size $V_0 = \int_{\mathcal{V}}{\rm d}^3x$ and restrict integrations of the canonical form and the Lagrangian to this region.

If only homogeneous configurations are considered, the size and position of this region should not matter. The restricted canonical form is

$$\int_{\mathcal{V}}{\rm d}^3x\dot{\phi}(x)p_\phi(x) = V_0\dot{\bar{\phi}}\bar{p}_\phi \tag{52}$$

for homogeneous configurations $\phi(x) = \bar{\phi}$ and $p_\phi(x) = \bar{p}_\phi$. The momentum of $\bar{\phi}$ is therefore not equal to $\bar{p}_\phi$, but to $\bar{p} = V_0\bar{p}_\phi$. The standard Hamiltonian of a scalar field, just like other Hamiltonians for instance of gravitational degrees of freedom, then depends on V_0 when it is restricted to canonically conjugate minisuperspace configurations. For a scalar field, we have

$$\int_{\mathcal{V}}{\rm d}^3x\left(\frac{1}{2}\dot{\bar{\phi}}^2 + W(\bar{\phi})\right) = \frac{V_0}{2}\bar{p}_\phi^2 + V_0W(\bar{\phi}) = \frac{1}{2}\frac{\bar{p}^2}{V_0} + V_0W(\bar{\phi}) \tag{53}$$

and the dependence on V_0 is not just by a multiplicative factor. While the classical theory does not depend on the choice of V_0, the quantum theory does, for instance via V_0-dependent effective potentials.

In [1], it was shown that the V_0-dependent semiclassical corrections in minisuperspace effective potentials of a scalar-field theory are related to infrared contributions to field-theory effective potentials. The choice of V_0 then has physical relevance in the number of infrared modes included in the minisuperspace model, but it is a property of the minisuperspace truncation rather than of physical interactions. It is therefore difficult to justify interpretations of minisuperspace effects in quantitative terms.

In order to test this question in our spin system, we now derive a continuum theory from which our minimal minisuperspace model can be obtained by using homogeneous configurations. Generalizations corresponding to the interacting minisuperspace models will also be considered. As we will see, for this kind of spin systems the nature of averaging regions is less problematic than for canonical field theories. There may therefore be an advantage in deriving minisuperspace models directly from discrete quantum theories.

A. Continuum models

Starting with the unreduced theory, we introduce two continuum fields $\hat{\mathbf{J}}_{\rm h}(x)$ and $\hat{\mathbf{J}}_{\rm v}(x)$ where x runs through the entire length of our graph. In terms of the measure provided by the choice of x, we introduce the coordinate distance between two vertices i at $x = x_i$ and $i+1$ at $x = x_{i+1}$ by $\delta = \int_{x_i}^{x_{i+1}} {\rm d}x$, assumed to be independent of i. The total length of the graph is $L_0 = \int_{x_1}^{x_{N+1}} {\rm d}x = N\delta$. For integer values of $x = x_i$, we identify $\hat{\mathbf{J}}_{\rm v}(x_i) = \hat{\mathbf{J}}_{{\rm v},i} = \hat{\mathbf{J}}_{2i}$, $\hat{\mathbf{J}}_{{\rm h}1,i} = \hat{\mathbf{J}}_{2i-1}$ and $\hat{\mathbf{J}}_{{\rm h}2,i} = \hat{\mathbf{J}}_{2i+1}$.

1. *Minimal model*

For the minimal model, we do not treat $\hat{\mathbf{J}}_{{\rm h}1,i}$ and $\hat{\mathbf{J}}_{{\rm h}2,i}$ as independent fields but rather view them as one horizontal field evaluated at different positions $x_i \pm \delta/2$: We set $\hat{\mathbf{J}}_{\rm h}(x_i - \delta/2) = \hat{\mathbf{J}}_{{\rm h}1,i}$ and $\hat{\mathbf{J}}_{\rm h}(x_i + \delta/2) = \hat{\mathbf{J}}_{{\rm h}2,i}$. The leading terms in a continuum limit of the Hamiltonian and the constraint can then be obtained by an expansion in δ up to second order. Derivatives by x appear in the process, and will be denoted by a prime.

The constraints are

$$\begin{aligned} \hat{C}^a(x_i) &= -\hat{J}^a_{\rm h}(x_i - \delta/2) + \hat{J}^a_{\rm v}(x_i) + \hat{J}^a_{\rm h}(x_i + \delta/2) && (54) \\ &= \hat{J}^a_{\rm v}(x_i) + \delta \hat{J}^a_{\rm h}(x_i)' \,. && (55) \end{aligned}$$

The Hamiltonian is

$$\begin{aligned} \hat{H} &= \alpha\delta_{ab} \sum_{i=1}^N \Big(-\hat{J}^a_{\rm h}(x_i-\delta/2)\hat{J}^b_{\rm v}(x_i) + \hat{J}^a_{\rm h}(x_i-\delta/2)\hat{J}^b_{\rm h}(x_i+\delta/2) + \hat{J}^a_{\rm v}(x_i)\hat{J}^b_{\rm h}(x_i+\delta/2)\Big) \\ &= \alpha \sum_{i=1}^N \left(\hat{J}_{\rm h}(x_i)^2 + \delta \hat{\mathbf{J}}_{\rm v}(x_i)\cdot\hat{\mathbf{J}}_{\rm h}(x_i)' + \frac{1}{4}\delta^2 (\hat{\mathbf{J}}_{\rm h}(x_i)\cdot\hat{\mathbf{J}}_{\rm h}(x_i)'' - (\hat{J}_{\rm h}(x_i)')^2)\right) \,. && (56) \end{aligned}$$

Solving the constraint, the Hamiltonian becomes

$$\hat{H} = \alpha \sum_{i=1}^N \left(\hat{J}_{\rm h}(x_i)^2 + \frac{1}{4}\delta^2 (\hat{\mathbf{J}}_{\rm h}(x_i)\cdot\hat{\mathbf{J}}_{\rm h}(x_i)'' - 5(\hat{J}_{\rm h}(x_i)')^2)\right) \quad (57)$$

and gives rise to the continuum limit

$$\begin{aligned} \hat{H}^{(1)}_{\rm cont} &= \frac{\alpha}{\delta}\int {\rm d}x \left(\hat{J}_{\rm h}^2 + \frac{1}{4}\delta^2(\hat{\mathbf{J}}_{\rm h}\cdot\hat{\mathbf{J}}''_{\rm h} - 5(\hat{J}'_{\rm h})^2)\right) && (58) \\ &= \frac{\alpha}{\delta}\int {\rm d}x \left(\hat{J}_{\rm h}^2 - \frac{3}{2}\delta^2(\hat{J}'_{\rm h})^2\right) + \frac{1}{4}\alpha\delta \hat{\mathbf{J}}_{\rm h}\cdot\hat{\mathbf{J}}'_{\rm h}|_\partial && (59) \end{aligned}$$

with a boundary term denoted by a subscript ∂. The continuum limit for the reduced Hamiltonian (4) is the same as (59).

The minisuperspace model obtained from this continuum Hamiltonian is

$$H^{(1)}_{\rm mini} = \frac{\alpha}{\delta} L_0 \hat{J}_{\rm h}^2 = \alpha N \hat{J}_{\rm h}^2 \,. \quad (60)$$

It is identical with the Hamiltonian in our minimal minisuperspace model. Any reference to the averaging length L_0 can be expressed in terms of the number N of vertices of the fundamental discrete theory, which has physical meaning free of truncation choices. Therefore, there are no such problems as may be related to the appearance of V_0 in minisuperspace models derived from classical continuum theories.

2. *Interacting minisuperspace models*

If we do distinguish between $\hat{\mathbf{J}}_{{\rm h}1}(x_i)$ and $\hat{\mathbf{J}}_{{\rm h}2}(x_i)$, we obtain the continuum Hamiltonian

$$\hat{H}^{(2)}_{\rm cont} = \frac{\alpha}{\delta}\int {\rm d}x \left(-\hat{\mathbf{J}}_{{\rm h}1}\cdot\hat{\mathbf{J}}_{\rm v} + \hat{\mathbf{J}}_{{\rm h}1}\cdot\hat{\mathbf{J}}_{{\rm h}2} + \hat{\mathbf{J}}_{\rm v}\cdot\hat{\mathbf{J}}_{{\rm h}2}\right) \quad (61)$$

with constraint

$$\hat{C} = -\hat{\mathbf{J}}_{\rm h1} + \hat{\mathbf{J}}_{\rm v} + \hat{\mathbf{J}}_{\rm h2} = 0\,. \tag{62}$$

Solving the constraint gives the Hamiltonian

$$\hat{H}^{(2)}_{\rm cont} = \frac{\alpha}{\delta}\int {\rm d}x\left(-\hat{J}^2_{\rm h1} + 3\hat{\mathbf{J}}_{\rm h1}\cdot\hat{\mathbf{J}}_{\rm h2} - \hat{J}^2_{\rm h2}\right)\,, \tag{63}$$

from which we obtain the interacting minisuperspace Hamiltonian $\hat{H}^{(2)}_{\rm mini}$.

For the second interacting model, we do not distinguish between $\hat{\mathbf{J}}_{\rm h1}(x_i)$ and $\hat{\mathbf{J}}_{\rm h2}(x_i)$, but introduce two averaging regions of length $L_0/2$ each in which we have the constant fields $\hat{\mathbf{J}}_1$ and $\hat{\mathbf{J}}_2$, respectively. Starting with the first continuum theory with Hamiltonian (59), we obtain

$$\hat{H}^{(3)}_{\rm hom} = \frac{\alpha}{\delta}\frac{L_0}{2}\left(\hat{J}^2_1 + \hat{J}^2_2\right) + \frac{1}{4}\alpha\delta\left(\hat{\mathbf{J}}_1 - \hat{\mathbf{J}}_2\right)\cdot\hat{\mathbf{J}}'\,. \tag{64}$$

The last term, originating from the boundary term, needs further discussion. We have left a $\hat{\mathbf{J}}'$ in the equation, still referring to an inhomogeneous continuum field. It gives us the difference between the two averaged fields $\hat{\mathbf{J}}_1$ and $\hat{\mathbf{J}}_2$. It is infinite if we have two constant fields taking different values in the two regions, but it is multiplied with the spacing δ which goes to zero in the continuum limit. We can regularize this product to a finite number by defining it as

$$\lim_{\delta\to 0}(\delta\hat{\mathbf{J}}') = \lim_{\delta\to 0}\left(\delta\frac{\hat{\mathbf{J}}_2 - \hat{\mathbf{J}}_1}{\delta}\right) = \hat{\mathbf{J}}_2 - \hat{\mathbf{J}}_1\,. \tag{65}$$

The minisuperspace Hamiltonian is then equal to

$$\hat{H}^{(3)}_{\rm mini} = \frac{\alpha}{\delta}\frac{L_0}{2}\left(\hat{J}^2_1 + \hat{J}^2_2\right) - \frac{1}{4}\alpha\left(\hat{\mathbf{J}}_1 - \hat{\mathbf{J}}_2\right)^2 \tag{66}$$

$$= \alpha(N/2 - 1/4)\left(\hat{J}^2_1 + \hat{J}^2_2\right) + \frac{1}{2}\alpha\hat{\mathbf{J}}_1\cdot\hat{\mathbf{J}}_2\,. \tag{67}$$

Unlike what we saw for $\hat{H}^{(2)}_{\rm mini}$, this result is not identical with the previous derivation (26) from the discrete theory. However, if we slightly modify our boundary regularization by introducing an additional factor of six in the definition (65), the minisuperspace Hamiltonians are the same. The initial disagreement is a result only of the fact that the boundary regularization is ambiguous, which is necessary for $\hat{H}^{(3)}_{\rm mini}$ derived from the continuum theory, but not for $\hat{H}^{(2)}_{\rm mini}$. Notice that the final agreement is non-trivial, because changing the factor in (65) modifies the interacting as well as non-interacting terms in the resulting $\hat{H}^{(3)}_{\rm mini}$, which then both agree with (26).

V. EFFECTIVE ANALYSIS OF THE DISCRETE THEORY

So far, we have analyzed different minisuperspace models, their ground states and effective potentials, as well as relations with continuum theories. Minisuperspace models of quantum gravity are used to analyze homogeneous solutions, which necessarily have long-range correlations as seen from the fundamental theory because distant degrees of freedom are identified. Non-zero fluctuations of a single minisuperspace variable are therefore the same as long-range correlations in the fundamental theory.

Homogeneity and long-range correlations can easily be achieved in the ground state of the fundamental theory. However, discrete ground states are not suitable as nearly classical geometries of a discrete model of quantum gravity, in which degrees of freedom must be excited in order to have non-degenerate geometries. We therefore need excited states with approximate homogeneity and long-range correlations, which poses a question very different from just finding ground states. Long-range correlations related to homogeneity easily build up us a system settles down to its ground state, but it is not guaranteed that this can happen also in an excited state of an isolated system (such as the whole universe) which does not have a drain for surplus energy. In our discrete model, the evolution of long-range correlations can be studied in qualitative terms.

A. Effective dynamics

For the full or reduced discrete theories, we have large systems of coupled equations generated by the effective Hamiltonians

$$
\begin{aligned}
H_{\rm eff} \;&=\; \alpha\delta_{ab}\sum_{i=1}^{N}\big(-\langle\hat{J}^a_{2i-1}\rangle\langle\hat{J}^b_{2i}\rangle+\langle\hat{J}^a_{2i-1}\rangle\langle\hat{J}^b_{2i+1}\rangle+\langle\hat{J}^a_{2i}\rangle\langle\hat{J}^b_{2i+1}\rangle-\Delta(J^a_{2i-1}J^b_{2i})\\
&+\;\Delta(J^a_{2i-1}J^b_{2i+1})+\Delta(J^a_{2i}J^b_{2i+1})\big)
\end{aligned}
\tag{68}
$$

and

$$
\begin{aligned}
H_{\rm eff,red} \;&=\; \alpha\delta_{ab}\sum_{i=1}^{N}\big(-\langle\hat{J}^a_{2i-1}\rangle\langle\hat{J}^b_{2i-1}\rangle+3\langle\hat{J}^a_{2i-1}\rangle\langle\hat{J}^b_{2i+1}\rangle-\langle\hat{J}^a_{2i+1}\rangle\langle\hat{J}^b_{2i+1}\rangle-\Delta(J^a_{2i-1}J^b_{2i-1})\\
&+\;3\Delta(J^a_{2i-1}J^b_{2i+1})-\Delta(J^a_{2i+1}J^b_{2i+1})\big)\,,
\end{aligned}
\tag{69}
$$

respectively. The Poisson brackets between expectation values and moments are as in (33), except that the subscript takes values in a larger set.

For the unreduced system, we have effective constraints in addition to the effective Hamiltonian. A single constraint operator generates infinitely many effective constraints because it restricts not only expectation values but also the associated moments [40, 41]. In states annihilated by the constraint, we have

$$
C^a_i := \langle\hat{C}^a_i\rangle = -\langle\hat{J}^a_{2i-1}\rangle+\langle\hat{J}^a_{2i}\rangle+\langle\hat{J}^a_{2i+1}\rangle = 0 \tag{70}
$$

as well as

$$
C^a_{i,J^b_j} := \langle(\hat{J}^b_j-\langle\hat{J}^b_j\rangle)\hat{C}^a_i\rangle = -\Delta(J^b_jJ^a_{2i-1})+\Delta(J^b_jJ^a_{2i})+\Delta(J^b_jJ^a_{2i+1}) = 0 \tag{71}
$$

and higher-order constraints.

For the boundary expectation values, the equations of motion are

$$
\frac{{\rm d}\langle\hat{J}^a_1\rangle}{{\rm d}t} = \alpha\,\epsilon^{abc}\left(\langle\hat{J}^b_3\rangle\langle\hat{J}^c_1\rangle+\Delta(J^b_3J^c_1)-\langle\hat{J}^b_2\rangle\langle\hat{J}^c_1\rangle-\Delta(J^b_2J^c_1)\right) \tag{72}
$$

and

$$
\frac{{\rm d}\langle\hat{J}^a_{2N+1}\rangle}{{\rm d}t} = \alpha\,\epsilon^{abc}\left(\langle\hat{J}^b_{2N}\rangle\langle\hat{J}^c_{2N+1}\rangle+\Delta(J^b_{2N}J^c_{2N+1})-\langle\hat{J}^b_{2N-1}\rangle\langle\hat{J}^c_{2N+1}\rangle-\Delta(J^b_{2N-1}J^c_{2N+1})\right). \tag{73}
$$

Implementing the constraint, we obtain

$$
\frac{{\rm d}\langle\hat{J}^a_1\rangle}{{\rm d}t} = 2\,\alpha\,\epsilon^{abc}\left(\langle\hat{J}^b_3\rangle\langle\hat{J}^c_1\rangle+\Delta(J^b_3J^c_1)\right) \tag{74}
$$

and

$$
\frac{{\rm d}\langle\hat{J}^a_{2N+1}\rangle}{{\rm d}t} = 2\,\alpha\,\epsilon^{abc}\left(\langle\hat{J}^b_{2N-1}\rangle\langle\hat{J}^c_{2N+1}\rangle+\Delta(J^b_{2N-1}J^c_{2N+1})\right). \tag{75}
$$

The reduced equations for internal spins are

$$
\frac{{\rm d}\langle\hat{J}^a_{2i+1}\rangle}{{\rm d}t} = 2\,\alpha\,\epsilon^{abc}\left(\langle\hat{J}^b_{2i-1}\rangle\langle\hat{J}^c_{2i+1}\rangle+\Delta(J^b_{2i-1}J^c_{2i+1})+\langle\hat{J}^b_{2i+3}\rangle\langle\hat{J}^c_{2i+1}\rangle+\Delta(J^b_{2i+3}J^c_{2i+1})\right) \tag{76}
$$

where $i=\{2,\cdots,N-1\}$. The equation of motion generated by the effective version of the reduced quantum Hamiltonian (69) are

$$
\frac{{\rm d}\langle\hat{J}^{{\rm red},a}_{2i+1}\rangle}{{\rm d}t} = 3\,\alpha\,\epsilon^{abc}\left(\langle\hat{J}^b_{2i-1}\rangle\langle\hat{J}^c_{2i+1}\rangle+\Delta(J^b_{2i-1}J^c_{2i+1})+\langle\hat{J}^b_{2i+3}\rangle\langle\hat{J}^c_{2i+1}\rangle+\Delta(J^b_{2i+3}J^c_{2i+1})\right). \tag{77}
$$

As in the case of operator equations, the equations can be mapped into each other by a constant rescaling of the time coordinate. We note that these equations are exact as no truncation by moments has been necessary. However, equations of motion for second-order moments depend on third-order moments and have to be truncated for a self-contained semiclassical approximation to first order in $\hbar$.

Analytical solutions of these non-linear equations for large N are difficult to find. However, it is possible to analyze some general properties of interest in the context of long-range correlations. In particular, we are interested in correlations between spins at the two boundaries of the graph, which are certainly long-range for large N. We will assume that initially there are no correlations between different spins, but they will build up over time as the system evolves.

The reduced equation of motion for the correlation $\Delta(J_1^a J_{2i+1}^b)$ is

$$\frac{\mathrm{d}\Delta(J_1^a J_{2i+1}^b)}{\mathrm{d}t} = 2\,\alpha \left\{ \epsilon^{acd}\left(\langle \hat{J}_1^d\rangle \Delta(J_3^c J_{2i+1}^b) + \langle \hat{J}_3^c\rangle \Delta(J_1^d J_{2i+1}^b)\right) + \epsilon^{bef}\left(\langle \hat{J}_{2i-1}^e\rangle \Delta(J_1^a J_{2i+1}^f) + \langle \hat{J}_{2i+1}^f\rangle \Delta(J_1^a J_{2i-1}^e)\right) \right\}. \tag{78}$$

If we start with an uncorrelated state, the initial conditions are such that the only non-zero initial values are the expectation value of $\hat{J}_1$ and $\hat{J}_{2N+1}$ and their variances chosen such that uncertainty relations are respected. For interior spins, only the fluctuations are assumed non-zero (although they would be allowed to be zero for a general spin system).

The expectation values and second-order moments of the state for all spins are coupled to one another through the evolution equations. One may solve them perturbatively by orders of α. To zeroth order, all expectation values and moments are constant and no correlations build up. To first order, using (78), $\mathrm{d}\Delta(J_1J_3)/\mathrm{d}t$ has a non-zero contribution of the form $\alpha\langle\hat{J}_1\rangle\Delta(J_3J_3)$ and after some time $\Delta(J_1J_3)$ is non-zero to first order in α. The expectation value $\langle\hat{J}_3\rangle$ has a time derivative of the form (76) or (77) with one non-zero term of the form $\alpha\Delta(J_1J_3)$ and after some time is non-zero to second order in α. We can iterate this procedure and generate non-zero expectation values on all links, as well as non-zero correlations between the links. The first step has generated a covariance $\Delta(J_1J_3) = \mathcal{O}(\alpha\hbar)$ because we used one solution perturbative in α and one fluctuation. For the next step, moving up to $\Delta(J_1J_5)$, we first need to generate a non-zero $\langle\hat{J}_3\rangle$ and $\Delta(J_3J_5)$, using one additional fluctuation and repeated perturbative solutions. We therefore obtain a non-zero $\Delta(J_1J_5) = \mathcal{O}(\alpha^4\hbar^2)$. By iteration, we obtain a non-zero $\Delta(J_1J_{2i+1}) = \mathcal{O}(\alpha^{3i-2}\hbar^i)$. For long-range correlations between the boundary spins, we need to apply the procedure up to $i = N/2$, after which all spins are correlated from non-zero seed expectation values on both boundaries; see Figs. 3 and 4. A small number of iterations, $M \ll N$, does not lead to strong long-range correlations. The building-up of long-range correlations from local dynamics can therefore be seen only to high perturbative orders, or non-perturbatively, and it requires high orders in an $\hbar$-expansion.

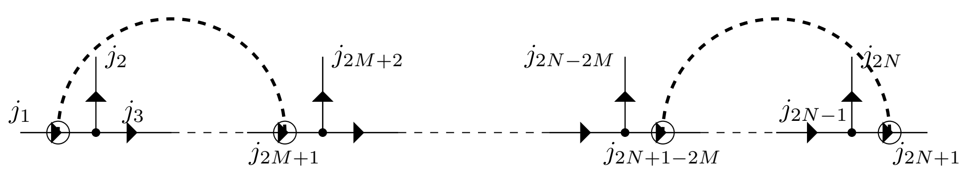

FIG. 3: Correlations between the farthest spin sets for $\hat{\mathbf{J}}_1$ and $\hat{\mathbf{J}}_{2N+1}$ after solving the equations M times iteratively for a graph with N vertices for $2M \le N+1$. The semi-circles represent non-zero covariances $\Delta(J_1^a J_{2M+1}^b)$ and $\Delta(J_{2N+1}^a J_{2N+1-2M}^b)$.

B. Dynamical stability for small number of vertices

We now return to the question of dynamical stability of states of the spin chain as an isolated system. We will consider discrete theories with small number of vertices and choose initial values corresponding to the different ground states found in our minisuperspace models. The effective evolution equations can be solved perturbatively and indicate dynamical stability in some cases. (We will not analyze stability under perturbations of initial configurations.) Although we present explicit solutions only for small numbers of vertices, the relevant features are generic and can be seen also for large numbers of vertices, but it is more cumbersome to produce explicit expressions.

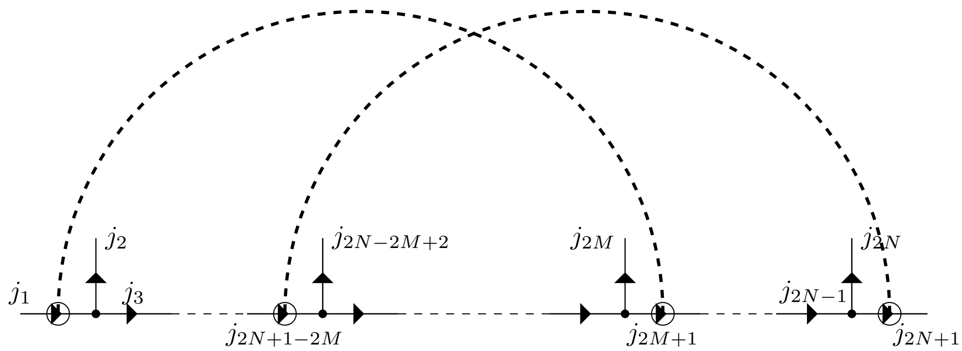

FIG. 4: Correlations between the farthest spin sets for $\hat{\mathbf{J}}_1$ and $\hat{\mathbf{J}}_{2N+1}$ after solving the equations M times iteratively for a graph with N vertices for $N < 2M$. The semi-circles represent non-zero covariances $\Delta(J_1^a J_{2M+1}^b)$ and $\Delta(J_{2N+1}^a J_{2N+1-2M}^b)$.

1. Single-vertex graph

For $N = 1$, we have two horizontal spins which we can identify with the spins in our two interacting minisuperspace models, or view as a theory beyond the minimal minisuperspace model. In this case, correlations between the boundary spins are next-neighbour correlations. The reduced equations of motion are

$$\frac{\mathrm{d}\langle \hat{J}_1^x \rangle}{\mathrm{d}t} = 2\,\alpha \left(\langle \hat{J}_1^z \rangle \langle \hat{J}_3^y \rangle - \langle \hat{J}_1^y \rangle \langle \hat{J}_3^z \rangle + \Delta(J_1^z J_3^y) - \Delta(J_1^y J_3^z) \right) \tag{79}$$

and cyclic for the y and z components. For the second horizontal spin, $\mathbf{J}_3$, we have

$$\frac{\mathrm{d}\langle \hat{J}_3^x \rangle}{\mathrm{d}t} = 2\,\alpha \left(\langle \hat{J}_1^y \rangle \langle \hat{J}_3^z \rangle - \langle \hat{J}_1^z \rangle \langle \hat{J}_3^y \rangle + \Delta(J_1^y J_3^z) - \Delta(J_1^z J_3^y) \right) = -\frac{\mathrm{d}\langle \hat{J}_1^x \rangle}{\mathrm{d}t} \tag{80}$$

and again cyclic for the y and z components. The equality $\mathrm{d}\langle \hat{\mathbf{J}}_1 \rangle/\mathrm{d}t = -\mathrm{d}\langle \hat{\mathbf{J}}_3 \rangle/\mathrm{d}t$ can be read off from the equations of motion, but it also follows directly from the conservation of (2). The vertical spin can be obtained from the horizontal ones by the simple constraint $\langle \hat{\mathbf{J}}_2 \rangle = \langle \hat{\mathbf{J}}_1 \rangle - \langle \hat{\mathbf{J}}_3 \rangle$.

We will first solve the corresponding classical equations, which are as above but with zero covariances. We proceed by perturbation theory with respect to α, so that we have constant spins to zeroth order. Upon repeatedly inserting lower-order solutions in the equations of motion, we proceed up to second order and obtain

$$\mathbf{J}_1^{(2)}(t) = \mathbf{A}_1 + 2\,\alpha\,\mathbf{B}_1\,t + 4\,\alpha^2\,\mathbf{C}_1 t^2 + \cdots \tag{81}$$

$$\mathbf{J}_3^{(2)}(t) = \mathbf{A}_3 + 2\,\alpha\,\mathbf{B}_3\,t + 4\,\alpha^2\,\mathbf{C}_3 t^2 + \cdots\,, \tag{82}$$

where $\mathbf{A}_i$ are free constant vectors. These approximate solutions are valid up to $t \sim 1/\alpha$, respecting the perturbative regime.

The remaining coefficients

$$B_1^x = A_1^z A_3^y - A_1^y A_3^z \tag{83}$$

and

$$C_1^x = -A_1^y B_3^z - B_1^y A_3^z + A_1^z B_3^y + B_1^z A_3^y\,, \tag{84}$$

and cyclic for the y and z components, are also constant but strictly related to $\mathbf{A}_i$. From (80), we have that $\mathbf{B}_1 = -\mathbf{B}_3$ and $\mathbf{C}_1 = -\mathbf{C}_3$. Therefore,

$$\begin{aligned} C_1^x &= (A_1^z A_3^x - A_1^x A_3^z) A_3^z - A_1^y (A_1^x A_3^y - A_1^y A_3^x) - (A_1^x A_3^y - A_1^y A_3^x) A_3^y + A_1^z (A_1^z A_3^x - A_1^x A_3^z) \\ &= -A_1^x \left((A_3^y)^2 + (A_3^z)^2 \right) + A_3^x \left((A_1^y)^2 + (A_1^z)^2 \right) + A_1^z A_3^z \,(A_3^x - A_1^x) + A_1^y A_3^y \,(A_3^x - A_1^x) \\ &= A_3^x\,|\mathbf{A}_1|^2 - A_1^x\,|\mathbf{A}_3|^2 + (\mathbf{A}_1 \cdot \mathbf{A}_3)\,(A_3^x - A_1^x)\,, \end{aligned} \tag{85}$$

which enables the vector coefficient $\mathbf{C}_3$ expressing purely in terms of constant vectors $\mathbf{A}_i = A_i \hat{A}_i$:

$$\begin{aligned} \mathbf{C}_3 = -\mathbf{C}_1 &= A_3^2 \mathbf{A}_1 - A_1^2 \mathbf{A}_3 + (\mathbf{A}_1 \cdot \mathbf{A}_3)(\mathbf{A}_1 - \mathbf{A}_3) \\ &= A_1 A_3 \left[(A_3 + (\hat{A}_1 \cdot \hat{A}_3) A_1) \hat{A}_1 - (A_1 + (\hat{A}_1 \cdot \hat{A}_3) A_3) \hat{A}_3 \right] . \end{aligned} \tag{86}$$

As one possible choice of initial conditions, we could impose that all spins other than the boundary ones (that is, only the vertical spin in the present model) are zero. Therefore, for a single vertex, $\mathbf{J}_2(0) = 0$ and $\mathbf{A}_1 = \mathbf{A}_3$. Equations (83) and (84) then imply that $\mathbf{B}_i = 0 = \mathbf{C}_i$, and all spins remain constant in time. This result, although it is classical, agrees with the trivial dynamics in our minimal minisuperspace model.

We now include moment terms and find solutions of the quantum theory, again perturbative in α. In order to obtain information about the boundary correlations, we should compute quantities such as $\Delta(J_1^x J_3^y)$ from

$$\begin{aligned} \frac{\mathrm{d}\Delta(J_1^x J_3^y)}{\mathrm{d}t} = 2\,\alpha \big(&- \langle \hat{J}_1^y \rangle \Delta(J_3^y J_3^z) + \langle \hat{J}_3^y \rangle \Delta(J_1^z J_3^y) + \langle \hat{J}_1^z \rangle \Delta((J_3^y)^2) - \langle \hat{J}_3^z \rangle \Delta(J_1^y J_3^y) \\ &+ \langle \hat{J}_1^z \rangle \Delta(J_1^x J_3^x) - \langle \hat{J}_1^x \rangle \Delta(J_1^x J_3^z) - \langle \hat{J}_3^z \rangle \Delta((J_1^x)^2) + \langle \hat{J}_3^x \rangle \Delta(J_1^x J_1^z) \big) . \end{aligned} \tag{87}$$

We will also need to consider

$$\begin{aligned} \frac{\mathrm{d}\Delta(J_1^y J_3^z)}{\mathrm{d}t} = 2\,\alpha \big(&- \langle \hat{J}_1^z \rangle \Delta(J_3^z J_3^x) + \langle \hat{J}_3^z \rangle \Delta(J_1^x J_3^z) + \langle \hat{J}_1^x \rangle \Delta((J_3^z)^2) - \langle \hat{J}_3^x \rangle \Delta(J_1^z J_3^z) \\ &+ \langle \hat{J}_1^x \rangle \Delta(J_1^y J_3^y) - \langle \hat{J}_1^y \rangle \Delta(J_1^y J_3^x) - \langle \hat{J}_3^x \rangle \Delta((J_1^y)^2) + \langle \hat{J}_3^y \rangle \Delta(J_1^y J_1^x) \big) . \end{aligned} \tag{88}$$

and

$$\begin{aligned} \frac{\mathrm{d}\Delta(J_1^x J_3^z)}{\mathrm{d}t} = 2\,\alpha \big(&- \langle \hat{J}_1^y \rangle \Delta((J_3^z)^2) + \langle \hat{J}_3^y \rangle \Delta(J_1^z J_3^z) + \langle \hat{J}_1^z \rangle \Delta(J_3^y J_3^z) - \langle \hat{J}_3^z \rangle \Delta(J_1^y J_3^z) \\ &+ \langle \hat{J}_1^x \rangle \Delta(J_1^x J_3^y) - \langle \hat{J}_1^y \rangle \Delta(J_1^x J_3^x) - \langle \hat{J}_3^x \rangle \Delta(J_1^x J_1^y) + \langle \hat{J}_3^y \rangle \Delta((J_1^x)^2) \big) . \end{aligned} \tag{89}$$

For generic initial conditions, these covariances will have the same quadratic form to second order in α, using non-zero initial fluctuations. However, in some specific cases the covariances remain constant, corresponding to stable initial configurations. In particular, we are interested in whether our classical solutions (81) are perturbatively stable within a semiclassical treatment of the quantum dynamics. We must then test whether the covariance terms in (79) change the behavior.

As before, we first assume that the initial expectation values are such that $\mathbf{J}_2(0) = 0$, or $\mathbf{A}_1 = \mathbf{A}_3$. Moreover, we assume fluctuations and covariances as we found them for the corresponding ground state in the minimal minisuperspace model, given by (37): $\Delta((J^x)^2) = \Delta((J^y)^2) = \frac{1}{2}\hbar |\langle \hat{J}^z \rangle|$, now for both horizontal spins in the single-vertex model. This result had been derived by assuming the spin expectation values to point in the z-direction, which we will also do now. Moreover, we have initially zero covariances between components of the two spins.

Assuming the spin expectation values to point in the z-direction leaves only three non-zero terms in (87), two of which vanish for zero initial covariances. We are left with $\langle \hat{J}_1^z \rangle \Delta((J_3^y)^2) - \langle \hat{J}_3^z \rangle \Delta((J_1^y)^2)$. This difference is zero initially because the fluctuations and expectation values on the two horizontal links are the same. Therefore, $\mathrm{d}\Delta(J_1^x J_3^y)/\mathrm{d}t = 0$ and this covariance remains zero to the orders considered here. Similarly, (88) and (89) remain zero, and the covariance terms in (79) do not contribute for this choice of initial values. The configuration corresponding to the minimal minisuperspace model is therefore dynamically stable within the single-vertex model.

For the ground-state configurations of the interacting minisuperspace models we also obtain perturbative stability, but the arguments are slightly different in the case of antiparallel alignment. In (87), the fluctuation terms no longer cancel out because $\langle \hat{J}_3^z \rangle = -\langle \hat{J}_1^z \rangle$. However, there are now two non-zero covariance terms in (87) because $\Delta(J_1^x J_3^x) = \Delta(J_1^y J_3^y) = -\frac{1}{2}\hbar |\langle \hat{J}^z \rangle|$ from (47), where $\langle \hat{J}^z \rangle$ on the right could now refer to either $\hat{J}_1^z$ or $\hat{J}_3^z$ because their absolute values are equal. We now have four non-zero individual terms in (87), but they all cancel out for the given initial values. Again, we have dynamical stability of the ground state.

Before we move on to two vertices, we confirm the ground-state covariances for a spin-1/2 system. For $\gamma < 0$ and spins pointing in the z-direction, the ground state $|J_1^z, J_3^z\rangle = |1/2, 1/2\rangle$ is uncorrelated and has zero spin-spin covariance $\Delta(J_1^x J_3^x)$ as used. For $\gamma < 0$, the ground state is the singlet $2^{-1/2}(|1/2, -1/2\rangle - |-1/2, 1/2\rangle$, which is correlated and leads to $\langle \hat{J}_1^x \hat{J}_3^x \rangle = -\frac{1}{4}\hbar^2$ by standard calculations. Since $\langle \hat{J}_1^x \rangle = 0 = \langle \hat{J}_3^x \rangle$ in this state, we have $\Delta(J_1^x J_3^x) = -\frac{1}{4}\hbar^2 = -\frac{1}{2}\hbar |\langle \hat{J}_3^z \rangle|$ as derived in (47).

2. Two-vertex graph and beyond

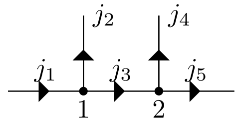

FIG. 5: A one-dimensional two-vertex graph.

For the two-vertex graph, the evolution equations of the boundary spins at the edges are of a similar form as in the case of a single-vertex graph:

$$\frac{\mathrm{d}\langle\hat{J}_1^x\rangle}{\mathrm{d}t} = 2\,\alpha\left(\langle\hat{J}_1^z\rangle\langle\hat{J}_3^y\rangle - \langle\hat{J}_1^y\rangle\langle\hat{J}_3^z\rangle + \Delta(J_1^zJ_3^y) - \Delta(J_1^yJ_3^z)\right) \tag{90}$$

$$\frac{\mathrm{d}\langle\hat{J}_5^x\rangle}{\mathrm{d}t} = 2\,\alpha\left(\langle\hat{J}_3^y\rangle\langle\hat{J}_5^z\rangle - \langle\hat{J}_3^z\rangle\langle\hat{J}_5^y\rangle + \Delta(J_3^yJ_5^z) - \Delta(J_3^zJ_5^y)\right) \tag{91}$$

but now they couple to internal spins. For the internal horizontal spin, $\hat{\mathbf{J}}_3$, we have

$$\begin{aligned}\frac{\mathrm{d}\langle\hat{J}_3^x\rangle}{\mathrm{d}t} &= 2\,\alpha\left((\langle\hat{J}_1^y\rangle + \langle\hat{J}_5^y\rangle)\langle\hat{J}_3^z\rangle - (\langle\hat{J}_1^z\rangle + \langle\hat{J}_5^z\rangle)\langle\hat{J}_3^y\rangle + \Delta(J_1^yJ_3^z) - \Delta(J_1^zJ_3^y) + \Delta(J_3^zJ_5^y) - \Delta(J_3^yJ_5^z)\right)\\ &= -\frac{\mathrm{d}\langle\hat{J}_1^x\rangle}{\mathrm{d}t} - \frac{\mathrm{d}\langle\hat{J}_5^x\rangle}{\mathrm{d}t}\,.\end{aligned} \tag{92}$$

Classical solutions up to second order in α are

$$\mathbf{J}_1^{(2)}(t) = \mathbf{A}_1 + 2\,\alpha\,\mathbf{B}_1\,t + 4\,\alpha^2\,\mathbf{C}_1t^2 + \cdots \tag{93}$$

$$\mathbf{J}_5^{(2)}(t) = \mathbf{A}_5 + 2\,\alpha\,\mathbf{B}_5\,t + 4\,\alpha^2\,\mathbf{C}_5t^2 + \cdots\,. \tag{94}$$

With (92), the solution for the internal spin is

$$\mathbf{J}_3^{(2)}(t) = \mathbf{A}_3 + 2\,\alpha\,\mathbf{B}_3\,t + 4\,\alpha^2\,\mathbf{C}_3t^2 + \cdots\,, \tag{95}$$

where $\mathbf{B}_3 = -(\mathbf{B}_1 + \mathbf{B}_5)$ and $\mathbf{C}_3 = -(\mathbf{C}_1 + \mathbf{C}_5)$ follow from the conversation of the total spin. The coefficients $\mathbf{C}_3$ can be obtained by replacing $\mathbf{A}_1 \to \mathbf{A}_1 + \mathbf{A}_5$ in (86):

$$\mathbf{C}_3 = A_3^2\,(\mathbf{A}_1 + \mathbf{A}_5) - |\mathbf{A}_1 + \mathbf{A}_5|^2\mathbf{A}_3 + \left[(\mathbf{A}_1 + \mathbf{A}_5)\cdot\mathbf{A}_3\right](\mathbf{A}_1 + \mathbf{A}_5 - \mathbf{A}_3)\,. \tag{96}$$

For the vertical spins $\mathbf{J}_2$ and $\mathbf{J}_4$ we then have

$$\mathbf{J}_2^{(2)}(t) = \mathbf{J}_1^{(2)}(t) - \mathbf{J}_3^{(2)}(t) = \mathbf{A}_1 - \mathbf{A}_3 + 2\,\alpha\,(2\mathbf{B}_1 + \mathbf{B}_5)\,t + 4\,\alpha^2\,(2\mathbf{C}_1 + \mathbf{C}_5)t^2 + \cdots \tag{97}$$

$$\mathbf{J}_4^{(2)}(t) = \mathbf{J}_3^{(2)}(t) - \mathbf{J}_5^{(2)}(t) = \mathbf{A}_3 - \mathbf{A}_5 - 2\,\alpha\,(\mathbf{B}_1 + 2\,\mathbf{B}_5)\,t - 4\,\alpha^2\,(\mathbf{C}_1 + 2\mathbf{C}_5)t^2 + \cdots\,. \tag{98}$$

We need equations of motion for covariances in order to extend the classical solutions to the semiclassical regime. These equations for correlations of neighboring spins are very similar to the equations of the single-vertex case, but we can now also have changing values of more distant spins, such as

$$\begin{aligned}\frac{\mathrm{d}\Delta(J_1^xJ_5^y)}{\mathrm{d}t} = 2\,\alpha\big(&-\langle\hat{J}_1^y\rangle\Delta(J_3^zJ_5^y) + \langle\hat{J}_3^y\rangle\Delta(J_1^zJ_5^y) + \langle\hat{J}_1^z\rangle\Delta(J_3^yJ_5^y) - \langle\hat{J}_3^z\rangle\Delta(J_1^yJ_5^y)\\ &-\langle\hat{J}_3^x\rangle\Delta(J_1^xJ_5^z) - \langle\hat{J}_5^z\rangle\Delta(J_1^xJ_3^x) + \langle\hat{J}_3^z\rangle\Delta(J_1^xJ_5^x) + \langle\hat{J}_5^x\rangle\Delta(J_1^xJ_3^z)\big)\end{aligned} \tag{99}$$

for the boundary spins of the two-vertex graph. For spin expectation values pointing in the z-direction, there are four potentially non-zero terms, $\langle\hat{J}_1^z\rangle\Delta(J_3^yJ_5^y) - \langle\hat{J}_5^z\rangle\Delta(J_1^xJ_3^x) + \langle\hat{J}_3^z\rangle(\Delta(J_1^xJ_5^x) - \Delta(J_1^yJ_5^y))$. The covariances are zero unless we have a state with antiparallel orientation of neighboring spins. The four remaining terms then cancel out because $\langle\hat{J}_1^z\rangle = -\langle\hat{J}_3^z\rangle = \langle\hat{J}_5^z\rangle$ and $\Delta(J_1^xJ_5^x) = \Delta(J_1^yJ_5^y)$.

At this point we have to be careful when we compare minisuperspace configurations with the exact ground state(s) of the two-vertex model, which is an odd-number spin chain and has strong finite-size effects. Diagonalizing the Hamiltonian $\alpha(\hat{\mathbf{J}}_1\cdot\hat{\mathbf{J}}_3+\hat{\mathbf{J}}_3\cdot\hat{\mathbf{J}}_5)$ in the spin-1/2 case leads to the degenerate ground states $\psi_1 = 6^{-1/2}(|-1/2,1/2,1/2\rangle - 2|1/2,-1/2,1/2\rangle + |1/2,1/2,-1/2\rangle)$ and $\psi_2 = 6^{-1/2}(|-1/2,-1/2,1/2\rangle - 2|-1/2,1/2,-1/2\rangle + |1/2,-1/2,-1/2\rangle)$.

Choosing the first state to be specific, one can then compute the expectation values $\langle \hat{J}_1^z\rangle = \frac{1}{3}\hbar = \langle \hat{J}_5^z\rangle$ and $\langle \hat{J}_3^z\rangle = -\frac{1}{6}\hbar$. These values are next-neighbor antiparallel, but do not obey $\langle \hat{J}_1^z\rangle = -\langle \hat{J}_3^z\rangle = \langle \hat{J}_5^z\rangle$. Moreover, we have the covariances $\Delta(J_1^xJ_3^x) = -\frac{1}{6}\hbar^2 = \Delta(J_1^yJ_3^y)$ which do not obey (47), and we have $\Delta(J_1^zJ_3^z) = -\frac{1}{9}\hbar^2$. There are also distant covariances such as $\Delta(J_1^xJ_5^x) = \frac{1}{12}\hbar^2 = \Delta(J_1^yJ_5^y)$ and $\Delta(J_1^zJ_5^z) = -\frac{1}{36}\hbar^2$. Although these values do not show the generic antiparallel behavior, one can still see that all terms in (99) cancel out.

The expressions for $\mathbf{B}_i$ and $\mathbf{C}_i$ in terms of $\mathbf{A}_i$ are very similar to those in (83) and (84), just with different labels. (The relation (85) for $\mathbf{C}_i$ in terms of $\mathbf{A}_i$, however, has a different form for multiple vertices because it has been derived for the single-vertex graph using $\mathbf{B}_3 = -\mathbf{B}_1$.) For ground-state configurations of minisuperspace models we therefore obtain the same cancelations as in the single-vertex model because these considerations depend only on the expectation values and moments of horizontal spins on neighboring links. In the two-vertex model, one can choose initial conditions such that all vertical spins have zero expectation values and covariances with any other spins. Therefore, $\mathbf{J}_4^{(2)}(0) = \mathbf{J}_2^{(2)}(0) = 0$ and $\mathbf{A}_1^{(0)} = \mathbf{A}_3^{(0)} = \mathbf{A}_5^{(0)}$. This implies that $\mathbf{B}_i = 0 = \mathbf{C}_i$, and all spins remain constant in time as in the single-vertex.

The same pattern is then realized also for more vertices, and we conclude that the various ground states are dynamically stable. The minisuperspace models with Hamiltonians $\hat{H}^{(1)}_{\rm mini}$ and $\hat{H}^{(2)}_{\rm mini}$ predict the same ground-state configurations as the discrete theory, and these minisuperspace states are therefore stable within the discrete model. The minisuperspace model with Hamiltonian $\hat{H}^{(3)}_{\rm mini}$, however, is unstable. It not only predicts a ground-state configuration that does not agree with any fundamental ground state, it also has unstable dynamics when its ground-state configuration is embedded in a fundamental spin chain. Unlike in the other two models, there are then three neighboring spins $\mathbf{J}_{2i_{\rm c}-3}$, $\mathbf{J}_{2i_{\rm c}-1}$ and $\mathbf{J}_{2i_{\rm c}+1}$ around the central vertex at $i_{\rm c} = (N+1)/2$, such that $\langle \hat{\mathbf{J}}_{2i_{\rm c}-3}\rangle = \langle \hat{\mathbf{J}}_{2i_{\rm c}-1}\rangle = -\langle \hat{\mathbf{J}}_{2i_{\rm c}+1}\rangle$, and no cancellations happen in (99).

VI. POSSIBLE IMPLICATIONS FOR QUANTUM COSMOLOGY

We have analyzed four different minisuperspace models of a spin system related to the Heisenberg spin chain. The first model resembles the traditional construction of quantum-cosmology models in which only homogeneous degrees od freedom without spatial variation are considered. In the present context, such a model does not capture the dynamical nature of coupled spins. The second model is a more recent proposal to apply mathematical constructions of condensate states to quantum cosmology. The dynamics is then quite different from a traditional minisuperspace model. In particular, a non-trivial dynamics is now realized even though the degrees of freedom included in the model are the same. The remaining models incorporate additional degrees of freedom in two different ways. They both lead to non-trivial dynamics and in this sense improve the reduction. However, details of ground states and stability are very different in the two models, indicating that good knowledge of the fundamental dynamics is important for a successful construction of minisuperspace models. This conclusion is our first result in an application to quantum cosmology: Traditional minisuperspace models start with a reduction of the classical theory, and then quantize by using some ideas related to candidates for full quantum gravity. But if they do not directly address the full dynamics, they may be in danger of missing crucial information, just as our model Hamiltonian $\hat{H}^{(4)}_{\rm mini}$ does compared with $\hat{H}^{(3)}_{\rm mini}$.

Our successful Hamiltonian $\hat{H}^{(3)}_{\rm mini}$ is similar to the sublattice Hamiltonians introduced in [28] for Heisenberg spin chains. The detailed analysis of this paper showed that such an approximation, for given chain length N, is better for larger spin lengths s on the chain. If a similar statement is true for the dynamics of quantum cosmology, it would indicate that a coarse-graining procedure applied before symmetry reduction could improve models of quantum cosmology, for such a procedure would combine the small fundamental spins of $s = 1/2$ to systems of larger spin lengths. Coarse graining in gravitational, and in particular background-independent theories, is not well-understood, but proposals have been made for instance in [4, 5]. It is also encouraging that sublattice structures can be found in spin systems with more than one dimensions [42, 43].

Our minisuperspace Hamiltonian $\hat{H}^{(4)}_{\rm mini}$ is based on a construction similar to the separate-universe approximation of classical cosmology. Its failure to model properties of ground states and stability indicates that it is not a good quantum approximation for all kinds of fundamental dynamics. Its lack of stability is of particular concern, resulting from the fact that in this model the minisuperspace ground state does not correspond to the ground state of the discrete theory. We therefore have provided an explicit example of important fundamental properties not captured by a minisuperspace model. Such models are unstable if energy can be exchanged with an environment, and one might conclude that they are unreliable.

However, not just ground states but also excited states may be stable in an isolated system if no energy can be exchanged with an environment. This is the situation usually realized in models of quantum cosmology, where the state represents the whole universe with nothing outside. The stability of excited states then results in a large variety of candidates for homogeneous configurations. Nevertheless, some caution toward such minisuperspace states used in quantum cosmology is still required: Our spin system can be taken as a model for quantum space, in which near homogeneity should be possible and stable under evolution. However, there should also be matter, with additional degrees of freedom that could be placed on the same graph used for our spin chain but representing a different system of degrees of freedom. There could then be energy exchange between the spin system analyzed here and the new matter system. We would be back at the question whether a homogeneous spin configuration, representing quantum space, can be stable within the coupled system if it does not capture the correct ground state. The question of how matter is coupled to quantum space therefore seems important in the context of the emergence and stability of correlated quantum-cosmology states.

VII. DISCUSSION

We have analyzed a discrete spin model with different methods used in recent years in canonical quantum gravity. Our aim is to test the latter, rather than revealing new properties of spin systems in general. We have found new results in three different classes: minisuperspace truncations, effective theories, and dynamical long-range correlations.

We have derived different minisuperspace models directly from the discrete theory, which is a new procedure compared with the usual construction of minisuperspace models by quantizing homogeneous configurations of a continuum theory. Several novel features could be seen, for instance the existence of different minisuperspace models of the same discrete theory, paralleling the existence of different continuum limits of one discrete theory. In our specific constructions, starting with the discrete theory has the advantage that no problems related to infrared scales of traditional minisuperspace models occur. We have seen that it can be of advantage to keep more degrees of freedom in a minisuperspace model than simple homogeneous configurations would suggest, in particular when non-trivial dynamical properties should be obtained. Good knowledge of the fundamental theory is required in order to select a reliable minisuperspace model. Alternatively, non-trivial dynamics can be obtained by using condensate states, as employed also in cosmological models of group-field theories [19–26].

In several examples of our minisuperspace models, we have computed canonical effective potentials and equations and found good agreement with known ground-state energies and configurations. Our results provide further support for the canonical effective methods proposed for quantum gravity in [36, 37], with an extension to the computation of effective potentials in [44].

We have also analyzed the discrete spin system directly, with an emphasis on properties that should be important for the dynamical building-up of long-range correlations as they are likely to be relevant for the dynamical emergence of states that may be described by minisuperspace models, a question related to the continuum limit of discrete quantum gravity. Our analysis, based on rather general properties of the underlying equations of motion, suggests that such features can only be seen in a full non-perturbative treatment of the dynamics. An application of our results in a quantum-cosmological context has further highlighted important questions which are usually not addressed in minisuperspace constructions, related for instance to coarse graining and stability.

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