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Black Hole Ringing, Quasinormal Modes, and Light Rings

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Modelling of gravitational waves from binary black hole inspiral has played an important role in the recent observations of such signals. The late-stage ringdown phase of the gravitational waveform is often associated with the null particle orbit (“light ring”) of the black hole spacetime. With simple models we show that this link between the light ring and spacetime ringing is based more on the history of specific models than on an actual constraining relationship. We also show, in particular, that a better understanding of the dissociation of the two may be relevant to the astrophysically interesting case of rotating (Kerr) black holes.

Introduction—Black hole ringing or ringdown (BHR) has been important in recent gravitational wave identifications [1, 2]. The late time waveform is typically identified with a quasinormal (QN) frequency (ω_{QN}). This is an association that took root in the research community almost a half century ago [3–5], and one that is frequently appropriate. But this is not always the case, and it has the great potential to be misleading. In this paper we want to point out the several physical and mathematical elements that are bundled together in the currently accepted viewpoint, to disentangle these elements, and to emphasize the potential for confusion and its relevance to current research.

First, QN frequencies are the complex eigenvalues of single frequency modes; in our case, the modes of black hole perturbations. BHR describes the damped (due to outgoing radiation) oscillation of a black hole. “Light rings” (LRs) are the orbits of massless particles in a spacetime. These three topics often overlap, but they are fundamentally independent.

Current viewpoints are rooted in the history of the field. In 1970 Vishveshwara [3] did computer simulations that revealed a characteristic damped oscillation of quadrupole perturbations of a Schwarzschild hole. In the early 70s, Price [6], working with a simple toy model for black hole mathematics, found complex poles of the transmission function that yielded characteristic damped oscillations. Soon after, Press [4], by computationally evolving perturbations, found damped Schwarzschild oscillations for perturbations of high multipole moment ℓ . Of particular importance to the subject’s history is the paper by Goebel [5], in which he demonstrated that Press’s results could be understood in terms of the angular velocity and Lyapunov exponent for the orbits of massless particles near the LR. The argument was heuristic, but was very appealing, and gave excellent approximations, even for only moderate values of ℓ . In the early 70s this LR interpretation was frequently mentioned in papers [7, 8] on black hole radiation.

Chandrasekhar and Detweiler [10] seem to have been the first to treat QN phenomena as unusual eigenmodes,

and to compute the Schwarzschild black hole’s ω_{QN} values as complex eigenvalues. (The fact that they can be complex is possible because they are eigenvalues of a non-self-adjoint problem [11].)

By the late 70s the idea of QNR seems to have become common in considerations of black hole sources of radiation [12], with LRs as part of the conceptual background.

One possibility of confusion lay in the fact that black hole studies encompass two “potentials” with very different meaning. The analysis of null orbits in the Schwarzschild spacetime can be understood in terms of an “effective potential” for radial motion [13]. The peak of this potential indicates the location of an unstable light ring, and the curvature at the peak determines the Lyapunov exponent at that ring. By contrast, there is also a “curvature potential,” that arises in the wave equation for perturbation multipoles, such as the Zerilli equation [14]. In 1985 Schutz and Will [15], following on related work by Ferrari and Mashhoon [16], demonstrated that good approximations for values of ω_{QN} could be found by applying the WKB approximation to the peak of the curvature potential. The WKB is understood to be a high frequency or eikonal approximation, conceptually linking this method to the null geodesics of the spacetime, and thereby to the LRs. In the high ℓ application on which Schutz and Will focused, the curvature potential was dominated by its quasi-classical centrifugal part. The result was the approximate equivalence of the location of the peaks of the effective potential and the curvature potential, and a bolstering of the apparent link between QNR and LRs. Yet other approximations, such as the “optical geometry” approach [17], have been based even more directly on the eikonal limit.

The outcome of this multi-decade long general association of black hole QNR and LRs is that it has been in routine use by the community in the context of some problems (see Ref. [18, 19] and references therein), and is considered as an idea that may aid in understanding the phenomenology of the generation of gravitational waves in black hole binary inspiral (see Ref. [20] and references therein). Moreover, it plays an important role [21] in the

context of the effective-one-body (EOB) models [22] that are currently in use by the gravitational observatories to generate the waveform template banks that are needed for the signal searches.

In this paper our aim is to disentangle BHR, QNR and LRs, and to show why such a clarification might be important to understanding the phenomenology of gravitational waves from binary black hole inspiral. Our clarification will include the use of models very intentionally designed to break the connection of the kinematic (LR) and wave (QNR) aspects of oscillations. Since it is the WKB method that shows the connection between these two aspects, it is to be expected that the WKB method fails for these models.

Simple Models—Let us first take on the association between the damped oscillations of fields in a spacetime, and QNR. The disconnect between BHR and QN frequencies was first (to our knowledge) and most dramatically (in our opinion) demonstrated by Nollert [24] who replaced the BH mathematical problem (more specifi-

cally, the curvature potential) with an approximation using a set of steps. The result was a problem with a vastly different spectrum of QN frequencies, but almost identical BHR. Other such models have also been presented in the recent research literature [23, 25].

Turning to the more tangled issue of QNR and LRs, we shall exploit the convenience of gravity-free spherically symmetric wormholes as simple examples. The metric for such a spacetime is

$$ds^2 = -dt^2 + dx^2 + r^2(x) (d\theta^2 + \sin^2 \theta d\phi^2). \quad (1)$$

All of the properties of this wormhole spacetime are contained in the function $r(x)$.

It is straightforward to show that the function $V_{\text{eff}} \equiv 1/r^2(x)$ serves as an effective potential in the same sense as the effective potential for the Schwarzschild spacetime [13]. In particular a circular null orbit requires that $dr/dx = 0$, and such an orbit is unstable if $d^2r/dx^2 > 0$. For our first example, we choose our $r(x)$, and thereby choose the effective potential V_{eff} , to be given by

$$V_{\text{eff}}(x) \equiv \frac{1}{r^2(x)} = \frac{\pi}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}} + \frac{k}{2} \right) \text{Tan}^{-1} \left(1 - \sqrt{2}x^2 \right) - \left(\frac{1}{\sqrt{2}} + \frac{k}{2} \right) \text{Tan}^{-1} \left(1 + \sqrt{2}x^2 \right). \quad (2)$$

This function is symmetric in x and has the property that $r \rightarrow x + \mathcal{O}(1/x)$ as $x \rightarrow \pm\infty$, thus the spacetime is asymptotically flat. This wormhole has a minimum radius at $x = 0$, where there is an unstable LR. If $k < 2$, this is the only LR, but for $k > 2$ there are two other unstable LRs located symmetrically around the central LR.

The effective potential determines the curvature potential for a given multipole. For a scalar multipole perturbation, $\Psi = r^{-1}\psi(x, t)Y_{\ell, m}(\theta, \phi)$, the sourceless wave equation $g^{\mu\nu}\Psi_{,\mu;\nu} = 0$ takes the form

$$\frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial x^2} + V_{\text{curv}}(x)\psi = 0, \quad (3)$$

where the curvature potential V_{curv} turns out to be

$$V_{\text{curv}} = \ell(\ell + 1)V_{\text{eff}}(x) - \frac{1}{2V_{\text{eff}}} \left[\frac{d^2 V_{\text{eff}}}{dx^2} - \frac{3}{2V_{\text{eff}}} \left(\frac{dV_{\text{eff}}}{dx} \right)^2 \right]. \quad (4)$$

A particular example is shown in Fig. 1: the effective and curvature potentials for $k = 1.95$ and 2.05 . The top graph shows V_{eff} for each of the k values. Of particular importance is the fact that for the smaller k value there is only the $x = 0$ central LR; for the larger k there are additional unstable LRs at $x \approx \pm 1.12$. The lower graph shows the $\ell = 3$ curvature potentials; these potentials determine the values of ω_{QN} . The plot shows that the curvature potentials are qualitatively the same,

even quantitatively similar, independent of the number of LRs.

Since the curvature potential determines the oscillations of the spacetime, we would expect that the two models in Fig. 1 will have very similar oscillations. Figure 2 shows that this expectation is met. The oscillations shown, furthermore, fit very well to waveforms with the QN values $\omega_{\text{QN}} = 2.440 + i0.3203$ (for $k = 1.95$) and $\omega_{\text{QN}} = 2.256 + i0.2526$ (for $k = 2.05$) found by an eigenvalue search in the complex plane adopted from that of Ref. [10]. The waveforms are clearly qualitatively similar, and quantitatively not very different. They are, furthermore, clearly QNR.

It is of some interest to apply the Schutz-Will [15] WKB approximation to the models we consider using the formula [15]

$$\omega_{\text{QN}}^2 = V_{\text{curv}} - i\left(n + \frac{1}{2}\right) \sqrt{-2d^2 V_{\text{curv}}/dx^2}, \quad (5)$$

where it is understood that V_{curv} is to be evaluated at the peak. In doing this it is important to keep in mind that the limitation of that approximation is that, in an equation with the form

$$\frac{\partial^2 \psi}{\partial x^2} + (\omega^2 + V_{\text{curv}}(x))\psi = 0, \quad (6)$$

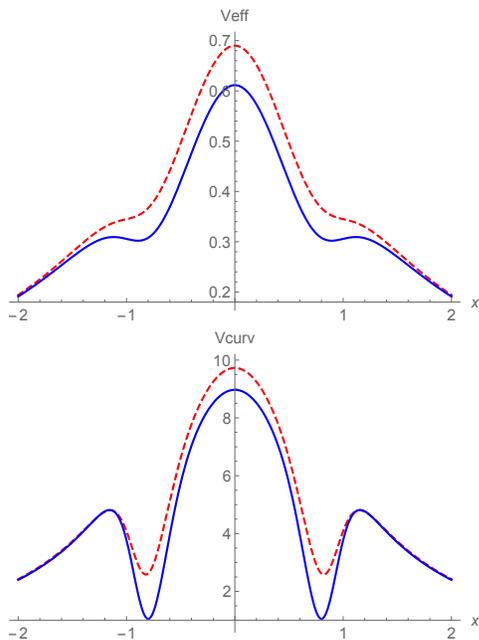


FIG. 1: The effective and $\ell = 3$ curvature potentials for models with $k = 1.95$ (dashed) and $k = 2.05$ (solid).

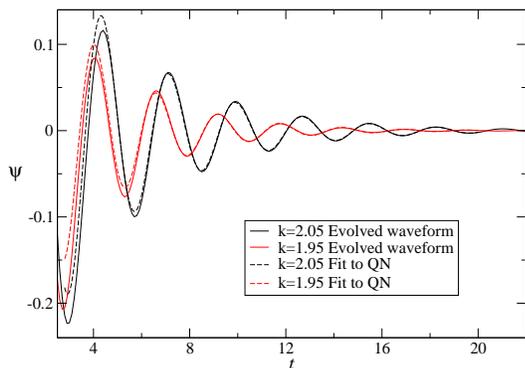


FIG. 2: Waveforms for the Eq. (2) model, with k parameters 1.95 and 2.05. The solid curves show the computed evolution of initial data, while the dashed curves show the fit to the least damped QN oscillation of each model.

the relationship

$$\left| \frac{dV_{\text{curv}}(x)}{dx} \right| \ll [\omega^2 + V_{\text{curv}}(x)]^{3/2} \quad (7)$$

holds throughout the region to which the approximation is applied [9]. We can simplify this slightly by using the fact that the value of the WKB frequency will always be roughly equal to the peak of the curvature potential and this we can approximate a lower bound on the right side of Eq. (7) by $|\omega|^3$.

In applying this criterion to the models illustrated in Fig. 1 we can see immediately that the worst violation occurs around $x \approx \pm 0.63$ where $|dV_{\text{curv}}(x)/dx|$ is 20.3 and 16.2, for the $k = 2.05$ and $k = 1.95$ cases, respectively. The values of $|\omega|^3$, on the other hand, are 11.5 and 14.5 for the $k = 2.05$ and $k = 1.95$ cases, respectively, so the condition for validity of the WKB approximation is badly violated. The actual values given by the Schutz-Will approximation is in error by around 50%, which is smaller than what one might have guessed.

For another example, we choose a scalar dipole ($\ell = 1$), and k values 0.2 and 0.8. The curvature potentials are shown in Fig. 3. The location of the larger peak should be noted. For $k = 0.8$ the larger peak is at $x = 0$; for $k = 0.2$, they are at $x \approx \pm 1$.

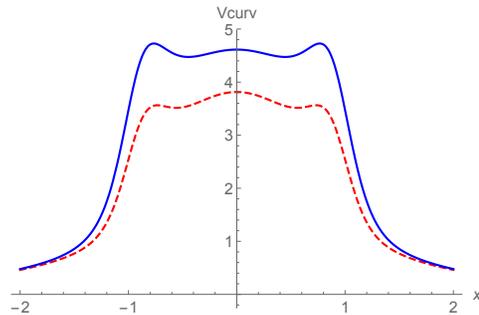


FIG. 3: The curvature potentials for our wormhole models with $k = 0.8$ (dashed) and $k = 0.2$ (solid).

With a search in the frequency plane using the method of Chandrasekhar and Detweiler [10] we have been able to achieve considerable precision in determining the ω_{QN} for these two cases. Figure 4 shows the evolution of initial data in each of the backgrounds, and shows that the fit of the late-time evolved waveforms to these ω_{QN} is excellent. There is, then, no question that the ω_{QN} are meaningful, and that they describe the waveform that develops in these spacetimes. Table I compares these ω_{QN} to values given by WKB and LR approaches. The second column gives true least damped ω_{QN} , the frequencies found by the above mentioned frequency domain eigenvalue search, which are also the frequencies that fit the evolved waveform. The third column and fourth columns give WKB results. The third column gives the result of applying this for the peak at $x = 0$; the fourth column gives the result for the peaks at $|x| \approx 1$. Lastly, the fifth column is the prediction of the ω_{QN} based on the LR analysis given by Cardoso *et al.* [19] (see, in particular, Eqs. (2) and (40) of that reference). It is worth emphasizing again, that this LR analysis is based on the kinematic potential, *not* the curvature potential. We have applied that LR analysis at $x = 0$, the location of the only LR in these models. (Recall that the peaks in V_{curv} at $x \approx \pm 1$ are not related to LRs; for the functions $r(x)$ given by Eq. (2) models with $k < 2$ have only the central LR.)

k	Actual	WKB at $x = 0$	WKB at $x \approx \pm 1$	LR
0.2	$2.3638 + i 0.2901$	$2.1646 + i 0.2686$	$2.3243 + i 0.8214$	$1.4368 + i 0.3480$
0.8	$2.1364 + i 0.3633$	$1.9841 + i 0.3501$	$1.9966 + i 0.6502$	$1.2622 + i 0.3962$

TABLE I: Computed QN frequencies from different approaches, as described in the text, for the $k = 0.2, 0.8$ wormhole model cases. It is clear that while the WKB approximation performs reasonably well (especially for the real part) and LR approach performs very poorly.

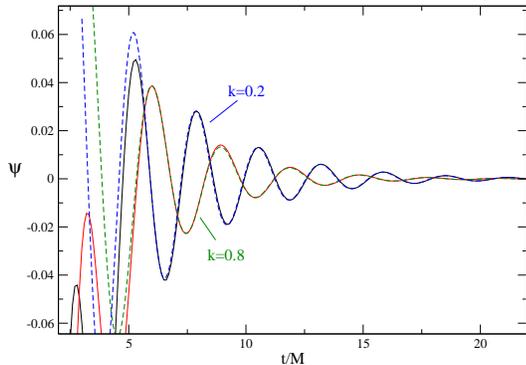


FIG. 4: The ringdown phase of the time-domain scalar field waveforms in our example wormhole spacetimes for $k = 0.8$ and $k = 0.2$, matched to our precision QNR results from the frequency plane eigenvalue search.

The results in this table are quite telling. The WKB approximation applied at the $x = 0$ peak gives reasonable approximations for the true values. The WKB approximation applied to the peaks gives more or less equally good approximations for the real part of the ω_{QN} , but due to the narrowness of the peaks, gives values of the damping that are significantly too high. It is interesting that for the $k = 0.2$ model, the peaks at $x \approx \pm 1$ are higher than at $x = 0$, and yet the WKB approximation at $x = 0$ is better. An intuitive explanation for this is that the wavelength of the relevant mode is much wider than the entire potential; thus, the results are likely to be relatively insensitive to the small scale peaks.

What is most important to notice is that the approximation based on the LR is far from correct. This is not surprising in retrospect, since the kinematic potential given by Eq. (2) is dramatically different from the curvature potential in Eq. (4), shown in Fig. 3. Much of the published literature in this context is in the “geometric optics” i.e., large ℓ , limit wherein the curvature potential is dominated by the centrifugal term. We have considered here curvature potentials that are not the single-peak potentials for which the LR argument works well. And it is the curvature potential that governs waves, and hence

determines everything about the QN phenomena for a spacetime.

The previous examples help to weaken the link between LRs and oscillations in curved spacetimes. Next, we weaken that even further with a model defined by

$$V_{\text{eff}} = \frac{1}{r^2(x)} = \frac{e^{-2x/3}}{4 + e^{-2x/3}} + \frac{1}{12 + x^2 + x^4}, \quad (8)$$

and pictured in Fig. 5. As the figure shows, this wormhole approaches a cylinder (constant radius $r = 1$) asymptotically, as $x \rightarrow -\infty$. What is most important, and immediately apparent in the top plot in Fig. 5, is that there is no LR, since there is no point at which $dr/dx = 0$. Despite this, the associated curvature potential for monopole waves, shown in Fig. 5 (bottom) has the peaked form that suggests that the spacetime will have QNR. Figure 6

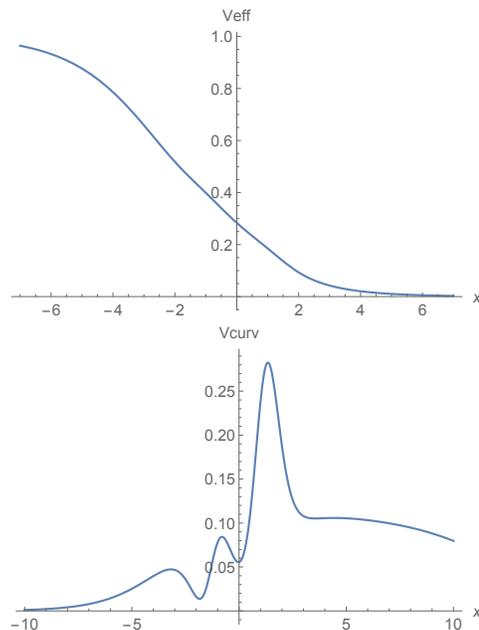


FIG. 5: The effective potential (top), and $\ell = 0$ curvature potential (bottom) for scalar waves in a spacetime defined by Eq. (8). This model has no LR, but exhibits QNR.

shows that this is indeed the case. The solid curve shows the result of the computer evolution of scalar initial data. That evolved waveform is compared to the fit to a QN

oscillation with $\omega_{\text{QN}} = 0.356 + i0.060$, the value found with a search in the complex plane for the least damped monopolar QN mode. The figure leaves no doubt that this spacetime, despite the absence of *any* LR, exhibits QN ringing [26].

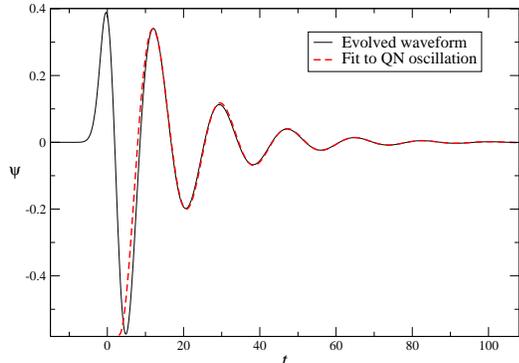


FIG. 6: Evolution of initial data for a wormhole model with no LR.

For this “no LR” model, we can apply the WKB approximation to the major peak of the curvature potential at $x \approx 1.35$. We expect the WKB model to fail, and it is no surprise that it fails badly, giving the $0.692731 + i0.4442$ which is off by a considerable factor from the correct value.

Possible relevance to Kerr ringdown—Our goal in this section is to show that the unreliable LR/QNR association may mislead research in the phenomenology of binary inspiral. As a plausible example of the failure of this association we present, in Fig. 7, the result of a particle perturbation evolution code [27] representing a scalar-charged particle spiraling into a Kerr black hole with $a/M = 0.9$. Two curves are presented. One shows the scalar radiation generated by a prograde geodesic (“forward”) equatorial orbit for per-particle-mass energy $E = 0.84$, and angular momentum $L/M = 2.1$. The second curve is the radiation from the (non-geodesic) “reversed” orbit resulting from the reversal of the angular direction of the “forward” orbit. Both orbits start with the particle located at $r = 2.5M$. The “junk radiation” attending the birth of the particle quickly dissipates, and is irrelevant to our considerations here. From the published results of Berti *et al.* [28] we have that for such a black hole, the least damped quadrupole scalar ω_{QN} are $(0.78164 + i0.06929)/M$, for $m = +2$, and $(0.38780 + i0.09379)/M$ for $m = -2$. The figure shows the excellent fit of the late-time forward/reverse radiation to the $m = +2/-2$ modes respectively.

It is intuitively appealing that the retrograde, reversed orbit generates $m = -2$ QNR, and it is tempting to associate this with the retrograde LR. But there is an impor-

tant barrier to this association. QNR is a phenomenon of the curvature potential, and the excitation of QNR by an infalling particle can be traced to the passage of the particle past a feature of the curvature potential [20]. In the common viewpoint this is equivalent to passing the LR (see Ref. [20] and references therein). This viewpoint does not appear to apply to the example in Fig. 7. For $a/M = 0.9$, the retrograde LR is at $r = 3.91027M$, and our particle source, whether in forward or reverse orbit, starts at $r = 2.5M$ and hence spends no time at the retrograde LR, the location that one might associate with the retrograde QNR. Note that the prograde LR is at $1.55785M$. The short vertical bar in Fig. 7 shows the time, $\sim 1070M$ at which the particle reaches this prograde LR. One can interpret this to mean that the radiation from the forward orbit is associated with the prograde LR, but how does one explain the fact that the reverse-orbit radiation starts at around the same time?

One plausible answer is simply that the QNR/LR association must be treated with caution and even skepticism.

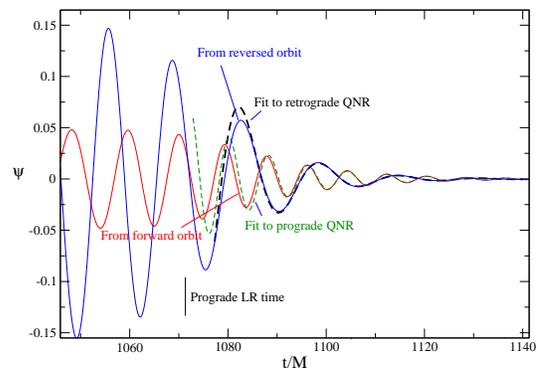


FIG. 7: Ringdown stage scalar field waveforms from a test particle with scalar charge falling into a Kerr black hole with $a/M = 0.9$ on a prograde geodesic, and also on a retrograde “reversed” non-geodesic path. It is clear that the prograde case excites the $+m$ QNR, while in the retrograde case the $-m$ mode is excited.

Conclusions—In many cases the oscillations of a black hole, or other spacetime, is a manifestation of QNR, a complex eigenmode for radiation in the spacetime. In the development of black hole perturbation studies quasinormal ringing has been successfully linked to the orbits of null particles, the LRs. We have shown that this is a weak link by presenting models in which the QNR is clearly not linked to such LRs, even a model in which there is very clear QNR in a spacetime with no LR.

We also noted that this dissociation of QNR and LRs may explain some of the phenomenology of binary inspiral radiation, especially in connection with the QN mode

that has been thought to be related to the retrograde light ring.

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