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One-loop pseudo-Goldstone masses in the minimal $SO(10)$ Higgs model

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We calculate the prominent perturbative contributions shaping the one-loop scalar spectrum of the minimal renormalizable non-supersymmetric $SO(10)$ Higgs model whose unified gauge symmetry is spontaneously broken by an adjoint scalar. Focusing on its potentially realistic $45 \oplus 126$ variant in which the rank is reduced by a VEV of the 5-index antisymmetric self-dual tensor, we provide a thorough analysis of the corresponding Coleman-Weinberg one-loop effective potential, paying particular attention to the masses of the potentially tachyonic pseudo-Goldstone bosons (PGBs) transforming as $(1, 3, 0)$ and $(8, 1, 0)$ under the Standard Model gauge group. The results confirm the assumed existence of extended regions in the parameter space supporting a locally stable SM-like quantum vacuum inaccessible at the tree-level. The effective potential (EP) tedium is compared to that encountered in the previously studied $45 \oplus 16$ $SO(10)$ Higgs model where the polynomial corrections to the relevant pseudo-Goldstone masses turn out to be easily calculable within a very simplified purely diagrammatic approach.

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I. INTRODUCTION

The upcoming generation of very large volume detectors such as Hyper-K [1] and/or DUNE [2] is not only a blessing for the neutrino community, but it is also likely to provide a great deal of information to other branches of particle physics research. Concerning, in particular, the possible baryon number non-conservation signals such as proton decay, the sensitivity of the current searches may be improved by as much as one order of magnitude, reaching up to about 10^{35} years for the proton lifetime.

Unfortunately, this steady progress is not matched by any significant developments on the theory side. As a matter of fact, the existing proton lifetime estimates – usually made in the context of grand unified theories (GUTs) [3], the most economical scheme for addressing these issues in the standard quantum field theory context – are typically plagued by theoretical uncertainties stretching over many orders of magnitude, see, for instance, Table II in [4] and references therein. Needless to say, this is way too poor to make any real benefit from the expected experimental sensitivity improvements (unless we were lucky and a clear signal of baryon number violation was observed; however, even in such a case it would be extremely difficult to distinguish among even the simplest models, let alone more complicated settings).

There are two general reasons behind this unsatisfactory situation:

1. The main quantities governing the proton lifetime estimates in GUTs, in particular the unification scale M_{GUT} (which, in the non-supersymmetric context enters the rates quartically) and the flavour structure of the relevant baryon and lepton number violating currents, are very difficult to estimate with good-enough accuracy from just the low-energy data we have access to. As for the former, at least a two-loop renormalization-group-equation (RGE) analysis is necessary to keep the error in M_{GUT} at a reasonable level which, however, assumes a detailed knowledge of the relevant threshold corrections (and, hence, the theory spectrum); for the latter one often needs information that is inaccessible even in principle at the electroweak scale, such as, for instance, the shape of the right-handed (RH) rotations in flavour space.
2. Since the unification scale turns out to be only a few orders of magnitude below the Planck scale M_{Pl} , the (a-priori unknown) effects of the M_{Pl} -suppressed higher-dimensional operators need not be negligible [5–7]; in practice, they often turn out to be comparable in size to those of the one-loop thresholds and, as such, the associated uncertainties tend to ruin the efforts to go beyond the leading order in precision anyway.

Nevertheless, there seems to exist an exception to these empirical restrictions, namely, the minimal renormalizable $SO(10)$ grand unified theory [8–10] in which the unified gauge symmetry is spontaneously broken by the 45-dimensional scalar. This choice turns out to be rather special as it inhibits the most dangerous class of the leading order (i.e., $d = 5$) gravity-induced operators and, hence, also the corresponding theoretical uncertainties in the determination of M_{GUT} .

Remarkably enough, this scenario has not been considered for more than 30 years since its first formulation at the beginning of 1980s due to notorious tachyonic instabilities [11–13] appearing in its scalar sector along essentially all potentially realistic symmetry breaking chains; it was only in 2010 that these were shown to be just artefacts of the tree-level approach [14] or rather that a region of parameter space exists, where the tree-level contribution is suppressed with respect to the loop corrections and that the theory there may be fully consistent at the quantum level.

To this end, the simplest version of the relevant Higgs model in which the rank of the gauge group is reduced by a 16-dimensional scalar field has been thoroughly studied in the same work. However, it turns out that the $45 \oplus 16$ scenario can hardly support a potentially realistic theory because it is unclear how it could accommodate the electroweak data (namely, the weak mixing angle) together with (a variant of) the seesaw mechanism for the neutrino masses. This is namely due to the fact that the seesaw requires two $B - L$ breaking vacuum expectation value (VEV) insertions (recall that the Standard Model singlet in 16 carries only one unit of $B - L$); this, however, calls for the $B - L$ breaking to occur at a relatively large scale (in the 10^{14} GeV ballpark) which is generically difficult to reconcile with the gauge unification constraints¹. For the same reason, the renormalizable alternative to seesaw mechanism by Witten [15] does not work either due to the extra two-loop suppression. Furthermore, it is very problematic to get any firm grip on the flavour structure of this model as any potentially realistic variant of its Yukawa sector relies on a number of contributions from non-renormalizable operators.

Therefore, the most promising scenario of this kind includes one copy of the 126-dimensional representation in the scalar sector instead of the spinorial 16; its main virtue is that it can support the standard seesaw mechanism, as well

¹ This is almost obvious in the minimally fine-tuned scenarios; however, admitting accidentally light extra scalars in the $45 \oplus 16$ model does not seem to work either as there are simply no fields around that may affect significantly the running of the strong coupling to the extent achieved, e.g., by the $(8, 2, +\frac{1}{2})$ scalar in the $45 \oplus 126$ setting.

as (upon adding another 10-dimensional scalar representation to the flavour sector) a potentially realistic (yet simple) Yukawa pattern at the renormalizable level and, thus, avoid most of the aforementioned complications.

The first attempt to study the quantum version of the $45 \oplus 126$ model was undertaken in the works [10, 16, 17] where it was shown that, under several simplifying assumptions, there are extended regions in its parameter space that can support a stable Standard Model (SM) vacuum, accommodate all the SM data and, at the same time, maintain compatibility with the existing proton lifetime constraints. Remarkably, this can all be attained with only a single fine-tuning of the model parameters ensuring one specific heavy scalar in the desert² [16, 17]. The main drawback of these early studies lies in the fact that, out of all relevant quantum corrections emerging at one loop, only the simplest universal type has been considered in order to stabilize the scalar sector of the theory with minimum efforts. Hence, those results should be regarded as only approximate and the situation clearly calls for a more complete treatment.

In the current paper we partly fill this gap by calculating in great detail the leading one-loop corrections to the masses of the scalar multiplets transforming as $(1, 3, 0)$ and $(8, 1, 0)$ under the $SU(3)_c \times SU(2)_L \times U(1)_Y$ SM gauge group in the $45 \oplus 126$ Higgs model. We focus our attention solely on these two fields as they are the principal culprits causing the notorious tree-level tachyonic instabilities mentioned above and, thus, their quantum-level behaviour is of our primary concern. In this sense, a thorough analysis of the relevant radiative corrections represents the first and minimal step towards any reliable phenomenological analysis of this scenario in the future.

The work is structured as follows: after a short recapitulation of the tree-level shape of the model's scalar spectrum in Section II we use the effective potential techniques to calculate the zero-momentum one-loop corrections to the masses of $(1, 3, 0)$ and $(8, 1, 0)$ (Section III) and cross-check our results by means of two basic methods: first, by inspecting the relevant formulae in various limits where the spectrum takes specific known shapes and, second, by confirming the coefficient of the simplest $SO(10)$ -invariant contribution with the direct diagrammatic calculation which, for this term, is relatively easy. Furthermore, we use these results to provide a sample point in the parameter space that is not only free from all the aforementioned pathologies but, at the same time, may even support a potentially realistic GUT scenario (including neutrino masses); we also add several comments on the methods of implementation of the results in a future numerical analysis. Most of the technicalities are deferred to a set of Appendices. Then we conclude.

II. THE MINIMAL $SO(10)$ HIGGS MODEL

In what follows, we shall use the symbols ϕ_{ij} and Σ_{ijklm} (with all Latin indices running from 1 to 10) for the components of the 45-dimensional adjoint and the 126-dimensional self-dual 5-index antisymmetric irreducible tensor representations of the $SO(10)$ gauge group, respectively. Note that in the real basis of the $SO(10)$ both these structures are fully antisymmetric in all their indices and that Σ obeys $\Sigma_{ijklm} = -\frac{i}{5!}\varepsilon_{ijklmnopqr}\Sigma_{nopqr}$ provided $\varepsilon_{12345678910} = +1$. Unlike ϕ , Σ is a complex representation and we shall denote the complex conjugated object by Σ^* . For more on the notation regarding these representations, see Appendix A. The decompositions of ϕ and Σ into irreducible representations of the SM gauge group are listed in Table I.

A. The $SO(10)$ symmetric Lagrangian in the unbroken phase

The normalization of the component fields in ϕ and Σ follows the usual convention which fixes the kinetic part of the relevant Lagrangian, $\mathcal{L} = \mathcal{L}_{kin} - V_0$, to the form

$$\begin{aligned} \mathcal{L}_{kin} = & \frac{1}{4}(F_{\mu\nu})_{ij}(F^{\mu\nu})_{ij} + \frac{1}{4}(D_\mu\phi)^\dagger_{ij}(D^\mu\phi)_{ij} + \\ & + \frac{1}{5!}(D_\mu\Sigma)^\dagger_{ijklm}(D^\mu\Sigma)_{ijklm}, \end{aligned} \quad (1)$$

where

$$(F^{\mu\nu})_{ij} = \partial^\mu(A^\nu)_{ij} - \partial^\nu(A^\mu)_{ij} - ig[A^\mu, A^\nu]_{ij}, \quad (2)$$

and a summation over the repeated Latin indices is implicit (for remaining definitions of the used quantities, see Appendix A 1). This yields the ‘‘standard’’ kinetic terms for the relevant SM components including coefficients $\frac{1}{2}$ and 1 for real and complex fields, respectively. With this at hand, the (renormalizable) scalar potential reads

$$V_0(\phi, \Sigma, \Sigma^*) = V_{45}(\phi) + V_{126}(\Sigma, \Sigma^*) + V_{\text{mix}}(\phi, \Sigma, \Sigma^*), \quad (3)$$

² Given that in all potentially realistic settings identified there, the seesaw scale σ turned out to be relatively close to M_{GUT} , one may even view the situation with the accidentally light scalar S as if the ‘usual’ fine-tuning in σ was ‘traded’ for that in the scalar’s mass m_S .

TABLE I. All types of SM representations R of scalar fields in the $45 \oplus 126$ Higgs model. The \mathbb{R}/\mathbb{C} column denotes whether the representation is real or complex (implicitly, for a complex R , there exists an inequivalent conjugate representation \bar{R}), the hash sign $\#$ denotes the multiplicity of R (and consequently the dimension of the corresponding block in the full scalar mass matrix) and the dagger \dagger indicates the presence of a would-be Goldstone mode. There are in general 33 Goldstones contained in 5 different SM multiplets corresponding to the same number of broken $SO(10)$ generators. The “size” column enumerates the real degrees of freedom in the representation R (reflected in the number of equivalent blocks with identical eigenvalues in the properly reordered full mass matrix). Summing the size \times multiplicity over all blocks yields $\sum_{i=a}^s \#_i \times \text{size}_i = 45 + 126 + 126 = 297$ real degrees of freedom in total. There are 19 different SM representations, out of which 11 appear only in one copy, 5 are 2-fold degenerate, 2 are 3-fold degenerate and 1 appears even 4 times (the singlet block is 4×4 ; two real singlets and one complex, which can acquire non-zero VEVs $\sqrt{3}\omega_b$, $\sqrt{2}\omega_r$ and $\sqrt{2}\sigma$, respectively). Hence, there are in principle 31 different eigenvalues; since 5 of them are Goldstone bosons, one is left with only 26 non-vanishing (and different) eigenvalues. The G_{422} column denotes the origin of each SM representation within the corresponding representations of the Pati-Salam $SU(4)_c \times SU(2)_L \times SU(2)_R$ subgroup of $SO(10)$. (Note however, that none of the representations actually contains a singlet under Pati-Salam, so there can be no intermediate physical phase characterized by the Pati-Salam G_{422} gauge symmetry).

label	$R \sim G_{321}$	\mathbb{R}/\mathbb{C}	$\#$	size	$R \subseteq G_{422}$	$\subseteq SO(10)$
a	$(1, 3, 0)$	\mathbb{R}	1	3	$(1, 3, 1)$	ϕ
b	$(8, 1, 0)$	\mathbb{R}	1	8	$(15, 1, 1)$	ϕ
c	$(3, 2, -\frac{5}{6})$	\mathbb{C}	1^\dagger	12	$(6, 2, 2)$	ϕ
d	$(1, 1, +2)$	\mathbb{C}	1	2	$(\bar{10}, 1, 3)$	Σ
e	$(1, 3, -1)$	\mathbb{C}	1	6	$(10, 3, 1)$	Σ
f	$(\bar{3}, 1, +\frac{4}{3})$	\mathbb{C}	1	6	$(\bar{10}, 1, 3)$	Σ
g	$(3, 3, -\frac{1}{3})$	\mathbb{C}	1	18	$(10, 3, 1)$	Σ
h	$(6, 3, +\frac{1}{3})$	\mathbb{C}	1	36	$(10, 3, 1)$	Σ
i	$(\bar{6}, 1, -\frac{4}{3})$	\mathbb{C}	1	12	$(\bar{10}, 1, 3)$	Σ
j	$(\bar{6}, 1, -\frac{1}{3})$	\mathbb{C}	1	12	$(\bar{10}, 1, 3)$	Σ
k	$(\bar{6}, 1, +\frac{2}{3})$	\mathbb{C}	1	12	$(\bar{10}, 1, 3)$	Σ
l	$(1, 2, +\frac{1}{2})$	\mathbb{C}	2	4	$(15, 2, 2), (15, 2, 2)^*$	Σ, Σ^*
m	$(3, 2, +\frac{7}{6})$	\mathbb{C}	2	12	$(15, 2, 2), (15, 2, 2)^*$	Σ, Σ^*
n	$(8, 2, +\frac{1}{2})$	\mathbb{C}	2	32	$(15, 2, 2), (15, 2, 2)^*$	Σ, Σ^*
o	$(\bar{3}, 1, +\frac{1}{3})$	\mathbb{C}	3	6	$(6, 1, 1), (6, 1, 1)^*, (\bar{10}, 1, 3)$	Σ, Σ^*, Σ
p	$(1, 1, +1)$	\mathbb{C}	2^\dagger	2	$(1, 1, 3), (\bar{10}, 1, 3)$	ϕ, Σ
q	$(\bar{3}, 1, -\frac{2}{3})$	\mathbb{C}	2^\dagger	6	$(15, 1, 1), (\bar{10}, 1, 3)$	ϕ, Σ
r	$(3, 2, +\frac{1}{6})$	\mathbb{C}	3^\dagger	12	$(6, 2, 2), (15, 2, 2), (15, 2, 2)^*$	ϕ, Σ, Σ^*
s	$(1, 1, 0)$	\mathbb{R}	2	$\left. \begin{matrix} 1 \\ 1 \end{matrix} \right\} 4^\dagger$	$(15, 1, 1), (1, 1, 3),$	ϕ, ϕ
		\mathbb{C}	1		$(\bar{10}, 1, 3)$	Σ

provided

$$V_{45} = -\frac{\mu^2}{4}(\phi\phi)_0 + \frac{a_0}{4}(\phi\phi)_0(\phi\phi)_0 + \frac{a_2}{4}(\phi\phi)_2(\phi\phi)_2, \quad (4)$$

$$\begin{aligned}
V_{126} = & -\frac{\nu^2}{5!}(\Sigma\Sigma^*)_0 + \frac{\lambda_0}{(5!)^2}(\Sigma\Sigma^*)_0(\Sigma\Sigma^*)_0 + \\
& + \frac{\lambda_2}{(4!)^2}(\Sigma\Sigma^*)_2(\Sigma\Sigma^*)_2 + \frac{\lambda_4}{(3!)^2(2!)^2}(\Sigma\Sigma^*)_4(\Sigma\Sigma^*)_4 + \\
& + \frac{\lambda'_4}{(3!)^2}(\Sigma\Sigma^*)_{4'}(\Sigma\Sigma^*)_{4'} + \frac{\eta_2}{(4!)^2}(\Sigma\Sigma)_2(\Sigma\Sigma)_2 + \\
& + \frac{\eta_2^*}{(4!)^2}(\Sigma^*\Sigma^*)_2(\Sigma^*\Sigma^*)_2,
\end{aligned} \tag{5}$$

$$\begin{aligned}
V_{\text{mix}} = & \frac{i\tau}{4!}(\phi)_2(\Sigma\Sigma^*)_2 + \frac{\alpha}{2 \cdot 5!}(\phi\phi)_0(\Sigma\Sigma^*)_0 + \\
& + \frac{\beta_4}{4 \cdot 3!}(\phi\phi)_4(\Sigma\Sigma^*)_4 + \frac{\beta'_4}{3!}(\phi\phi)_{4'}(\Sigma\Sigma^*)_{4'} + \\
& + \frac{\gamma_2}{4!}(\phi\phi)_2(\Sigma\Sigma)_2 + \frac{\gamma_2^*}{4!}(\phi\phi)_2(\Sigma^*\Sigma^*)_2,
\end{aligned} \tag{6}$$

where $(\phi\phi)_0 \equiv \phi_{ij}\phi_{ij}$, $(\phi\phi)_2 \equiv (\phi\phi)_{jk} \equiv \phi_{ij}\phi_{ik}$, $(\Sigma\Sigma^*)_0 \equiv \Sigma_{ijklm}\Sigma^*_{ijklm}$, $(\Sigma\Sigma^*)_2 \equiv (\Sigma\Sigma^*)_{mn} \equiv \Sigma_{ijklm}\Sigma^*_{ijkln}$, $(\Sigma\Sigma^*)_4 \equiv (\Sigma\Sigma^*)_{lmno} \equiv \Sigma_{ijklm}\Sigma^*_{ijkno}$, and analogously if Σ or Σ^* is replaced with its conjugate. The invariant contractions among these expressions read

$$\begin{aligned}
(\Sigma\Sigma^*)_2(\Sigma\Sigma^*)_2 &= (\Sigma\Sigma^*)_{mn}(\Sigma\Sigma^*)_{mn}, \\
(\Sigma\Sigma^*)_4(\Sigma\Sigma^*)_4 &= (\Sigma\Sigma^*)_{lmno}(\Sigma\Sigma^*)_{lmno}, \\
(\Sigma\Sigma^*)_{4'}(\Sigma\Sigma^*)_{4'} &= (\Sigma\Sigma^*)_{lmno}(\Sigma\Sigma^*)_{lnmo}, \\
(\phi)_2(\Sigma\Sigma^*)_2 &= \phi_{mn}(\Sigma\Sigma^*)_{mn}, \\
(\phi\phi)_4(\Sigma\Sigma^*)_4 &= \phi_{lm}\phi_{no}(\Sigma\Sigma^*)_{lmno}, \\
(\phi\phi)_{4'}(\Sigma\Sigma^*)_{4'} &= \phi_{lm}\phi_{no}(\Sigma\Sigma^*)_{lnmo}, \\
(\phi\phi)_2(\Sigma\Sigma)_2 &= (\phi\phi)_{jk}(\Sigma\Sigma)_{jk}.
\end{aligned} \tag{7}$$

Note that there are 3 parameters with a positive dimension of mass $\{\mu, \nu, \tau\}$ in V_0 , 9 dimensionless real parameters $\{a_0, a_2, \lambda_0, \lambda_2, \lambda_4, \lambda'_4, \alpha, \beta_4, \beta'_4\}$ and 2 dimensionless complex parameters $\{\eta_2, \gamma_2\}$. The minus signs in front of μ^2 and ν^2 and the various symmetry factors in other terms are mere convenience. Note also that the coefficient of the μ^2 term has been fixed in a different way than in [10, 16–18]; the slight advantage of the current notation is the fact that in the symmetric phase $-\mu^2$ and $-\nu^2$ are exactly the squares of the (tree-level) physical masses of the SM fields in ϕ and Σ , respectively. In what follows, we shall use Φ as a generic symbol denoting all scalar components at play, i.e., $\Phi \equiv (\phi, \Sigma, \Sigma^*)$.

B. Spontaneous $SO(10)$ symmetry breaking

There are 3 SM singlets in the scalar sector: 2 real in ϕ and 1 complex in Σ . In what follows, we shall denote their potentially non-vanishing VEVs by

$$\begin{aligned}
\langle(1, 1, 1, 0)_\phi\rangle &\equiv \sqrt{3}\omega_b, \quad \langle(1, 1, 3, 0)_\phi\rangle \equiv \sqrt{2}\omega_r, \\
\langle(1, 1, 3, +2)_\Sigma\rangle &= \langle(1, 1, 3, -2)_{\Sigma^*}\rangle^* \equiv \sqrt{2}\sigma.
\end{aligned} \tag{8}$$

The multiplets above were written in the $SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ language and the corresponding fields are assumed to be canonically normalized. The VEVs ω_b and ω_r are real while σ is a VEV of a complex scalar singlet and, hence, it can be complex. Note that there is a freedom to redefine the overall phase of Σ in such a way that σ can be made real; alternatively, the same transformation can be used to absorb the phase of γ_2 in equation (6), thus reducing γ_2 to a real parameter. In the latter case σ can be complex (and, hence, it may be convenient to keep track of the relevant complex conjugation as we shall do in what follows).

Assuming no correlations among the VEVs above, the $SO(10)$ gauge symmetry gets spontaneously broken down to the SM group $SU(3)_c \times SU(2)_L \times U(1)_Y$. Special symmetry breaking patterns can be attained in various limits as listed in Table II. From the phenomenological perspective, however, it is sensible to consider predominantly the case with $|\sigma| \ll \max\{|\omega_b|, |\omega_r|\}$ in which σ plays the role of an intermediate (seesaw) scale, while the dominant $\omega_{r,b}$ corresponds to the unification scale.

TABLE II. Residual gauge symmetries (in a self-explanatory notation) attained for various configurations of the VEVs defined in eq. (8). The last column corresponds to the alternative ‘flipped’ embedding of the SM hypercharge into the $SU(5)' \times U(1)_{Z'}$ subgroup of the $SO(10)$, cf. [19, 20]. Note also that for $\omega_b = \omega_r$ the breaking to SM cannot occur.

	$\omega_b \neq 0, \omega_r \neq 0$	$\omega_b = 0, \omega_r \neq 0$	$\omega_b \neq 0, \omega_r = 0$	$\omega_b = \omega_r \neq 0$	$\omega_b = -\omega_r \neq 0$
$\sigma = 0$	$3_c 2_L 1_R 1_{B-L}$	$4_c 2_L 1_R$	$3_c 2_L 2_R 1_{B-L}$	$5 1_Z$	$5' 1_{Z'}$
$\sigma \neq 0$	$3_c 2_L 1_Y$	$3_c 2_L 1_Y$	$3_c 2_L 1_Y$	5	$3_c 2_L 1_Y$

The tree-level vacuum stationarity conditions translating among these VEVs and the massive parameters of the potential read (see also [10])

$$\mu^2 = (12a_0 + 2a_2)\omega_b^2 + (8a_0 + 2a_2)\omega_r^2 + 2a_2\omega_b\omega_r + 4(\alpha + \beta'_4)|\sigma|^2, \quad (9)$$

$$\nu^2 = 3(\alpha + 4\beta'_4)\omega_b^2 + 2(\alpha + 3\beta'_4)\omega_r^2 + 12\beta'_4\omega_b\omega_r + 4\lambda_0|\sigma|^2 + a_2\frac{\omega_b\omega_r}{|\sigma|^2}(\omega_b + \omega_r)(3\omega_b + 2\omega_r), \quad (10)$$

$$\tau = 2\beta'_4(3\omega_b + 2\omega_r) + a_2\frac{\omega_b\omega_r}{|\sigma|^2}(\omega_b + \omega_r). \quad (11)$$

Note that there are potentially problematic terms containing $|\sigma|^2$ in the denominator in the latter two conditions that may ruin the perturbative expansion whenever the relevant expression exceeds significantly the GUT scale³ (i.e., the maximum of $\omega_{b,r}$). Hence, in realistic settings one should assume that

$$a_2\frac{\omega_b\omega_r}{|\sigma|^2}(\omega_b + \omega_r) \ll M_{\text{Pl}}. \quad (12)$$

C. The tree-level spectrum

With this information at hand, the tree-level scalar and gauge spectra of the $45 \oplus 126$ $SO(10)$ Higgs model under consideration can be readily obtained. Since 45 is a real representation and 126 is complex, the total number of real degrees of freedom in the scalar sector is 297.

For later convenience, it is useful to arrange the second derivatives of V_0 into a (297-dimensional) Hermitian matrix

$$\begin{aligned} \mathbf{M}_{\mathbf{S}}^2(\Phi) &\equiv \mathbf{M}_{\mathbf{S}}^2(\phi, \Sigma, \Sigma^*) = \partial\partial^* V_0(\phi, \Sigma, \Sigma^*) \\ &= \begin{pmatrix} V_{\phi\phi} & V_{\phi\Sigma^*} & V_{\phi\Sigma} \\ V_{\Sigma\phi} & V_{\Sigma\Sigma^*} & V_{\Sigma\Sigma} \\ V_{\Sigma^*\phi} & V_{\Sigma^*\Sigma^*} & V_{\Sigma^*\Sigma} \end{pmatrix}, \end{aligned} \quad (13)$$

with sub-blocks indicating the types of fields with respect to which the relevant derivatives are taken. In the SM vacuum (characterized by one of the four relevant VEV configurations in the 2nd row of Table II) this matrix encodes the tree-level scalar spectrum of the model and, as such, it may be brought into block-diagonal form; each block has a size equal to the number of same-type SM irreducible representations the states of the block are coming from (*multiplicity*), and the blocks repeat with a degeneracy equal to the number of states in the representation (*size*), see Table I. Note that the fields of our main interest, i.e., the pseudo-Goldstones $(1, 3, 0)$ and $(8, 1, 0)$, are then fully contained in the $V_{\phi\phi}$ sector of $\mathbf{M}_{\mathbf{S}}^2(\Phi)$. The complete structure of $\mathbf{M}_{\mathbf{S}}^2(\Phi)$ in the (block-diagonal) SM basis evaluated in the SM vacuum is given in Appendix A 2, see also [10]. In order to conform to the needs of the subsequent quantum-level analysis the notation here has been slightly amended⁴ with respect to that used in [10]; see, in particular, definitions (A38)–(A44).

In a similar manner one can define the 45-dimensional field-dependent mass matrix for gauge bosons $\mathbf{M}_{\mathbf{G}}^2(\Phi)$, see Appendix A 1. Since we do not consider the breaking of the Standard Model gauge group, this matrix evaluated in the SM vacuum has 12 massless modes corresponding to the gluons, the W^\pm , Z^0 and the photon.

³ To this end, let us note that several of the tree-level scalar sector mass-squares calculated in Appendix A 2 are linear in the combination (11) so the scalar spectrum would be badly distorted if the condition (12) was not satisfied.

⁴ Besides the overall compactness of the results obtained in Sect. III the new notation facilitates their cross-checking in various limits corresponding to enhanced gauge symmetries, in particular, those listed in Table II.

1. The masses of the $(1, 3, 0)$ and $(8, 1, 0)$ pseudo-Goldstone bosons

The multiplets of our main interest in the current study are the two scalar pseudo-Goldstone bosons (cf. [14]) transforming as $(1, 3, 0)$ and $(8, 1, 0)$ under the $SU(3)_c \times SU(2)_L \times U(1)_Y$ of the SM. Their masses, at the tree level, are (see Appendix A 2 for the notation)

$$M_a^2 = 2a_2(\omega_r - \omega_b)(\omega_b + 2\omega_r) \equiv +2a_2\omega'w_{[1,2]}^1, \quad (14)$$

$$M_b^2 = 2a_2(\omega_b - \omega_r)(\omega_r + 2\omega_b) \equiv -2a_2\omega'w_{[2,1]}^1. \quad (15)$$

The two states are referred to as pseudo-Goldstone bosons, because the tree-level masses are vanishing as the parameter $a_2 \rightarrow 0$ (this is no longer true though for higher order corrections). Given these mass expressions, the scalar spectrum can be non-tachyonic if and only if

$$a_2 > 0 \quad \text{and} \quad -2 < \frac{\omega_b}{\omega_r} < -\frac{1}{2}, \quad (16)$$

i.e., in the vicinity of the flipped $SU(5)$ limit (the last column in Table II). This means, however, that the symmetry-breaking chains supporting, at the tree level, a locally convex minimum (i.e., a potentially stable SM-like vacuum) all feature an approximate $SU(5)' \times U(1)_{Z'}$ -symmetric scalar spectrum clustering around the superheavy breaking scale $|\omega_r| \approx |\omega_b|$, at odds with the gauge unification constraints (at least in the minimally fine-tuned scenarios, i.e., those obeying the minimal survival hypothesis, cf. [8, 21–23]). This, obviously, disqualifies the minimal (and minimally fine-tuned) setting from any potentially realistic model building, at least at the lowest order in perturbation theory. This conclusion applies even to the situation when the minimal Higgs model is further extended by an extra 10-dimensional scalar multiplet in order to support a viable Yukawa sector with at least two different complex symmetric matrices of Yukawa couplings – the extra 10_H does not alter the symmetry breaking pattern by more than just an admixture of its weak doublet component within the electroweak Higgs and an extra set of superheavy doublets and color triplets clustered around the $SO(10)$ breaking scale.

Barring, for the sake of this study, the option of non-minimally fine-tuned settings, the only chance to bring the current scenario back from oblivion seems to be a careful inspection of its quantum structure. The hope is that higher order effects may disentangle the overly strong correlation between the two pseudo-Goldstone masses above, at least in case that the tree-level contributions happen to be accidentally small; this option has been identified (but never inspected in detail) in previous studies like [14]. A detailed calculation of the relevant radiative contributions to the tree-level mass relations (14) and (15) is the scope of the next section.

III. ONE-LOOP PSEUDO-GOLDSTONE MASSES IN THE MINIMAL $SO(10)$ HIGGS MODEL

A. One-loop scalar masses from the effective potential

In this section we review some of the technical aspects of the effective potential formalism we adopt for the computation of the desired scalar masses at the 1-loop level.

1. Scalar mass matrix at the one-loop level

In the effective potential approach there are in general two types of effects contributing to the scalar masses at the one-loop level (i.e., at the order characterized by one power of the generic $\hbar/16\pi^2$ suppression factor), namely:

1. The usual one-loop corrections to the two-point 1PI Green's functions whose roots in the true vacuum define the pole masses of the scalar excitations.
2. The quantum shift of the vacuum which justifies the use of a simplified perturbation theory in which there are no degenerate one-point vertices in the interaction part of the Lagrangian density (aka “tadpole cancellation”).

Up to the first power in \hbar (ultimately set to $\hbar \rightarrow 1$), the relevant combination of the two effects governing the \hbar -expansion of the one-loop scalar mass matrix \mathbf{M} around its tree-level value $\partial^2 V_0|_{v_0}$ (at zero external momentum, as implicitly assumed within the effective potential approach) reads formally

$$\mathbf{M}_{ab}^2 \equiv \partial_a \partial_b V|_v = \partial_a \partial_b V_0|_{v_0 + \hbar v_1} + \hbar \partial_a \partial_b V_1|_{v_0} + \mathcal{O}(\hbar^2), \quad (17)$$

where $V = \sum_{i=0}^{\infty} V_i \hbar^i = V_0 + \hbar V_1 + \mathcal{O}(\hbar^2)$ is the Coleman-Weinberg effective potential [24] and $v = \sum_{i=0}^{\infty} v_i \hbar^i = v_0 + \hbar v_1 + \mathcal{O}(\hbar^2)$ denotes the true quantum vacuum of the theory determined from the stationary point condition

$$\partial_a V|_v = \partial_a V_0|_v + \hbar \partial_a V_1|_v = 0, \quad (18)$$

and expanded around its tree level value⁵ v_0 satisfying the condition $\partial_a V_0|_{v_0} = 0$. Besides the tree-level contribution V_0 discussed at length in the previous section, cf. (3), the one-loop scalar potential (in the $\overline{\text{MS}}$ renormalization scheme) is given by [25]

$$V_1(\Phi, \mu_r) = \frac{1}{64\pi^2} \text{Tr} \left[\mathbf{M}_{\mathbf{S}}^4(\Phi) \left(\log \frac{\mathbf{M}_{\mathbf{S}}^2(\Phi)}{\mu_r^2} - \frac{3}{2} \right) + 3\mathbf{M}_{\mathbf{G}}^4(\Phi) \left(\log \frac{\mathbf{M}_{\mathbf{G}}^2(\Phi)}{\mu_r^2} - \frac{5}{6} \right) \right], \quad (19)$$

where $\mathbf{M}_{\mathbf{S}}^2(\Phi)$ and $\mathbf{M}_{\mathbf{G}}^2(\Phi)$ are the tree-level field-dependent scalar and gauge mass matrices introduced in Sect. II C (with boldface always indicating a matrix structure) and μ_r is the relevant renormalization scale.

The quantum-level contribution to the stationary point condition (18) is then given by

$$\begin{aligned} \partial_a V_1 = & -\frac{1}{32\pi^2} \text{Tr} [\mathbf{M}_{\mathbf{S}}^2 \partial_a \mathbf{M}_{\mathbf{S}}^2 + \mathbf{M}_{\mathbf{G}}^2 \partial_a \mathbf{M}_{\mathbf{G}}^2] + \\ & + \frac{1}{64\pi^2} \text{Tr} \left[\{ \mathbf{M}_{\mathbf{S}}^2, \partial_a \mathbf{M}_{\mathbf{S}}^2 \} \log \frac{\mathbf{M}_{\mathbf{S}}^2}{\mu_r^2} + \right. \\ & \left. + 3 \{ \mathbf{M}_{\mathbf{G}}^2, \partial_a \mathbf{M}_{\mathbf{G}}^2 \} \log \frac{\mathbf{M}_{\mathbf{G}}^2}{\mu_r^2} \right]; \end{aligned} \quad (20)$$

note that we have dropped all the brackets denoting the implicit dependence of the mass matrices on the scalar fields of the model.

Due to the general non-commutativity of $\mathbf{M}_{\mathbf{S},\mathbf{G}}^2$ with their own first derivatives, the second derivative of the formula (19) is far more involved⁶:

$$\begin{aligned} \partial_a \partial_b V_1 = & -\frac{1}{32\pi^2} \text{Tr} [\partial_a \mathbf{M}_{\mathbf{S}}^2 \partial_b \mathbf{M}_{\mathbf{S}}^2 + \mathbf{M}_{\mathbf{S}}^2 \partial_a \partial_b \mathbf{M}_{\mathbf{S}}^2 + \\ & + \partial_a \mathbf{M}_{\mathbf{G}}^2 \partial_b \mathbf{M}_{\mathbf{G}}^2 + \mathbf{M}_{\mathbf{G}}^2 \partial_a \partial_b \mathbf{M}_{\mathbf{G}}^2] + \\ & + \frac{1}{64\pi^2} \text{Tr} \left[(\{ \partial_a \mathbf{M}_{\mathbf{S}}^2, \partial_b \mathbf{M}_{\mathbf{S}}^2 \} + \{ \mathbf{M}_{\mathbf{S}}^2, \partial_a \partial_b \mathbf{M}_{\mathbf{S}}^2 \}) \times \right. \\ & \quad \times \log \frac{\mathbf{M}_{\mathbf{S}}^2}{\mu_r^2} + \mathbf{S}_{ab} \Big] + \\ & + \frac{3}{64\pi^2} \text{Tr} \left[(\{ \partial_a \mathbf{M}_{\mathbf{G}}^2, \partial_b \mathbf{M}_{\mathbf{G}}^2 \} + \{ \mathbf{M}_{\mathbf{G}}^2, \partial_a \partial_b \mathbf{M}_{\mathbf{G}}^2 \}) \times \right. \\ & \quad \times \log \frac{\mathbf{M}_{\mathbf{G}}^2}{\mu_r^2} + \mathbf{G}_{ab} \Big]. \end{aligned} \quad (21)$$

Here

$$\begin{aligned} \mathbf{S}_{ab} &= \Upsilon \left(\frac{\mathbf{M}_{\mathbf{S}}^2}{\mu_r^2}, \partial_a \mathbf{M}_{\mathbf{S}}^2, \partial_b \mathbf{M}_{\mathbf{S}}^2 \right), \\ \mathbf{G}_{ab} &= \Upsilon \left(\frac{\mathbf{M}_{\mathbf{G}}^2}{\mu_r^2}, \partial_a \mathbf{M}_{\mathbf{G}}^2, \partial_b \mathbf{M}_{\mathbf{G}}^2 \right), \end{aligned} \quad (22)$$

are expressed via a matrix function Υ including an infinite series of nested commutators

$$\begin{aligned} \Upsilon(\mathbf{A}, \mathbf{A}_a, \mathbf{A}_b) = & \sum_{m=1}^{\infty} (-1)^{m+1} \frac{1}{m} \sum_{k=1}^m \binom{m}{k} \{ \mathbf{A}, \mathbf{A}_a \} \times \\ & \times \underbrace{[\mathbf{A}, \dots [\mathbf{A}, \mathbf{A}_b] \dots]}_{(k-1) \times \text{commutator}} (\mathbf{A} - \mathbb{1})^{m-k}, \end{aligned} \quad (23)$$

⁵ In our $45 \oplus 126$ model that corresponds to the values of mass parameters defined in eqs. (9)–(11).

⁶ The main source of complication here are, namely, the derivatives of the matrix logs, see Appendix C.

where the first commutator bracket is just \mathbf{A}_b , the second is $[\mathbf{A}, \mathbf{A}_b]$, the third $[\mathbf{A}, [\mathbf{A}, \mathbf{A}_b]]$ and so on. The general strategy for dealing with the formula (23), together with a brief discussion of the shape of the results it yields, is given in Section III A 2 and Appendix C.

Let us also remark that there are no such issues for a single derivative of the matrix logarithm in the expression (20) because of the cyclic property of the overall trace which admits a resummation of the Taylor series expanded expression into a simple polynomial form.

2. Dealing with the nested commutators

In this section we describe several tricks that facilitate dealing with the nested commutators⁷, focusing mainly on their numerical evaluation. As for a full analytic account, it turns out in our case that the scalar sector contribution is only practical to write down in special cases as, for instance, the one discussed in Section III A 3. For all the derivations and proofs of the expressions in use see Appendix C.

The object of our main interest, i.e., the trace of Υ evaluated in the tree-level vacuum, cf. (21)–(22), may be further simplified using the identity (see Appendix C)

$$\begin{aligned} \text{Tr } \Upsilon(\mathbf{A}, \mathbf{A}_a, \mathbf{A}_b) &= \sum_{i,j; \lambda_i \neq \lambda_j} M_{ji}^a M_{ij}^b \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} \log \frac{\lambda_i}{\lambda_j} + \\ &+ \sum_{i,j; \lambda_i = \lambda_j} 2M_{ji}^a M_{ij}^b, \end{aligned} \quad (24)$$

where λ_i are eigenvalues of the matrix \mathbf{A} with corresponding orthonormal eigenvectors \mathbf{v}_i , while M^a and M^b are the matrices \mathbf{A}_a and \mathbf{A}_b rotated into the orthonormal eigenbasis of \mathbf{A} :

$$M_{ij}^a = \mathbf{v}_i^\dagger \mathbf{A}_a \mathbf{v}_j, \quad M_{ij}^b = \mathbf{v}_i^\dagger \mathbf{A}_b \mathbf{v}_j. \quad (25)$$

Let us note that this approach is completely general and applicable (at least numerically) to any form of the matrices \mathbf{A} , \mathbf{A}_a and \mathbf{A}_b . In that sense, it is superior to the method used previously in, e.g., ref. [26] which assumed a simple geometric behaviour of the nested commutators from a certain value of the k index in eq. (23) onwards – unfortunately, unlike in the case of the simpler $45 \oplus 16$ model studied previously in [14] (or the Abelian Higgs model, cf. [26]) where this was indeed the case, the situation in the $45 \oplus 126$ Higgs model is more complicated.

Another point worth making here concerns the visual difference between the two contributions in eq. (24) – the former structure, belonging to a set of non-degenerate eigenvalues, suggests a log-type behaviour while the latter tends to yield non-log terms (in fact, in most cases even polynomials). Since the same two types of terms emerge also from the “non-commutator” parts of the basic formula (21), the distinction between log and non-log terms is in fact very handy and we shall use it in the next section. However, one needs to be careful when it comes to limits in which the character of the spectrum changes qualitatively, i.e., when the degeneracies increase. Indeed, if two formerly non-degenerate eigenvalues $\lambda_{i,j}$ become equal in a certain limit, one has

$$\lim_{\lambda_i \rightarrow \lambda_j} (\lambda_i + \lambda_j) \frac{\log \lambda_i - \log \lambda_j}{\lambda_i - \lambda_j} = 2, \quad (26)$$

and hence a term of the first type becomes formally a second-type contribution, cf. Section III B 1.

3. One-loop masses of the $(1, 3, 0)$ and $(8, 1, 0)$ scalars in the $45 \oplus 126$ Higgs model

The one-loop masses (at zero momentum, in $\overline{\text{MS}}$ renormalization scheme) of the $(1, 3, 0)$ and $(8, 1, 0)$ scalars can be written as

$$\begin{aligned} M_{a,1\text{-loop}}^2 &= M_a^2 + \Delta_a^{G[\text{poly}]} + \Delta_a^{G[\log]} + \\ &+ \Delta_a^{S_{FIN}[\text{poly}]} + \Delta_a^{S_{INF}[\text{poly}]} + \Delta_a^{S[\log]}, \end{aligned} \quad (27)$$

$$\begin{aligned} M_{b,1\text{-loop}}^2 &= M_b^2 + \Delta_b^{G[\text{poly}]} + \Delta_b^{G[\log]} + \\ &+ \Delta_b^{S_{FIN}[\text{poly}]} + \Delta_b^{S_{INF}[\text{poly}]} + \Delta_b^{S[\log]}, \end{aligned} \quad (28)$$

⁷ This concerns especially the \mathbf{S}_{ab} structure defined in eq. (22) which comes from the huge scalar sector; the gauge contribution proportional to $\text{Tr } \mathbf{G}_{ab}$ is at least for both PGBs of our main interest relatively simple since then $[\mathbf{A}, \mathbf{A}_b] = 0$ and the infinite series of nested commutators in \mathbf{G}_{ab} reduces to $\{\mathbf{A}, \mathbf{A}_a\} \mathbf{A}_b \mathbf{A}^{-1}$. For other states in the spectrum one can often employ tricks described in [26].

where the first terms on the RH sides are their tree-level values, cf. (14)–(15) and the Δ symbols correspond to different types of one-loop contributions calculated from \hbar part of $\partial^2 V_0|_{v_0+\hbar v_1}$ and formulae (21)–(23). They are sorted with respect to their origin (gauge with superscript G and scalar with superscript S) and their mathematical form (polynomial [poly] and logarithmic [log]) as follows:

1. The $\Delta^{G[\text{poly}]}$ contributions contain all the polynomial-type terms from diagrams with gauge bosons running in the loops. As such, these contributions are all proportional to g^4 , where g is the $SO(10)$ gauge coupling constant. They generally come from both terms on the RH side of eq. (17), i.e. the polynomial part of the 1-loop vacuum substituted into $\partial^2 V_0$ and the polynomial terms of $\partial^2 V_1$ given in eq. (21) including the second type of terms in the expression for the nested commutator series (24) corresponding to degenerate eigenvalues of M_G^2 .
2. The $\Delta^{G[\text{log}]}$ terms contain all the logarithmic terms from the diagrams with gauge bosons running in the loops. As before, there are g^4 -proportional factors in front of the logs whose arguments are the masses of the massive gauge bosons corresponding to the broken generators of $SO(10)$. Again, these come from both parts of expression (17) including, in this case, the “non-degenerate” contributions from eq. (24).
3. The $\Delta^{S_{FIN}[\text{poly}]}$ terms contain all polynomial contributions from the scalars running in the loop, except for those coming from the nested commutator series, i.e., they are fully contained in the $\partial^2 V_0$ factor in (17) and the “finite” part of the formula (21). These terms are homogenous quadratic polynomials in the parameters of the scalar potential (3).
4. The $\Delta^{S_{INF}[\text{poly}]}$ pieces denote the polynomial scalar contributions coming solely from the infinite series term S_{ab} in eq. (21) in which they emerge from the scalar spectrum degeneracies due to the residual Standard Model gauge symmetry.
5. Finally, the $\Delta^{S[\text{log}]}$ structure labels all the logarithmic contributions associated to the graphs with scalars running in the loop. These again come from both parts of the expression (17), including the “non-degenerate” contributions from eq. (24). The coefficients in front of the logs are homogenous quadratic polynomials of the tree-level scalar potential parameters, while their arguments contain the (squared) tree-level masses of the relevant massive scalars in the loops.

Note that the $\Delta^{S_{INF}[\text{poly}]}$ terms have been singled out because these are rather difficult to calculate analytically and the results in a closed form are cumbersome; the same applies also to $\Delta^{S[\text{log}]}$. We shall thus not present them in their full complexity, but rather in a simplified form that they attain in the limit⁸

$$\sigma \rightarrow 0, \quad a_2 \rightarrow 0, \quad \gamma_2 \rightarrow 0, \quad \frac{a_2}{|\sigma|^2} = \text{const.} \quad (29)$$

Let us also remark that this setting is not an arbitrary choice but rather a physically well-motivated approximation to the general case: as far as the $\sigma \rightarrow 0$ limit is concerned, the “delayed breaking” of the $U(1)_{B-L}$ obtained in such a situation corresponds to the potentially realistic seesaw scale with the RH neutrino masses well below M_{GUT} ; the $a_2 \rightarrow 0$ and $\gamma_2 \rightarrow 0$ limits are, on the contrary, suggested by the (simplified) preceding studies [16, 17] as the only situation in which a fully non-tachyonic spectrum compatible with the gauge unification constraints seems to be attainable.

The full analytic form (modulo the aforementioned limit (29) adopted for simplicity reasons for the $\Delta^{S_{INF}[\text{poly}]}$ and $\Delta^{S[\text{log}]}$ pieces) of the one-loop corrections entering formulae (27)–(28) is given in Appendix B.

4. Going to the mass shell

As we have already mentioned, the formulae (27)–(28) with the Δ factors given in Appendix B encode the (zero-momentum) masses of the two pseudo-Goldstone bosons of our interest in the $\overline{\text{MS}}$ renormalization scheme. A direct use of these results is, however, not so straightforward. The main reason is that there are peculiar infra-red (IR) divergences due to a certain number of zero eigenvalues in the arguments of logs in the $\Delta^{S,G[\text{log}]}$ terms emerging from

⁸ Note that in the $\sigma \rightarrow 0, \gamma_2 \rightarrow 0$ limit the Σ -self interaction terms do not contribute to the 1-loop masses of the fields coming solely from ϕ . Hence, one can neglect the $\lambda_0, \lambda_2, \lambda_4, \lambda'_4$ and η_2 terms in the potential V_0 from the beginning (in fact, η_2 is absent from M_a^2 and M_b^2 even at the level of field dependent tree-level mass matrix $M_\Sigma^2(\Phi)$). The reason is that all the off-diagonal blocks in the mass matrix (13) in vacuum vanish in this limit. Then the only mixing within the 126-dimensional $V_{\Sigma\Sigma^*}$ block is among the states in M_ϕ^2 belonging either to $(6, 1, 1)$ or $(\overline{10}, 1, 3)$ of $SU(4)_c \times SU(2)_L \times SU(2)_R$.

the tree-level field-dependent mass matrices $\mathbf{M}_{\mathbf{S},\mathbf{G}}^2(\Phi)$ in (21) evaluated at the tree-level vacuum⁹. Obviously, these are associated with the Goldstone modes whose propagators, in the Landau gauge, have poles at $p^2 = 0$.

A minimal and natural solution to this issue is provided by the transition from the zero-momentum to the on-shell regime in which the physical masses are given as a solution of the secular equation

$$\det [p^2 - \mathbf{M}^2 - \mathbf{\Sigma}(p^2) + \mathbf{\Sigma}(0)] = 0, \quad (30)$$

where \mathbf{M}^2 is the matrix of the second derivatives of the effective potential in the vacuum calculated above and $\mathbf{\Sigma}(p^2)$ denotes the corresponding matrix of the scalar fields' self-energies (in any scheme; the scheme dependence of $\mathbf{\Sigma}$ drops out of the difference above). Note also that, by definition, $\mathbf{\Sigma}(0)$ is nothing but the loop part of \mathbf{M}^2 .

In principle, the transition from the zero-momentum to the on-shell masses is highly non-trivial as it includes the full structure of $\mathbf{\Sigma}(p^2)$. However, given the scope of this study, i.e., to provide a robust description of the heavy spectrum for a future two-loop RG analysis¹⁰, the effects of $\mathbf{M}^2 + \mathbf{\Sigma}(p^2) - \mathbf{\Sigma}(0)$ in the calculation of the masses of the fields of our main interest, i.e., the pseudo-Goldstone bosons tachyonic at the tree-level, may still be reasonably well approximated by the contributions from \mathbf{M}^2 alone, as long as their pole masses stay somewhat below those of the heavy ‘‘GUT-scale’’ fields (M) circulating in the relevant loops. This may be readily seen from the momentum expansion of the typical scalar-field contribution to $\mathbf{\Sigma}(p^2) - \mathbf{\Sigma}(0)$:

$$\mathbf{\Sigma}(p^2) - \mathbf{\Sigma}(0) = \frac{1}{16\pi^2} \left(c_1 p^2 + c_2 \frac{p^4}{M^2} + \dots \right), \quad (31)$$

where c_i are numerical $\mathcal{O}(1)$ coefficients with i denoting the power of p^2 in the numerators of the corresponding terms. Substituting this into (30) and solving for p^2 in the regime in which the tree-level contribution to \mathbf{M}^2 is absent or strongly suppressed (with respect to the dominant 1-loop contribution of the order $M^2/16\pi^2$), the on-shell mass, i.e., the physical root of (30), obeys $m_{\text{phys}}^2 = \mathcal{O}(M^2/16\pi^2)$. Hence, $\mathbf{\Sigma}(p^2) - \mathbf{\Sigma}(0) = \mathcal{O}[M^2/(16\pi^2)^2] = \mathcal{O}(m_{\text{phys}}^4/M^2)$ which is clearly subleading with respect to the leading contribution from \mathbf{M}^2 .

The only exception to this simple reasoning is the case when some of the sub-blocks of the tree-level scalar mass matrix in the arguments of the log terms contain Goldstone-mode zeros which are not regulated by the corresponding zero pre-factors. Such an IR divergence is then compensated only by the $\mathbf{\Sigma}(0)$ term in eq. (30) which, however, is much easier to calculate than the full-fledged $\mathbf{\Sigma}(p^2)$; alternatively, one can just discard such IR divergences at the leading order in the perturbative expansion.

In summary, for those fields whose masses are dominated by the one-loop corrections, there is no need to deal with the self-energy at the leading order and the IR-regulated zero-momentum mass expressions derived from the effective potential are sufficient as inputs of a two-loop RGE analysis.

B. Consistency checks

Given the high complexity of the results presented in Appendix B we find it convenient to supply a set of consistency checks concerning their behaviour in several limits corresponding to an enhanced gauge symmetry when the character of the spectrum changes qualitatively.

1. Limits

There are two specific limits in which one can anticipate the form of the one-loop results (27) and (28) on the symmetry basis, corresponding to the standard and the flipped $SU(5) \times U(1)$ scenarios, attained in the regimes $\omega_b = \pm\omega_r$ (and $\sigma = 0$), respectively, see Table II.

The ‘‘standard’’ $SU(5) \times U(1)$ limit $\omega_r \rightarrow \omega_b$: In this limit, the one-loop triplet and octet masses (27) and (28) should vanish as they do at the tree level, see (14) and (15); the reason is that they become members of an $SU(5) \times U(1)$ multiplet which, in the SM vacuum, contains also a Goldstone mode $(3, 2, -\frac{5}{6}) + h.c.$, cf. Sect. III B 2.

⁹ Remarkably, all the spurious IR divergences happen to disappear from the triplet and octet Δ factors in the limit (29).

¹⁰ This, in the usual situation, requires just the tree-level masses inserted into the relevant one-loop matching formulae [27, 28]; however, in models which possess a (meta)stable vacuum supporting a non-tachyonic spectrum only at the loop level, the critical (i.e., potentially tachyonic) sectors of the spectrum require regularization by means of radiative corrections.

In order to see this, it is convenient to substitute $\omega_r = \omega_b + \kappa$ into the relevant formulae in Appendix B and then take the $\kappa \rightarrow 0$ limit. Note that the contributions of the logarithmic and polynomial type in (27) and (28) do not need to vanish separately due to the aforementioned metamorphosis (26) of some of the logarithmic terms into polynomial form. The behaviour of the individual contributions to $M_{a,b,1\text{-loop}}^2$ is sketched in the following scheme:

$$\begin{aligned}
& M_{a,\text{tree}}^2, M_{b,\text{tree}}^2 \xrightarrow{\kappa \rightarrow 0} 0, \\
& \Delta_a^{S_{FIN}[\text{poly}]}, \Delta_b^{S_{FIN}[\text{poly}]} \xrightarrow{\kappa \rightarrow 0} 0, \\
& \Delta_a^{G[\text{poly}]} + \Delta_a^{G[\log]}, \Delta_b^{G[\text{poly}]} + \Delta_b^{G[\log]} \xrightarrow{\kappa \rightarrow 0} 0, \\
& \Delta_a^{S_{INF}[\text{poly}]} + \Delta_a^{S[\log]}, \Delta_b^{S_{INF}[\text{poly}]} + \Delta_b^{S[\log]} \xrightarrow{\kappa \rightarrow 0} 0.
\end{aligned} \tag{32}$$

The proof of these equalities is slightly complicated by the fact that all the pre-factors of the log terms tend to blow up in the $\kappa \rightarrow 0$ limit, see Appendices B 1 and B 2, which renders the individual log-type contributions divergent. The obvious trick is to group together the logs whose arguments converge to the same limit and use the identity

$$\begin{aligned}
& \lim_{\kappa \rightarrow 0} \left[\sum_x \frac{A_x(\kappa)}{\kappa} \log(m_0 + c_x \kappa + \mathcal{O}(\kappa^2)) \right] \\
& = \lim_{\kappa \rightarrow 0} \left[\frac{\sum_x A_x(\kappa)}{\kappa} \right] \log(m_0) + \sum_x \frac{c_x A_x(0)}{m_0},
\end{aligned} \tag{33}$$

where x sums over a group of indices with the same argument m_0 (for $\kappa \rightarrow 0$) in the logarithm – for the scalar contributions¹¹, for instance, the indices belong to the following 6 groups:

$$\begin{aligned}
\{x\} : \quad & \{l_2, o_1\}; \{p, q, r_1\}; \{e, k, r_2\}; \{d, h, i, m_1, n_2, o_3\}; \\
& \{f, g, j, l_1, m_2, n_1, o_2\}; \{s\}.
\end{aligned} \tag{34}$$

$A_x(\kappa)$ are analytic functions of κ and c_x is the first expansion coefficient in κ of the logarithm arguments. Only terms with a non-zero constant part in $A_x(\kappa)$ produce non-vanishing polynomial pieces in the $\kappa \rightarrow 0$ limit, i.e. only the logs with divergent prefactors can give rise to a polynomial contribution. The behaviour (32) can be viewed as a rather non-trivial consistency check of the results because the constant and linear terms in κ in the sums $\sum_x A_x(\kappa)$ must vanish for each group of indices x to annihilate the logarithmic loop contributions, while the divergent parts must cancel the original polynomial pieces, as they indeed do.

The “flipped” $SU(5) \times U(1)$ limit $\omega_r \rightarrow -\omega_b$: In this case, the symmetry group is $SU(5)' \times U(1)'$, cf. Table II, the hypercharge is a linear combination of one of the Cartans of $SU(5)'$ and the extra $U(1)'$ charge. The triplet $(1, 3, 0)$ and the octet $(8, 1, 0)$ again become part of the same 24-dimensional gauge multiplet (this time together with $(3, 2, +\frac{1}{6}) + h.c.$). This is easily seen from the form of the M_r^2 matrix (A36) which becomes diagonal in this limit, with the 11-entry therein reducing to the same expression as M_a^2 and M_b^2 (in the same limit). In contrast with the “standard” $SU(5)$ case above, the $(3, 2, +\frac{1}{6}) + h.c.$ multiplet is not a would-be Goldstone boson (in the SM vacuum), so the octet and triplet masses should be equal but non-vanishing.

As before, it is convenient to implement this limit by means of the κ parameter ($\omega_r = -\omega_b + \kappa$) with $\kappa \rightarrow 0$ and, for the scalar contributions, it is useful to group the logarithmic terms according to the scheme¹²:

$$\begin{aligned}
\{x\} : \quad & \{d\}; \{l_1, o_1\}; \{f, m_1, p\}; \{e, i, m_2\}; \\
& \{g, j, l_2, n_2, o_2, q, r_2\}; \{h, k, n_1, o_3, r_1\}; \{s\}.
\end{aligned} \tag{35}$$

The results assume the following form:

¹¹ Note that there are 22 different log terms in the scalar sector, while 4 such terms are generated by the gauge interactions, cf. Appendices B 1 and B 2.

¹² These groupings are listed in the same ordering as the log terms in equation (40), with all IR divergences coming from the next to last grouping of indices, and the last grouping having no contribution. As we have argued in Sect. III A 4, the IR divergences disappear in the on-shell formula (30).

$$M_{a,\text{tree}}^2, M_{b,\text{tree}}^2 \xrightarrow{\kappa \rightarrow 0} 4a_2\omega_b^2, \quad (36)$$

$$\Delta_a^{G[\text{poly}]}, \Delta_b^{G[\text{poly}]} \xrightarrow{\kappa \rightarrow 0} \frac{17}{32\pi^2} g^4 \omega_b^2, \quad (37)$$

$$\Delta_a^{G[\log]}, \Delta_b^{G[\log]} \xrightarrow{\kappa \rightarrow 0} \frac{3}{32\pi^2} g^4 \omega_b^2 \log[g^2 \omega_b^2 / \mu_r^2], \quad (38)$$

$$\Delta_b^{S_{FIN}[\text{poly}]}, \Delta_b^{S_{FIN}[\text{poly}]} \xrightarrow{\kappa \rightarrow 0} -\frac{1}{8\pi^2} (96a_0a_2 + 76a_2^2 + 560|\gamma_2|^2 - 5\beta_4^2 + 60\beta_4\beta_4' - 100\beta_4'^2) \omega_b^2, \quad (39)$$

$$\begin{aligned} & \Delta_a^{S_{INF}[\text{poly}]} + \Delta_a^{S[\log]}, \quad \Delta_b^{S_{INF}[\text{poly}]} + \Delta_b^{S[\log]} \xrightarrow{\kappa \rightarrow 0} \frac{5}{8\pi^2} F(7; 0, 0; 9\omega_b^2, 12\omega_b^2, 60\omega_b) + \\ & + \frac{1}{16\pi^2} \left(F(-4; 0, 64\omega_b; 0, 0, -240\omega_b^2) \log[4\omega_b f(1; 0, -6\omega_b)/\mu_r^2] + \right. \\ & + F(1; -12\omega_b, -4\omega_b; -51\omega_b^2, 12\omega_b^2, 4\omega_b^2) \log[6\beta_4\omega_b^2/\mu_r^2] + \\ & + F(-9; -34\omega_b, 72\omega_b; 2\omega_b^2, 172\omega_b^2, -132\omega_b^2) \log[2\omega_b f(1; \omega_b, -4\omega_b)/\mu_r^2] + \\ & + F(-29; -20\omega_b, -148\omega_b; 0, -40\omega_b^2, -68\omega_b^2) \log[-4\omega_b f(1; 0, 2\omega_b)/\mu_r^2] + \\ & + F(6; 66\omega_b, 156\omega_b; 39\omega_b^2, -24\omega_b^2, 96\omega_b^2) \log[-2\omega_b f(1; -\omega_b, 0)/\mu_r^2] + \\ & \left. + \text{IR divergent log terms} \right), \quad (40) \end{aligned}$$

where the F function is defined in eq. (A44). Let us note that the “log-to-polynomial metamorphosis” is less frequent here than in the previous case because only few of (exclusively scalar) log prefactors blow up in the “flipped” $SU(5)$ limit. In fact, only the O_j , O_{l_2} , O_{n_2} , O_{o_2} , T_g , T_{o_2} terms (see Appendix B) contribute to the polynomial part of eq. (40).

2. Exact Goldstone bosons

There is another relatively cheap consistency check of the method used for the calculation of the various $\Delta_{a,b}$ factors governing the leading one-loop contributions to the PGB masses of our interest: all would-be Goldstone modes associated with the gauge fields from the $SO(10)/SU(3) \times SU(2) \times U(1)$ coset should be massless at all orders in perturbation theory. We managed to demonstrate that this is the case at least for the most accessible of these fields, namely, the $(3, 2, -\frac{5}{6}) + h.c.$ scalar which is contained solely in the scalar 45 of the $SO(10)$ and, hence, no mixing with the scalar 126 needs to be considered.

We have checked explicitly that it indeed remains massless even at the 1-loop level, i.e., that each of its Δ_c -contributions – defined along the lines of eqs. (27) and (28) – individually vanishes: the log and the polynomial terms of the gauge contribution, as well as the polynomial terms and log terms of the scalar contribution. In this case, the polynomial parts from scalars $\Delta_c^{S_{FIN}[\text{poly}]}$ and $\Delta_c^{S_{INF}[\text{poly}]}$ even vanish separately.¹³

Moreover, the structure of all the Δ_c contributions is such that it allows for a simple numerical check even outside the “analytic simplification domain” (29) where it likewise yields zero for all randomly chosen values of σ , γ_2 and a_2 .

3. Diagrammatics

Last, but not least, it is relatively straightforward to calculate some of the leading polynomial parts of the one-loop corrections to the masses of the $(1, 3, 0)$ and $(8, 1, 0)$ PGBs by means of the standard perturbative expansion. This applies, in particular, to the τ^2 -proportional (i.e., $SO(10)$ invariant) contribution, cf. eqs. (B7) and (B8); the other leading polynomial terms, namely, those proportional to ω_r^2 , ω_b^2 and/or $\omega_r\omega_b$ turn out to be easily accessible only in the simplified version of the minimal model with 16 instead of 126 in the Higgs sector. In what follows, we shall first comment briefly on the salient points of the corresponding calculation in the sample $45 \oplus 16$ Higgs model and then turn our attention to the $45 \oplus 126$ scenario of our current interest.

¹³ If one expresses the scalar mass matrix in a reordered basis, in which \mathbf{M}_S^2 acquires a block diagonal form, where each block consists of states with degenerate mass, it is easy to see explicitly that $\Delta_c^{S_{INF}[\text{poly}]} = 0$ without even referring to any special limit. The block structure alone allows us in that case to avoid the computation of complicated analytical forms of eigenvectors (modulo those of Goldstone modes, which are relatively simple) and automatically discards the nested commutator’s polynomial contribution to the 1-loop mass of the would-be Goldstone pair $(3, 2, -\frac{5}{6}) \oplus (\bar{3}, 2, +\frac{5}{6})$.

TABLE III. β^2 -proportional parts of the polynomial corrections to the masses of the pseudo-Goldstone bosons in the simplified $45 \oplus 16$ scenario in various limits.

$4\pi^2\Delta_\Phi$	$\omega_r = 0$	$\omega_b = 0$	$\omega_r = -\omega_b \equiv \omega$
$\Phi = (1, 3, 0)$	$2\beta^2\omega_b^2$	$2\beta^2\omega_r^2$	$5\beta^2\omega^2$
$\Phi = (8, 1, 0)$	$3\beta^2\omega_b^2$	$\beta^2\omega_r^2$	$5\beta^2\omega^2$

The method: In the simplest scalar theory context with a pair of scalar fields Φ and ϕ with only the latter developing a VEV, it is straightforward to show that the leading order one-loop contribution to the mass (squared) of Φ can be formally written as [29]

$$\Delta_\Phi = \text{[diagram: blob with two external legs and dots]} - \frac{1}{\langle\phi\rangle} \text{[diagram: blob with one external leg and dots]}, \quad (41)$$

where the graphs denote the sums of the one-loop contributions to the two-point and one-point functions with appropriate external legs Φ , respectively, while the dots between the crossed lines correspond to all possible insertions¹⁴ of the VEVs of ϕ . In the classical “ $\lambda\phi^4$ ” context” these structures can be formally expanded as

$$\text{1-point: } \text{[diagram: blob with one external leg]} \ni \text{[diagram: tadpole]} + \text{[diagram: tadpole with loop]} + \dots, \quad (42)$$

$$\text{2-point: } \text{[diagram: blob with two external legs]} \ni \text{[diagram: bubble]} + \text{[diagram: bubble with loop]} + \text{[diagram: bubble with tadpole]} + \dots, \quad (43)$$

where the symbols with “empty” blobs stand for the usual Feynman diagrams of a given topology. There are a few points worth making here:

1. Only some of the one-loop topologies above will generate a polynomial contribution to Δ_Φ ; for example, the first two displayed contributions to the 2-point function (43) yield a polynomial contribution, while the third does not.
2. The undisplayed terms in the remainder of series (42) and (43) correspond to diagrams with higher and higher number of insertions of (pairs of) VEVs and, as such, they may generate a power-series-like structure of similar polynomial contributions; if the interactions are simple enough, the quotients of such power series may be identified and the series themselves may be eventually summed up in a closed form.
3. Note that if there is simultaneously a trilinear vertex at play, many more topologies become available; this will lead to “mixed” contributions proportional not only to the VEVs but also to the dimensionful trilinear vertex coupling (such as τ in the $SO(10)$ context of our interest). However, as long as one is interested in either the pure VEV-squared or the pure τ^2 contribution to Δ_Φ , it is sufficient to focus on the relevant sub-series with only one kind of interactions (quartic or trilinear, respectively) connecting the VEV legs to the main loop.

Diagrammatics in the $45 \oplus 16$ scenario, the β^2 -proportional polynomial piece: This all said, let us first turn our attention to the $SO(10)$ Higgs model featuring a simplified set of scalars transforming as $45 \oplus 16$, see, e.g., reference [14]. There are again three convenient limits in this setting corresponding to the assumed single VEV situation, namely $\omega_r = 0$ (the $3_c 2_L 2_R 1_{B-L}$ limit), $\omega_b = 0$ (the $4_c 2_L 1_R$ limit) and $\omega_r = -\omega_b$ (the flipped $SU(5)$ limit), in which the formalism above may be quite easily applied and clusters of graphs with contributions behaving as a power series can be identified in the 1-point and 2-point Green’s function expansions.

For instance, focusing solely on the quartic interaction governed by the β coupling (see [14] for its structure), the power-like behaviour of the polynomial contributions on the RHS of eqs. (42) and (43) may be readily inferred. The combinations (41) of the sums of the corresponding power series are given in Table III. Remarkably, the information thus obtained in the three different limits above is just enough to reconstruct all three coefficients c_i^Φ of the expected form of the β^2 -proportional polynomial one-loop contribution to Δ_Φ , namely $\Delta_\Phi = \beta^2(c_1^\Phi\omega_r^2 + c_2^\Phi\omega_r\omega_b + c_3^\Phi\omega_b^2)$.

¹⁴ Obviously, the calculation is performed in the unbroken phase formalism in which the VEV is kept in the interaction part of the Lagrangian.

TABLE IV. The shape of a potentially viable scalar spectrum (all masses in units of ω_b) at the leading order (corresponding to the one-loop level expressions (27)–(28) for the critical pseudo-Goldstones and tree-level formulae for the other scalars, respectively) computed in the limit (29). The underlying parameters were chosen as follows: $a_2 = -19.7|\sigma|^2/\omega_b^2$ was determined by the vacuum stationarity condition (11) and the other parameters assumed the values $\omega_r/\omega_b = 0.2$ (hence abandoning the flipped $SU(5)$ scenario required at tree level, cf. (16)), $\tau/\omega_b = -2$, $\mu_r/\omega_b = 2$, $\beta_4 = \beta'_4 = 0.4$, $a_0 = 0.2$ and $g = 0.7$. The scalar potential parameters α , λ_0 , λ_2 , λ_4 , λ'_4 and η_2 can remain unspecified since, at the leading order, they do not contribute to the above masses (as explained in Section III A 3). Note also that the loop-induced pseudo-Goldstone masses are much lighter than the bulk of the scalar spectrum, which justifies the simple transition from zero-momentum to on-shell-momentum masses advocated in Sect. III A 4.

m_d	m_e	m_f	m_g	m_h	m_i	m_j	m_k	m_{l_1}	m_{l_2}	m_{m_1}	m_{m_2}	m_{n_1}	m_{n_2}	m_{o_1}	m_{o_2}	m_{o_3}	m_p	m_q	m_{r_1}	m_{r_2}	m_s	$M_{a,1\text{-loop}}$	$M_{b,1\text{-loop}}$
1.9	3.7	3.3	3.7	3.0	3.5	3.6	3.7	3.6	3.5	2.6	3.7	3.3	3.5	3.2	3.7	3.1	1.4	2.9	2.3	3.7	2.2	0.27	0.35

Putting all this together, the polynomial pieces of the scalar-loop-generated corrections to the masses of the pseudo-Goldstone bosons $(1, 3, 0)$ and $(8, 1, 0)$ in the simplified $45 \oplus 16$ scenario read (in the same notation as before)

$$\begin{aligned}\Delta_a^{S[\text{poly}],\beta^2} &= \frac{1}{4\pi^2}\beta^2 (2\omega_r^2 - \omega_r\omega_b + 2\omega_b^2) , \\ \Delta_b^{S[\text{poly}],\beta^2} &= \frac{1}{4\pi^2}\beta^2 (\omega_r^2 - \omega_r\omega_b + 3\omega_b^2) ,\end{aligned}\tag{44}$$

which, indeed, coincides with the results of the existing effective potential analysis [14].

Diagrammatics in the $45 \oplus 16$ scenario, the τ^2 -proportional polynomial piece: Since the interaction of our interest here is trilinear, there are no τ -proportional polynomial contributions popping up from the two-point part (43) of formula (41) and, hence, it is sufficient to consider only the τ -proportional tadpoles (42). The summation of the relevant parts of the corresponding power series yields a universal (i.e., $SO(10)$ invariant) contribution

$$\Delta_a^{S[\text{poly}],\tau^2} = \Delta_b^{S[\text{poly}],\tau^2} = \frac{\tau^2}{4\pi^2} ,\tag{45}$$

which, as before, coincides with that obtained by the effective potential approach [14]. Hence, at least for the one-loop polynomial corrections to the PGB triplet and octet masses, the purely diagrammatic approach admits an efficient cross-check of the EP results.

Diagrammatics in the $45 \oplus 126$ scenario, the τ^2 -proportional polynomial piece: Finally, let us attempt to evaluate some of the $\Delta_a^{S[\text{poly}]}$ and $\Delta_b^{S[\text{poly}]}$ terms in the $45 \oplus 126$ model of our main interest. Unfortunately, the presence of two types of quartic interactions between a pair of 45s and two 126s, i.e., the β_4 and β'_4 terms in the scalar potential (6), complicates the combinatorics of the VEV insertions in diagrams (41) to such an extent that here we have managed to calculate just the universal τ^2 -proportional factor

$$\Delta_a^{S[\text{poly}],\tau^2} = \Delta_b^{S[\text{poly}],\tau^2} = \frac{35\tau^2}{8\pi^2} ,\tag{46}$$

which, indeed, is identical to that obtained by the effective potential method earlier in this study, cf. Sect. III A 3 and formulae (B7) and (B8).

Finally, it may be interesting to note that the parametrically higher level of complexity encountered in the $45 \oplus 126$ model in the relevant Feynman diagram combinatorics is reflected in the effective potential approach by the behaviour of the nested commutators (23): the interplay between β_4 and β'_4 was indeed the main obstacle to writing the results of the EP calculation of Sect. III A 3 in a more compact form, which would be otherwise possible if one of these couplings was zero. In that respect, the simplicity of the diagrammatic calculation in the $16 \oplus 45$ case can be attributed to the vanishing of the relevant nested commutators in the EP approach to this model.

C. Viability of the minimal $SO(10)$ Higgs model at the one-loop level

With all this information at hand one can finally re-address the central question – whether the radiative corrections can really provide a regularization of the supposedly fatal tachyonic instabilities of the tree-level scalar spectrum.

Clearly, the first condition to be fulfilled is that there should exist a domain in the parameter space where the loop contributions to the masses of the pseudo-Goldstone triplet and octet scalars, cf. (27)–(28), are large enough to compete with the problematic tree-level expressions therein and where all the other scalar-sector masses-squares are positive.

The former assumption is obviously attained in the regime when a_2 is sufficiently small. Remarkably enough, this is not only an option that one may choose at will, but rather a crucial ingredient of any perturbative account, see inequality (12). In this respect, *in the perturbative regime supporting a standard seesaw mechanism, the quantum corrections exceed the tree level contribution and might hence regularize the notorious tachyonic instabilities of the scalar spectrum*. As for the latter, there seems to be no simple analytic parametrization of the non-tachyonicity domain for the rest of the scalar spectrum. Hence, we provide a numerical example of a point in parameter space where the *entire scalar spectrum is perturbative and non-tachyonic*, see Table IV. Note that the situation therein corresponds to the simplified setting (29) in which $SO(10)$ gets first broken to $SU(3)_c \times SU(2)_L \times U(1)_R \times U(1)_{B-L}$ whose subsequent breaking to the SM gauge group is arbitrarily delayed (due to tiny σ) and, thus, there is an extra virtually massless mode in M_s alongside the usual massless would-be Goldstones in matrices $\{M_c, M_p, M_q, M_r, M_s\}$, with multiplicities 1 for each.

To this end, it is also worth noting that there is yet another field for which the radiative corrections may be important, namely, the lighter singlet scalar spanned predominantly on the upper-left 2×2 block of matrix (A37). The point is that this state also becomes accidentally light in the potentially realistic $a_2 \ll 1$ and $|\sigma| \ll \max\{|\omega_r|, |\omega_b|\}$ regime. However, unlike the two pseudo-Goldstones of our main interest here, this state still receives tree-level mass contributions from σ and, moreover, it is innocent from the gauge running perspective. Hence, we defer a thorough scrutiny of this issue to the future phenomenological analysis.

IV. CONCLUSIONS AND OUTLOOK

In this work, we have calculated the one-loop corrections to the masses of a pair of scalar fields (transforming as $(1, 3, 0)$ and $(8, 1, 0)$ under the SM gauge group) in the spectrum of the minimal non-supersymmetric $45 \oplus 126$ $SO(10)$ Higgs model which, at the tree level, cause a notorious tachyonic instability in all of its potentially realistic vacuum configurations. That required considerable efforts due to large representations involved, presence of matrix logarithms and infinite series of nested commutators. The calculation confirms the former expectation made on qualitative grounds in [14] that the quantum effects can stabilize the phenomenologically viable vacua of the model at the one-loop level. Hence, the $45 \oplus 126$ framework may be revived as a basis of a full-fledged $SO(10)$ GUT construction that may be worth a further scrutiny concerning, in particular, the fundamental signal of gauge unifications – the proton lifetime. To this end, the current framework exhibits a particular robustness to various kinds of theoretical uncertainties, essentially unattainable in other popular GUT scenarios, which makes it very special when it comes to the exploitation of the information accessible in future megaton-scale experiments such as Hyper-Kamiokande or DUNE.

The consistency of the effective potential approach adopted in this study has been demonstrated by a number of explicit cross-checks of the results, including a thorough inspection of several of their enhanced-symmetry limits, a semi-analytic proof of the absence of a one-loop mass term for a selected would-be Goldstone mode, as well as a partial reconstruction of their purely polynomial parts by means of standard diagrammatic methods.

Future prospects

For minimal $SO(10)$ GUTs featuring a $45 \oplus 126$ scalar sector, a detailed understanding of the scalar spectrum behaviour (and in particular of its critical components) is of course vital for any future phenomenological analysis going beyond the first rather simplified attempts [10, 17]. The obvious goal here is to provide *really robust* estimates of the attainable proton lifetime with at least the leading theoretical uncertainties well under control, hopefully even within the expected “sensitivity improvement window” of the upcoming facilities. For that sake, a detailed analysis of the unification constraints including a two-loop renormalization group evolution of the gauge couplings is a particularly important element which, as an input, among other things, requires exactly the information supplied by this study.

On the practical side, that demands an extensive numerical simulation which would go even beyond the limit (29) in which the analytic results have been displayed in this work (due to the paramount complexity of the full results which, however, are also available). This will also facilitate the calculation of the radiative corrections to the mass of the third member of the potentially dangerous pseudo-Goldstone boson family which, in the physically interesting $\sigma < \max\{\omega_r, \omega_b\}$ regime, can be identified among the SM singlets (A37). In spite of its practical irrelevance for the gauge running, it may still represent an extra source of tachyonic instabilities that a decisive scrutiny of the minimal

model under consideration should not neglect. This, however, is beyond the scope of the current work and will be elaborated on elsewhere.

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Appendix A: The tree-level spectrum

1. Gauge bosons

The scalar sector of the model is spanned on a 45-dimensional 2-index antisymmetric real representation ϕ_{ij} and a 126-dimensional 5-index antisymmetric self-dual complex representation Σ_{ijklm} , defined as¹⁵

$$\Sigma_{ijklm} = \frac{1}{\sqrt{2}} \left(\phi_{ijklm} - \frac{i}{5!} \epsilon_{ijklmabcde} \phi_{abcde} \right), \quad (\text{A1})$$

where ϕ_{ijklm} and $\epsilon_{ijklmabcde}$ are the 252-dimensional 5-index antisymmetric tensor and the completely antisymmetric Levi-Civita tensor with the positive signature $\epsilon_{12345678910} = +1$, respectively. Under the infinitesimal $SO(10)$ transformations these objects change as

$$\phi_{ij} \rightarrow \phi_{ij} + i [\varphi, \phi]_{ij}, \quad (\text{A2})$$

$$\begin{aligned} \Sigma_{ijklm} \rightarrow \Sigma_{ijklm} + i (\varphi_{ia} \Sigma_{ajklm} + \varphi_{jb} \Sigma_{ibklm} + \\ + \varphi_{kc} \Sigma_{ijclm} + \varphi_{ld} \Sigma_{ijkdm} + \varphi_{me} \Sigma_{ijkle}), \end{aligned} \quad (\text{A3})$$

where

$$\varphi_{ij} \equiv \frac{1}{2} \varphi_{\alpha\beta} (\hat{T}^{\alpha\beta})_{ij}, \quad (\text{A4})$$

and $\varphi_{\alpha\beta}$ are the infinitesimal antisymmetric real parameters, while $\hat{T}^{\alpha\beta}$ are the generators in the fundamental representation of $SO(10)$ defined as:

$$(\hat{T}^{\alpha\beta})_{ij} \equiv -\frac{i}{\sqrt{2}} (\delta_{\alpha i} \delta_{\beta j} - \delta_{\alpha j} \delta_{\beta i}), \quad (\text{A5})$$

that satisfy the $SO(10)$ algebra

$$[\hat{T}^{\alpha\beta}, \hat{T}^{\gamma\delta}] = \frac{i}{\sqrt{2}} (\delta^{\alpha\gamma} \hat{T}^{\beta\delta} + \delta^{\beta\delta} \hat{T}^{\alpha\gamma} - \delta^{\beta\gamma} \hat{T}^{\alpha\delta} - \delta^{\alpha\delta} \hat{T}^{\beta\gamma}). \quad (\text{A6})$$

Consequently they are then canonically normalized to Dynkin index 1

¹⁵ This self-duality projection holds only when indices are in the real basis of the fundamental representation 10 of $SO(10)$.

TABLE V. SM representations of gauge bosons, using definitions from Table I. There are two kinds of contributions to the tree level masses of gauge bosons: $\Delta_{\mathbf{M}_G^2}^\phi$ coming from the kinetic term of the 45-dimensional Higgs (due to the VEVs of the SM singlets $\sqrt{3}\omega_b$ and $\sqrt{2}\omega_r$) and $\Delta_{\mathbf{M}_G^2}^\Sigma$ from the kinetic term of the 126-dimensional representations (due to $\sqrt{2}\sigma$ and $\sqrt{2}\sigma^*$). The final masses M_G^2 are the sums of both, producing 12 massless gauge bosons, corresponding to 12 unbroken generators of the SM; the dagger \dagger denotes the presence of a broken generator and consequently a massive gauge boson. We kept both contributions $\Delta_{\mathbf{M}_G^2}^\phi$ and $\Delta_{\mathbf{M}_G^2}^\Sigma$ separate only to make various limits more evident. For that reason we also indicated which representations of the Pati-Salam gauge group G_{422} individual SM representations descend from. It is then easy to check that for $|\sigma| = 0$ in the $SU(5) \times U(1)_Z$ ($\omega_r = \omega_b$), flipped $SU(5)' \times U(1)_{Z'}$ ($\omega_r = -\omega_b$), left-right G_{3221} ($\omega_r = 0$), G_{421} ($\omega_b = 0$) and G_{3211} ($\omega_r, \omega_b \neq 0$) limits, all the states within the same multiplets have the same masses and we get 25, 25, 15, 19 and 13 massless gauge bosons, respectively. The two singlets (1, 1, 0) in different Pati-Salam representations mix; the two given expressions are the two eigenvalues of the corresponding 2×2 mass matrix.

$R \sim G_{321}$	\mathbb{R}/\mathbb{C}	#	size	$R \subseteq G_{422}$	$\Delta_{\mathbf{M}_G^2}^\phi$	$\Delta_{\mathbf{M}_G^2}^\Sigma$
(8, 1, 0)	\mathbb{R}	1	8	(15, 1, 1)	0	0
(1, 3, 0)	\mathbb{R}	1	3	(1, 3, 1)	0	0
(1, 1, 0)	\mathbb{R}	2^\dagger	1	(15, 1, 1), (1, 1, 3)	0; 0	$10 g^2 \sigma ^2$; 0
(1, 1, +1)	\mathbb{C}	1^\dagger	2	(1, 1, 3)	$2g^2 \omega_r^2$	$2g^2 \sigma ^2$
(3, 1, $+\frac{2}{3}$)	\mathbb{C}	1^\dagger	6	(15, 1, 1)	$2g^2 \omega_b^2$	$2g^2 \sigma ^2$
(3, 2, $+\frac{1}{6}$)	\mathbb{C}	1^\dagger	12	(6, 2, 2)	$\frac{1}{2} g^2 (\omega_r + \omega_b)^2$	$2g^2 \sigma ^2$
(3, 2, $-\frac{5}{6}$)	\mathbb{C}	1^\dagger	12	(6, 2, 2)	$\frac{1}{2} g^2 (\omega_r - \omega_b)^2$	0

$$\text{Tr} \left(\hat{T}^{\alpha\beta} \hat{T}^{\gamma\delta} \right) = \delta_{\alpha\gamma} \delta_{\beta\delta} - \delta_{\alpha\delta} \delta_{\beta\gamma}, \quad (\text{A7})$$

and, hence, the corresponding gauge coupling follows the usual $SU(5)$ / Standard Model normalization convention. The Latin and the Greek indices all run from 1 to 10, while the summation over repeating indices is assumed everywhere. The gauge bosons in the adjoint representation

$$(A_\mu)_{ij} \equiv \frac{1}{2} A_\mu^{\alpha\beta} (\hat{T}^{\alpha\beta})_{ij}, \quad (\text{A8})$$

then transform as

$$(A_\mu)_{ij} \rightarrow (A_\mu)_{ij} + i [\varphi, A_\mu]_{ij} + \frac{1}{g} (\partial_\mu \varphi)_{ij}. \quad (\text{A9})$$

Their mass term $\mathcal{L} \supset \frac{1}{2} \mathbf{M}_G^2_{(\alpha\beta)(\gamma\delta)} A_\mu^{(\alpha\beta)} A^{\mu(\gamma\delta)}$ then originates from the kinetic terms for the scalar fields

$$\mathcal{L}_{kin} \supset \frac{1}{4} (D_\mu \phi_{ij})^* D^\mu \phi_{ij} + \frac{1}{5!} (D_\mu \Sigma_{ijklm})^* D^\mu \Sigma_{ijklm}, \quad (\text{A10})$$

where the covariant derivatives are defined as

$$D_\mu \phi_{ij} \equiv \partial_\mu \phi_{ij} - i g [A_\mu, \phi]_{ij}, \quad (\text{A11})$$

$$\begin{aligned} D_\mu \Sigma_{ijklm} \equiv & \partial_\mu \Sigma_{ijklm} - i g \{ (A_\mu)_{ia} \Sigma_{ajklm} + \\ & + (A_\mu)_{jb} \Sigma_{ibklm} + (A_\mu)_{kc} \Sigma_{ijclm} + \\ & + (A_\mu)_{ld} \Sigma_{ijkdm} + (A_\mu)_{me} \Sigma_{ijkle} \}, \end{aligned} \quad (\text{A12})$$

and the subscripts $(i_1 \dots i_n)$ stand for the $\binom{10}{n}$ ordered n-tuples of indices. For instance, $(\alpha\beta)$ and $(\gamma\delta)$ represent the 45 ordered pairs of indices that form the basis in which the 45×45 -dimensional gauge boson mass matrix \mathbf{M}_G^2 is

expressed as

$$\begin{aligned}
\mathbf{M}_{\mathbf{G}(\alpha\beta)(\gamma\delta)}^2 &= \frac{g^2}{2} \left\{ \frac{1}{2} \left[\hat{T}^{(\alpha\beta)}, \langle \phi \rangle \right]_{ij} \left[\hat{T}^{(\gamma\delta)}, \langle \phi \rangle \right]_{ji} + \right. \\
&+ \delta^{\alpha\gamma} \langle \Sigma_{\beta(jklm)}^* \rangle \langle \Sigma_{\delta(jklm)} \rangle + \delta^{\beta\delta} \langle \Sigma_{\alpha(jklm)}^* \rangle \langle \Sigma_{\gamma(jklm)} \rangle - \\
&- \delta^{\alpha\delta} \langle \Sigma_{\beta(jklm)}^* \rangle \langle \Sigma_{\gamma(jklm)} \rangle - \delta^{\beta\gamma} \langle \Sigma_{\alpha(jklm)}^* \rangle \langle \Sigma_{\delta(jklm)} \rangle + \\
&+ \langle \Sigma_{\alpha\delta(klm)}^* \rangle \langle \Sigma_{\beta\gamma(klm)} \rangle + \langle \Sigma_{\beta\gamma(klm)}^* \rangle \langle \Sigma_{\alpha\delta(klm)} \rangle - \\
&- \left. \langle \Sigma_{\alpha\gamma(klm)}^* \rangle \langle \Sigma_{\beta\delta(klm)} \rangle - \langle \Sigma_{\beta\delta(klm)}^* \rangle \langle \Sigma_{\alpha\gamma(klm)} \rangle \right\} + \\
&+ \begin{pmatrix} \alpha \leftrightarrow \gamma \\ \beta \leftrightarrow \delta \end{pmatrix}.
\end{aligned} \tag{A13}$$

Except for the 2×2 block of singlets, the matrix $\mathbf{M}_{\mathbf{G}}^2|_v$ is already diagonal, which greatly simplifies the computation and final form of the nested commutator and logarithmic contributions to the scalar masses at 1 loop. Its eigenvalues, representing the actual physical masses of the gauge bosons, are collected in Table V. The 12 zero modes correspond to the 12 generators of G_{321} . The tree-level masses of the GUT-scale gauge bosons are

$$M_G^2(3, 2, -\frac{5}{6}) = \frac{1}{2} g^2 (\omega_r - \omega_b)^2, \tag{A14}$$

$$M_G^2(3, 2, +\frac{1}{6}) = 2 g^2 \left(\frac{1}{4} (\omega_r + \omega_b)^2 + |\sigma|^2 \right), \tag{A15}$$

$$M_G^2(3, 1, +\frac{2}{3}) = 2 g^2 (\omega_b^2 + |\sigma|^2), \tag{A16}$$

$$M_G^2(1, 1, +1) = 2 g^2 (\omega_r^2 + |\sigma|^2), \tag{A17}$$

$$M_G^2(1, 1, 0) = 10 g^2 |\sigma|^2. \tag{A18}$$

2. Scalars

The tree level mass matrices of the various scalar fields belonging to ϕ , Σ and Σ^* representations (whose decompositions and possible mixings are explained in Table I; the states of a given SM rep. R in that table correspond to basis states for rows in the mass matrices below, all in the same order, while columns have a basis of conjugate states \bar{R}) are

$$M_a^2 \equiv M_S^2(1, 3, 0) = +2a_2\omega'w_{[1,2]}^1, \quad (\text{A19})$$

$$M_b^2 \equiv M_S^2(8, 1, 0) = -2a_2\omega'w_{[2,1]}^1, \quad (\text{A20})$$

$$M_c^2 \equiv M_S^2(3, 2, -\frac{5}{6}) = 0, \quad (\text{A21})$$

$$M_d^2 \equiv M_S^2(1, 1, +2) = 4f(-\omega_r; 0, 6\omega_b\omega_r) + 8s_{[1,1,4]}^2, \quad (\text{A22})$$

$$M_e^2 \equiv M_S^2(1, 3, -1) = 2f(-w_{[3,1]}^1; 0, 2\omega_r w_{[3,1]}^1) + 8s_{[2,3,2]}^2, \quad (\text{A23})$$

$$M_f^2 \equiv M_S^2(\bar{3}, 1, +\frac{4}{3}) = 2f(-w_{[1,2]}^1; \omega_b^2, 4\omega_b w_{[1,2]}^1) + 4s_{[3,3,4]}^2, \quad (\text{A24})$$

$$M_g^2 \equiv M_S^2(3, 3, -\frac{1}{3}) = 2f(-w_{[2,1]}^1; \omega_b^2, 2\omega w_{[2,1]}^1) + 4s_{[3,3,4]}^2, \quad (\text{A25})$$

$$M_h^2 \equiv M_S^2(6, 3, +\frac{1}{3}) = 2f(-\omega; 0, 2\omega w_{[2,1]}^1) + 8s_{[1,1,4]}^2, \quad (\text{A26})$$

$$M_i^2 \equiv M_S^2(\bar{6}, 1, -\frac{4}{3}) = 4f(-\omega_b; 0, 2\omega_b w_{[1,2]}^1) + 8s_{[1,1,4]}^2, \quad (\text{A27})$$

$$M_j^2 \equiv M_S^2(\bar{6}, 1, -\frac{1}{3}) = 2f(-w_{[2,1]}^1; \omega_r^2, 2\omega w_{[2,1]}^1) + 4s_{[3,3,4]}^2, \quad (\text{A28})$$

$$M_k^2 \equiv M_S^2(\bar{6}, 1, +\frac{2}{3}) = 4f(-\omega; 0, 2\omega\omega_b) + 8s_{[2,3,2]}^2. \quad (\text{A29})$$

$$M_l^2 \equiv M_S^2(1, 2, +\frac{1}{2}) = \begin{pmatrix} f(-3\omega; \frac{1}{2}w_{[7,-4,1]}^2, 3\omega w_{[3,1]}^1) + 4s_{[3,3,4]}^2 & -2\gamma_2\omega\omega' \\ -2\gamma_2^*\omega\omega' & f(-w_{[3,1]}^1; \frac{1}{2}w_{[7,4,1]}^2, 3\omega w_{[3,1]}^1) + 8s_{[1,1,-2]}^2 \end{pmatrix}, \quad (\text{A30})$$

$$M_m^2 \equiv M_S^2(3, 2, +\frac{7}{6}) = \begin{pmatrix} f(-w_{[1,3]}^1; \frac{1}{2}\omega'^2, w_{[1,3]}^1 w_{[5,1]}^1) + 8s_{[1,1,4]}^2 & -2\gamma_2\omega\omega' \\ -2\gamma_2^*\omega\omega' & f(-w_{[5,1]}^1; \frac{1}{2}\omega^2, w_{[1,3]}^1 w_{[5,1]}^1) + 4s_{[3,3,4]}^2 \end{pmatrix}, \quad (\text{A31})$$

$$M_n^2 \equiv M_S^2(8, 2, +\frac{1}{2}) = \begin{pmatrix} f(-3\omega; \frac{1}{2}\omega^2, 3\omega w_{[3,1]}^1) + 4s_{[3,3,4]}^2 & -2\gamma_2\omega\omega' \\ -2\gamma_2^*\omega\omega' & f(-w_{[3,1]}^1; \frac{1}{2}\omega'^2, 3\omega w_{[3,1]}^1) + 8s_{[1,1,4]}^2 \end{pmatrix}, \quad (\text{A32})$$

$$M_o^2 \equiv M_S^2(\bar{3}, 1, +\frac{1}{3}) = \begin{pmatrix} f(-2\omega; w_{[1,0,1]}^2, 4\omega w_{[2,1]}^1) + 8s_{[1,1,2]}^2 & -4\gamma_2^*\omega\omega' & 2\sqrt{2}(\beta_4\omega_b\omega_r - 8s_{[0,0,1]}^2) \\ -4\gamma_2\omega\omega' & f(-2w_{[2,1]}^1; w_{[1,0,1]}^2, 4\omega w_{[2,1]}^1) + 4s_{[3,3,4]}^2 & 0 \\ 2\sqrt{2}(\beta_4\omega_b\omega_r - 8s_{[0,0,1]}^2) & 0 & 2f(-\omega; w_{[1,0,1]}^2, 2\omega w_{[2,1]}^1) + 8s_{[1,1,0]}^2 \end{pmatrix}. \quad (\text{A33})$$

$$M_p^2 \equiv M_S^2(1, 1, +1) = \begin{pmatrix} -2a_2\omega\omega_b + 2s_{[1,-2]}'^2 & 2\sigma f(1; -\omega_r, -2w_{[3,1]}^1) \\ 2\sigma^* f(1; -\omega_r, -2w_{[3,1]}^1) & -2\omega_r f(1; -\omega_r, -2w_{[3,1]}^1) \end{pmatrix}, \quad (\text{A34})$$

$$M_q^2 \equiv M_S^2(\bar{3}, 1, -\frac{2}{3}) = \begin{pmatrix} -2a_2\omega\omega_r + 2s_{[1,-2]}'^2 & -2\sigma^* f(1; -\omega_b, -4\omega) \\ -2\sigma f(1; -\omega_b, -4\omega) & -2\omega_b f(1; -\omega_b, -4\omega) \end{pmatrix}, \quad (\text{A35})$$

$$M_r^2 \equiv M_S^2(3, 2, +\frac{1}{6}) = \begin{pmatrix} -4a_2\omega_b\omega_r + 2s_{[1,-2]}'^2 & \sigma f(2; -\omega, -2w_{[5,3]}^1) & 4\gamma_2\sigma\omega' \\ \sigma^* f(2; -\omega, -2w_{[5,3]}^1) & -\frac{1}{2}\omega f(2; -\omega, -2w_{[5,3]}^1) & -2\gamma_2\omega\omega' \\ 4\gamma_2^*\sigma^*\omega' & -2\gamma_2^*\omega\omega' & f(-w_{[5,3]}^1; \frac{1}{2}\omega'^2, \omega w_{[5,3]}^1) + 8s_{[2,3,2]}^2 \end{pmatrix}, \quad (\text{A36})$$

$$M_s^2 \equiv M_S^2(1, 1, 0) = \begin{pmatrix} 24a_0\omega_b^2 - 2a_2\omega'w_{[2,1]}^1 - 12s_{[0,1]}'^2 & -\sqrt{6}(-8a_0\omega_b\omega_r + 4s_{[0,1]}'^2) & \sqrt{6}\sigma(2\alpha\omega_b + f(1; 0, -4\omega)) & \sqrt{6}\sigma^*(2\alpha\omega_b + f(1; 0, -4\omega)) \\ -\sqrt{6}(-8a_0\omega_b\omega_r + 4s_{[0,1]}'^2) & 16a_0\omega_r^2 + 2a_2\omega'w_{[1,2]}^1 - 8s_{[0,1]}'^2 & 2\sigma(2\alpha\omega_r + f(1; 0, -2w_{[3,1]}^1)) & 2\sigma^*(2\alpha\omega_r + f(1; 0, -2w_{[3,1]}^1)) \\ \sqrt{6}\sigma^*(2\alpha\omega_b + f(1; 0, -4\omega)) & 2\sigma^*(2\alpha\omega_r + f(1; 0, -2w_{[3,1]}^1)) & 4\lambda_0|\sigma|^2 & 4\lambda_0\sigma^{*2} \\ \sqrt{6}\sigma(2\alpha\omega_b + f(1; 0, -4\omega)) & 2\sigma(2\alpha\omega_r + f(1; 0, -2w_{[3,1]}^1)) & 4\lambda_0\sigma^2 & 4\lambda_0|\sigma|^2 \end{pmatrix}. \quad (\text{A37})$$

In the expressions above (and in section III B 1) we have used the definitions

$$w_{[a_n, \dots, a_0]}^n := \sum_{i=0}^n a_{n-i} \omega_b^{n-i} \omega_r^i, \quad (\text{A38})$$

$$\omega := \omega_b + \omega_r = w_{[1,1]}^1, \quad (\text{A39})$$

$$\omega' := -\omega_b + \omega_r = w_{[-1,1]}^1, \quad (\text{A40})$$

$$s_{[a_1, a_2, a_3]}^2 := (\lambda_2 a_1 + \lambda_4 a_2 + \lambda'_4 a_3) |\sigma|^2, \quad (\text{A41})$$

$$s_{[a_1, a_2]}'^2 := (\beta_4 a_1 + \beta'_4 a_2) |\sigma|^2, \quad (\text{A42})$$

$$f(z_1; z_2, z_3) := \tau z_1 + \beta_4 z_2 + \beta'_4 z_3, \quad (\text{A43})$$

$$F(z_1; z_2, z_3; z_4, z_5, z_6) := \tau^2 z_1 + \tau(\beta_4 z_2 + \beta'_4 z_3) + \beta_4^2 z_4 + \beta_4 \beta'_4 z_5 + \beta_4'^2 z_6. \quad (\text{A44})$$

The rationale behind this notation is the following:

1. Labels a_i enclosed in square brackets in subscripts of functions w^n , s^2 and s'^2 are integers parametrising the coefficients of the corresponding polynomial expressions, z_i 's enclosed in round brackets are arguments of functions f and F ; note that not all of them have the same mass dimension d because τ is a $d = 1$ parameter while β_4 and β'_4 are dimensionless. Variables with the same mass dimensions are separated with commas and groups with different mass dimensions are separated with semicolons.
2. Symbols ω and ω' denote the sum and difference of the VEVs ω_r and ω_b , respectively; these were introduced mainly for their relevance in the standard and/or the flipped $SU(5) \times U(1)$ limits.
3. In various limits of increased symmetry the RH sides of the equalities above simplify to

$$\sigma \rightarrow 0 \Rightarrow s_{[a_1, a_2, a_3]}^2 \rightarrow 0, \quad s_{[a_1, a_2]}'^2 \rightarrow 0, \quad (\text{A45})$$

$$\omega_r \rightarrow 0 \Rightarrow w_{[a_n, \dots, a_0]}^n \rightarrow a_n \omega_b^n \quad (\text{A46})$$

$$\omega_b \rightarrow 0 \Rightarrow w_{[a_n, \dots, a_0]}^n \rightarrow a_0 \omega_r^n, \quad (\text{A47})$$

$$\omega_r \rightarrow +\omega_b \Rightarrow w_{[a_n, \dots, a_0]}^n \rightarrow \sum_i a_{n-i} \omega_b^n, \quad (\text{A48})$$

$$\omega_r \rightarrow -\omega_b \Rightarrow w_{[a_n, \dots, a_0]}^n \rightarrow \sum_i (-1)^i a_{n-i} \omega_b^n. \quad (\text{A49})$$

For more details on these limits the reader is kindly referred to Table II and the discussion in Section III B 1.

A few other comments regarding the tree-level spectrum are worth making here:

1. There are 3 equations for the vacuum configuration of the 3 independent VEVs in the $45 \oplus 126$ model (let us reiterate that ω_r and ω_b are real while σ can be complex). These quantities can be traded for the 3 mass parameters (μ^2, ν^2, τ) via the eqs. (9)–(11). For the sake of simplicity, we shall sometimes use a hybrid notation in which the τ parameter may appear in the relevant expressions alongside the three VEVs above, see eqs. (A19)–(A37); however, in practice, the third condition of eq. (11) should always be used to eliminate τ .
2. The tree level scalar spectrum of this model has been previously calculated in [10]; our results agree with those given there with the only exception of the numerical coefficients in front of λ_4 terms, which have now been corrected.
3. With no special correlation among the three VEVs above, the $SO(10) \rightarrow G_{321}$ breaking is achieved and one should end up with 33 massless would-be Goldstone modes identified with the broken $SO(10)$ generators corresponding to massive gauge bosons in eqs. (A14)–(A18). There is indeed a zero eigenvalue in each of the following mass matrices above: M_c^2 , M_p^2 , M_q^2 , M_r^2 and M_s^2 , which together with the dimension of each of the states and their conjugated counterparts (as indicated in the column “size” of Table I) gives the desired number of would-be Goldstone modes.

Appendix B: The one-loop contributions to the pseudo-Goldstone boson masses

In this Appendix, we present the full mathematical form of the Δ -terms entering the relevant formulae for the one-loop pseudo-Goldstone triplet $(1, 3, 0)$ and octet $(8, 1, 0)$ masses (27) and (28). For that sake, it is convenient to define the symbols:

$$C = -\pi^2 \omega', \quad R^2 = \sqrt{\omega_b^4 + 34 \omega_b^2 \omega_r^2 + \omega_r^4}. \quad (\text{B1})$$

1. Gauge boson contributions

Here we present the 1-loop contributions to the scalar masses (27)–(28) coming from the gauge bosons running in the loops. All the terms in this subsection have been computed in full generality, i.e., they are valid for arbitrary values of all parameters. They agree¹⁶ with the previous results for the $45 \oplus 16$ [14, 30] and $45 \oplus 126$ [17] models — the gauge sector polynomial contribution is the same in both models for those states which come solely from 45.

¹⁶ The agreement is up to the definition of g ; we use the canonical normalization with Dynkin index 1 for the generators in the fundamental representation, while the cited works use the normalization 2.

$$\Delta_a^{G[\text{poly}]} = \frac{g^4}{16\pi^2} w_{[19,1,16]}^2, \quad \Delta_b^{G[\text{poly}]} = \frac{g^4}{16\pi^2} w_{[22,1,13]}^2, \quad (\text{B2})$$

$$\begin{aligned} \Delta_a^{G[\log]} = \frac{3g^4}{16C} & \left(-8\omega_b (|\sigma|^2 + \omega_b^2) \log \left[\frac{2g^2 (|\sigma|^2 + \omega_b^2)}{\mu_r^2} \right] + 4\omega_r (|\sigma|^2 + \omega_r^2) \log \left[\frac{2g^2 (|\sigma|^2 + \omega_r^2)}{\mu_r^2} \right] + \right. \\ & \left. + \left(4|\sigma|^2 w_{[2,-1]}^1 + \omega^2 w_{[5,-4]}^1 \right) \log \left[\frac{2g^2 (|\sigma|^2 + \frac{1}{4}\omega^2)}{\mu_r^2} \right] - 2\omega'^3 \log \left[\frac{\frac{1}{2}g^2 \omega'^2}{\mu_r^2} \right] \right), \end{aligned} \quad (\text{B3})$$

$$\begin{aligned} \Delta_b^{G[\log]} = \frac{3g^4}{32C} & \left(-4 \left(|\sigma|^2 w_{[3,1]}^1 + \omega_b^2 w_{[1,3]}^1 \right) \log \left[\frac{2g^2 (|\sigma|^2 + \omega_b^2)}{\mu_r^2} \right] + 8\omega_r (|\sigma|^2 + \omega_r^2) \log \left[\frac{2g^2 (|\sigma|^2 + \omega_r^2)}{\mu_r^2} \right] + \right. \\ & \left. + \left(4|\sigma|^2 w_{[3,-1]}^1 + \omega^2 w_{[7,-5]}^1 \right) \log \left[\frac{2g^2 (|\sigma|^2 + \frac{1}{4}\omega^2)}{\mu_r^2} \right] - \omega'^3 \log \left[\frac{\frac{1}{2}g^2 \omega'^2}{\mu_r^2} \right] \right). \end{aligned} \quad (\text{B4})$$

2. Scalar contributions

Here we list the scalar-loop-induced contributions to the masses (27)–(28). The $\Delta^{S_{\text{FIN}}[\text{poly}]}$ structures are computed for arbitrary values of parameters, while $\Delta^{S_{\text{INF}}[\text{poly}]}$ and $\Delta^{S[\log]}$ are only given in the limit (29). The tree-level value of τ from eq. (11) should eventually be inserted into the expressions below.

$$\Delta_a^{S_{\text{FIN}}[\text{poly}]} = -\frac{\omega' w_{[1,2]}^1}{16\pi^2} (96a_0 a_2 + 76a_2^2 - 5(\beta_4 - 10\beta_4')(\beta_4 - 2\beta_4') + 560|\gamma_2|^2), \quad (\text{B5})$$

$$\Delta_b^{S_{\text{FIN}}[\text{poly}]} = +\frac{\omega' w_{[2,1]}^1}{16\pi^2} (96a_0 a_2 + 76a_2^2 - 5(\beta_4 - 10\beta_4')(\beta_4 - 2\beta_4') + 560|\gamma_2|^2). \quad (\text{B6})$$

The polynomial parts of the nested commutator contributions computed in the limit (29) are

$$\Delta_a^{S_{\text{INF}}[\text{poly}]} = \frac{1}{8\pi^2} F(35; 0, 0; 3w_{[6,0,5]}^2, 60\omega_r^2, 60w_{[4,0,1]}^2), \quad (\text{B7})$$

$$\Delta_b^{S_{\text{INF}}[\text{poly}]} = \frac{1}{24\pi^2} F(105; 0, 0; 216\omega_b^2 \frac{\omega_r^4}{R^4} + w_{[71,0,32]}^2, 180\omega_b^2, 60w_{[7,0,8]}^2), \quad (\text{B8})$$

and the log terms read

$$\Delta_a^{S[\log]} \equiv \sum_{x \in X} T_x \log(m_x^2/\mu_r^2), \quad \Delta_b^{S[\log]} \equiv \sum_{x \in X} O_x \log(m_x^2/\mu_r^2), \quad (\text{B9})$$

where x runs over the set of indices $X = \{d, e, f, g, h, i, j, k, l_1, l_2, m_1, m_2, n_1, n_2, o_1, o_2, o_3, p, q, r_1, r_2\}$.

The T_x and O_x prefactors are

$$T_d = \frac{1}{2C} F\left(\omega_r; 0, -4\omega_r w_{[3,-1]}^1; 0, 0, 12\omega_b \omega_r w_{[3,-2]}^1\right), \quad (\text{B10})$$

$$T_e = \frac{1}{8C} F\left(-w_{[7,5]}^1; 2\omega' w_{[3,1]}^1, -2w_{[3,20,1]}^2; 0, -4\omega' \omega_r w_{[3,1]}^1, 4w_{[18,-3,17,4]}^3\right), \quad (\text{B11})$$

$$T_f = \frac{1}{2C} F\left(w_{[1,2]}^1; 2\omega_b \omega_r, -2w_{[1,2]}^1 w_{[3,-2]}^1; -\omega_b^3, -2\omega_b^2 w_{[1,6]}^1, 8\omega_b w_{[1,-2]}^1 w_{[1,2]}^1\right), \quad (\text{B12})$$

$$T_g = \frac{1}{8C} F\left(-9\omega_r; \frac{3}{\omega} w_{[-11,3,8,2]}^3, -18w_{[-4,2,1]}^2; -\frac{6}{\omega} \omega_b^2 \omega' w_{[3,1]}^1, 6w_{[9,2,-8,-2]}^3, -12w_{[4,3,-9,-4]}^3\right), \quad (\text{B13})$$

$$T_h = \frac{1}{4C} F\left(3w_{[3,-1]}^1; 6\omega \omega', -6w_{[3,0,5]}^2; 0, -12\omega \omega' w_{[2,1]}^1, -12w_{[4,3,-9,-4]}^3\right), \quad (\text{B14})$$

$$T_i = \frac{1}{C} F\left(-\omega_b; 0, 8\omega_b \omega_r; 0, 0, 4\omega_b w_{[1,-2]}^1 w_{[1,2]}^1\right), \quad (\text{B15})$$

$$T_j = \frac{1}{4C} F\left(-w_{[2,1]}^1; -\omega_r w_{[12,5]}^1, 2\omega_r w_{[2,1]}^1; 6\omega_r^3, 2\omega_r w_{[12,19,6]}^2, 4\omega \omega_b w_{[2,1]}^1\right), \quad (\text{B16})$$

$$T_k = \frac{1}{C} F\left(\omega; 0, 4\omega \omega_r; 0, 0, -4\omega \omega_b w_{[1,2]}^1\right), \quad (\text{B17})$$

$$T_{l_1} = \frac{1}{32C} F\left(4w_{[2,1]}^1; w_{[19,42,-13]}^2, -2w_{[3,2,7]}^2; -\frac{1}{3} w_{[95,-48,3,2]}^3, -4\omega_b w_{[29,30,-5]}^2, -4w_{[9,12,1,2]}^3\right), \quad (\text{B18})$$

$$T_{l_2} = \frac{1}{16C} F\left(-2\omega; \frac{1}{6} w_{[27,58,23]}^2, w_{[15,10,7]}^2; -\frac{1}{6} w_{[11,48,39,10]}^3, -2w_{[11,14,13,4]}^3, -2w_{[9,12,1,2]}^3\right), \quad (\text{B19})$$

$$T_{m_1} = \frac{1}{32C} F\left(4w_{[5,9]}^1; \omega' w_{[15,-31]}^1, -2w_{[81,134,9]}^2; -4\omega'^3, -2\omega' w_{[25,-83,10]}^2, 4w_{[79,108,-1,-18]}^3\right), \quad (\text{B20})$$

$$T_{m_2} = \frac{1}{32C} F\left(-4w_{[16,5]}^1; w_{[-45,26,23]}^2, 2w_{[-51,94,41]}^2; 2\omega^2 w_{[5,-4]}^1, 2w_{[7,84,7,-22]}^3, 4w_{[79,108,-1,-18]}^3\right), \quad (\text{B21})$$

$$T_{n_1} = \frac{1}{2C} F\left(2w_{[2,1]}^1; -\frac{1}{2} \omega w_{[5,1]}^1, w_{[-3,-2,-7]}^2; \frac{1}{3} \omega^2 w_{[5,-4]}^1, \omega w_{[11,1,-6]}^2, -2w_{[9,12,1,2]}^3\right), \quad (\text{B22})$$

$$T_{n_2} = \frac{1}{12C} F\left(-12\omega; -\omega' w_{[27,-11]}^1, 6w_{[15,10,7]}^2; -4\omega'^3, 6\omega' w_{[7,11,-2]}^2, -12w_{[9,12,1,2]}^3\right), \quad (\text{B23})$$

$$T_{o_1} = \frac{1}{32C} F\left(4\omega; 3w_{[1,-2,-5]}^2 - \frac{1}{R^2} w_{[-9,98,68,-62,13]}^4, -8\omega w_{[3,1]}^1; w_{[-7,36,-23,12]}^3 - \frac{3}{R^2} w_{[3,-28,38,-64,19,-4]}^5, \right. \\ \left. -6w_{[3,4]}^1 w_{[1,-2,-1]}^2 - \frac{2}{R^2} w_{[19,-186,-200,90,37,-12]}^5, 16\omega \omega_b w_{[2,1]}^1\right), \quad (\text{B24})$$

$$T_{o_2} = \frac{1}{16C} F\left(-2w_{[2,1]}^1; \frac{1}{\omega} w_{[93,43,-1,1]}^3, 4\omega_r w_{[2,1]}^1; \frac{\omega_b}{\omega} w_{[-23,1]}^1 w_{[1,0,1]}^2, -2\omega_b w_{[91,42,-3]}^2, 8\omega \omega_b w_{[2,1]}^1\right), \quad (\text{B25})$$

$$T_{o_3} = \frac{1}{32C} F\left(4\omega; 3w_{[1,-2,-5]}^2 + \frac{1}{R^2} w_{[-9,98,68,-62,13]}^4, -8\omega w_{[3,1]}^1; w_{[-7,36,-23,12]}^3 + \frac{3}{R^2} w_{[3,-28,38,-64,19,-4]}^5, \right. \\ \left. -6w_{[3,4]}^1 w_{[1,-2,-1]}^2 + \frac{2}{R^2} w_{[19,-186,-200,90,37,-12]}^5, 16\omega \omega_b w_{[2,1]}^1\right), \quad (\text{B26})$$

$$T_p = \frac{1}{8C} F\left(\omega_r; -3\omega_r^2, -2\omega_r w_{[6,1]}^1; 2\omega_r^3, 2\omega_r^2 w_{[9,2]}^1, 12\omega_b \omega_r w_{[3,1]}^1\right), \quad (\text{B27})$$

$$T_q = \frac{1}{4C} F\left(-\omega_b; 3\omega_b^2, 2\omega_b w_{[3,4]}^1; -2\omega_b^3, -2\omega_b^2 w_{[5,6]}^1, -8\omega \omega_b w_{[1,2]}^1\right), \quad (\text{B28})$$

$$T_{r_1} = \frac{1}{32C} F\left(4w_{[2,-1]}^1; -3\omega w_{[7,-5]}^1, -2w_{[27,2,-1]}^2; 2\omega^2 w_{[5,-4]}^1, 2\omega w_{[43,-7,-14]}^2, 4w_{[19,12,11,6]}^3\right), \quad (\text{B29})$$

$$T_{r_2} = \frac{1}{32C} F\left(-4w_{[1,3]}^1; -\omega' w_{[9,23]}^1, 2w_{[39,-38,-33]}^2; -4\omega'^3, 2\omega' w_{[35,-1,-2]}^2, 4w_{[19,12,11,6]}^3\right), \quad (\text{B30})$$

$$T_s = 0, \quad (\text{B31})$$

$$O_d = \frac{1}{2C} F\left(\omega_r; 0, 4\omega_r w_{[-3,1]}^1; 0, 0, 12\omega_b \omega_r w_{[3,-2]}^1\right), \quad (\text{B32})$$

$$O_e = \frac{1}{8C} F\left(-3w_{[3,1]}^1; 0, -6w_{[3,-1]}^1 w_{[3,1]}^1; 0, 0, 36\omega_b \omega_r w_{[3,1]}^1\right), \quad (\text{B33})$$

$$O_f = \frac{1}{16C} F\left(w_{[9,15]}^1; -2w_{[3,-9,-2]}^2, -48w_{[1,1,-1]}^2; -2\omega_b^2 w_{[1,3]}^1, -4\omega_b^2 w_{[1,27]}^1, 4w_{[13,-9,-48,-4]}^3\right), \quad (\text{B34})$$

$$O_g = \frac{1}{16C} F\left(-9\omega; 6w_{[9,3,1]}^2, 36\omega_b^2; -6\omega_b^2 w_{[1,3]}^1, -12w_{[4,12,6,1]}^3, 12w_{[1,3,6,2]}^3\right), \quad (\text{B35})$$

$$O_h = \frac{1}{16C} F\left(3w_{[9,-1]}^1; 18\omega\omega', -12w_{[6,3,7]}^2; 0, -36\omega\omega' w_{[2,1]}^1, 12w_{[-7,-3,24,10]}^3\right), \quad (\text{B36})$$

$$O_i = \frac{1}{16C} F\left(-w_{[27,5]}^1; 12\omega_b\omega', 4w_{[15,39,10]}^2; 0, -24\omega_b\omega' w_{[1,2]}^1, 4w_{[17,-45,-48,-20]}^3\right), \quad (\text{B37})$$

$$O_j = \frac{1}{16C} F\left(-3w_{[1,3]}^1; -\frac{2}{\omega} w_{[6,27,4,-5]}^3, 12w_{[5,-2,-1]}^2; \frac{6}{\omega} \omega_r^2 w_{[1,4,-1]}^2, 4w_{[6,27,3,2]}^3, 4w_{[-7,-3,24,10]}^3\right), \quad (\text{B38})$$

$$O_k = \frac{1}{16C} F\left(w_{[21,11]}^1; 12\omega\omega', 4w_{[15,21,-4]}^2; 0, -24\omega\omega_b\omega', -4w_{[31,33,12,20]}^3\right), \quad (\text{B39})$$

$$O_{l_1} = \frac{1}{288C} F\left(54\omega; \frac{3}{\omega_b} w_{[79,66,-57,-8]}^3, -54\omega w_{[3,-1]}^1; -\frac{1}{\omega_b} w_{[350,-361,114,-7,-4]}^4, \right. \\ \left. -\frac{12}{\omega_b} w_{[75,76,-33,-18,-2]}^4, -108\omega_r w_{[3,4,1]}^2\right), \quad (\text{B40})$$

$$O_{l_2} = \frac{1}{288C} F\left(-18w_{[3,1]}^1; \frac{1}{\omega\omega_b} w_{[147,361,105,27,8]}^4, 18w_{[3,1]}^1 w_{[3,5]}^1; -\frac{1}{\omega\omega_b} w_{[7,4,1]}^2 w_{[14,39,-3,4]}^3, \right. \\ \left. -\frac{12}{\omega_b} w_{[21,76,21,6,2]}^4, -108\omega\omega_r w_{[3,1]}^1\right), \quad (\text{B41})$$

$$O_{m_1} = \frac{1}{32C} F\left(2w_{[9,19]}^1; 3\omega' w_{[3,-11]}^1, 2w_{[-75,-150,1]}^2; -\omega'^3, -2\omega' w_{[13,-89,4]}^2, 4w_{[76,135,-24,-19]}^3\right), \quad (\text{B42})$$

$$O_{m_2} = \frac{1}{32C} F\left(-6w_{[11,3]}^1; w_{[-15,18,1]}^2, 6w_{[-23,38,13]}^2; \omega^2 w_{[7,-5]}^1, 2\omega_b w_{[43,54,-21]}^2, 4w_{[76,135,-24,-19]}^3\right), \quad (\text{B43})$$

$$O_{n_1} = \frac{1}{12C} F\left(9w_{[3,1]}^1; \frac{\omega}{\omega_b} w_{[-29,14,1]}^2, -72\omega_r^2; \frac{\omega^2}{6\omega_b} w_{[56,-47,-1]}^2, -\frac{\omega}{\omega_b} w_{[-69,31,17,1]}^3, -18w_{[9,9,-3,1]}^3\right), \quad (\text{B44})$$

$$O_{n_2} = \frac{1}{12C} F\left(-3w_{[3,5]}^1; -\frac{\omega'}{3\omega\omega_b} w_{[39,31,1,1]}^3, 12w_{[9,3,4]}^2; -\frac{\omega'^3}{6\omega\omega_b} w_{[20,23,-1]}^2, \frac{\omega'}{\omega_b} w_{[39,19,13,1]}^3, -18w_{[9,9,-3,1]}^3\right), \quad (\text{B45})$$

$$O_{o_1} = \frac{1}{32C} F\left(2w_{[3,1]}^1; -3R^2 + \frac{16}{R^6} \omega^3 \omega'^3 \omega_r^2 - \frac{6}{R^2} \omega\omega' w_{[1,-16,1]}^2 - 3w_{[1,0,5]}^2, -8w_{[3,3,2]}^2; \right. \\ \left. -\frac{12}{R^6} \omega^2 \omega'^3 \omega_r^2 w_{[1,0,1]}^2 + w_{[-1,30,-18,7]}^3 - \frac{3}{R^2} w_{[1,-26,19,-45,18,-3]}^5, \right. \\ \left. -\frac{32}{R^6} \omega^3 \omega'^3 \omega_r^2 w_{[2,1]}^1 + 6w_{[-2,1,10,5]}^3 - \frac{2}{R^2} w_{[2,-1]}^1 w_{[5,-92,-146,-28,9]}^4, 8w_{[1,3,6,2]}^3\right), \quad (\text{B46})$$

$$O_{o_2} = \frac{1}{16C} F\left(-3\omega; \frac{1}{\omega} w_{[1,-5]}^1 w_{[3,12,5]}^2, 12\omega_b^2; \frac{1}{\omega} w_{[1,1,12,1,13]}^4, 2w_{[2,3,54,27]}^3, 4w_{[1,3,6,2]}^3\right), \quad (\text{B47})$$

$$O_{o_3} = \frac{1}{32C} F\left(2w_{[3,1]}^1; 3R^2 - \frac{16}{R^6} \omega^3 \omega'^3 \omega_r^2 + \frac{6}{R^2} \omega\omega' w_{[1,-16,1]}^2 - 3w_{[1,0,5]}^2, -8w_{[3,3,2]}^2; \right. \\ \left. \frac{12}{R^6} \omega^2 \omega'^3 \omega_r^2 w_{[1,0,1]}^2 + w_{[-1,30,-18,7]}^3 + \frac{3}{R^2} w_{[1,-26,19,-45,18,-3]}^5, \right. \\ \left. \frac{32}{R^6} \omega^3 \omega'^3 \omega_r^2 w_{[2,1]}^1 + 6w_{[-2,1,10,5]}^3 + \frac{2}{R^2} w_{[2,-1]}^1 w_{[5,-92,-146,-28,9]}^4, 8w_{[1,3,6,2]}^3\right), \quad (\text{B48})$$

$$O_p = \frac{1}{8C} F\left(\omega_r; -3\omega_r^2, -2\omega_r w_{[6,1]}^1; 2\omega_r^3, 2\omega_r^2 w_{[9,2]}^1, 12\omega_b \omega_r w_{[3,1]}^1\right), \quad (\text{B49})$$

$$O_q = \frac{1}{16C} F\left(-w_{[3,1]}^1; 6\omega\omega_b, 8w_{[3,3,1]}^2; -2\omega_b^2 w_{[1,3]}^1, -4\omega_b w_{[7,9,6]}^2, -4w_{[11,21,12,4]}^3\right), \quad (\text{B50})$$

$$O_{r_1} = \frac{1}{32C} F\left(2w_{[3,-1]}^1; -w_{[15,6,-9]}^2, -2w_{[21,6,1]}^2; \omega^2 w_{[7,-5]}^1, 2\omega w_{[31,-1,-8]}^2, 4w_{[16,15,12,5]}^3\right), \quad (\text{B51})$$

$$O_{r_2} = \frac{1}{32C} F\left(-2w_{[3,5]}^1; -3\omega' w_{[13,3]}^1, 2w_{[21,-30,-23]}^2; -\omega'^3, -2\omega' w_{[1,-29,-20]}^2, 4w_{[16,15,12,5]}^3\right), \quad (\text{B52})$$

$$O_s = 0. \quad (\text{B53})$$

The m_x^2 are the eigenvalues of the scalar mass matrices in eqs. (A19)–(A37), computed in the limit (29):

$$m_d^2 := f(-4\omega_r; 0, 24\omega_b\omega_r), \quad (\text{B54})$$

$$m_e^2 := f(-2w_{[3,1]}^1; 0, 4\omega_r w_{[3,1]}^1), \quad (\text{B55})$$

$$m_f^2 := f(-2w_{[1,2]}^1; 2\omega_b^2, 8\omega_b w_{[1,2]}^1), \quad (\text{B56})$$

$$m_g^2 := f(-2w_{[2,1]}^1; 2\omega_b^2, 4\omega w_{[2,1]}^1), \quad (\text{B57})$$

$$m_h^2 := f(-2\omega; 0, 4\omega w_{[2,1]}^1), \quad (\text{B58})$$

$$m_i^2 := f(-4\omega_b; 0, 8\omega_b w_{[1,2]}^1), \quad (\text{B59})$$

$$m_j^2 := f(-2w_{[2,1]}^1; 2\omega_r^2, 4\omega w_{[2,1]}^1), \quad (\text{B60})$$

$$m_k^2 := f(-4\omega; 0, 8\omega\omega_b), \quad (\text{B61})$$

$$m_{l_1}^2 := f(-3\omega; \frac{1}{2}w_{[7,-4,1]}^2, 3w_{[3,4,1]}^2), \quad (\text{B62})$$

$$m_{l_2}^2 := f(-w_{[3,1]}^1; \frac{1}{2}w_{[7,4,1]}^2, 3w_{[3,4,1]}^2), \quad (\text{B63})$$

$$m_{m_1}^2 := f(-w_{[1,3]}^1; \frac{1}{2}\omega'^2, w_{[1,3]}^1 w_{[5,1]}^1), \quad (\text{B64})$$

$$m_{m_2}^2 := f(-w_{[5,1]}^1; \frac{1}{2}\omega^2, w_{[1,3]}^1 w_{[5,1]}^1), \quad (\text{B65})$$

$$m_{n_1}^2 := f(-3\omega; \frac{1}{2}\omega^2, 3\omega w_{[3,1]}^1), \quad (\text{B66})$$

$$m_{n_2}^2 := f(-w_{[3,1]}^1; \frac{1}{2}\omega'^2, 3\omega w_{[3,1]}^1), \quad (\text{B67})$$

$$m_{o_1}^2 := f(-2\omega; \frac{3}{2}w_{[1,0,1]}^2 + \frac{R^2}{2}, 4\omega w_{[2,1]}^1), \quad (\text{B68})$$

$$m_{o_2}^2 := f(-2w_{[2,1]}^1; w_{[1,0,1]}^2, 4\omega w_{[2,1]}^1), \quad (\text{B69})$$

$$m_{o_3}^2 := f(-2\omega; \frac{3}{2}w_{[1,0,1]}^2 - \frac{R^2}{2}, 4\omega w_{[2,1]}^1), \quad (\text{B70})$$

$$m_p^2 := f(-2\omega_r; 2\omega_r^2, 4\omega_r w_{[3,1]}^1), \quad (\text{B71})$$

$$m_q^2 := f(-2\omega_b; 2\omega_b^2, 8\omega\omega_b), \quad (\text{B72})$$

$$m_{r_1}^2 := f(-\omega; \frac{1}{2}\omega^2, \omega w_{[5,3]}^1), \quad (\text{B73})$$

$$m_{r_2}^2 := f(-w_{[5,3]}^1; \frac{1}{2}\omega'^2, \omega w_{[5,3]}^1), \quad (\text{B74})$$

$$m_s^2 := 8a_0 w_{[3,0,2]}^2. \quad (\text{B75})$$

The labels in the subscript of masses correspond to those defined in Table I and they are further enumerated when there is more than 1 non-vanishing eigenvalue in the relevant sector.

The structure of the results is as anticipated:

1. The tree-level masses of the pseudo-Goldstone triplet and octet are proportional to a_2 ; the desired dominance of the 1-loop contribution to the masses of these two states (and them alone) requires this parameter to be small.
2. The results above (in particular, the $\Delta^{SINF[\text{poly}]}$ and $\Delta^{S[\log]}$ terms) are written in a simplified form corresponding to the limit (29)¹⁷. Note, however, that in full generality a_2 is strongly correlated to $|\sigma|^2$ and τ (which has to be well below M_{Pl} in order to retain the perturbative regime). At one loop, it is sufficient to insert the tree-level expression (11) for τ into the 1-loop Δ -terms.
3. In the limit (29), the largest blocks of the scalar mass matrix evaluated in the vacuum are 2×2 and the only one which is not diagonalized trivially is M_o^2 , see (A33). This is where the square-root (B1) may arise and it does so in $\Delta_b^{SINF[\text{poly}]}$ due to the non-commutativity of the relevant M_o^2 block with the corresponding part of the derivatives of the field dependent mass matrix (requiring diagonalization and normalization of the eigenmodes as suggested by eqs. (24)–(25)). These purely polynomial terms emerge solely from the degeneracy of the scalar spectrum in the SM symmetry limit along the lines described in Sect. III A 2. Furthermore, rational functions including the R -factor also appear in the coefficients T_x and O_x of logarithmic contributions $\Delta_a^{S[\log]}$ and $\Delta_b^{S[\log]}$ (in particular, in terms with $x \in \{o_1, o_3\}$).
4. The logarithmic terms in $\Delta^{G[\log]}$ and $\Delta^{S[\log]}$ have arguments of the form m^2/μ_r^2 , where m is a mass of a physical particle (gauge boson or scalar) and μ_r is the renormalization scale. Only terms corresponding to particles with non-vanishing masses turn out to be present, i.e., there are no IR divergences in the results (for $\Delta^{S[\log]}$, the limit (29) was used; in further limits of the regular- and flipped- $SU(5)$ case IR divergences may appear, see comments in Sections III A 4 and III B 1). From the list of massive particles, all have terms present except for two exceptions: the remaining gauge singlet (A18) contribution in $\Delta^{G[\log]}$ and the massive singlet present in M_s^2 (B75) is missing from $\Delta^{S[\log]}$, i.e. $T_s = O_s = 0$. Note that the use of limit (29) gives an extra would-be Goldstone singlet in M_s^2 (because the symmetry is increased), while states a , b and another singlet in M_s^2 become massless due to their pseudo-Goldstone nature; the limit gives 22 massive fields instead of 26 which, given the absence of m_s^2 , yields 21 contributions to $\Delta^{S[\log]}$, see eq. (B9).

¹⁷ Since the largest block in the mass matrix one needs to diagonalize is ‘only’ 4×4 (and in addition even contains a massless would-be Goldstone mode), it is possible to write down the results also in the situation when the limit (29) is not imposed. However, as we plan to use these mainly for the sake of a future numerical study where the diagonalization of such structures is trivial, there is no real reason to do that here.

Appendix C: The effective potential and nested commutators – further details

In this Appendix we intend to clarify several technical points related to the evaluation of the derivatives of the effective potential, and especially, the traces of the infinite series of nested commutators (23).

The derivation of the basic formulae (20)–(23) for the derivatives of the one-loop effective potential (19) is straightforward, see also [26]. Essentially, the matrix logarithm is expanded into a power series around the identity matrix; in doing so a particular attention should be paid to the ordering of the individual factors as matrix derivatives do not necessarily commute with the original matrix. Although the cyclic property of the trace makes this issue irrelevant for the first order derivatives of V_1 , cf. (20), it is of paramount importance for second order. In the latter case, it is convenient to move all derivative factors to one side of the expression by repeatedly using the commutator rule $\mathbf{A}\mathbf{B} = \mathbf{B}\mathbf{A} + [\mathbf{A}, \mathbf{B}]$ where \mathbf{B} is an arbitrary square matrix with appropriate dimension. Hence, powers of the basic “binomial structure” $(\mathbf{A} - \mathbb{1})$ can be isolated from the relevant expressions with pre-factors in the form of nested commutators, cf. (23).

A direct term-by-term evaluation of this series may be rather complicated due to the fact that there is no general connection among \mathbf{A} , \mathbf{A}_a and \mathbf{A}_b (which correspond to different orders in the Taylor expansion of the field-dependent mass matrices $\mathbf{M}_{\mathbf{S}, \mathbf{G}}^2(\Phi)$ around the tree-level vacuum) and, on top of that, the expected $a \leftrightarrow b$ symmetry is not apparent on the RHS of the formula (23). The method described below and used to deal with this issue yields not only an interesting analytic insight but it also provides a relatively simple procedure for the analytic evaluation that becomes almost trivial in case of a numerical treatment.

The argument goes as follows: any of the $n \times n$ matrices \mathbf{A} , \mathbf{A}_a and \mathbf{A}_b can be viewed as an element of the space of endomorphisms $\text{End}(\mathbb{R}^n)$. Since \mathbf{A} is proportional to the mass matrix, it is Hermitian and hence diagonalizable; its spectral decomposition then reads $\mathbf{A} = \sum_{i=1}^n \lambda_i \mathbf{v}_i \mathbf{v}_i^\dagger$ with dimensionless eigenvalues $\lambda_i = m_i^2/\mu_r^2$ corresponding to the normalized eigenvectors \mathbf{v}_i forming an orthonormal basis in \mathbb{R}^n , $\mathbf{v}_i^\dagger \mathbf{v}_j = \delta_{ij}$. Consequently, the matrices $\mathbf{v}_i \mathbf{v}_j^\dagger$ form an orthonormal basis of the n^2 -dimensional space of $\text{End}(\mathbb{R}^n)$, $\text{Tr}[\mathbf{v}_i \mathbf{v}_j^\dagger (\mathbf{v}_k \mathbf{v}_l^\dagger)^\dagger] = \delta_{ik} \delta_{jl}$. Notice that the commutator with \mathbf{A} , defined as $\text{Ad}_{\mathbf{A}}(\mathbf{B}) := [\mathbf{A}, \mathbf{B}]$, is a linear operator on the space of matrices $\text{End}(\mathbb{R}^n)$ and, hence $\text{Ad}_{\mathbf{A}} \in \text{End}^2(\mathbb{R}^n)$. Crucially, the set of matrices $\{\mathbf{v}_i \mathbf{v}_j^\dagger\}$ forms a complete eigenbasis for the $\text{Ad}_{\mathbf{A}}$ operator, with respective real eigenvalues $\lambda_i - \lambda_j$. The nested commutators with \mathbf{A} are then easily evaluated on these basis elements:

$$\underbrace{[\mathbf{A}, [\mathbf{A}, \dots [\mathbf{A}, \mathbf{v}_i \mathbf{v}_j^\dagger]]]}_{k-1 \text{ times}} = \text{Ad}_{\mathbf{A}}^{k-1}(\mathbf{v}_i \mathbf{v}_j^\dagger) = (\lambda_i - \lambda_j)^{k-1} \mathbf{v}_i \mathbf{v}_j^\dagger. \quad (\text{C1})$$

Expanding now \mathbf{A}_a and \mathbf{A}_b in the $\{\mathbf{v}_i \mathbf{v}_j^\dagger\}$ eigenbasis as $\mathbf{A}_a = \sum_{i,j=1}^n M_{ij}^a \mathbf{v}_i \mathbf{v}_j^\dagger$, $\mathbf{A}_b = \sum_{i,j=1}^n M_{ij}^b \mathbf{v}_i \mathbf{v}_j^\dagger$, with coefficients M_{ij}^a and M_{ij}^b (which can be alternatively viewed as the matrix elements of \mathbf{A}_a and \mathbf{A}_b rewritten in the eigenbasis of \mathbf{A} , cf. eq. (25)), one arrives at

$$\begin{aligned} \Upsilon(\mathbf{A}, \mathbf{A}_a, \mathbf{A}_b) &= \sum_{p,q=1}^n \sum_{i,j=1}^n M_{pq}^a M_{ij}^b \sum_{m=1}^{\infty} (-1)^{m+1} \frac{1}{m} \sum_{k=1}^m \binom{m}{k} \{\mathbf{A}, \mathbf{v}_p \mathbf{v}_q^\dagger\} \text{Ad}_{\mathbf{A}}^{k-1}(\mathbf{v}_i \mathbf{v}_j^\dagger) (\mathbf{A} - \mathbb{1})^{m-k} \\ &= \sum_{p,q=1}^n \sum_{i,j=1}^n M_{pq}^a M_{ij}^b \mathbf{v}_p \mathbf{v}_q^\dagger \mathbf{v}_i \mathbf{v}_j^\dagger (\lambda_p + \lambda_q) \times \begin{cases} \frac{\log \lambda_i - \log \lambda_j}{\lambda_i - \lambda_j} & \text{for } \lambda_i \neq \lambda_j \\ \frac{1}{\lambda_j} & \text{for } \lambda_i = \lambda_j \end{cases}. \end{aligned} \quad (\text{C2})$$

In case of degenerate eigenvalues, i.e. $\lambda_i = \lambda_j$, only the $k = 1$ term gives a non-zero contribution. With this at hand the trace of Υ is obtained readily:

$$\begin{aligned} \text{Tr } \Upsilon(\mathbf{A}, \mathbf{A}_a, \mathbf{A}_b) &= \sum_{i,j; \lambda_i \neq \lambda_j} M_{ji}^a M_{ij}^b \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} \log \frac{\lambda_i}{\lambda_j} + \\ &\quad + \sum_{i,j; \lambda_i = \lambda_j} 2M_{ji}^a M_{ij}^b, \end{aligned} \quad (\text{C3})$$

and one arrives at the formula (24). Note also that the $a \leftrightarrow b$ symmetry becomes manifestly apparent in this form.

Eq. (C3) gives some interesting insights into the contribution of the nested commutator series to the masses:

1. Notice that there are two types of contributions in equation (C3), polynomial and logarithmic. The arguments of the logs are the eigenvalues of $\mathbf{A} = \mathbf{M}_{\mathbf{S}, \mathbf{G}}^2/\mu_r^2$, so they give rise to exactly the same types of logs as the usual

matrix logarithms in the rest of eq. (21) and thus analytic results are again subject to the same limitation – the diagonalization procedure, which for sufficiently complicated matter content cannot be worked out analytically (but this represents no problem in numerical treatment). From this perspective, the result for \mathfrak{T} is as good as could be hoped for.

2. The polynomial contribution is due to the degeneracy of the eigenvalues of \mathbf{A} , which is generally the case of any non-Abelian gauge theory with non-trivial matter content. More symmetry implies more degeneracy, and thus more terms of the polynomial type. If one turns off certain VEVs in a spontaneously broken scenario, the degeneracy of the masses increases and further logarithmic contributions transform into polynomials in that limit. Some explicit examples of such behaviour can be found in Section III B 1.
3. Notice that the result is invariant under the rescaling $\lambda_k \rightarrow \kappa \lambda_k$ with a common factor κ for all k , and thus does not explicitly depend on the renormalization scale μ_r (unlike other terms in the 1 loop correction to the mass).
4. The form of the result (C3) admits for a simple isolation of the spurious IR divergences due to the vanishing log arguments corresponding to Goldstone bosons, cf. Sect. III A 4.

Finally, let us remark that the ordering of the basic operations that we did in dealing with the second derivatives of the EP, i.e., capturing first all the analytic complication into a series of nested commutators and *only then* going to the mass basis, is not fundamental. An alternative approach in which the transition to the mass basis is performed much sooner has been recently advocated in [31]; the main benefit of that scheme is namely its universality for all derivative orders.

Nevertheless, there is a clear rationale in favour of the (perhaps somewhat less universal) former approach in the situation of our interest: working with the commutator series offers a more direct comparison with the diagrammatic method employed in Section III B 3 because the nested commutators reflect all the combinatorial difficulties in summing diagrams in which the insertion of VEVs and an interaction vertex do not commute.

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