Membrane paradigm for Einstein-Gauss-Bonnet gravity

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The membrane paradigm for Einstein-Gauss-Bonnet gravity

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We construct the membrane paradigm for black objects in Einstein-Gauss-Bonnet gravity in spacetime dimensions ≥ 5. As in the case of general relativity, the horizon can be modelled as a membrane endowed with fluid-like properties. We derive the stress-tensor for this membrane fluid and study the perturbation around static backgrounds with constant curvature horizon cross-section, for which the stress tensor can be regularized with the usual redshift factor, and expressed in the form of a Newtonian viscous fluid with pressure, shear viscosity and bulk viscosity. We evaluate the transport coefficients for black holes with constant curvature horizons and negative or zero cosmological constant. For the black brane geometry our result for the ratio of shear viscosity to entropy density agrees with that obtained previously in different frameworks.

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I. INTRODUCTION

Gravitation, being the manifestation of the curvature of spacetime, affects the causal structure of the spacetime. This can lead to the existence of regions that are causally inaccessible to a class of observers. An example of such a region is the portion of spacetime inside the event horizon of a black hole, which is causally disconnected from any outside observer. Hence, the relevant physics for the observers outside the hole must be independent of what is happening inside the hole. This observation forms the basis of the membrane paradigm for black holes.

In membrane paradigm [1, 2] the interaction of the black hole with the outside world is modelled by replacing the black hole by a membrane of fictitious fluid "living" on the horizon. Interaction of the black hole with the outside world is then captured by the (theory dependent) transport coefficients of the fluid. For example, the electromagnetic interaction of the black hole is described by endowing the horizon with conductivity. This formalism provides an intuitive and elegant understanding of the physics of the event horizon in terms of a simple non-relativistic language, and also serves as an efficient computational tool useful in dealing with some astrophysical problems. After the advent of holography, the membrane paradigm took a new life in which the membrane fluid, at least in the case of planar horizons, captures some aspects of the long wavelength description of the strongly coupled quantum field theory at a finite temperature [3].

The original membrane paradigm [1, 2] was constructed for black holes in general relativity. The membrane fluid has the shear viscosity, $\eta = 1/16\pi G$. Dividing this by the Bekenstein-Hawking entropy density, $s = 1/4G$, gives a dimensionless number, $\eta/s = 1/4\pi$. The calculation which leads to this ratio relies only on the dynamics and the thermodynamics of the horizon in classical general relativity. But, interestingly enough, it was found in [3] that the same ratio is obtained in the holographic description of the hydrodynamic limit of the strongly coupled $\mathcal{N} = 4$, $U(N)$ gauge theory at finite temperature which, is dual to general relativity in the limit of large N and large λ_t, where N is the number of colors and λ_t is the 't Hooft coupling. The authors of [4] conjectured that this ratio is a universal lower bound for all the materials, this is called the KSS bound. The relationship between the membrane paradigm calculations and the holographically derived KSS bound was explained as a consequence of the trivial renormalization group (RG) flow from IR to UV in the boundary gauge theory as one moves the outer cutoff surface from the horizon to the boundary of spacetime [5, 6]. The universality of this bound, how it might be violated, and the triviality of the RG flow in the long wavelength limit at the level of the linear response, were first clarified in [5].

The general theory of relativity, which is based upon the Einstein-Hilbert action functional, is the simplest theory of gravity one can write guided by the principle of diffeomorphism invariance while containing only the time derivatives of second order in the equation of motion. Although such a simple choice of the action functional has so far been adequate to explain all the experimental and observational results, there is no reason to believe that this choice is fundamental. Indeed, it is expected on various general grounds that the low energy limit of any quantum theory of gravity will contain higher derivative correction terms. In fact, in string theory the low energy effective action generically contains terms which are higher order in curvature due to the stringy (α') corrections. In the context of holographic duality, such α' modifications correspond to the corrections due to finite λ_t. The specific form of these terms depends ultimately on the detailed features of the quantum theory. From the classical point of view, a simple modification of the Einstein-Hilbert action is to include the higher order curvature terms preserving the diffeomorphism invariance and still leading to an equation of motion containing no more than second order time derivatives. In fact this generalization is unique [7] and goes by the name of Lanczos-Lovelock gravity, of which the lowest order correction (second order in curvature) appears as the Gauss-Bonnet (GB) term in spacetime dimensions $D > 4$. Einstein-Gauss-Bonnet (EGB) gravity is free from ghosts [8, 9] (when realized around flat space), and leads to a well-defined initial value problem. Black hole solutions in EGB gravity have been studied extensively and are found to have various interesting features [10, 11]. The entropy of these black holes is no longer proportional to the area of the horizon but contains a curvature dependent term [12–15]. Hence, unlike in general relativity, the entropy density of the horizon in EGB gravity is not a constant but depends on the horizon curvature. Now, the form of the membrane stress tensor in the fluid model of the horizon is also theory dependent and therefore the transport coefficients of the membrane fluid will change due to the presence of the

GB term in the action. Hence, it is of interest to investigate the membrane paradigm and calculate the transport coefficients for the membrane fluid in the EGB gravity. For planar horizons the violation of the KSS bound due to the GB term in the action has already been shown in [16] and [17] using other methods. Also, the arguments that there really are string theories that violate the bound were presented in [18] and [19].

In this paper, we extend to EGB gravity the action principle formalism of the membrane paradigm as constructed in [20]. We first derive the membrane stress-energy tensor on the stretched horizon for EGB theory. After restriction to the linearized perturbations of static black backgrounds with horizon cross-section of constant curvature, followed by "regularization", we express the membrane stress tensor in the form of a Newtonian viscous fluid described by certain transport coefficients. Our main result is the horizon curvature dependent membrane transport coefficients in equations (51)-(53), and the ratio of shear viscosity and entropy density (η/s) in equation (54),

$$\frac{\eta}{s} = \frac{1}{4\pi} \frac{\left[1 - 2\frac{D-1}{D-3}\frac{\lambda}{l^2}\right] r_+^4 + 2(D-5)\lambda k (r_+^2 + \lambda k)}{(r_+^2 + 2\lambda k)\left[r_+^2 + 2\frac{D-2}{D-4}\lambda k\right]}. \tag{1}$$

Here the horizon is a $D-2$ dimensional space of constant curvature $(D-2)(D-3)k/r_+^2 \quad (k = 0, \pm 1)$ and "Schwarzschild" radius r_+, $\lambda = (D-3)(D-4)\alpha$, where α is the GB coupling constant, and the cosmological constant is $\Lambda = -(D-1)(D-2)/2l^2$. The result for black holes in asymptotically flat spacetime follows by taking the limit $l \to 0$. In particular, for black brane ($k = 0$) in AdS, the ratio is (55)

$$\frac{\eta}{s} = \frac{1}{4\pi}\left[1 - 2\frac{(D-1)}{(D-3)}\frac{\lambda}{l^2}\right]. \tag{2}$$

The presence of the GB term violates the KSS bound, $\eta/s \geq 1/4\pi$, for any $\alpha > 0$. This matches with the result found in [16], which used the real-time AdS/CFT calculation of shear viscosity. Ref.[16] also applied the method of [4] to calculate the ratio η/s by relating it to the shear mode diffusion constant on the stretched horizon. The calculation of the shear mode diffusion constant involves an integral that runs from the stretched horizon to infinity. In our membrane paradigm approach, by contrast, the ratio is calculated at the horizon membrane, while the asymptotic spacetime plays no role.

This paper is organized as follows: in section II we present the geometric setup of the membrane paradigm. In section III, we review the action based membrane paradigm approach for the black holes in general relativity. This is then generalized in section IV where we construct the membrane paradigm for the black objects in the EGB gravity. In that section we obtain the stress tensor for the membrane fluid and we derive the expressions for the transport coefficients of the fluid. In section V we evaluate these transport coefficients for black holes with constant curvature horizons and negative or zero cosmological constant. Finally, we conclude with a summary and discussion in section VI.

We adopt the metric signature $(-, +, +, +, ...)$ and our sign conventions are same as those of MTW [21]. All the symbols used in the main body of the paper are defined when introduced for the first time. For the convenience of the reader we have also included a table in the appendix summarizing these symbols and their meanings.

II. GEOMETRIC SET-UP

In this section we elaborate on the geometric set-up necessary to construct the membrane paradigm. We include it in this paper to keep it self-contained but the interested reader can find a detailed discussion in the monograph [1].

The event horizon, H, of the black hole in D spacetime dimensions, is a $(D-1)$-dimensional null hypersurface generated by the null geodesics l^a. We choose a non-affine parameterization such that the null generators satisfy the geodesic equation $l^a \nabla_a l^b = \kappa\, l^b$, where κ is a constant non-affine coefficient.

For a stationary spacetime, l^a coincides with the null limit of the time-like Killing vector and κ can then be interpreted as the surface gravity of the horizon.

Next we introduce a time-like surface positioned just outside H which is called the stretched horizon and denoted by $\mathcal{H}_s$. One can think of $\mathcal{H}_s$ as the world-tube of a family of fiducial observers just outside the black hole horizon. The four velocity of these fiducial observers is denoted by u^a. Just as H is generated by the null congruence l^a, $\mathcal{H}_s$ is generated by the time-like congruence u^a. The unit normal to $\mathcal{H}_s$ is denoted by n^a and is taken to point away from the horizon into the bulk. We relate the points on $\mathcal{H}_s$ and H by ingoing light rays parametrized by an affine parameter γ, such that $\gamma = 0$ is the position of the horizon and $(\partial/\partial\gamma)^a\, l_a = -1$ on the horizon. Then, in the limit $\gamma \to 0$, when the stretched horizon approaches the true one, $u^a \to \delta^{-1} l^a$ and $n^a \to \delta^{-1} l^a$ where $\delta = \sqrt{2\kappa\gamma}$ [1]. If the black hole is stationary, and l^a is the horizon generating Killing vector with surface gravity κ, then δ is the norm of the Killing vector, $\delta = |l|$.

The induced metric h_{ab} on $\mathcal{H}_s$ can be expressed in terms of the spacetime metric g_{ab} and the covariant normal n_a as $h_{ab} = g_{ab} - n_a n_b$. Similarly, the induced metric γ_{ab} on the $(D-2)$-dimensional space-like cross-section of $\mathcal{H}_s$ orthogonal to u_a is given by $\gamma_{ab} = h_{ab} + u_a u_b$. The extrinsic curvature of $\mathcal{H}_s$ is defined as $K^a_b = h^c_b \nabla_c n^a$. Using the limiting behavior of n^a and u^a it is easy to verify that in the limit that $\delta \to 0$ various components of the extrinsic curvature behave as [1]

$$\begin{aligned} \text{As } \delta \to 0 : K^u_u &= K^a_b u_a u^b = g \sim \frac{\kappa}{\delta}, \\ K^u_A &= K^a_b u_a \gamma^b_B = 0, \\ K^A_B &= K^a_b \gamma^A_a \gamma^b_B \sim \frac{k^A_B}{\delta}, \\ K &= K_{ab} g^{ab} \sim \frac{(\theta + \kappa)}{\delta}, \end{aligned} \tag{3}$$

where θ is the expansion scalar of l^a and k^A_B is the extrinsic curvature of the $(D–2)$-dimensional space-like cross-section[22]of the true horizon H. Note that apriori the projection of the extrinsic curvature of $\mathcal{H}_s$ on the cross-section of $\mathcal{H}_s$ has nothing to do with the extrinsic curvature of the cross-section (orthogonal to u^a) *as embedded in* $\mathcal{H}_s$, i.e., there is, in general, no relationship between the pull-back of $\nabla_a n_b$ and $\nabla_a u_b$ to the cross-section of the stretched horizon. However, in the null limit $(\delta \to 0)$ both u^a and n^a map to the same null vector l^a and we have $K^A_B \to \delta^{-1} k^A_B$. Finally, we decompose k_{AB} into its trace-free and trace-full part as

$$k_{AB} = \sigma_{AB} + \frac{1}{(D-2)} \theta\, \gamma_{AB}, \tag{4}$$

where σ_{AB} is the shear of l^a. It is clear from equation (3) that in the null limit, various components of the extrinsic curvature diverge and we need to regularize them by multiplying by a factor of δ. The physical reason behind such infinities is that, as the stretched horizon approaches the true one, the fiducial observers experience more and more gravitational blue shift; on the true horizon, the amount of blue shift is infinite.

This completes the description of our geometric set-up. Next, we review the derivation of the black hole membrane paradigm in standard Einstein gravity.

III. THE MEMBRANE PARADIGM IN EINSTEIN GRAVITY

In this section we construct the membrane paradigm in Einstein gravity in four spacetime dimensions. Our construction will closely follow the action approach of [20]. Our purpose is to fix the notation and emphasize the points in the construction which will be of importance for the corresponding construction in the EGB gravity. We will highlight the steps which will be different in the EGB case and where one has to make assumptions. For the construction of the membrane stress tensor we will work exclusively with differential forms and only in the end do we go back to the metric formalism.

In the rest of this paper, unless otherwise explicitly written, we will work with the units such that $16\pi G = 1$. The small Roman letters on the differential forms are the Lorentz indices while in the space-time tensors we will not differentiate between the Lorentz and the world indices, this being understood that one can always use the vierbeins to convert the indices from the Lorentz to the world and vice-versa.

In the Cartan formalism, the Einstein-Hilbert lagrangian is written in terms of the vector valued vierbein one-form e^a, related to the metric by $g = \eta_{ab} e^a \otimes e^b$, and the Lorentz Lie-algebra valued torsion-free connection one-form $\omega^{ab}(e)$ defined by the equation $\mathcal{D}e^a = de^a + \omega^a{}_b \wedge e^b = 0$. The action is then

$$S_{EH} = \frac{1}{2!}\int_M \Omega^{ab} \wedge e^c \wedge e^d \epsilon_{abcd} \pm \frac{1}{2!}\int_{\partial M} \theta^{ab} \wedge e^c \wedge e^d \epsilon_{abcd}, \quad (5)$$

where, Ω^{ab} is the curvature of the torsion-free connection given by $\Omega^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}$, and θ^{ab} is the second fundamental form on the boundary ∂M of M, [23] [24]. It is related to the extrinsic curvature by

$$\theta^{ab} = (n.n)(n^a K^b_c - n^b K^a_c)e^c. \quad (6)$$

In our case, the boundary ∂M of M consists of the outer boundary at spatial infinity and the inner boundary at the streched horizon, $\mathcal{H}_s$. Variation of the action with respect to e^a can be separated into the contribution from $\omega(e)$ and the rest. Variation with respect to ω of the bulk part of the action yields a total derivative which, after the integration by parts, gives a contribution identical to the negative of the variation of the boundary part. Thus the variation of the action with respect to ω vanishes identically. In the absence of the inner boundary, variation of S_{EH} with respect to the vierbein, e^a, under the Dirichlet boundary condition (holding the vierbein fixed on the outer boundary) yields the equation of motion for the vierbein. But when the inner boundary is present, there is no natural way to fix the vierbein there. The physical reason for this is simply that the horizon acts as a boundary only for a class of obervers, and surely the metric is not fixed there. However, because it is a causal boundary the dynamics outside the horizon is not affected by what happens inside. Hence, for the consideration of the outside dynamics one can imagine some fictitious matter living on the stretched horizon whose contribution to the variation of the action cancels that of the inner boundary. This is the basic idea of the construction of stress-tensor *à la* Brown and York, [25]. Since we will be interested in the boundary term on the stretched horizon which is a time-like hypersurface, from now on we will put $n.n = 1$.

Hence, variation of the action with respect to the vierbein gives

$$\delta S_{EH} = \int_M \Omega^{ab} \wedge e^c \wedge \delta e^d \epsilon_{abcd} + \int_{\partial M} \theta^{ab} \wedge e^c \wedge \delta e^d \epsilon_{abcd}. \quad (7)$$

The bulk term just gives the equation of motion. Using the Dirichlet boundary condition on the outer boundary and the expression of θ given in equation (6), the surviving contribution of the variation of the total action $S_{total} = S_{EH} + S_{matter}$ comes solely from the inner boundary and is given by

$$\delta S_{total} = \int_{\mathcal{H}_s} K^b_m e^m \wedge e^c \wedge \delta e^d \epsilon_{bcd}, \quad (8)$$

where $\epsilon_{bcd} = -n^a \epsilon_{abcd}$. This surviving contribution can be interpreted as due to a fictitious matter source residing on $\mathcal{H}_s$ whose stress tensor is given by

$$t^{ab} = 2(Kh^{ab} - K^{ab}). \quad (9)$$

In terms of t_{ab} the on-shell variation of S_{EH} becomes

$$\delta S_{EH} = -\frac{1}{2}\int_{\mathcal{H}_s} t_{ab}\delta h^{ab} + \frac{1}{2}\int_M T_{ab}\delta g^{ab}, \quad (10)$$

where, $T_{ab} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{matter}}{\delta g^{ab}}$ is the external matter's stress energy tensor. Now consider the variation δ_ξ induced by a vector field ξ which is arbitrary in the bulk and on the boundary behaves in a prescribed fashion. We take ξ to be such that it is tangential to the inner-boundary and vanishes on the all the other boundaries. Then the diffeomorphism invariance of the theory ensures that

$$\begin{aligned}\delta_\xi S_{EH} &= -\frac{1}{2}\int_{\mathcal{H}_s} t_{ab}\delta_\xi h^{ab} + \frac{1}{2}\int_M T_{ab}\delta_\xi g^{ab} = 0\\ &\Rightarrow -\int_{\mathcal{H}_s} t_{ab}D^a\xi^b + \int_M T_{ab}\nabla^a\xi^b = 0,\end{aligned} \tag{11}$$

where, D^a is the covariant derivative of the induced metric on the inner boundary $\mathcal{H}_s$, and ∇^a is the covariant derivative of the spacetime metric. Using integration by parts and the afore-mentioned conditions on ξ^a, equation (11) gives

$$D^a t_{ab} = -T_{ac}n^a h^c{}_b, \tag{12}$$

where the negative sign on the right hand side arises because we have chosen n^a as pointing away from the stretched horizon into the bulk. The right hand side of this equation can be interpreted as the flux of external matter crossing the horizon from the bulk. Then equation (12) has the interpretation of the continuity equation satisified by the fictitious matter living on the stretched horizon.

At this stage, we would like to point out a difference between our approach and that of ref.[20]. In [20], the Gibbons-Hawking boundary term is considered to be only on the outer boundary. Therefore, one needs to show that a certain contribution containing the derivatives of the variation of the metric on the stretched horizon vanishes in the limit as the stretched horizon reaches the true horizon. In our approach there is no such requirement because we have the boundary term on the entire boundary which includes the stretched horizon. Since both the approaches finally yield the same horizon stress tensor, there is no difference in the physics in our approach and that of ref.[20]. However, an added advantage of our approach is that it also works beyond general relativity and, in particular, in any Lovelock gravity. If one does not include the Gibbons-Hawking term for Lovelock gravity, one needs to show that all the extra terms in the variation of the action vanish when the limit to the true horizon is taken. We have simply avoided this difficulty by adding the boundary term on the stretched horizon as well. Consequently, although our approach and that of ref.[20] ultimately give the same result for the horizon stress tensor, we believe the inclusion of the Gibbons-Hawking term makes the calculations easier. Also, it is worth pointing out that if we were to put the membrane at some finite distance from the horizon then we need the Hawking-Gibbons term in order to derive the stress tensor and the approach of ref.[20] just won't work. We believe that our approach is conceptually transparent and computationally simpler than the one in [20].

We have derived the form of the membrane stress tensor for the particular case of $D = 4$ spacetime dimensions, but it is easy to check that the form of the stress tensor in equation (9) remains unchanged for a general D-dimensional spacetime. Then the components of the membrane stress tensor t_{ab}, evaluated on the stretched horizon in the basis (u^a, x^A) are given by

$$\begin{aligned}t_{uu} &= \rho = -2\theta_s,\\ t^{AB} &= 2\left(-\sigma_s^{AB} + \frac{(D-3)}{(D-2)}\theta_s\gamma^{AB} + g\,\gamma^{AB}\right),\end{aligned} \tag{13}$$

where $g = \kappa/\delta$. In deriving this expression, we have replaced K^{AB} by the expression: $\sigma_s^{AB} + \frac{\theta_s}{(D-2)}\gamma^{AB}$, where θ_s is the expansion and σ_s^{AB} is the shear of the congruence generated by the time-like vector field u^a on $\mathcal{H}_s$. As pointed out in [1, 2], this replacement is valid only up to $\mathcal{O}(\delta)$. Since we are ultimately interested in the limit $\delta \to 0$, any $\mathcal{O}(\delta)$ error does not contribute. Although this is certainly true for general relativity, for the EGB gravity such $\mathcal{O}(\delta)$ terms will play an important role and they actually contribute in the limit that the stretched horizon becomes the true horizon.

The particular form of the components of the membrane stress tensor in equation (13) has an interpretation: the fictitious matter on the stretched horizon can be regarded as a $(D-2)$-dimensional viscous fluid [26] with the energy density and transport coefficients given by

$$\begin{aligned}
\text{Energy density} : \ & \rho_s = -2\theta_s, \\
\text{Pressure} : \ & p_s = 2g, \\
\text{Shear Viscosity} : \ & \eta_s = 1, \\
\text{Bulk Viscosity} : \ & \zeta_s = -2\left(\frac{D-3}{D-2}\right).
\end{aligned} \tag{14}$$

Hence the entire t_{ab} on the stretched horizon can be expressed as,

$$t_{ab} = \rho_s u_a u_b + \gamma_a^A \gamma_b^B \left(p_s \gamma_{AB} - 2\eta_s \sigma_{s\,AB} - \zeta_s \theta_s \gamma_{AB}\right). \tag{15}$$

Substituting these quantities in the conservation equation (12) then gives the evolution equation for the energy density,

$$\mathcal{L}_u \rho_s + \rho_s \theta_s = -p_s \theta_s + \zeta_s \theta_s^2 + 2\eta_s \sigma_s^2 + T_{ab} n^a u^b. \tag{16}$$

The evolution equation matches exactly with energy conservation equation of a viscous fluid. We stress the fact that equation (16) is a direct consequence of the conservation equation (12) and the form of the stress tensor in equation (15). In fact, in any theory of gravity once we can express the stress tensor of the fictitious matter obeying the conservation equation on the stretched horizon in a form analogous to the one in equation (15), the conservation equation will automatically imply an evolution equation of the form (16). Notice that the conservation equation is only valid on-shell, which means that the equations of motion of the theory have to be satisfied for it to hold.

From the analysis of section II it is evident that as the stretched horizon approaches the true horizon the membrane stress tensor in equation (13) diverges as δ^{-1} due to the large blue shift near the horizon. This divergence is regulated by simply multiplying it by δ. This limiting and regularization procedure then yields a stress tensor of a fluid living on the cross sections of the horizon itself in terms of the quantities intrinsic to the horizon. This stress tensor is

$$t^{(H)}_{AB} = \left(p\gamma_{AB} - 2\eta\sigma_{AB} - \zeta\theta\gamma_{AB}\right), \tag{17}$$

and the energy density and the transport coefficients become,

$$\begin{aligned}
\text{Energy density} : \ & \rho = -2\theta, \\
\text{Pressure} : \ & p = 2\kappa, \\
\text{Shear Viscosity} : \ & \eta = 1, \\
\text{Bulk Viscosity} : \ & \zeta = -2\left(\frac{D-3}{D-2}\right)\eta,
\end{aligned} \tag{18}$$

where θ is the expansion of the null generator of the true horizon as discussed in section II. Similarly, the regularization of the evolution equation (16) gives

$$\mathcal{L}_l \rho + \rho\theta = -p\theta + \zeta\theta^2 + 2\eta\sigma^2 + T_{ab} l^a l^b, \tag{19}$$

which is just the Raychaudhuri equation of the null congruence generating the horizon. It should be noted that our approach for deriving the evolution equation is different from the one used in [20]. In principle, one can just take the Lie derivative of the energy density with respect to the generator of the horizon to obtain the Raychaudhuri equation as in [20] and then use the Einstein equation to replace the curvature dependence in terms of the matter energy-momentum tensor. We have followed an indirect approach in which we derive the evolution equation via the continuity equation (12) which is valid on-shell, i.e., the equation of motion has already been used in its derivation. This approach is particularly useful when the equations of motion are complicated as in the EGB gravity and it becomes difficult to replace the curvature dependence in terms of the matter energy-momentum tensor.

For general relativity, the Bekenstein-Hawking entropy of the horizon is $4\pi\mathcal{A}$, where $\mathcal{A}$ is the area of the horizon, hence the ratio of the shear viscosity to the entropy density is,

$$\frac{\eta}{s} = \frac{1}{4\pi}. \tag{20}$$

Note that this is a dimensionless constant independent of the parameters of the horizon. Comparing this with the KSS bound, $\eta/s \geq 1/4\pi$, we see that the bound is saturated in general relativity. Interestingly, for any gravity theory with a Lagrangian depending on the Ricci scalar only, the value of this ratio is same as that in general relativity and therefore the KSS bound is saturated in these theories [27].

Another important fact is that the bulk viscosity associated with the horizon is negative. Clearly, the fluid corresponding to the horizon is not an ordinary fluid, and as explained in [1], its negative bulk viscosity does not entail an instability, because the definition of the horizon imposes a future boundary condition precluding such behavior.

IV. THE MEMBRANE PARADIGM IN EINSTEIN-GAUSS-BONNET GRAVITY

EGB gravity is a natural generalization of general relativity which includes terms higher order in curvature but in just such a way that the equation of motion remains second order in time.

The action of the theory is given by

$$S_{total} = S_{EH} + \alpha S_{GB} + S_{matter} \tag{21}$$

where, S_{EH} and S_{matter} are the contribution of the Einstein-Hilbert and the matter, respectively, while S_{GB} is the Gauss-Bonnet addition to the action. From the analysis in the section III we know how to take care of the S_{EH}. So, we can exclusively work with the GB term now. The GB contribution to the total action, in $D = 5$, is given by [28] [23]

$$S_{GB} = \textstyle\int_M \Omega^{ab} \wedge \Omega^{cd} \wedge e^f \epsilon_{abcdf} + 2\int_{\partial M} \theta^{ab} \wedge (\Omega - \frac{2}{3}\theta \wedge \theta)^{cd} \wedge e^f \epsilon_{abcdf}. \tag{22}$$

As in the case of general relativity discussed in section III the variation of S_{GB} with respect to the connection ω vanishes identically. Variation with respect to the vierbien e^a under the Dirichlet boundary condition yields the equation of motion. Using the torsion-free condition $\mathcal{D}e^a = 0$ on the connection, this equation can be shown to be the same as that obtained in the metric formalism. In the presence of the inner boundary at the stretched horizon, $\mathcal{H}_s$, we need the variation of the boundary part of the S_{GB} due to the variation of the vierbein on the inner boundary. This is obtained from equation (22), which after variation with respect to e^a and *then* using the relations $\theta^{ab} = (n^a K^b_c - n^b K^a_c)e^c$ and $\Omega^{ab} = \frac{1}{2}R^{ab}{}_{mn} e^m \wedge e^n$, gives

$$\delta S_{GB}|^{bndy} = 4\int_{\partial M} K^s_a \left(\frac{1}{2}h^c_p h^d_q h^r_m h^s_n R^{pq}{}_{rs} + \frac{2}{3}K^c_m K^d_n\right) 4!\, \delta^{[amnb]}_{s\,c\,d\,f}\, \delta e^f_b. \tag{23}$$

The projections of the spacetime Riemann tensor can be written in terms of the Riemann tensor intrinsic to $\mathcal{H}_s$ and the extrinsic curvature of $\mathcal{H}_s$ using the Gauss-Codazzi equation

$$h^c_p h^d_q h^r_a h^s_b R^{pq}{}_{rs} = \hat{R}^{cd}{}_{ab} - K^c_a K^d_b + K^c_b K^d_a, \tag{24}$$

where $\hat{R}^{cd}{}_{ab}$ is the Riemann tensor intrinsic to $\mathcal{H}_s$. Thus the variation of the boundary term can be written in terms of the quantities intrinsic to the boundary,

$$\delta S_{GB}|^{bndy} = 4\int_{\partial M} K^s_a \left(\frac{1}{2}\hat{R}^{cd}{}_{mn} - \frac{1}{3}K^c_m K^d_n\right) 4!\, \delta^{[amnb]}_{s\,c\,d\,f}\, \delta e^f_b. \tag{25}$$

This can be evaluated to be

$$\delta S_{GB}|^{bndy} = 4\int_{\partial M} \left(2K_{mn}\hat{P}^{amn}{}_b - J^a_b + \frac{1}{3}J\delta^a_b\right) \delta e^b_a, \tag{26}$$

where we have defined,

$$\hat{P}^{amn}{}_b = \hat{R}^{amn}{}_b + 2\hat{R}^{n[m}h_b^{a]} + 2\hat{R}_b^{[a}h^{m]n} + \hat{R}h^{n[a}h_b^{m]}, \tag{27}$$

$$J_b^a = K^2 K_b^a - K^{cd}K_{cd}K_b^a + 2K_c^a K_d^c K_b^d - 2KK_c^a K_b^c, \tag{28}$$

$$J = K^3 - 3KK^{cd}K_{cd} + 2K_b^a K_c^b K_a^c. \tag{29}$$

As in the case of general relativity, we can interpret the variation δS_{total} as due to a fictitious matter living on the membrane whose stress energy tensor is given by the coefficient of δe_b^a. Thus from the equation (26) we can read off the membrane stress tensor due to the GB term (now including the GB coupling α' and a negative sign arising from the fact that we are defining the stress tensor with covariant indices). The total stress tensor for the the membrane, including the contribution (9) coming from the Einstein-Hilbert action, is

$$t_{ab} = 2\left(Kh_{ab} - K_{ab}\right) - 4\alpha\left(2K_{mn}\hat{P}_a{}^{mn}{}_b - J_{ab} + \frac{1}{3}Jh_{ab}\right). \tag{30}$$

Although, we have derived the membrane stress tensor for the particular case of $D = 5$, the result can be easily generalized to arbitrary dimensions and the form in equation (30) remains unchanged. By the same arguments as discussed in the case of general relativity, this t_{ab} also satisfies the continuity equation (12). This can also be verified explicitly by taking the divergence of t_{ab} and using the appropriate projections of the equation of motion [29].

Note that a crucial difference between the membrane stress tensors for general relativity and EGB gravity is that in the former the stress tensor is linear in the extrinsic curvature while in the latter the stress tensor contains terms cubic in the extrinsic curvature of the stretched horizon. Since, as we take the limit to the true horizon, the extrinsic curvature of the stretched horizon diverges as δ^{-1}, one would expect higher order divergences in the case of EGB gravity coming from the contribution of the GB term to the stress tensor. The regularization procedure used to tame the divergence coming from the Einstein-Hilbert term involves multiplication with δ, which does not tame the cubic order divergence coming from the GB term. Clearly one needs either a new regularization procedure or some well-motivated prescription which justifies neglect of the terms that lead to higher order divergences in the limit when the stretched horizon approaches the true one. In this paper, we will adopt the latter approach.

We will restrict attention to background geometries which are static so that the expansion and the shear of the null generators of the true horizon are zero. Next, we will consider some arbitrary perturbation of this background which may arise due to the flux of matter flowing into the horizon. As a result, the horizon becomes time dependent and acquires expansion and shear. We will assume this perturbation of the background geometry to be small so that we can work in the linear order of perturbation and ignore all the higher order terms. Essentially, our approximation mimics a slow physical-process version of the dynamics of the horizon [30]. This essentially means that we are restricting ourselves to the terms proportional to the first derivative of the observer's four velocity u^a which plays the role of the velocity field for the fluid. We will discard all higher order derivatives of the velocity except the linear one so that we can write the membrane stress tensor as a Newtonian viscous fluid. In this limited setting we will see that the only divergence that survives is of $\mathcal{O}(\delta^{-1})$ which can be regularized in the same fashion as in the case of general relativity. Therefore, when we encounter a product of two quantities X and Y, we will always express such a product as,

$$XY \approx \mathring{X}\,\mathring{Y} + \mathring{X}\,\delta Y + \mathring{Y}\,\delta X, \tag{31}$$

where $\mathring{X}$ is the value of the quantity X evaluated on the static background and δX is the perturbed value of X linear in perturbation.

In order to implement this scheme, we first define a quantity Q_{ab}, whose importance will be apparent later, as

$$Q_{ab} = KK_{ab} - K_{ac}K_b^c, \tag{32}$$

in terms of which, we write

$$J_{ab} = K_{ab}Q - 2K_{ac}Q^c_b, \tag{33}$$

where Q is the trace of Q_{ab}.

Now we observe the following facts. First, the components of the extrinsic curvature of $\mathcal{H}_s$ in the backgrounds that we are interested in is $\mathcal{O}(\delta)$. In particular, for the static spacetimes one can choose the cross-sections of the $\mathcal{H}_s$ such that the pull-back of the extrinsic curvature to these cross-sections is

$$K_{AB} = \frac{\delta}{r}\gamma_{AB}. \tag{34}$$

Here "r" is defined by this equation; for the specific metrics considered below it will coincide with the radial coordinate in a particular coordinate system. Secondly, for these backgrounds Q_{ab} and Q defined as above are finite on $\mathcal{H}_s$ and remain finite in the limit as $\mathcal{H}_s$ reaches the true horizon. In fact, Q_{AB} for the background, in the limit that $\mathcal{H}_s$ approaches the true horizon, is simply $Q_{AB} = \frac{\kappa}{r}\gamma_{AB}$. Finally, the linearized Q_{AB} and Q have the most singular terms given by

$$\delta Q_{AB} \sim \frac{1}{\delta^2}\kappa\left(\sigma_{AB} + \frac{\theta}{(D-2)}\gamma_{AB}\right), \tag{35}$$

$$\delta Q \sim \frac{1}{\delta^2}2\kappa\theta. \tag{36}$$

Notice that the perturbation of the non affine coefficient κ can always be gauged away by choosing a suitable parametrization of the horizon. This remark means that we can put $\delta\kappa$ equal to zero. So, without any loss of generality, we set κ equal to the surface gravity of the background black-geometry. Now, as an illustration of the perturbation scheme mentioned in the equation (31) and the reason for the definition of the Q_{ab}, we use the facts mentioned in the previous paragraph to evaluate a term $K_{AC}Q^C_D$ contributed by J_{AD} which we encounter in the projection of t_{ab} on the cross-section of $\mathcal{H}_s$. We evaluate this term as follows :

$$\begin{aligned}
K_{AC}Q^C_D &\approx \mathring{K}_{AC}\mathring{Q}^C_D + \delta K_{AC}\mathring{Q}^C_D + \mathring{K}_{AC}\delta Q^C_D \\
&= \frac{\delta}{r}\gamma_{AC}\mathring{Q}^C_D + \frac{1}{\delta}\left(\sigma_{AC} + \frac{\theta}{(D-2)}\gamma_{AC}\right)\mathring{Q}^C_D + \frac{\delta}{r}\gamma_{AC}\frac{1}{\delta^2}\kappa\left(\sigma^C_D + \frac{\theta}{(D-2)}\gamma^C_D\right) \\
&\sim \frac{1}{\delta}\left(\sigma_{AC}\mathring{Q}^C_D + \frac{\theta}{(D-2)}\mathring{Q}_{AD} + \frac{\kappa}{r}\sigma_{AD} + \frac{\kappa}{r}\frac{\theta}{(D-2)}\gamma_{AD}\right) \\
&\sim \frac{1}{\delta}\left(\frac{\kappa}{r}\sigma_{AD} + \frac{\theta}{(D-2)}\frac{\kappa}{r}\gamma_{AD} + \frac{\kappa}{r}\sigma_{AD} + \frac{\kappa}{r}\frac{\theta}{(D-2)}\gamma_{AD}\right).
\end{aligned} \tag{37}$$

In the steps above we have dropped the terms which are of $\mathcal{O}(1)$ or $\mathcal{O}(\delta)$ because after regularization (i.e., multiplying by δ) those terms will make no contribution. In this way one sees that our method of approximation is consistent. At the linear order in perturbation the only divergence that comes up in the membrane stress energy tensor is of $\mathcal{O}(\delta^{-1})$ and therefore the whole stress tensor can be regularized simply by multiplying with δ exactly as in general relativity. The sample calculation above elaborates on how it is done for one particular term. Using the Gauss-Codacci equation and the Raychaudhari equation it can be shown that the projections of the curvature of the $\mathcal{H}_s$ on the cross-section of $\mathcal{H}_s$ is equal to the curvature of the cross-section in the limit $\delta \to 0$.

Before writing down the stress tensor there is one more restriction that we are going to put on the background geometry. We require that the cross-section of the horizon of the background geometry be a space of constant curvature, i.e., ${}^{(D-2)}\mathring{R}_{ABCD} = c\,(\gamma_{AC}\gamma_{BD} - \gamma_{AD}\gamma_{BC})$, where c is a constant of dimension $length^{-2}$ which is related to the intrinsic Ricci scalar of the horizon cross-section as

$$c = \frac{\mathcal{R}}{(D-3)(D-2)}, \qquad \mathcal{R} \equiv {}^{(D-2)}\mathring{R}. \tag{38}$$

This assumption regarding the intrinsic geometry of the horizon cross-section is necessary for the stress tensor to be of the form of an isotropic viscous fluid. Note that this assumption rules out, e.g., the topological black holes in anti-de Sitter in which the horizon cross-section is a non-constant curvature Einstein space [31].

Using these observations and approximations, the contribution of different terms in the membrane stress energy tensor due to the GB term in the action are obtained as :

$$\begin{aligned}
K_{mn}\hat{P}_a{}^{mn}{}_b\gamma^a_A\gamma^b_B &\sim -\frac{1}{\delta}\frac{(D-4)}{2(D-2)}\mathcal{R}\kappa\gamma_{AB} + \frac{1}{\delta}\frac{(D-5)(D-4)}{2(D-3)(D-2)}\mathcal{R}\left[\sigma_{AB} - \frac{(D-3)}{(D-2)}\theta\gamma_{AB}\right],\\
J_{ab}\gamma^a_A\gamma^b_B &\sim \frac{1}{\delta}\sigma_{AB}(D-4)\frac{2\kappa}{r} + \frac{1}{\delta}\gamma_{AB}\frac{(D-3)}{(D-2)}\frac{4\kappa\theta}{r},\\
J &\sim \frac{1}{\delta}(D-3)\frac{6\kappa\theta}{r},\\
K_{mn}\hat{P}_a{}^{mn}{}_b u^a u^b &\sim -\frac{1}{\delta}\frac{(D-4)}{2(D-2)}\mathcal{R}\kappa\theta,\\
J_{ab}u^a u^b &\sim -\frac{1}{\delta}(D-3)\frac{2\kappa\theta}{r}.
\end{aligned} \tag{39}$$

All the steps required to obtain the regularized membrane stress tensor are laid out now. Our perturbative strategy and the restriction that the horizon's cross-section be the space of constant curvature then yields the energy density and membrane stress tensor as

$$\rho = -2\theta - 4\frac{D-4}{D-2}\alpha\mathcal{R}\theta, \tag{40}$$

$$t^{(H)}_{AB} = p\,\gamma_{AB} - 2\eta\,\sigma_{AB} - \zeta\theta\,\gamma_{AB}, \tag{41}$$

where,

$$p = 2\kappa + 4\frac{D-4}{D-2}\alpha\mathcal{R}\kappa, \tag{42a}$$

$$\eta = 1 - 4(D-4)\alpha\left[\frac{\kappa}{r} - \frac{(D-5)}{2(D-3)(D-2)}\mathcal{R}\right], \tag{42b}$$

$$\zeta = -2\frac{D-3}{D-2}\eta. \tag{42c}$$

Note that the ratios $p/\rho = -\kappa/\theta$ and $\zeta/\eta = -2\frac{D-3}{D-2}$ are the same as the ones that hold in general relativity (18).

We stress that while we are working only at the linear order in the perturbations, the α corrections in the transport coefficients given in equations (42) are non-perturbative. This simply means that the theory is exactly the EGB theory and the background spacetimes of interest are the exact static solutions of this theory. This approach differs from some of the other work in the literature, see for example [32], where one considers the effect of the GB term in the action as a small perturbation of general relativity.

The continuity equation (12) yields the equation describing the evolution of the energy density along the null generators of the horizon. Note that we are actually applying the linearized continuity equation, and in order to derive the evolution equation we have to keep the induced metric on the stretched horizon fixed. As in the case of general relativity, we first write down the equation on the stretched horizon, keeping the terms which are linear in perturbations and have $\mathcal{O}(\delta^{-1})$ divergence in t_{ab}, which gives an $\mathcal{O}(\delta^{-2})$ divergence in the continuity equation. This is regularized by multiplying the whole equation by δ^2. This again has the same form as that in the equation (16) with ρ now given by (40), and p, η and ζ given by (42).

V. ADS BLACK HOLE BACKGROUNDS AND THEIR VARIOUS LIMITS

In this section we will calculate the transport coefficients and the ratio η/s for the membrane fluid around EGB black hole backgrounds with a negative cosmological constant $\Lambda = -(D-1)(D-2)/2l^2$ in the action, and horizon cross section of either positive, zero, or negative constant curvature. The limit $l \to \infty$ will yield also asymptotically flat solutions. The stable static solutions of this type in $D \geq 5$ dimensions is given by the metric [33]

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 h_{ij} dx^i dx^j, \tag{43}$$

where h_{ij} is the metric of a $D-2$ dimensional space of constant curvature $(D-2)(D-3)k$ and volume Σ_k, with $k = 0, \pm 1$, and

$$f(r) = k + \frac{r^2}{2\lambda}\left[1 - \sqrt{1 + \frac{4\lambda M}{(D-2)\Sigma_k r^{D-1}} - \frac{4\lambda}{l^2}}\right], \tag{44}$$

where

$$\lambda := (D-3)(D-4)\alpha. \tag{45}$$

The asymptotic AdS radius L is defined by $\lim_{r\to\infty} f(r) = r^2/L^2$, and is related to the cosmological constant length parameter l in the Lagrangian by

$$L^2/l^2 = \tfrac{1}{2}\left(1 + \sqrt{1 - 4\lambda/l^2}\right) \tag{46}$$

The mass M is related to the horizon radius r_+ by

$$M = (D-2)\Sigma_k r_+^{D-3}\left(k + \frac{\lambda k^2}{r_+^2} + \frac{r_+^2}{l^2}\right), \tag{47}$$

the surface gravity is

$$\kappa = \frac{(D-1)r_+^4 + (D-3)kl^2 r_+^2 + (D-5)\lambda k^2 l^2}{2l^2 r_+ (r_+^2 + 2\lambda k)}, \tag{48}$$

and the entropy density is

$$s = 4\pi\left(1 + \frac{(D-2)}{(D-4)}\frac{2\lambda k}{r_+^2}\right). \tag{49}$$

We remind the reader that we are using units with $16\pi G = 1$.[34] Note that a hyperbolic ($k = -1$) horizon exists only if $\Lambda \neq 0$ and/or $\lambda \neq 0$, given the requirement that the mass (47) be positive. The solution with $k = 1$ was first found in ref.[10]. The cases $k = 0$ and -1 were studied in ref.[33]. Solutions for particular value of the coupling, $\lambda = l^2/4$, were also studied in refs.[35–38].

Before we can evaluate the transport coefficients (42), we must determine the parameter "r" in the formula $K_{AB} = (\delta/r)\gamma_{AB}$ (34) for the extrinsic curvature of the horizon cross section. As explained in Sec. II, when the background is stationary with Killing vector ξ, we have $\delta = |\xi|$. Consider then a metric of the form (43), but with the coefficient g_{rr} of dr^2 an independent function to begin with. The extrinsic curvature of the horizon in a $t = 0$ slice is half the Lie derivative along the unit normal,

$$K_{AB} = \tfrac{1}{2}\mathcal{L}_n \gamma_{AB} = \tfrac{1}{2}\frac{d(r^2)}{ds}\frac{1}{r^2}\gamma_{AB} = \frac{1}{\sqrt{g_{rr}}\,r}\gamma_{AB}. \tag{50}$$

In the coordinate system of (43), $1/\sqrt{g_{rr}} = \sqrt{g_{tt}} = \delta$, so in fact "$r$" in the formula for K_{AB} is the same as the r coordinate.

Using equations (42,45,48), and $\mathcal{R} = (D-2)(D-3)k/r_+^2$, we find for the various transport coefficients

$$p = \frac{(D-1)r_+}{l^2} + \frac{(D-3)k}{r_+} + \frac{(D-5)\lambda k^2}{r_+^3}, \tag{51}$$

$$\eta = 1 + \frac{2\lambda\big[-2k - (D-1)r_+^2/l^2 + (D-5)\lambda k^2/r_+^2\big]}{(D-3)(r_+^2 + 2\lambda k)}, \tag{52}$$

$$\zeta = -2\frac{(D-3)}{(D-2)}\,\eta. \tag{53}$$

The ratio of shear viscosity and entropy density is given by

$$\frac{\eta}{s} = \frac{1}{4\pi}\frac{\left[1 - 2\frac{D-1}{D-3}\frac{\lambda}{l^2}\right] r_+^4 + 2(D-5)\lambda k(r_+^2 + \lambda k)}{(r_+^2 + 2\lambda k)\left[r_+^2 + 2\frac{D-2}{D-4}\lambda k\right]}, \tag{54}$$

which reduces to the GR value $1/4\pi$, when $\lambda = 0$. In particular, for planar horizons k=0, and we get

$$\frac{\eta}{s} = \frac{1}{4\pi}\left[1 - 2\frac{(D-1)}{(D-3)}\frac{\lambda}{l^2}\right], \tag{55}$$

which matches with the value found by other methods in the literature, see for example [16] where one of the calculations utilizes the Kaluza-Klein reduction to express the transverse metric perturbation in D-dimensions as the vector potential in $(D-1)$-dimensions and then uses the membrane pardigm results for the electromagnetic interaction of the black hole in $(D-1)$-dimensions. It is evident that the KSS bound is violated for any $\lambda > 0$. The shear viscosity becomes negative when $\lambda/l^2 > (D-3)/2(D-1)$. It was pointed out in [16] and [5] that the gravitons in the theory become strongly coupled as $\lambda/l^2 \to (D-3)/2(D-1)$, and for $\lambda/l^2 > (D-3)/2(D-1)$ the theory becomes unstable.

VI. SUMMARY AND DISCUSSION

In this paper we have employed a perturbative scheme to derive the horizon membrane stress tensor and the transport coefficients for the membrane fluid in Einstein-Gauss-Bonnet gravity. We used the action principle formalism to determine the membrane stress tensor on the stretched horizon. Our derivation is slightly different from the one given in [20], since we include the Gibbons-Hawking boundary term at infinity as well as on the stretched horizon. As a result, the contribution from the bulk part under the variation with respect to ω vanishes automatically. (In [20] it was argued that, without the Gibbons-Hawking term, these contributions vanish in the limit when the stretched horizon approaches the true horizon.) Since the original posting of a draft of this paper, our method has been generalized to the higher order Lovelock theories [39].

The membrane stress tensor in EGB gravity has terms cubic in the extrinsic curvature, and in the limit that the stretched horizon approaches the true horizon these terms are cubically divergent in δ^{-1}. We avoided dealing with the cubic divergences by studying the perturbations about the background static black geometries. We found that restricting to the linear order in perturbations on the stretched horizon, the membrane stress tensor is in fact only linearly divergent. Therefore, at the linear order, the divergence structure of the GB contribution to the stress tensor is identical to that of the Einstein contribution. Hence the whole membrane stress tensor could be regularized in the same way as in general relativity: simply by mutliplying the stress tensor by δ. The source of the divergences can be traced to the fact that the time direction along the stretched horizon becomes null in the limit $\delta \to 0$, so a finite passage of proper time extends over an infinite affine parameter of the null geodesics generating the horizon, leading to divergent time derivatives. These divergences could perhaps be avoided by working with the affine parameter directly on the true horizon, rather than proper time on the stretched horizon If so, one could treat the nonlinear membrane dynamics without resorting to any perturbative scheme.

In order to write the membrane stress tensor in the form of an isotropic viscous fluid we restricted the background geometries to those having a constant curvature horizon cross-section. It should be possible to lift this restriction by modeling the horizon as an anisotropic fluid with tensorial transport coefficients. The transport coefficients of the membrane fluid for the horizon in EGB gravity are given in equations (42). These coefficients all receive contributions from the GB coupling, but we observe that the relation between the shear viscosity and bulk viscosity (42c) is the same as it is in GR. This also holds for the higher order Lovelock theories [39], and for theories whose Lagrangian is an arbitrary function of the Ricci scalar [27]. In general relativity it is implied by the relative contributions of expansion and shear in the Raychaudhuri equation. Given the assumption that the unperturbed horizon is maximally symmetric, perhaps the persistence of this relationship can be traced back to the structure of the Raychaudhuri equation in all these cases. We also note that the ratio of the pressure to the energy density of the membrane fluid in EGB, and indeed in all Lovelock theories [39], is the same as it is in GR.

In section V we evaluated the pressure, and bulk and shear viscosities for the perturbations of AdS black holes with flat or curved horizons. Results for black holes in flat space can be obtained by taking the limit $l \to 0$. In particular, the ratio η/s of shear viscosity η to entropy density s is given by Eq. (54). For the black brane solution of EGB gravity, the membrane paradigm indeed gives a value for the ratio η/s that agrees with the literature. For example, in [16] the ratio is calculated using two different methods: first, using a holographic calculation of the linear response function, and second, using a Kaluza-Klein compactified version of the membrane paradigm. In agreement with that reference we find that the KSS bound ($\eta/s = 1/4\pi$) is violated for any value of positive GB coupling. It is worth mentioning that while our result for the ratio η/s agrees with that of [19] our results for η and s separately do not agree. This can be attributed to the metric field redefinition in [19], together with an inherent ambiguity in defining the spatial volume element of the boundary gauge theory, which is a CFT and thus has no intrinsic scale. However, the dimensionless ratio η/s is evidently invariant under these changes.

In gauge/gravity duality, thermodynamic quantities pertaining to black hole solutions in the gravity theory, e.g., free energy, temperature, and entropy etc., correspond to those describing the thermal state of the boundary gauge theory. It is therefore natural to expect that there would be some relation between the hydrodynamics of the boundary theory and the long wavelength perturbations of the black hole solution. At least for planar black hole solutions, which have no length scale intrinsic to the horizon, one would expect a hydrodynamic correspondence with the boundary gauge theory. Indeed, such a relation was discovered, and is known as the fluid gravity correspondence [40]. Some transport coefficients describing the boundary fluid can also be calculated by the membrane paradigm. As shown in [5], when the bulk theory is general relativity, the low-frequency limit of the linear response of the boundary theory to certain perturbations (eg. shear, external current etc.) can be calculated by that of the membrane-paradigm fluid living on the horizon of the black-brane. The corresponding analysis when the bulk theory is EGB gravity has not been carried out, as far as we are aware.

It is worth noting here that the negative bulk viscosity of the horizon membrane fluid does not correspond to the value of the bulk viscosity for the boundary theory fluid. The reason for this mismatch is that in the horizon membrane calculation one is not really making a systematic expansion in derivatives of fluid velocity. The study of the vacuum perturbation of a static black hole show [41] that the shear is second order in perturbation, while the expansion is third order. Therefore, while the shear viscosity of the boundary theory fluid can be directly read from the shear term of the membrane stress tensor, the bulk viscosity of the boundary fluid is, in fact, zero. Perturbations due to matter, however, induce a lower order non-zero expansion and yield a non-zero bulk viscosity [42]. The value of the bulk viscosity thus obtained has been shown, in general relativity, to agree with the bulk viscosity of the boundary fluid [41, 43].

Finally, as mentioned above, we have found that, unlike in GR, the membrane transport coefficients in EGB gravity depend on the curvature of the horizon. It is interesting to ask, then, under what circumstances, if any, this curvature dependence can tell us something about the transport coefficients in a dual CFT. For example, the membrane transport coefficients might provide a boundary condition for the flow equation determining the CFT transport coefficients, as discussed in [5] (see also [44]). In order for the membrane fluid to be related to the hydrodynamic limit of the CFT, we expect that the horizon curvature length scale must be much longer than the thermal length scale. This condition can be met for a large black hole ($r_+ \gg l$). It can also be met when the horizon curvature is induced by the presence of an inhomogeneous background field, as in the setting of [45]. It would be interesting to determine whether, under such circumstances, the membrane paradigm transport coefficients can indeed be related to those of a dual CFT.

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VII. APPENDIX: TABLE OF IMPORTANT SYMBOLS AND THEIR MEANINGS

D denotes the spacetime dimension. The metric signature is $(-,+,+,...)$.

Symbol	**Meaning**
H	True horizon, a $(D-1)$-dimensional null-hypersurface
$\mathcal{H}_s$	Stretched horizon, a $(D-1)$-dimensional time-like hypersurface with tangent u^a and normal n^a
$(a,b,c,...)$	Spacetime indices
$(A,B,C,...)$	Indices on the $(D-2)$-dimensional cross-section of the true/stretched horizon
l^a	Null generator of the true horizon parametrized by a non-affine parameter. Obeys the geodesic equation: $l^a\nabla_a l^b = \kappa\, l^b$
h_{ab}	Induced metric on the stretched horizon
γ_{ab}	Induced metric on the cross-section of the stretched horizon, which in the null limit is identified with the metric on the cross-section of the true horizon
K_{ab}	Extrinsic curvature of the stretched horizon defined as, $K_{ab} = \frac{1}{2}\mathcal{L}_n h_{ab}$
k_{AB}	Extrinsic curvature of the cross-section of the true horizon defined as $k_{AB} = \frac{1}{2}\mathcal{L}_{l^a}\gamma_{AB}$
θ, θ_s	Expansion of the true/strectched horizon
σ^{ab}, σ_s^{ab}	Shear of the true/stretched horizon
$\hat{R}_{abcd}$	Riemann tensor intrinsic to the stretched horizon
${}^{(D-2)}\mathring{R}_{ABCD}$	Intrinsic Riemann tensor of the $(D-2)$-dimensional cross section of the stretched horizon in the background geometry, which in the null limit is identified with the intrinsic Riemann tensor of the cross-section of the true horizon
${}^{(D-2)}\mathring{R} = \mathcal{R}$	Intrinsic Ricci scalar of the $(D-2)$-dimensional cross section of the stretched horizon in the background geometry, which in the null limit is identified with the intrinsic Ricci scalar of the cross-section of the true horizon
δ	The parameter which measures the deviation of the stretched horizon from the true horizon (equal to norm of horizon generating Killing vector when on a static background)
c	Defined as $c = \frac{{}^{(D-2)}\mathring{R}}{(D-3)(D-2)}$ (For the planar horizon, $c = 0$)
α	Gauss-Bonnet coupling constant
λ	Constant of dimension *length*^{2} related to the Gauss-Bonnet coupling α as $\lambda = (D-3)(D-4)\alpha$

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