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# Pion Distribution Amplitude and Quasi-Distributions

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We extend our analysis of quasi-distributions onto the pion distribution amplitude. Using the formalism of parton virtuality distribution amplitudes (VDAs), we establish a connection between the pion transverse momentum dependent distribution amplitude (TMDA)  $\Psi(x, k_\perp^2)$  and the pion quasi-distribution amplitude (QDA)  $Q_\pi(y, p_3)$ . We build models for the QDAs from the VDA-based models for soft TMDAs, and analyze the  $p_3$  dependence of the resulting QDAs. As there are many models claimed to describe the primordial shape of the pion DA, we present the  $p_3$ -evolution patterns for models producing some popular proposals: Chernyak-Zhitnitsky, flat and asymptotic DAs. Our results may be used as a guide for future studies of the pion distribution amplitude on the lattice using the quasi-distribution approach.

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## I. INTRODUCTION

The parton distribution functions (PDFs)  $f(x)$ , and two-body distribution amplitudes (DAs)  $\varphi(x)$  are related to matrix elements of bilocal operators on the light cone  $z^2 = 0$ , which prevents a straightforward calculation of these functions in a lattice gauge theory formulated in the Euclidean space. The usual way out is to calculate their moments. In particular, high precision lattice calculations of the second moment of the pion distribution amplitude  $\varphi_\pi(x)$  were reported in Ref. [1]. However, recently, X. Ji [2] suggested a method allowing to calculate PDFs and DAs as functions of  $x$ . To this end, he proposes to use purely space-like separations  $z = (0, 0, 0, z_3)$ .

The matrix elements of equal-time bilocal operators produce distributions  $Q(y, p_3)$  in the momentum  $p_3$  component (quasi-distributions). The crucial point is that they tend to the light-cone distributions  $f(y)$ ,  $\varphi(y)$  in the  $p_3 \rightarrow \infty$  limit. In case of PDFs, the results of lattice calculations of the parton quasi-distributions (PQDs) were reported in Refs. [3–8]. It is expected [9] that PQDs  $Q(y, p_3)$  should have a mild perturbative evolution [10–13] with respect to  $p_3$  for large  $p_3$ . However, the values of  $p_3$  used in the cited lattice calculations are not very large, and the observed strong variation of PQDs with  $p_3$  does not have a perturbative form.

In our recent paper [14] we have studied nonperturbative evolution of PQDs using the formalism of *virtuality distribution functions* [15, 16]. We found that PQDs can be obtained from the transverse momentum dependent distributions (TMDs)  $\mathcal{F}(x, k_\perp^2)$ . Then we built models for the nonperturbative evolution of PQDs using simple models for TMDs. Our results are in qualitative agreement with the  $p_3$ -evolution patterns obtained in lattice calculations [3–8] and also in diquark spectator models [17–19].

As emphasized in Ref. [14], because of the relation between PQDs and TMDs, the nonperturbative evolution of PQDs reflects the  $k_\perp$ -dependence of the TMDs  $\mathcal{F}(x, k_\perp^2)$ , and thus its study provides a new approach

to the investigation of the 3-dimensional structure of hadrons.

Our goal in the present paper is to perform a similar analysis of the pion *quasi-distribution amplitude* (QDA)  $Q_\pi(y, p_3)$  that produces the pion DA  $\varphi_\pi(y)$  in the large- $p_3$  limit. The basic ingredients of our analysis are *virtual distribution amplitudes* and *transverse momentum dependent amplitudes* introduced in Refs. [15, 16].

The paper is organized as follows. We start in Section 2 with an introductory overview of the basic concepts involved. First, we remind a covariant definition of the twist-2 pion distribution amplitude. After that, we discuss its definition within the light-front formalism. Then we outline the basics of the VDA/TMDA approach. In Section 3, we discuss the quasi-distribution amplitudes. In particular, we show that QDAs are completely determined by TMDAs through a rather simple transformation. Since the basic relations between the parton distributions are rather insensitive to complications brought by spin, in Section 3 we refer to a simple scalar model. In Section 4, we discuss modifications related to quark spin and gauge nature of gluons in quantum chromodynamics (QCD). In Section 5 we discuss the VDA-based models for soft TMDAs, and present our results for non-perturbative evolution of QDAs obtained in these models. The large- $p_3$  limit of perturbative evolution is discussed in Section 6. Our conclusions are given in Section 7.

## II. PION DISTRIBUTION AMPLITUDE

### A. Covariant Definition

The pion *distribution amplitude* (DA)  $\varphi_\pi(x, \mu^2)$  was originally introduced [20] as a function  $\varphi_\pi(x, \mu^2)$  whose  $x^n$  moments

$$f_n(\mu^2) = \int_0^1 x^n \varphi_\pi(x, \mu^2) dx \quad (1)$$

are given by reduced matrix elements of twist-2 local operators

$$\begin{aligned} i^{n+1} R_{\mu^2} \langle 0 | \bar{d}(0) \gamma_5 \{ \gamma_\nu D_{\nu_1} \dots D_{\nu_n} \} u(0) | \pi^+, P \rangle \\ = \{ P_\nu P_{\nu_1} \dots P_{\nu_n} \} f_n(\mu^2). \end{aligned} \quad (2)$$

As usual  $\{ \dots \}$  denotes the twist-2 projection of a Lorentz structure, i.e., symmetrization of indices and subtraction of traces. Since matrix elements of local operators with  $n > 0$  diverge, one needs to supply them by a renormalization procedure denoted above by  $R_{\mu^2}$ , with  $\mu^2$  being the renormalization scale. In QCD, the standard choice of  $R_{\mu^2}$  is based on the dimensional regularization and the modified minimal subtraction scheme  $\overline{\text{MS}}$ . As a result of such a renormalization, the zeroth moment  $f_0(\mu^2)$  does not have  $\mu^2$ -dependence since the anomalous dimension of the axial current is zero. Hence,  $f_0(\mu^2)$  for all  $\mu^2$  is equal to the pion decay constant  $f_\pi$

$$f_0(\mu^2) = \int_0^1 \varphi_\pi(x, \mu^2) dx = f_\pi \quad (3)$$

known experimentally,  $f_\pi \approx 130$  MeV.

This definition of DA is oriented on the use of the operator product expansion and a description of the pion in terms of the twist-2 DA  $\varphi_\pi(x, \mu^2)$  that gives the collinear distribution of the pion momentum  $p$  among its two valence constituents. The dependence of  $\varphi_\pi(x, \mu^2)$  on  $\mu^2$  is governed by perturbative evolution [21–23] and does not reflect the primordial (nonperturbative) pion’s structure in the direction transverse to  $p$ .

As is well-known, for very large  $\mu^2$ , the pion DA tends to the “asymptotic DA”  $\varphi_\pi^{\text{as}}(x) = 6f_\pi x(1-x)$  [24]. In general,  $\varphi_\pi(x, \mu^2)$  may differ from its asymptotic form. Over the years, several forms were proposed for the pion DA “at low normalization point”, e.g., Chernyak-Zhitnitsky DA  $\varphi_\pi^{\text{CZ}}(x) = 30f_\pi x(1-x)(1-2x)^2$  [25], “flat DA”  $\varphi_\pi^{\text{flat}}(x) = f_\pi$  [26–30], “root DA”  $\varphi_\pi^{\text{root}}(x) = 8f_\pi \sqrt{x(1-x)}/\pi$  [31], etc.

## B. Light-Front Formalism Definition

A different definition [23] is used in the light-front (LF) quantization framework, where the pion distribution amplitude  $\phi_\pi(x, \mu^2)$  is understood as the  $k_\perp$ -integral

$$\phi_\pi(x, \mu^2) = \frac{\sqrt{6}}{(2\pi)^3} \int_{k_\perp^2 \leq \mu^2} \psi(x, k_\perp) d^2 k_\perp \quad (4)$$

of the light-front wave function (LFWF)  $\psi(x, k_\perp)$ . We intentionally use here a different notation  $\phi_\pi(x, \mu^2)$  to emphasize the fact that  $\psi(x, k_\perp)$  is an object of the *Hamiltonian light-front framework*, while the pion DA  $\varphi_\pi(x, \mu^2)$  in Eq. (1) is defined within the *covariant Lagrangian formulation* of the quantum field theory (QFT).

Another difference is the use of a straightforward cut-off  $k_\perp^2 \leq \mu^2$  rather than a more sophisticated  $\overline{\text{MS}}$ -like

subtraction. As a result,  $\phi_\pi(x, \mu^2)$  has a nonperturbative evolution with  $\mu^2$  even if the perturbative evolution is absent. Take a simple example  $\psi(x, k_\perp) \sim \phi(x) e^{-k_\perp^2/\Lambda^2}$ . Then the zeroth  $x$ -moment of  $\phi_\pi(x, \mu^2)$  has the  $\sim [1 - e^{-\mu^2/\Lambda^2}]$ -dependence, i.e. it is not constant, reaching  $f_\pi$  in the  $\mu^2 \rightarrow \infty$  limit only.

Of course, if one has in mind only the applications in which nonperturbative part of the  $\mu^2$ -dependence may be ignored, then  $\phi_\pi(x, \mu^2)$  of the LF definition is very similar to the covariantly defined  $\varphi_\pi(x, \mu^2)$ , and the difference between them may be treated as the use of different renormalization schemes.

As a matter of fact, in actual LF calculations one encounters LFWFs integrated to some process-dependent scale  $\mu$ , i.e. the choice of the renormalization prescription and the scale  $\mu$  is dictated by diagrams. Moreover, if the relevant  $\mu^2$ ’s are not extremely large, the simple example above shows that one may need to take into account the nonperturbative  $\mu^2$ -dependence of  $\phi_\pi(x, \mu^2)$  reflecting the transverse momentum behavior of the LFWF  $\psi(x, k_\perp)$ , i.e., the 3-dimensional structure of the pion, which may be essential for some processes.

In particular, the photon-pion transition form factor involves  $\phi_\pi(x, \mu^2 = x^2 Q^2)/[xQ^2]$ , i.e. LFWF  $\psi(x, k_\perp)$  integrated over  $k_\perp$  till  $xQ$  [32, 33]. As a result, the remaining  $x$ -integral in the LF formula has a finite  $Q^2 \rightarrow 0$  limit: the infrared small- $x$  divergence is eliminated by a cut-off provided by  $\phi(x, \mu^2 = x^2 Q^2)$ . On the other hand, a formula involving  $\overline{\text{MS}}$ -based DA  $\varphi(x, \mu^2)$  with a fixed scale  $\mu^2$  is singular in the  $Q^2 \rightarrow 0$  limit. One may question the applicability of the LF formula down to  $Q^2 = 0$ , but at least it does not give an infinite result for a quantity that is known to be finite. For this reason, the LF formula looks as a more attractive tool for modeling the form factor behavior at moderate  $Q^2$  than the perturbative QCD  $1/Q^2$  twist expansion.

Still, a problem with the LF formalism is that LFWFs are not directly connected with the usual objects of the covariant field theory, such as matrix elements of local or nonlocal operators.

In our papers [15, 16], we have developed the formalism of *virtuality distribution amplitudes* (VDAs) that is fully based on the covariant field theory concepts. In the VDA approach, the pion is described by the *transverse momentum dependent distribution amplitude* (TMDA) which has a direct connection with the objects of the covariant QFT. On the other hand, just like the LF wave functions, the TMDAs give a 3-dimensional description of the pion structure.

## C. Pion TMDA

To omit inessential complications related to spin, we illustrate the ideas underlying TMDAs using a simple example of a scalar theory. The key element of our ap-

proach [15] is the VDA representation

$$\langle 0|\psi(0)\psi(z)|p\rangle = \int_0^\infty d\sigma \int_0^1 dx \times \Phi(x, \sigma) e^{-ix(pz) - i\sigma(z^2 - i\epsilon)/4} \quad (5)$$

that basically reflects the fact that the matrix element  $\langle 0|\psi(0)\psi(z)|p\rangle$  depends on  $z$  through  $(pz)$  and  $z^2$ . It may be treated as a double Fourier representation with respect to these variables.

The main non-trivial feature of this representation is in its specific limits of integration over  $x$  and  $\sigma$ . They hold for any contributing Feynman diagram [16], so we assume that this property is true in general. Note that starting with the first loop, the diagram contributions are non-analytic in  $z^2$  due to  $\ln z^2$  factors, but the VDA representation, unlike the Taylor expansion in  $z^2$ , is valid nevertheless.

While the VDA representation is a fully covariant expression, it is convenient to use a frame in which the pion momentum  $p$  is purely longitudinal  $p = (E, \mathbf{0}_\perp, P)$ . Choosing some special cases of  $z$ , one can get representations for several parton functions, all in terms of one and the same universal VDA  $\Phi(x, \sigma)$ . In particular, choosing  $z$  on the light front  $z_+ = 0$  and with  $z_\perp = 0$  (i.e., taking  $z = z_-$ ) gives the twist-2 distribution amplitude  $\varphi(x)$

$$\langle 0|\psi(0)\psi(z_-)|p\rangle = \int_0^1 dx \varphi(x) e^{-ixp_+ z_-} \quad (6)$$

Comparing this relation with the VDA representation we have

$$\varphi(x) = \int_0^\infty \Phi(x, \sigma) d\sigma, \quad (7)$$

provided that the  $z^2 \rightarrow 0$  limit is finite, e.g. in the super-renormalizable  $\varphi^3$  theory. In the renormalizable  $\varphi^4$  theory, the function  $\Phi(x, \sigma)$  has a  $\sim 1/\sigma$  hard part, and the integral (7) is logarithmically divergent, reflecting the perturbative evolution of the DA in such a theory. In this case, one may arrange a regularization of the  $\sigma$ -integral characterized by some parameter  $\mu^2$ . Then  $\varphi(x) \rightarrow \varphi(x, \mu^2)$ .

Light-cone singularities are avoided if we choose a spacelike  $z$ , e.g., take  $z$  that has  $z_-$  and  $z_\perp$  components only. Then we can introduce the *transverse momentum dependent distribution amplitude*  $\Psi(x, k_\perp^2)$  as a Fourier transform

$$\langle 0|\psi(0)\psi(z_-, z_\perp)|p\rangle = \int_0^1 dx e^{-ixp_+ z_-} \times \int d^2 k_\perp \Psi(x, k_\perp^2) e^{i(k_\perp z_\perp)} \quad (8)$$

of the matrix element with respect to  $z_-$  and  $z_\perp$ . Because of the rotational invariance in  $z_\perp$  plane, TMDA depends on  $k_\perp^2$  only, the fact already reflected in the notation.

The TMDA may be written in terms of the VDA as

$$\Psi(x, k_\perp^2) = \frac{i}{\pi} \int_0^\infty \frac{d\sigma}{\sigma} \Phi(x, \sigma) e^{-i(k_\perp^2 - i\epsilon)/\sigma} \quad (9)$$

The integrated TMDA

$$\mathfrak{f}(x, \mu^2) \equiv \pi \int_0^{\mu^2} dk_\perp^2 \Psi(x, k_\perp^2) \quad (10)$$

is analogous to the  $\mu^2$ -dependent pion distribution amplitude  $\phi(x, \mu^2)$  of the LF formalism (but, of course, being an object of the covariant QFT,  $\mathfrak{f}(x, \mu^2)$  does not coincide with it). In terms of the VDA,

$$\mathfrak{f}(x, \mu^2) = \int_0^\infty d\sigma \left[ 1 - e^{-i(\mu^2 - i\epsilon)/\sigma} \right] \Phi(x, \sigma) \quad (11)$$

Since it is defined by a straightforward cut-off,  $\mathfrak{f}(x, \mu^2)$  evolves with  $\mu^2$  even if the limit  $\mu^2 \rightarrow \infty$  is finite, e.g. in a super-renormalizable theory. The evolution equation

$$\mu^2 \frac{d}{d\mu^2} \mathfrak{f}(x, \mu^2) = \pi \mu^2 \Psi(x, \mu^2) \quad (12)$$

follows from the definition (10). When the TMDA  $\Psi(x, k_\perp^2)$  vanishes faster than  $1/k_\perp^2$  (such a TMDA will be called “soft”), evolution essentially stops at large  $\mu^2$ .

In a renormalizable theory, it makes sense to treat  $\Phi(x, \sigma)$  as a sum of a soft part  $\Phi^{\text{soft}}(x, \sigma)$ , generating a nonperturbative evolution of  $\mathfrak{f}(x, \mu^2)$ , and a  $\sim 1/\sigma$  hard tail. To avoid nonperturbative evolution, one may choose an  $\overline{\text{MS}}$ -type construction, e.g. regularize the  $\sigma$ -integral in Eq. (7) by a  $\sigma^{-\epsilon}$  factor and then subtract  $1/\epsilon$  poles.

However, just like in the LF formalism, the objects that appear in actual calculations are exactly the integrated TMDAs rather than their  $\overline{\text{MS}}$ -type sisters. In particular, the photon-pion transition form factor is given in the VDA approach by the  $x$ -integral of  $\mathfrak{f}(x, \mu^2)/[xQ^2]$  taken at  $\mu^2 = xQ^2$  [15], i.e., it involves TMDA  $\Psi(x, k_\perp^2)$  integrated over  $k_\perp^2$  till  $xQ^2$ . As a result, the TMDA formula has a finite  $Q^2 \rightarrow 0$  limit. Furthermore, using simple models for soft TMDAs one can get a very close description of experimental data by the nonperturbative evolution of the integrated TMDA [16].

For very large  $\mu^2$ , the perturbative evolution dominates and eventually brings  $\mathfrak{f}(x, \mu^2)$  to its asymptotic form  $6f_\pi x(1-x)$ . The question, however, is what kind of shape  $\mathfrak{f}(x, \mu^2)$  has at low scales  $\mu \sim 1$  GeV, and also how this shape changes with  $\mu^2$ . As we have discussed, this nonperturbative  $\mu^2$ -evolution reflects the  $k_\perp$  dependence of the soft part of the pion TMDA.

Below, we shall see that there is another function, the pion *quasi-distribution amplitude*  $Q_\pi(y, P)$  whose  $P$ -dependence is also determined by the  $k_\perp$ -dependence of the pion TMDA. The quasi-distributions have been introduced recently by X. Ji [2] to facilitate a calculation of light-front functions (PDFs, DAs, etc.) on the lattice.

### III. QUASI-DISTRIBUTION AMPLITUDE

#### A. Definition

The basic proposal of Ref. [2] is to consider equal-time bilocal operators corresponding to  $z = (0, 0, 0, z_3)$  [or, for brevity,  $z = z_3$ ]. Incorporating the VDA representation, we have

$$\langle 0|\psi(0)\psi(z_3)|p\rangle = \int_0^\infty d\sigma \int_{-1}^1 dx \Phi(x, \sigma) e^{ixp_3 z_3 + i\sigma z_3^2/4}. \quad (13)$$

Using again the frame in which  $p = (E, 0_\perp, P)$ , and introducing the pion quasi-distribution amplitude through

$$\langle 0|\psi(0)\psi(z_3)|p\rangle = \int_{-\infty}^\infty dy Q_\pi(y, P) e^{-iyPz_3}, \quad (14)$$

we get a relation between QDA and VDA,

$$Q_\pi(y, P) = \sqrt{\frac{iP^2}{\pi}} \int_0^\infty \frac{d\sigma}{\sqrt{\sigma}} \int_0^1 dx \Phi(x, \sigma) e^{-i(x-y)^2 P^2/\sigma}. \quad (15)$$

It is easy to see that, for large  $P$ , we have

$$\sqrt{\frac{iP^2}{\pi\sigma}} e^{-i(x-y)^2 P^2/\sigma} = \delta(x-y) + \frac{\sigma}{4P^2} \delta''(x-y) + \dots \quad (16)$$

and  $Q_\pi(y, P \rightarrow \infty)$  tends to the integral (7) leading to  $\varphi_\pi(y)$ . This observation suggests that one may be able to extract the “light-cone” distribution amplitude  $\varphi_\pi(y)$  from the studies of the purely “space-like” function  $Q_\pi(y, P)$  for large  $P$  [2].

#### B. Evolution

Again, to study the  $P$ -evolution of  $Q_\pi(y, P)$  it makes sense to split  $\Phi(x, \sigma)$  into the soft part, for which the integral over  $\sigma$  is finite, and the hard tail that generates perturbative evolution.

The nonperturbative evolution of  $Q_\pi^{\text{soft}}(y, P)$  with respect to  $P$  has the area-preserving property. Namely, since

$$\int_{-\infty}^\infty dy e^{-i(x-y)^2 P^2/\sigma} = \sqrt{\frac{\pi\sigma}{iP^2}} \quad (17)$$

we have

$$\int_{-\infty}^\infty dy Q_\pi^{\text{soft}}(y, P) = \int_0^1 dx \varphi_\pi^{\text{soft}}(x) = f_\pi. \quad (18)$$

In other words,  $Q_\pi^{\text{soft}}(y, P)$  for any  $P$  has the same area normalization as  $\varphi_\pi^{\text{soft}}(x)$ . In this respect, the pion QDA pleasantly differs from the integrated TMDA  $\mathfrak{f}^{\text{soft}}(x, \mu^2)$  whose zeroth moment is  $\mu^2$ -dependent.

Similarly, we have the momentum sum rule

$$\int_{-\infty}^\infty dy y Q_\pi^{\text{soft}}(y, P) = \int_0^1 dx x \varphi_\pi^{\text{soft}}(x). \quad (19)$$

### C. Relation to TMDA

Comparing the VDA representation (16) for  $Q_\pi(y, P)$  with that for the TMDA  $\Psi(x, k_\perp^2)$  (9) (note that they are valid both for soft and hard parts) we conclude that

$$Q_\pi(y, P) = \int_{-\infty}^\infty dk_1 \int_0^1 dx P \Psi(x, k_1^2 + (x-y)^2 P^2). \quad (20)$$

Thus, the quasi-distribution amplitude  $Q_\pi(y, P)$  is completely determined by the form of the TMDA  $\Psi(x, k_\perp^2)$ .

This formula may be also obtained if one takes  $z = (0, z_1, 0, z_3)$  in the VDA representation and introduces the momentum  $k_1$  conjugate to  $z_1$ . Then

$$\begin{aligned} & \int_{-\infty}^\infty dy e^{iyPz_3} \langle 0|\psi(0)\psi(z_1, z_3)|p\rangle \\ &= \int_{-\infty}^\infty dk_1 e^{-ik_1 z_1} \int_0^1 dx \Psi(x, k_1^2 + (x-y)^2 P^2). \end{aligned} \quad (21)$$

Taking  $z_1 = 0$  gives Eq. (20). Furthermore, introducing the variable  $k_3 \equiv (x-y)P$ , we have

$$Q_\pi(y, P) = \int_{-\infty}^\infty dk_1 \int_{-yP}^{(1-y)P} dk_3 \Psi(y + k_3/P, k_1^2 + k_3^2). \quad (22)$$

Thus,  $Q_\pi(y, P)$  is given by an integral over a stripe of width  $P$  in the 2-dimensional  $(k_1, k_3)$  plane. When  $P \rightarrow \infty$  for a fixed nonzero  $y$ , the stripe covers the whole  $(k_1, k_3)$  plane. Moreover, for a soft TMDA  $\Psi(x, k^2)$  that rapidly decreases outside a region  $k^2 \lesssim \Lambda^2$ , only the values of  $k_3 \lesssim \Lambda$  are essential, and for large  $P$  one may approximate the first argument of the TMDA by  $y$ . Hence, the  $P \rightarrow \infty$  limit gives  $\varphi_\pi^{\text{soft}}(y)$ .

For comparison, the integrated TMDA  $\mathfrak{f}(y, \mu^2)$  is obtained by integrating  $\Psi(y, k_\perp^2)$  over a circle of radius  $\mu$  in the  $k_\perp$  plane. Again, the circle covers the whole plane when  $\mu \rightarrow \infty$ , and  $\mathfrak{f}^{\text{soft}}(y, \mu^2) \rightarrow \varphi_\pi^{\text{soft}}(y)$ .

Thus, while the patterns of the nonperturbative evolution of  $Q_\pi^{\text{soft}}(y, P)$  and  $\mathfrak{f}^{\text{soft}}(y, \mu^2)$  are different, they become more and more close for large  $P$  and  $\mu$ , eventually producing the same function  $\varphi_\pi^{\text{soft}}(y)$ .

## IV. QCD

### A. Spinor quarks

In spinor case, one deals with the matrix element

$$B^\alpha(z, p) \equiv \langle 0|\bar{\psi}(0)\gamma_5 \gamma^\alpha \psi(z)|p\rangle. \quad (23)$$

It may be decomposed into  $p^\alpha$  and  $z^\alpha$  parts:  $B^\alpha(z, p) = p^\alpha B_p(z, p) + z^\alpha B_z(z, p)$ , or in the VDA representation

$$\begin{aligned} B^\alpha(z, p) &= \int_0^\infty d\sigma \int_{-1}^1 dx \\ &\times [p^\alpha \Phi(x, \sigma) + z^\alpha Z(x, \sigma)] e^{-ix(pz) - i\sigma(z^2 - i\epsilon)/4}. \end{aligned} \quad (24)$$

If we take  $z = (z_-, z_\perp)$  in the  $\alpha = +$  component of  $\mathcal{O}^\alpha$ , the purely higher-twist  $z^\alpha$ -part drops out and we can introduce the TMDA  $\Psi(x, k_\perp^2)$  that is related to the VDA  $\Phi(x, \sigma)$  by the scalar formula (9).

In the QDA case, the easiest way to avoid the effects of the  $z^\alpha$  admixture is to take the time component of  $B^\alpha(z = z_3, p)$  and define

$$B^0(z_3, p) = p^0 \int_{-1}^1 dx Q_\pi(y, P) e^{iyPz_3} . \quad (25)$$

The connection between  $Q_\pi(y, P)$  and  $\Phi(x, \sigma)$  is given then by the same formula (15) as in the scalar case. As a result, we have the sum rules (18) and (19) corresponding to charge and momentum conservation. Furthermore, the quasi-distribution amplitude  $Q_\pi(y, P)$  is related to TMDA  $\Psi(x, k_\perp^2)$  by the scalar conversion formula (20).

## B. Gauge fields

In QCD, for  $\pi^+$  one should take the operator

$$\mathcal{O}^\alpha(0, z; A) \equiv \bar{d}(0) \gamma_5 \gamma^\alpha \hat{E}(0, z; A) u(z) \quad (26)$$

involving a straight-line path-ordered exponential

$$\hat{E}(0, z; A) \equiv P \exp \left[ ig z_\nu \int_0^1 dt A^\nu(tz) \right] \quad (27)$$

in the quark (adjoint) representation. As is well-known, its Taylor expansion has the same structure as that for the original  $\bar{\psi}(0) \gamma_5 \gamma^\alpha \psi(z)$  operator, with the only change that one should use covariant derivatives  $D^\nu = \partial^\nu - ig A^\nu$  instead of the ordinary  $\partial^\nu$  ones.

Again, the  $z^\alpha$  admixture is avoided if the pion quasi-distribution amplitude is defined through the time component of  $\mathcal{O}^\alpha$ . Then we have the same relation between the VDA and QDA as in the scalar case. Due to Eq. (18), this results in the area preserving property for the soft part

$$\int_{-\infty}^{\infty} dy Q^{\text{soft}}(y, P) = f_\pi . \quad (28)$$

Also, due to Eq. (19) we have the momentum sum rule

$$\int_{-\infty}^{\infty} dy (y - 1/2) Q_q^{\text{soft}}(y, P) = 0 . \quad (29)$$

Since the VDA  $\Phi(x, \sigma)$  is defined through the matrix element of a gauge-invariant operator, it is gauge-invariant also. For this reason, TMDA  $\Psi(x, k_\perp^2)$  is a gauge-invariant object as well. It should not be confused with the  $k_T$ -dependent (and gauge-dependent) “underintegrated distributions” that appear in perturbative loop calculations based on Sudakov decomposition of the integration momentum  $k$ .

## V. MODELS FOR SOFT PART

### A. Models

To get an idea about patterns of the nonperturbative evolution of the QDAs, we need some explicit models of the  $k_\perp$  dependence of soft TMDAs  $\Psi(x, k_\perp^2)$ . We will use here the same models as in our papers [14, 16]. While TMDAs are functions of two independent variables  $x$  and  $k_\perp^2$ , we take, for simplicity, the case of factorized models

$$\Psi(x, k_\perp^2) = \varphi_\pi(x) \psi(k_\perp^2) , \quad (30)$$

in which  $x$ -dependence and  $k_\perp$ -dependence appear in separate factors.

If we assume a Gaussian dependence on  $k_\perp$ ,

$$\Psi^G(x, k_\perp^2) = \frac{\varphi_\pi(x)}{\pi \Lambda^2} e^{-k_\perp^2/\Lambda^2} , \quad (31)$$

the conversion formula (20) results in

$$Q_\pi^G(y, P) = \frac{P}{\Lambda \sqrt{\pi}} \int_0^1 dx \varphi_\pi(x) e^{-(x-y)^2 P^2/\Lambda^2} . \quad (32)$$

In the space of impact parameters  $z_\perp$ , the Gaussian model gives a  $e^{-z_\perp^2 \Lambda^2/4}$  fall-off that is too fast for large  $z_\perp$ . As an alternative extreme case, we take a model with the  $1/(1 + z_\perp^2 \Lambda^2/4)$  dependence on  $z_\perp$ , whose fall-off at large  $z_\perp$  is too slow. It corresponds to the “slow” model for the TMDA

$$\Psi^S(x, k_\perp^2) = 2\varphi_\pi(x) \frac{K_0(2|k_\perp|/\Lambda)}{\pi \Lambda^2} \quad (33)$$

that has a logarithmic singularity for small  $k_\perp$  reflecting a too slow fall-off for large  $z_\perp$ . For the QDA, we have

$$Q_\pi^S(y, P) = \frac{P}{\Lambda} \int_0^1 dx \varphi_\pi(x) e^{-2|x-y|P/\Lambda} . \quad (34)$$

Note that the Gaussian model and the “slow” model have the same  $\sim (1 - z_\perp^2 \Lambda^2/4)$  behavior for small  $z_\perp$ , i.e. they correspond to the same value of the  $\langle 0 | \varphi(0) \partial^2 \varphi(0) | p \rangle$  matrix element (in the scalar case), provided that one takes the same value of  $\Lambda$  in both models. For large  $z_\perp$ , however, the Gaussian model has a fall-off that is too fast, while the fall-off of the “slow” model is too slow. Thus, they look like two extreme cases, and provide a good illustration of the nonperturbative evolution of the pion QDA, with expectation that other models would produce results somewhere in between these two cases.

### B. Numerical Results

To compare evolution patterns induced by the Gaussian and “slow” models, we take the Ansatz (30) with  $\varphi_\pi(x)$  having a drastic shape of the Chernyak-Zhitnitsky DA  $\varphi_\pi^{\text{CZ}}(x) = 30 f_\pi x(1-x)(1-2x)^2$ . As one can see

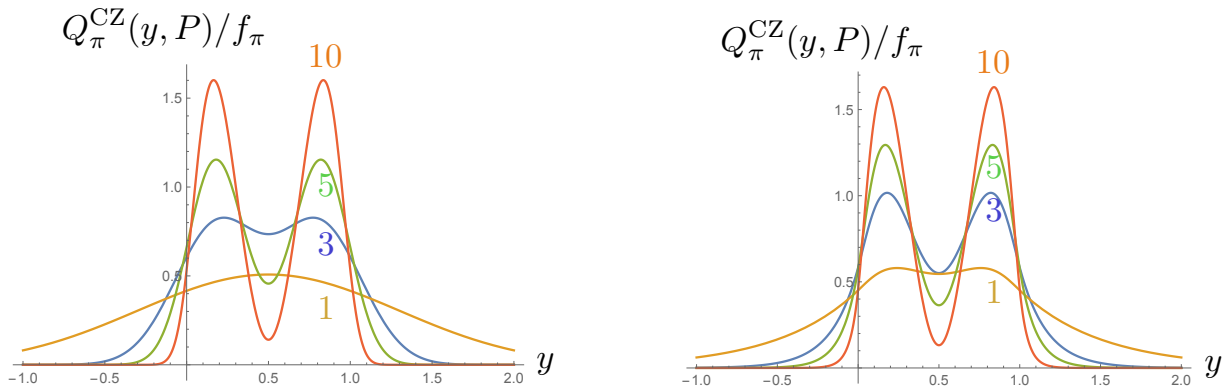


FIG. 1. Quasi-distribution amplitude  $Q_\pi^{\text{CZ}}(y, P)$  for  $P/\Lambda = 1, 3, 5, 10$  in the Gaussian (left) and “slow” models (right).

from Fig. 1, for  $P/\Lambda = 1$  the Gaussian model shows no indication of humps visible for higher  $P/\Lambda$  ratios. In the “slow” model, small humps are present even for  $P/\Lambda = 1$ . For high ratios  $P/\Lambda = 5$  and  $10$ , the two models give close results, with strong humps.

Assuming  $\Lambda \sim 0.6$  GeV suggested by the VDA-based fits of the photon-pion transition form factor in Ref. [16], we expect that  $P \sim 3$  GeV would be required to support (or rule out) the CZ-type shape of the pion DA.

It is also interesting to note that the nonperturbative evolution pattern here is exactly opposite to the perturbative one. In the latter case, the humps of the initially CZ-shaped DA become less pronounced as the normalization scale increases and eventually disappear, with the DA tending to the asymptotic  $\sim x(1-x)$  shape.

To compare patterns of the QDA’s nonperturbative evolution for different shapes of the limiting DA, we take three models for  $\varphi_\pi(x)$ : Chernyak-Zhitnitsky  $\varphi_\pi^{\text{CZ}}(x)$ , flat  $\varphi_\pi^{\text{flat}}(x) = f_\pi$  and asymptotic  $\varphi_\pi^{\text{as}}(x) = 6f_\pi x(1-x)$ . The results in the Gaussian and the “slow” models are rather similar. To avoid plotting too many graphs, we take, for definiteness, the “slow” model. Then, for the

flat limiting DA we have

$$\frac{1}{f_\pi} Q_\pi^{\text{S,flat}}(y, P) = \frac{P}{\Lambda} \int_0^1 dx e^{-2|x-y|P/\Lambda}. \quad (35)$$

This integral can be calculated analytically. Writing  $y = (1 + \eta)/2$  in terms of a symmetric variable  $\eta$ , we get

$$\begin{aligned} \frac{1}{f_\pi} Q_\pi^{\text{S,flat}}(y, P) = & \left(1 - e^{-P/\Lambda} \cosh(P\eta/\Lambda)\right) \theta(|\eta| \leq 1) \\ & + \sinh(P/\Lambda) e^{-P|\eta|/\Lambda} \theta(|\eta| \geq 1). \end{aligned} \quad (36)$$

Similar, but more lengthy expressions may be obtained for two other models. As one can see from Fig. 2, for small  $P = \Lambda$  we have very close curves. For larger  $P = 3\Lambda$  the difference becomes visible, and for large  $P = 5\Lambda$  and  $P = 10\Lambda$  the curves shown in Fig. 3 are distinctly different. In fact, the  $P = 10\Lambda$  curves are very close to their limiting forms. Again, the nonperturbative evolution pattern in case of the flat DA is opposite to the perturbative one: as  $P$  increases,  $Q_\pi^{\text{flat}}(y, P)$  broadens from a rather narrow function for  $P = \Lambda$  and becomes almost constant for  $P = 10\Lambda$ .

## VI. LEADING-ORDER HARD TAIL

The nonperturbative evolution of  $Q_\pi(y, P)$  essentially stops for  $P/\Lambda \gtrsim 20$ , and for larger values of  $P$  the dominant role is played by the perturbative evolution induced by the hard part. In our papers [15, 16], it was suggested to take a purely soft TMDA (or VDA) as a starting approximation, and then “generate” hard tail by adding one-gluon exchanges. The only new parameter is the overall factor  $\alpha_s$ , while the  $k_\perp$ -dependence of the hard tail of the TMDA  $\Psi(x, k_\perp^2)$  is completely determined by the soft part.

For large  $k_\perp$ , the generated hard part of the TMDA

has a  $\sim 1/k_\perp^2$  behavior, but its explicit functional form is much more complicated. In particular, it is finite in the  $k_\perp \rightarrow 0$  limit [16]. The infrared cut-off for the naive  $1/k_\perp^2$  extrapolation is provided by the finite size of the pion encoded in the parameters, like  $\Lambda$ , present in the soft TMDA. Postponing the analysis of the interplay between the nonperturbative and perturbative evolution for future studies, we just outline below the VDA treatment of the hard tail.

For large  $\sigma$ , the lowest-order (in  $\alpha_s$ ) hard tail has the form

$$\Phi^{\text{hard}}(x, \sigma) = \Delta(x)/\sigma, \quad (37)$$

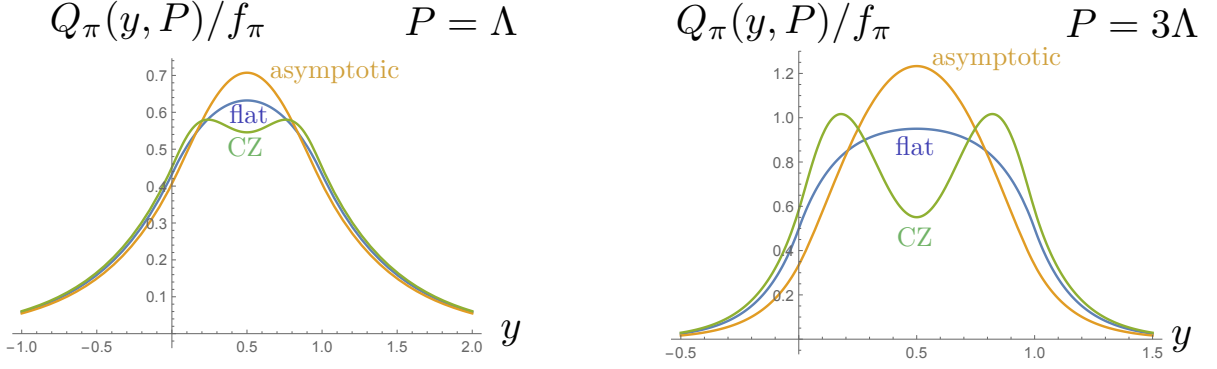


FIG. 2. Quasi-distribution amplitudes  $Q_\pi(y, P)$  in the “slow” model for  $P = \Lambda$  (left) and  $P = 3\Lambda$  (right) evolving to CZ, flat and asymptotic DAs.

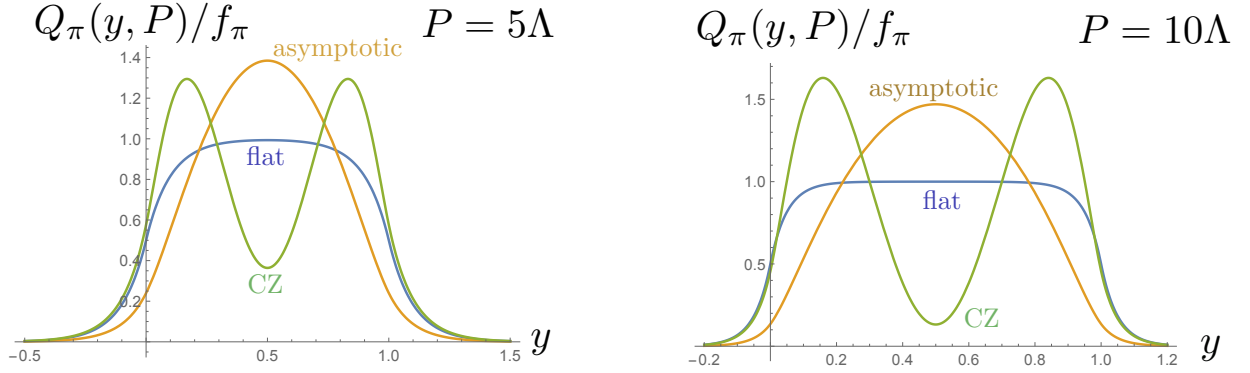


FIG. 3. Quasi-distribution amplitudes  $Q_\pi(y, P)$  in the “slow” model for  $P = 5\Lambda$  (left) and  $P = 10\Lambda$  (right) evolving to CZ, flat and asymptotic DAs.

with  $\Delta(x)$  given by

$$\Delta(x) = \int_0^1 dz V(x, z) \varphi_\pi^{\text{soft}}(z), \quad (38)$$

where  $V(x, z)$  is the perturbative evolution kernel [21–23]. The asymptotic form (37) corresponds to a  $\sim 1/k_\perp^2$  TMDA, which is singular for  $k_\perp = 0$ . As explained above, this singularity is absent in the exact (rather complicated) expression for the hard tail. For illustration purposes, we take now the simplest regularization  $1/k_\perp^2 \rightarrow 1/(k_\perp^2 + m^2)$ . It corresponds to the change  $1/\sigma \rightarrow e^{-im^2/\sigma}/\sigma$  in the hard part of VDA,

$$\Phi^{\text{hard}}(x, \sigma) \rightarrow \frac{\Delta(x)}{\sigma} e^{-im^2/\sigma}. \quad (39)$$

To proceed with the conversion formula, one needs the

integral over  $\sigma$

$$I(x, y, P) = \int_0^\infty \frac{d\sigma}{\sqrt{\pi\sigma}} \frac{P}{\sigma} e^{-(x-y)^2 P^2/\sigma - m^2/\sigma} = \frac{1}{\sqrt{(x-y)^2 + m^2/P^2}}. \quad (40)$$

This gives the hard part of the quasi-distribution amplitude

$$Q_\pi^{\text{hard}}(y, P) = \int_0^1 dx \frac{\Delta(x)}{\sqrt{(x-y)^2 + m^2/P^2}}. \quad (41)$$

It generates evolution with respect to  $P^2$  in the form

$$P^2 \frac{d}{dP^2} Q_\pi^{\text{hard}}(y, P) = \frac{m^2}{2P^2} \int_0^1 dx \frac{\Delta(x)}{[(x-y)^2 + m^2/P^2]^{3/2}}. \quad (42)$$



Taking the  $m/P \rightarrow 0$  limit we have

$$\begin{aligned} & \frac{m^2}{2P^2} \int_0^1 dx \frac{V(x, z)}{[(x-y)^2 + m^2/P^2]^{3/2}} \\ & = V(y, z) + \mathcal{O}(m^2/P^2), \end{aligned} \quad (43)$$

i.e. for large  $P^2$  the quasi-distribution amplitude evolves according to the perturbative evolution equation with respect to  $P^2$ . This evolution is completely determined by the form of the soft DA  $\varphi^{\text{soft}}(x)$ . When the model for the latter is fixed, the particular choice of the soft TMDA  $\Psi^{\text{soft}}(x, k_\perp^2)$  does not affect the form of the hard part and the perturbative  $P$ -evolution of the pion QDA  $Q(y, P)$ .

## VII. CONCLUSIONS

In this paper, we extended the approach of Ref. [14], where we have been dealing with the parton distribution functions, the basic ingredients of perturbative QCD analysis of hard inclusive processes. Now we have dealt with the pion distribution amplitude, the basic ingredient of hard exclusive processes involving the pion. We applied the formalism of virtuality distribution amplitudes to study the  $p_3$ -dependence of quasi-distribution amplitudes  $Q_\pi(y, p_3)$ .

Just like in Ref. [14], we have established a simple relation between QDAs and TMDAs that allows to derive models for QDAs from the models for TMDAs. Unlike the PDF case, there are many drastically different models claimed to describe the primordial shape of the pion DA. We have presented the  $p_3$ -evolution patterns for models producing some popular proposals: Chernyak-Zhitnitsky, flat and asymptotic DAs. Our results may be used as a guide for future studies of the pion distribution amplitude on the lattice using the quasi-distribution approach.

As our estimates show, one would need  $P$  of the order of a few GeV for the nonperturbative evolution to settle. It is natural to expect that perturbative evolution will be rather important at such scales. Thus, an interesting and technically challenging question for future studies is the interplay between the nonperturbative and perturbative evolution of the pion quasi-distribution amplitude.

Another interesting problem for future studies is the analysis of more complicated models for TMDAs, in particular, models with non-factorized  $k_\perp$  and  $x$ -dependence. As the simplest generalization of the models used in the present paper, one can take  $x$ -dependent functions  $\Lambda(x)$  instead of the constant values  $\Lambda$ .

## ACKNOWLEDGEMENTS

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