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Quasiparticle second-order viscous hydrodynamics from kinetic theory

Leonardo Tinti,[1] Amaresh Jaiswal,[2] and Radoslaw Ryblewski[3]

[1] *Department of Physics, The Ohio State University, Columbus, OH 43210-1117, USA*
[2] *GSI, Helmholtzzentrum für Schwerionenforschung, Planckstrasse 1, D-64291 Darmstadt, Germany*
[3] *Institute of Nuclear Physics, Polish Academy of Sciences, PL-31342 Kraków, Poland*

We present the derivation of second-order relativistic viscous hydrodynamics from an effective Boltzmann equation for a system consisting of quasiparticles of a single species. We consider temperature-dependent masses of the quasiparticles and devise a thermodynamically-consistent framework to formulate second-order evolution equations for shear and bulk viscous pressure corrections. The main advantage of this formulation is that one can consistently implement realistic equation of state of the medium within the framework of kinetic theory. Specializing to the case of one-dimensional purely-longitudinal boost-invariant expansion, we study the effect of this new formulation on viscous hydrodynamic evolution of strongly-interacting matter formed in relativistic heavy-ion collisions.

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I. INTRODUCTION

Relativistic fluid dynamics has been widely applied to study space-time evolution of the strongly-interacting, hot and dense matter created in ultra-relativistic heavy-ion collisions at the Relativistic Heavy-Ion Collider (RHIC) in BNL and the Large Hadron Collider (LHC) in CERN; for reviews see [1–5]. Multiple successes of the pioneering studies employing ideal fluid dynamics to describe experimental data at RHIC led to establishing the paradigm of perfect-fluidity of the created new phase of matter called the quark-gluon plasma (QGP) [6] and raised questions about its fast thermalization [7]; see also §1.3 in Ref. [8]. The universal existence of the viscous effects in nature [9–11] as well as the increasing precision of flow measurements at the LHC suggested necessity of inclusion of the dissipative effects in the fluid dynamic modelling and resulted in a rapid development of the formulation of relativistic viscous hydrodynamics [12–28]. Much of the research in the field of heavy-ion collisions has been devoted to the extraction of the thermodynamic and transport properties of the QGP medium encoded mainly in its equation of state, shear viscosity η, and bulk viscosity ζ [29–38].

The successes of viscous fluid dynamics in explaining a wide range of collective phenomena observed in high-energy heavy-ion collisions have been initially attributed to the proximity of the created QGP state to the local thermodynamic equilibrium. This assumption plays a key role in the formulation of the theory of relativistic hydrodynamics as it is usually constructed as an order-by-order expansion around equilibrium state in powers of thermodynamic gradients, where ideal hydrodynamics is of zeroth order. The first-order relativistic Navier-Stokes theory [39, 40] involves parabolic differential equations and therefore suffers from acausality and instability. The second-order Israel-Stewart theory [41–43] leads to hyperbolic equations which restores causality but may not guarantee stability [44, 45].

The procedure outlined above indirectly sets the formal applicability limits of the resulting theory, making it questionable when the dissipative corrections are large [46]. Interestingly, recent findings within the gauge/gravity duality framework [47], as well as, in the effective kinetic theory approach [48] show that, in practice, the relativistic viscous hydrodynamics behavior in heavy-ion collisions sets in quite early, even though the system is far from equilibrium – the phenomenon commonly known as the hydrodynamization of the system. These observations suggest that the applicability of the viscous hydrodynamics may be broader than previously expected.

In order to describe the collective behaviour of the QGP within fluid dynamics, one needs to incorporate its properties through the transport coefficients and equation of state. In principle, the two should be treated as an external input in the hydrodynamic equations, and extracted from the experimental data. As, in general, it is a highly non-trivial task, one usually follows different methodology. Since the full information on the properties of nuclear matter produced in heavy-ion collisions should follow from the fundamental theory of strong interactions, namely Quantum Chromodynamics (QCD), one may try to incorporate into hydrodynamics the results of ab-initio calculations of thermodynamic and transport quantities performed in the framework of lattice QCD (lQCD) [49–53]. However, it turns out that in contrast to the classical Navier-Stokes theory the form of the relativistic viscous hydrodynamics equations is not universal and strongly depends on the underlying microscopic theory used to derive them [12–28, 54–59]. Moreover, in the phenomenologically interesting regime the lQCD calculations of transport properties are still plagued by large uncertainties [60, 61].

In view of the above problems, one typically resorts to a simple microscopic theory, such as the kinetic theory, to derive the hydrodynamic evolution equations. Subsequently, in order to incorporate realistic properties of the strongly-interacting matter in the hydrodynamic evolution, equation of state and transport coefficients obtained from lQCD are implemented. However, the latter procedure inadvertently fixes the parameters of the microscopic theory, introducing effective interactions which were not taken into account in the derivation of the hydrodynamic evolution equations. These inconsistencies eventually lead to the violation of thermodynamic relations in such a system.

In the specific case of relativistic kinetic theory for a single particle species, the equation of state is fixed by the particle mass. Using this simple formalism it is not possible to fit exactly the temperature scaling of energy density and pressure given by lQCD as one may only select the most convenient effective mass. In order to improve the fit, one may consider temperature (medium) dependent particle masses, which, in a thermodynamically consistent framework, lead to a non-ideal equation of state [62–65]. This is achieved by introducing an extra contribution to the energy-momentum tensor which can be physically interpreted as a mean field. While this procedure is equivalent to the introduction of the notion of interacting quasiparticles in the microscopic theory, only in specific limits they can be considered actual quasiparticle excitation of the fundamental theory [63].

Once an effective quasiparticle kinetic theory is constructed which can reproduce any thermodynamically consistent equation of state, hydrodynamic evolution equations should be derived by coarse graining this theory. To the best of our knowledge, this has not yet been done. In this paper, we derive evolution equations for second-order viscous hydrodynamics for a system of single species of quasiparticles from an effective Boltzmann equation in the relaxation time approximation. Specializing to the case of transversally homogeneous and boost-invariant longitudinal expansion, we study the effect of this new formulation on the evolution of strongly interacting matter formed in high energy heavy-ion collisions.

II. QUASIPARTICLE THERMODYNAMICS

Let us consider a system of ideal (non-interacting) uncharged particles of a single species. Within kinetic theory, the equation of state of such a system depends parametrically only on the mass of the particle [66]. In order to describe an arbitrary equation of state, one can consider temperature-dependent masses of the particles, $m(T)$ [67]. Such an idea may be physically sound, for instance, when considering high-temperature QCD. In this case resummed perturbative calculations suggest that the system is made of partons with thermal masses $m(T) = g_s T$, where g_s is the strong coupling [68]. However, these particles do not correspond to any real excitations of the underlying fundamental theory, especially close to the crossover region. While introducing a temperature-dependent mass leads to some degree of freedom in choosing the equation of state of the system, unfortunately, it violates basic thermodynamic identities [62].

A possible way to restore thermodynamic consistency at global equilibrium is to introduce additional effective mean field through a bag function $B_0(T)$ to account for the fundamental interactions giving rise to the in-medium masses [62]. The latter may be included in the Lorentz covariant way by modifying the definition of the equilibrium energy-momentum tensor [63, 69–72]

$$T_0^{\mu\nu} = \int dP\, p^\mu p^\nu\, f_0 \,+ B_0(T)\, g^{\mu\nu}, \tag{1}$$

where $g^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ is the metric tensor, f_0 is the equilibrium distribution function, p^μ is the quasiparticle four-momentum and $B_0(T)$, called the bag pressure/energy, is a function which can be determined by requiring thermodynamic consistency. In the above equation, dP is the Lorentz covariant momentum integration measure defined as

$$\int dP = \int \frac{d^4p}{(2\pi)^4}\, 2\,\Theta(p\cdot t)\,(2\pi)\,\delta(p^2-m^2), \tag{2}$$

where Θ is the Heaviside step function, t^μ is an arbitrary time-like four-vector, $p\cdot t \equiv p^\mu g_{\mu\nu} t^\nu$ and $p^2 \equiv p\cdot p$.

The equilibrium energy density and pressure can be defined as

$$\mathcal{E}_0 = u\cdot T_0 \cdot u, \quad \mathcal{P}_0 = -\frac{1}{3}\Delta : T_0, \tag{3}$$

where $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ with $\Delta^{\mu\nu}u_\nu = 0$ and u^μ, which will be defined in Sec. III, is the fluid four-velocity satisfying $u^2 = 1$. We also introduced the following notation $A : B \equiv A^{\mu\nu}B_{\mu\nu}$ for the Frobenius product. The thermodynamic relation

$$\frac{d\mathcal{P}_0}{dT} = \frac{\mathcal{E}_0 + \mathcal{P}_0}{T}, \tag{4}$$

is guaranteed to be satisfied if [62]

$$dB_0 + m\, dm \int dP\ f_0 = 0. \tag{5}$$

The latter relation can be seen explicitly for the case of Maxwell-Boltzmann distribution [73],

$$f_0 = g\exp[-\beta(u\cdot p)], \tag{6}$$

where g is the degeneracy factor and $\beta \equiv 1/T$. In this case the equilibrium energy density and pressure is given by

$$\mathcal{E}_0 = \frac{gT^4z^2}{2\pi^2}\Big[3K_2(z) + zK_1(z)\Big] + B_0 \tag{7}$$

$$\mathcal{P}_0 = \frac{gT^4z^2}{2\pi^2}K_2(z) - B_0, \tag{8}$$

where $K_n(z)$ are the modified Bessel functions of second kind of order n and $z \equiv m/T$. The temperature derivative of pressure is given by

$$\begin{aligned}\frac{d\mathcal{P}_0}{dT} &= \frac{gT^3z^2}{2\pi^2}\left[4K_2(z) + zK_1(z) - \frac{dm}{dT}\,K_1(z)\right] - \frac{dB_0}{dT}\\ &= \frac{\mathcal{E}_0+\mathcal{P}_0}{T} - \left(\frac{dB_0}{dT} + m\,\frac{dm}{dT}\int dP f_0\right),\end{aligned} \tag{9}$$

which leads to Eq. (5) in order to satisfy the thermodynamic relation given in Eq. (4).

III. NON-EQUILIBRIUM AND BOLTZMANN EQUATION

A. Off-equilibrium mean field and the energy-momentum conservation

For the general non-equilibrium case, we propose the energy-momentum tensor of the form

$$T^{\mu\nu} = \int dP\, p^\mu p^\nu\, f \;+ B^{\mu\nu}, \tag{10}$$

where in equilibrium we require $B^{\mu\nu}|_{\rm eq} = B_0 g^{\mu\nu}$. With $f \to f_0$, the above equation reduces to Eq. (1) in equilibrium. We now split $B^{\mu\nu}$ into equilibrium and non-equilibrium parts,

$$B^{\mu\nu} = B_0\, g^{\mu\nu} + \delta B^{\mu\nu}, \tag{11}$$

where $\delta B^{\mu\nu}$ has to be fixed by requiring energy and momentum conservation. The dissipative quantities are defined as

$$\Pi \equiv -\frac{1}{3}\Delta : (T - T_0)\,, \tag{12}$$

$$\pi^{\mu\nu} \equiv \Delta^{\mu\nu}_{\alpha\beta}\left(T^{\alpha\beta} - T_0^{\alpha\beta}\right) = \Delta^{\mu\nu}_{\alpha\beta}\, T^{\alpha\beta}, \tag{13}$$

where Π is the bulk, $\pi^{\mu\nu}$ is the shear pressure correction, and $\Delta^{\mu\nu}_{\alpha\beta} = \frac{1}{2}(\Delta^\mu_\alpha\Delta^\nu_\beta + \Delta^\mu_\beta\Delta^\nu_\alpha - \frac{2}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta})$ is the symmetric traceless projector orthogonal to u^μ.

The energy-momentum conservation equation requires that the four divergence of energy-momentum tensor should vanish, i.e., $\partial_\mu T^{\mu\nu} = 0$. The four-divergence of $T^{\mu\nu}$ in Eq. (10) reads

$$\partial_\mu T^{\mu\nu} = \partial_\mu B^{\mu\nu} + \int \frac{d^4p}{(2\pi)^3}\, 2\Theta(p\cdot t)(p\cdot\partial)\delta(p^2 - m^2)\, p^\nu f + \int dP\, p^\nu (p\cdot\partial) f. \tag{14}$$

Rewriting the Dirac delta function in the above equation,

$$(p\cdot\partial)\delta(p^2 - m^2) = m\,(\partial_\mu m)\left[-\partial^\mu_{(p)}\delta(p^2 - m^2)\right], \tag{15}$$

where $\partial^\mu_{(p)}$ is the gradient with respect to the quasiparticle momenta, the energy-momentum conservation leads to

$$\partial_\mu B^{\mu\nu} + m\,\partial^\nu m \int dP f + \int dP\, p^\nu \left[(p\cdot\partial) f + m\,(\partial^\rho m)\,\partial^{(p)}_\rho\, f\right] = 0. \tag{16}$$

The Boltzmann equation for temperature-dependent particle masses can be written as [63, 74]

$$(p\cdot\partial) f + m\,(\partial^\rho m)\,\partial^{(p)}_\rho\, f = \mathcal{C}[f], \tag{17}$$

where $\mathcal{C}[f]$ is the collision kernel. Substituting the above equation in Eq. (16), we obtain

$$\partial_\mu B^{\mu\nu} + m\,\partial^\nu m \int dP f + \int dP\, p^\nu \mathcal{C}[f] = 0. \tag{18}$$

Note that the last term in the above equation corresponds to the first moment of the collision kernel, which may not vanish for a system of quasiparticles with temperature-dependent masses[1].

Using the equilibrium condition, Eq. (5), we write the four-momentum conservation requirement for non-equilibrium quantities

$$\partial_\mu \delta B^{\mu\nu} + m\,\partial^\nu m \int dP\,\delta f + \int dP\, p^\nu \mathcal{C}[f] = 0, \tag{19}$$

where $\delta f \equiv f - f_0$. In the present calculation, we assume the collision kernel to be given by the relaxation-time approximation [75],

$$\mathcal{C}[f] = -\frac{(u\cdot p)}{\tau_R}\delta f, \tag{20}$$

where τ_R is the relaxation time.

In order to have a well defined kinetic framework, it is necessary to define the fluid four-velocity u^μ and effective temperature T. Since in the present work we do not consider conserved charges, the natural choice for the fluid four-velocity is the Landau frame definition [75]

$$u_\mu T^{\mu\nu} = \mathcal{E} u^\nu. \tag{21}$$

It is important to note that in the present quasiparticle framework, the collision kernel defined in the relaxation-time approximation, Eq. (20), does not fix the local rest frame. Indeed, even in equilibrium, the mean field contribution to the energy and momentum B_0 can be significant. The quasiparticle excitations can exchange four-momentum with the mean fields and therefore they are not required to satisfy vanishing first moment of the collision kernel. Sources of four momentum from the kinetic contribution of $T^{\mu\nu}$ are acceptable as long as they are compensated with an opposite source on the dynamic $B^{\mu\nu}$ component.

Temperature of the system is defined through the matching condition

$$\mathcal{E} = \mathcal{E}_0 \quad \Rightarrow \quad \int dP (p\cdot u)^2 \delta f + u\cdot\delta B\cdot u = 0, \tag{22}$$

which is equivalent to fixing the effective local temperature T in order to reproduce the energy density of the system using the equilibrium relations. As a consequence,

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[1] Quasiparticles may not be the only carriers of energy and momentum.

the first-moment of the collision kernel in the relaxation-time approximation can be written as

$$-\frac{1}{\tau_R}\int dP\,(p\cdot u)p^\nu\delta f = \frac{1}{\tau_R}\,u_\mu\delta B^{\mu\nu}. \tag{23}$$

Therefore, the condition to satisfy energy and momentum conservation, Eq. (19), becomes

$$\partial_\mu\delta B^{\mu\nu} + m\,\partial^\nu m\int dP\,\delta f + \frac{1}{\tau_R}u_\mu\delta B^{\mu\nu} = 0. \tag{24}$$

After some straightforward algebra it is possible to show that for $\delta B^{\mu\nu}=0$, the above equation reduces to

$$-3\,m\,(\partial^\nu m)\,\Pi = 0. \tag{25}$$

Since we are interested in the case of temperature-dependent particle masses where bulk viscosity is non-vanishing, we conclude that, in general, $\delta B^{\mu\nu}\neq 0$ in order to keep the energy and momentum conserved.

We note that in Ref. [76] (see also Ref. [77]), the Authors also consider the out-of-equilibrium contribution, $\delta B^{\mu\nu}\neq 0$, in order to obtain a set of equations for anisotropic hydrodynamics within a quasiparticle kinetic framework. They assume $\delta B^{\mu\nu}$ to be proportional to the metric tensor, similar to the equilibrium case, for which the first moment of the collision kernel vanishes. While this is a convenient choice for anisotropic hydrodynamics at the leading order, where one can make certain approximations on the shape of the distribution function, in the present paper we prefer to have an extended algebra in order to fix $\delta B^{\mu\nu}$ by using only the constraints obtained from energy-momentum conservation. Moreover, in Refs. [76], the definition of effective temperature out of equilibrium fixes the kinetic contribution to the proper energy density (the quasiparticle contribution, opposed to the bag contribution, which can be interpreted as mean fields). Since this kind of quasiparticle do not correspond to any known excitation, we prefer not to use them to define the effective temperature. We therefore consider a standard Landau matching in which the effective temperature is defined using the full energy density.

B. Ansatz for $\delta B^{\mu\nu}$ and its evolution equations

In order to specify the form of $\delta B^{\mu\nu}$, we first note that, in general, the symmetry of $T^{\mu\nu}$ restricts $\delta B^{\mu\nu}$ to have ten independent components. However, the conservation of energy and momentum leads to only four constraints. Therefore one has to reduce the number of independent degrees of freedom in $\delta B^{\mu\nu}$ to four. In the present work, we make an *ansatz* for $\delta B^{\mu\nu}$ of the form,

$$\delta B^{\mu\nu} = b_0\,g^{\mu\nu} + u^\mu b^\nu + b^\mu u^\nu, \tag{26}$$

where b^μ is orthogonal to the fluid four-velocity, *i.e.* $u\cdot b=0$. Therefore, the matching condition, Eq. (22), leads to

$$\int dP(p\cdot u)^2\delta f = -b_0, \tag{27}$$

while the definition of bulk viscous pressure and shear-stress tensor in Eqs. (12) and (13) reduces to

$$\Pi = -\frac{1}{3}\Delta_{\alpha\beta}\int dP p^\alpha p^\beta\delta f - b_0, \tag{28}$$

$$\pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta}\int dP p^\alpha p^\beta\delta f. \tag{29}$$

It is interesting to observe that in the above equations, compared to its usual definition for a system of particles with constant masses, the definition of bulk viscous pressure receives a correction from $\delta B^{\mu\nu}$. On the other hand, the definition of shear-stress tensor remains unchanged.

For non-vanishing masses, Eq. (28) leads to

$$\int dP\,\delta f = -\frac{1}{m^2}\Big(3\Pi + 4b_0\Big). \tag{30}$$

Using the above relation along with Eq. (26) in the four-momentum conservation requirement for non-equilibrium quantities, Eq. (24), we get

$$\partial^\nu b_0 + \theta\,b^\nu + \dot b^\nu + (\partial\cdot b)u^\nu + (b\cdot\partial)u^\nu - \frac{\partial^\nu m}{m}\Big(3\Pi+4b_0\Big) + \frac{1}{\tau_R}\Big(b_0\,u^\nu + b^\nu\Big) = 0, \tag{31}$$

where we have defined $\dot A\equiv u\cdot\partial A$ and $\theta\equiv\partial\cdot u$. The projection along and orthogonal to u^μ leads to a set of relaxation-type equations for the components of $\delta B^{\mu\nu}$

$$\dot b_0 + \frac{b_0}{\tau_R} = \frac{\dot m}{m}\Big(3\Pi+4b_0\Big) + b\cdot\dot u - \partial\cdot b, \tag{32}$$

$$\dot b^{\langle\mu\rangle} + \frac{b^\mu}{\tau_R} = \frac{\nabla^\mu m}{m}\Big(3\Pi+4b_0\Big) - \nabla^\mu b_0 - \theta\,b^\mu - (b\cdot\partial)u^\mu, \tag{33}$$

where $\nabla^\mu\equiv\Delta^{\mu\nu}\partial_\nu$ and $\dot b^{\langle\mu\rangle}=\Delta^\mu_\nu\dot b^\nu$. It is interesting to note that in the above equations, the relaxation time for b_0 and b^μ is the same as the Boltzmann relaxation time.

In order to replace the $\dot m$ and $\nabla^\mu m$ by $\dot T$ and $\nabla^\mu T$ in Eqs. (32) and (33), we decompose the energy-momentum tensor in the hydrodynamic degrees of freedom,

$$T^{\mu\nu} = \mathcal{E}u^\mu u^\nu - (\mathcal{P}+\Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}, \tag{34}$$

where $\mathcal{P}=\mathcal{P}_0(T)$ is the hydrostatic pressure (the equilibrium contribution), which depends only on the effective temperature and it is not an additional degree of freedom. The projection of the energy-momentum conservation equation, $\partial_\mu T^{\mu\nu}=0$, along and orthogonal to u^μ leads to

$$\dot{\mathcal{E}} = -\,(\mathcal{E}+\mathcal{P})\,\theta - \Pi\,\theta + \pi:\sigma, \tag{35}$$

$$\nabla^\mu\mathcal{P} = (\mathcal{E}+\mathcal{P})\,\dot u^\mu + \Pi\,\dot u^\mu - \nabla^\mu\Pi + \Delta^\mu_\alpha\partial_\beta\pi^{\alpha\beta}. \tag{36}$$

where $\sigma^{\mu\nu}\equiv\Delta^{\mu\nu}_{\alpha\beta}\nabla^\alpha u^\beta$ is the stress tensor. Since one can always express $\mathcal{E}$ and $\mathcal{P}$ in terms of T using the matching condition, Eq. (22), and the equation of state, we can further rewrite the above equations as

$$\frac{\dot{T}}{T} = -\, c_s^2 \left(\theta + \frac{\Pi\, \theta - \pi : \sigma}{\mathcal{E} + \mathcal{P}} \right), \tag{37}$$

$$\frac{\nabla^\mu T}{T} = \left(\dot{u}^\mu + \frac{\Pi\, \dot{u}^\mu - \nabla^\mu \Pi + \Delta^\mu_\alpha \partial_\beta \pi^{\alpha\beta}}{\mathcal{E} + \mathcal{P}} \right), \tag{38}$$

where we have used the thermodynamic relation, Eq. (4), and the definition of squared speed of sound, $c_s^2 = d\mathcal{P}/d\mathcal{E}$.

Using Eqs. (37) and (38) to rewrite $\dot{m}/m = \kappa \dot{T}/T$ and $\nabla^\mu m/m = \kappa \nabla^\mu T/T$ and substituting in Eqs. (32) and (33), we get

$$\begin{aligned} \dot{b}_0 + \frac{b_0}{\tau_R} =&\ -\kappa c_s^2 \left(\theta + \frac{\Pi\, \theta - \pi : \sigma}{\mathcal{E} + \mathcal{P}} \right) \\ &\times \Big(3\Pi + 4 b_0 \Big) + b \cdot \dot{u} - (\partial \cdot b), \end{aligned} \tag{39}$$

$$\begin{aligned} \dot{b}^{\langle\mu\rangle} + \frac{b^\mu}{\tau_R} =&\ \kappa \left(\dot{u}^\mu + \frac{\Pi\, \dot{u}^\mu - \nabla^\mu \Pi + \Delta^\mu_\alpha \partial_\beta \pi^{\alpha\beta}}{\mathcal{E} + \mathcal{P}} \right) \\ &\times \Big(3\Pi + 4 b_0 \Big) - \nabla^\mu b_0 - \theta\, b^\mu - (b \cdot \partial) u^\mu, \end{aligned} \tag{40}$$

where we introduced $\kappa \equiv (T/m)(dm/dT)$. Note that, at first order in the gradient expansion, we get $b_0 = 0$ and $b^\mu = 0$, which is consistent with the results of Refs. [64, 65]. Up to second-order, we find

$$b_0 = -3\, \tau_R\, \kappa c_s^2\, \Pi\, \theta, \quad b^\mu = 3\, \tau_R\, \kappa\, \Pi\, \dot{u}^\mu. \tag{41}$$

IV. DISSIPATIVE EVOLUTION EQUATIONS

In order to derive second-order evolution equations for the dissipative quantities, we start with the Chapman-Enskog-like iterative solution of the Boltzmann equation, Eq. (17). The first-order solution is given by [63]

$$\delta f_1 = -\frac{\tau_R}{(u \cdot p)} \Big[(p \cdot \partial) f_0 + m\, (\partial_\rho m)\, \partial^\rho_{(p)} f_0 \Big]. \tag{42}$$

In the following, for simplicity, we restrict ourselves to classical Maxwell-Boltzmann distribution for the equilibrium distribution function, Eq. (6). Using Eqs. (37) and (38), one gets [63]

$$\begin{aligned} \delta f_1 =& \frac{f_0\, \beta \tau_R}{(u \cdot p)} \left[\left(\frac{1}{3} p^2 - m^2 \kappa\, c_s^2 \right) - \left(\frac{1}{3} - c_s^2 \right) (u \cdot p)^2 \right] \theta \\ &+ \frac{f_0\, \beta \tau_R}{(u \cdot p)} (p \cdot \sigma \cdot p). \end{aligned} \tag{43}$$

We can now proceed to obtain the first-order equations for the dissipative quantities. Using Eq. (43) in Eqs. (28) and (29), and keeping terms which are first-order in gradients, we get

$$\Pi = -\beta_\Pi\, \tau_R\, \theta, \quad \pi^{\mu\nu} = 2\, \beta_\pi \tau_R\, \sigma^{\mu\nu}, \tag{44}$$

where

$$\beta_\Pi = \frac{5}{3}\, \beta\, I_{3,2} - c_s^2\, (\mathcal{E} + \mathcal{P}) + \kappa c_s^2\, m^2 \beta\, I_{1,1}\ , \tag{45}$$

$$\beta_\pi = \beta I_{3,2}, \tag{46}$$

and the shear and bulk viscosities are given by the following relations $\beta_\pi \tau_R = \eta$ and $\beta_\Pi \tau_R = \zeta$, respectively.

In the above equations, the integral coefficients $I_{n,q}$ are defined as

$$I_{n,q} \equiv \frac{(-1)^q}{(2q+1)!!} \int dP\, (u \cdot p)^{n-2q}\, (p \cdot \Delta \cdot p)^q\, f_0. \tag{47}$$

Note that while the form of β_π in Eq. (46) is identical[2] to that obtained for the constant mass case [57], β_Π in Eq. (45) is different. Using Eq. (44) in Eq. (43), we obtain the first-viscous correction to the distribution function,

$$\begin{aligned} \delta f_1 =& -\frac{f_0 \beta}{(u \cdot p) \beta_\Pi} \\ &\times \left[\left(\frac{1}{3} p^2 - m^2 \kappa\, c_s^2 \right) - \left(\frac{1}{3} - c_s^2 \right) (u \cdot p)^2 \right] \Pi \\ &+ \frac{f_0 \beta}{2 (u \cdot p) \beta_\pi} (p \cdot \pi \cdot p). \end{aligned} \tag{48}$$

The above expression for δf_1 will be used to derive the second-order evolution equations for dissipative quantities.

For the formulation of second-order viscous hydrodynamics equations for $\pi^{\mu\nu}$ and Π, we adopt the method developed in Ref. [16]. We consider the co-moving derivative of dissipative quantities from Eqs. (28) and (29),

$$\dot{\Pi} = -\frac{1}{3}\, (u \cdot \partial) \int dP\, (p \cdot \Delta \cdot p)\, \delta f - \dot{b}_0, \tag{49}$$

$$\dot{\pi}^{\langle\mu\nu\rangle} = \Delta^{\mu\nu}_{\alpha\beta}\, (u \cdot \partial) \int dP\, p^{\langle\alpha} p^{\beta\rangle}\, \delta f, \tag{50}$$

where $X^{\langle\mu\nu\rangle} \equiv \Delta^{\mu\nu}_{\alpha\beta} X^{\alpha\beta}$ and $\dot{X}^{\langle\mu\nu\rangle} \equiv \Delta^{\mu\nu}_{\alpha\beta} \dot{X}^{\alpha\beta}$. Rearranging the Boltzmann equation, Eq. (17), in the relaxation-time approximation, Eq. (20), we get

$$\dot{\delta f} = -\dot{f}_0 - \frac{1}{(u \cdot p)} \Big[(p \cdot \partial) f + m\, (\partial_\rho m)\, \partial^\rho_{(p)} f \Big] - \frac{\delta f}{\tau_R} \tag{51}$$

Using $\delta f = \delta f_1$ in the right hand side of the above equation, substituting in Eqs. (49) and (50) and performing the integrations, we arrive at the second-order evolution equations for bulk viscous pressure and shear-stress tensor

$$\dot{\Pi} = -\frac{\Pi}{\tau_\Pi} - \beta_\Pi \theta - \delta_{\Pi\Pi} \Pi \theta + \lambda_{\Pi\pi} \pi : \sigma, \tag{52}$$

$$\begin{aligned} \dot{\pi}^{\langle\mu\nu\rangle} =& -\frac{\pi^{\mu\nu}}{\tau_\pi} + 2 \beta_\pi \sigma^{\mu\nu} + 2 \pi_\gamma^{\langle\mu} \omega^{\nu\rangle\gamma} - \tau_{\pi\pi} \pi_\gamma^{\langle\mu} \sigma^{\nu\rangle\gamma} \\ &- \delta_{\pi\pi} \pi^{\mu\nu} \theta + \lambda_{\pi\Pi} \Pi \sigma^{\mu\nu}, \end{aligned} \tag{53}$$

[2] In the sense that it corresponds to the simple substitution $\beta_\pi(m, T) \to \beta_\pi(m(T), T)$, from the fixed mass case to the quasi-particle one.

where $\omega^{\mu\nu} \equiv \frac{1}{2}(\nabla^\mu u^\nu - \nabla^\nu u^\mu)$ is the vorticity tensor.

The transport coefficients obtained here are:

$$\delta_{\Pi\Pi} = -\frac{5}{9}\chi - \left(1 - \kappa m^2 \frac{I_{1,1}}{I_{3,1}}\right) c_s^2 + \frac{1}{3}\frac{\beta\kappa c_s^2 m^2}{\beta_\Pi}\Bigg[\left(1-3c_s^2\right)\left(\beta I_{2,1} - I_{1,1}\right) - \left(1-3\kappa c_s^2\right) m^2 \left(\beta I_{0,1} + I_{-1,1}\right)\Bigg], \quad (54)$$

$$\lambda_{\Pi\pi} = \frac{\beta}{3\beta_\pi}\left(2I_{3,2} - 7I_{3,3}\right) - \left(1 - \kappa m^2 \frac{I_{1,1}}{I_{3,1}}\right) c_s^2, \quad (55)$$

$$\tau_{\pi\pi} = 2 - \frac{4\beta}{\beta_\pi} I_{3,3}, \quad (56)$$

$$\delta_{\pi\pi} = \frac{5}{3} - \frac{7}{3}\frac{\beta}{\beta_\pi} I_{3,3} - \frac{\beta}{\beta_\pi}\kappa c_s^2 m^2 \left(I_{1,2} - I_{1,1}\right), \quad (57)$$

$$\lambda_{\pi\Pi} = -\frac{2}{3}\chi, \quad (58)$$

where

$$\chi = \frac{\beta}{\beta_\Pi}\Bigg[\left(1-3c_s^2\right)\left(I_{3,2} - I_{3,1}\right) - \left(1-3\kappa c_s^2\right) m^2 \left(I_{1,2} - I_{1,1}\right)\Bigg]. \quad (59)$$

As expected, in the constant mass limit, $\kappa \to 0$, the above transport coefficients match exactly with those obtained in Ref. [57]. Note that functions $I_{n,q}$ used here are equivalent to $(-1)^q I^{(R)}_{N,q}$ from Ref. [57] with $n = N - R$.

The integral coefficients can be obtained in terms of modified Bessel functions of the second kind, $K_n(z)$,

$$I_{3,3} = \frac{gT^5 z^5}{210\pi^2}\left[\frac{1}{16}(K_5 - 11K_3 + 58K_1) - 4K_{i,1} + K_{i,3}\right], \quad (60)$$

$$I_{3,2} = \frac{gT^5 z^5}{30\pi^2}\left[\frac{1}{16}(K_5 - 7K_3 + 22K_1) - K_{i,1}\right], \quad (61)$$

$$I_{3,1} = \frac{gT^5 z^3}{2\pi^2} K_3 = T(\mathcal{P}_0 + \mathcal{E}_0) = T^2\mathcal{S}_0, \quad (62)$$

$$I_{2,1} = \frac{gT^4 z^2}{2\pi^2} K_2(z) = \mathcal{P}_0 + B_0 \quad (63)$$

$$I_{2,0} = \frac{gT^4 z^2}{2\pi^2}\left[3K_2(z) + zK_1(z)\right] = \mathcal{E}_0 - B_0 \quad (64)$$

$$I_{1,2} = \frac{gT^3 z^3}{30\pi^2}\left[\frac{1}{4}(K_3 - 9K_1) + 3K_{i,1} - K_{i,3}\right], \quad (65)$$

$$I_{1,1} = \frac{gT^3 z^3}{6\pi^2}\left[\frac{1}{4}(K_3 - 5K_1) + K_{i,1}\right], \quad (66)$$

$$I_{0,1} = \frac{gT^2 z^2}{6\pi^2}\left[\frac{1}{2}(K_2 - 3K_0) + K_{i,2}\right], \quad (67)$$

$$I_{-1,1} = \frac{gTz}{6\pi^2}\left[K_1 - 2K_{i,1} + K_{i,3}\right], \quad (68)$$

where the z-dependence of K_n is implicitly understood. Here the function $K_{i,m}$ is defined by the integral

$$K_{i,m}(z) = \int_0^\infty \frac{d\theta}{(\cosh\theta)^m}\exp(-z\cosh\theta), \quad (69)$$

which has the following property

$$\frac{d}{dz}K_{i,m}(z) = -K_{i,m-1}(z). \quad (70)$$

This identity can also be written in integral form

$$K_{i,m}(z) = K_{i,m}(0) - \int_0^z K_{i,m-1}(z')dz'. \quad (71)$$

We observe that by using the series expansion of $K_{i,0}(z) = K_0(z)$, the above recursion relation can be employed to evaluate $K_{i,m}(z)$ up to any given order in z.

V. IMPOSING LATTICE QCD EQUATION OF STATE

In order to solve the energy-momentum conservation equations, Eqs. (35) and (36), coupled to the relaxation-type equations for the dissipative quantities, Eqs. (52) and (53), one has to impose the equation of state of the system. This is necessary to fix the temperature dependence of the thermodynamic quantities like energy density and pressure, as well as the transport coefficients, Eqs. (45)-(46) and Eqs. (54)-(59). For this purpose it is sufficient to define the temperature dependence of the quasiparticle mass, $m(T)$, extracted for the microscopic theory in question. In this case one may follow the prescription employed in Refs. [63, 76].

We consider finite-temperature lQCD equation of state at zero baryon chemical potential computed by the Wuppertal-Budapest collaboration [51]. To correctly match the energy density and pressure given in Eqs. (7) and (8) with that of lQCD at asymptotically large temperatures (Stefan–Boltzmann limit), we fix the degeneracy factor to be

$$g = \frac{\pi^4}{180}\left(4(N_c^2 - 1) + 7N_c N_f\right), \quad (72)$$

where for the number of colors and flavors we choose $N_c = 3$ and $N_f = 3$, respectively. Using Eqs. (7) and (8) one finds that the equilibrium entropy density, $\mathcal{S}_0 = (\mathcal{E}_0 + \mathcal{P}_0)/T$, as expressed in Eq. (4), is independent of the bag pressure, B_0. Therefore it is the most convenient quantity which can be used to determine $m(T)$ by numerically solving

$$\frac{g}{2\pi^2}\left(\frac{m(T)}{T}\right) K_3\left(\frac{m(T)}{T}\right) = \frac{\mathcal{S}_0(T)}{T^3}\bigg|_{\rm lQCD}, \quad (73)$$

where right hand side of the above equation is evaluated using lQCD results [51].

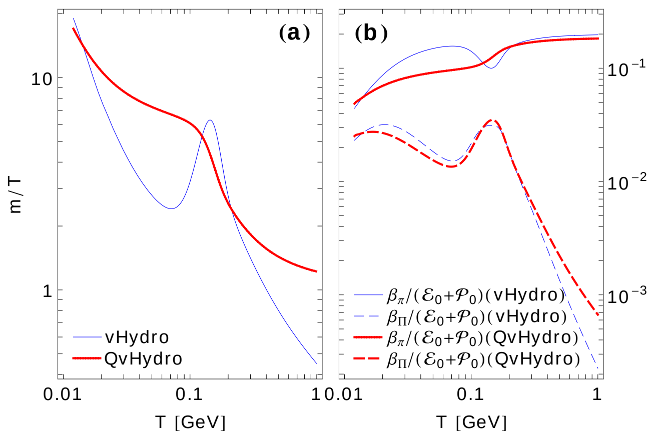

FIG. 1. (Color online) Temperature dependence of the quasiparticle mass (a) and the bulk, β_Π, and shear, β_π, first-order transport coefficients (see Eqs. (45) and (46)) scaled by $\mathcal{E}_0+\mathcal{P}_0$ (b) in the case of standard second-order viscous hydrodynamics with lQCD equation of state (vHydro) and the present formulation of quasiparticle second-order viscous hydrodynamics (QvHydro).

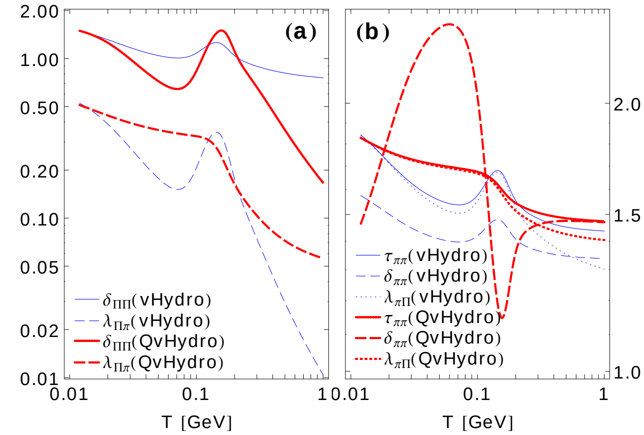

FIG. 2. (Color online) Temperature dependence of the bulk (a) and shear (b) second-order transport coefficients in the case of standard second-order viscous hydrodynamics with lQCD equation of state (vHydro) and the present formulation of quasiparticle second-order viscous hydrodynamics (QvHydro).

For numerical convenience, we use analytic fit to the lQCD results for the interaction measure (trace anomaly) [51]

$$\frac{\mathcal{I}_0(T)}{T^4} = \exp\Big[-\Big(\frac{h_1}{\hat{T}}+\frac{h_2}{\hat{T}^2}\Big)\Big] \times\Big[\frac{h_0}{1+h_3\hat{T}^2}+\frac{f_0\big[\tanh(f_1\hat{T}+f_2)+1\big]}{1+g_1\hat{T}+g_2\hat{T}^2}\Big], \tag{74}$$

with $\hat{T} \equiv T/(0.2\ \text{GeV})$. Following Refs. [76, 77], the parameters for the fit function are chosen as follows: $h_0 = 0.1396$, $h_1 = -0.18$, $h_2 = 0.035$, $f_0 = 2.76$, $f_1 = 6.79$, $f_2 = -5.29$, $g_1 = -0.47$, $g_2 = 1.04$, and $h_3 = 0.01$. Using Eq. (74), the equilibrium pressure is obtained by performing the integral

$$\frac{\mathcal{P}_0(T)}{T^4} = \int_0^T \frac{dT}{T}\frac{\mathcal{I}_0(T)}{T^4}\,, \tag{75}$$

while the energy density is calculated from the relation

$$\frac{\mathcal{E}_0(T)}{T^4} = 3\frac{\mathcal{P}_0(T)}{T^4}+\frac{\mathcal{I}_0(T)}{T^4}. \tag{76}$$

To obtain $B_0(T)$, one may use the relation expressing thermodynamic consistency, Eq. (5), which reduces to

$$\frac{dB_0(T)}{dT} = -\frac{gT^3z^2}{2\pi^2}K_1(z)\,\frac{dm}{dT}. \tag{77}$$

The above equation can be solved numerically using the boundary condition $B_0 = 0$ at $T \simeq 0$.

In order to compare with the usual viscous hydrodynamic results obtained using lQCD equation of state, we use the *standard* prescription of matching speed of sound squared c_s^2 obtained from kinetic theory and lQCD to extract $m(T)$. In this method, one commonly uses a small-m/T expansion for the second-order transport coefficients [37, 78]. It is important to note that this naive prescription is not only thermodynamically inconsistent but also incompatible with the small-m/T expansion; see Fig. 1 (a). However, using equation $c_s^2(m(T)/T) = c_s^2(T)|_{\text{lQCD}}$, one can extract $m(T)$, exactly, via the relation [57]

$$\frac{m(T)}{T} = \sqrt{\frac{3(\mathcal{E}_0+\mathcal{P}_0)}{c_s^2\,\mathcal{P}_0}\left(\frac{1}{3}-c_s^2\right)}, \tag{78}$$

where, $\mathcal{P}_0$ and $\mathcal{E}_0$ are taken from parametrization of the lQCD results, Eqs. (75) and (76). The extracted $m(T)$ is then used to evaluate the transport coefficients which appear in the viscous evolution equations (52) and (53). In the following, we refer to this prescription as the standard viscous hydrodynamics (vHydro).

In Fig. 1 (b) we show the temperature dependence of bulk, β_Π, and shear, β_π, first-order transport coefficients (see Eqs. (45) and (46)) scaled by $\mathcal{E}_0+\mathcal{P}_0$ for vHydro approach (thin blue lines) and for the present framework of quasiparticle viscous hydrodynamics (QvHydro) (thick red lines). We observe that although the quasiparticle mass is significantly different for the case of vHydro and QvHydro formulation (see Fig. 1 (a)), the first-order coefficients are quite similar. In Fig. 2 we show the similar comparison performed for the second-order transport coefficients. In this case we see that there are substantial differences in the two cases for all the transport coefficients. In the next Section, we study the effect of these differences in the case of one-dimensional boost-invariant expansion of the viscous QCD medium.

VI. LONGITUDINAL BJORKEN FLOW

In order to quantify the effect of the present formulation of quasiparticle second-order viscous hydrodynamics, we consider transversely homogeneous and purely-longitudinal boost-invariant, the so-called Bjorken, expansion [79]. The latter may be applicable to the early-time evolution of the viscous QCD matter in the very center of the heavy-ion collision. With this symmetry it is convenient to use Milne coordinates, $x^\mu = (\tau, x, y, \varsigma)$, where $\tau \equiv \sqrt{t^2 - z^2}$, $\varsigma \equiv \tanh^{-1}(z/t)$ and $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -\tau^2)$. One may check that in such a case the fluid becomes static, $u^\mu = (1, 0, 0, 0)$.

In this case the energy-momentum conservation equations, Eqs. (35) and (36), together with the equations for the dissipative quantities, Eqs. (52) and (53), reduce to

$$\dot{\mathcal{E}} = -\frac{1}{\tau}\left(\mathcal{E} + \mathcal{P} + \Pi - \pi\right) , \tag{79}$$

$$\dot{\Pi} + \frac{\Pi}{\tau_\Pi} = -\frac{\beta_\Pi}{\tau} - \delta_{\Pi\Pi}\frac{\Pi}{\tau} + \lambda_{\Pi\pi}\frac{\pi_s}{\tau} , \tag{80}$$

$$\dot{\pi}_s + \frac{\pi_s}{\tau_\pi} = \frac{4}{3}\frac{\beta_\pi}{\tau} - \left(\frac{1}{3}\tau_{\pi\pi} + \delta_{\pi\pi}\right)\frac{\pi_s}{\tau} + \frac{2}{3}\lambda_{\pi\Pi}\frac{\Pi}{\tau} , \tag{81}$$

where $\pi_s \equiv -\tau^2\pi^{\varsigma\varsigma}$. The transport coefficients appearing in the above equations are given in Eqs. (45)-(46) and Eqs. (54)-(59), and the integral coefficients are given in Eqs. (60)-(68).

We note that for the results presented here we choose equal bulk and shear relaxation times, $\tau_\pi = \tau_\Pi = \tau_R$. Identifying the first-order constitutive relations, Eq. (44), as the Navier–Stokes equations, we obtain

$$\tau_R = \frac{\bar{\eta}\mathcal{S}_0}{\beta_\pi}. \tag{82}$$

In the above equation, we have defined $\bar{\eta} \equiv \eta/\mathcal{S}$ and used $\mathcal{S} \simeq \mathcal{S}_0$. One can safely make this approximation, because $\mathcal{S} = \mathcal{S}_0 + \mathcal{O}(\delta^2)$, which leads to third-order corrections in Eqs. (80) and (81) that can be ignored.

To study the evolution of the viscous QCD matter, we numerically solve Eqs. (79)-(81) with the transport coefficients given in Eqs. (45)-(46) and Eqs. (54)-(59), and supplemented with the lQCD equation of state, as described in Section V. For the initial conditions we choose $T(\tau_i) = 0.6$ GeV, $\pi_s(\tau_i) = 0$ and $\Pi(\tau_i) = 0$, where the initial proper time is $\tau_i = 0.25$ fm. The initial temperature and thermalization time roughly correspond to those attained at the LHC. To study the effect of the equation of state in the entire physically interesting temperature range, and to compare with other results available in the literature, we perform the evolution to extremely late times, $\tau_f = 500$ fm. For the value of shear viscosity to entropy density ratio, we consider the lower bound $\eta/\mathcal{S} = 1/(4\pi)$. We compare our results (QvHydro) with

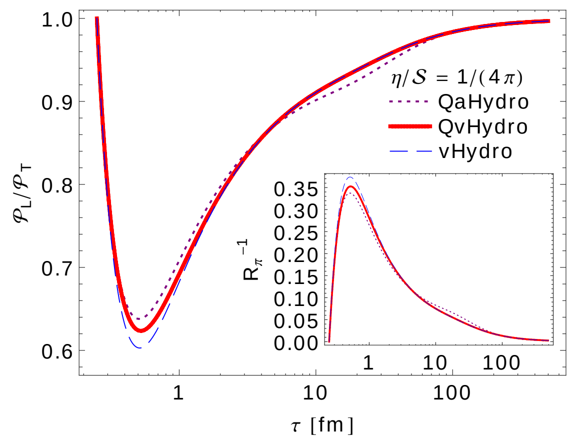

FIG. 3. (Color online) The proper-time evolution of the ratio of longitudinal pressure to transverse pressure. In the inset we present the respective proper-time evolution of the inverse Reynolds number $R_\pi^{-1} = \sqrt{\pi:\pi}/\mathcal{P}_0 = \sqrt{3/2}\,\pi_s/\mathcal{P}_0$.

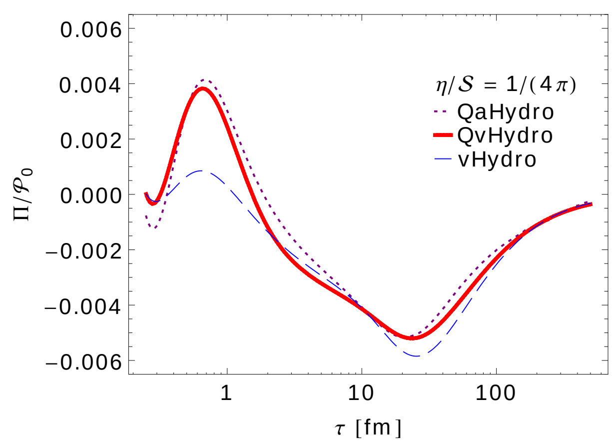

FIG. 4. (Color online) The proper-time dependence of the bulk viscous pressure scaled by the equilibrium pressure.

that obtained using quasiparticle anisotropic hydrodynamics (QaHydro) formulated in Ref. [76], and the standard second-order viscous hydrodynamics (vHydro).

In Fig. 3, we present the proper-time evolution of the ratio of longitudinal pressure, $\mathcal{P}_L = \mathcal{P}_0 + \Pi - \pi_s$, to transverse pressure, $\mathcal{P}_T = \mathcal{P}_0 + \Pi + \pi_s/2$, obtained using QaHydro (purple dotted curve), the present formulation of QvHydro (red solid curve) and vHydro (blue dashed curve). We observe that, at early times, the predictions concerning $\mathcal{P}_L/\mathcal{P}_T$ evolution within QvHydro and QaHydro are more in agreement compared to vHydro evolution. This feature is more evident in Fig. 4, where we show the proper-time dependence of the bulk pressure Π scaled by the equilibrium pressure for the three cases. We see that, at early times, QaHydro and QvHydro lead to a similar increase of the scaled bulk pressure as compared to the vHydro result. We note, that the proper-time evolution of temperature resulting from these three hydrodynamic formulations is the same up to the 0.5% accuracy.

VII. SUMMARY AND OUTLOOK

In this paper, we have presented a first derivation of the second-order relativistic viscous hydrodynamics for a system of quasiparticles of a single species from an effective Boltzmann equation. We allowed the quasiparticles to have temperature-dependent masses and devised a thermodynamically-consistent framework to formulate second-order evolution equations for the shear and bulk viscous pressure corrections. The formulation presented here is capable of accommodating an arbitrary equation of state, such as those obtained from lattice QCD calculations, within the framework of kinetic theory. It is important to maintain this consistency when one is using transport coefficients derived from kinetic theory for hydrodynamic simulations of QCD matter. Finally, we studied the effect of this new formulation in the case of one-dimensional purely-longitudinal boost-invariant expansion of viscous QCD medium formed in ultra-relativistic heavy-ion collisions.

Looking forward, it would be interesting to consider lQCD equation of state with non-zero chemical potential in the present calculation. This will require current conservation equation and evolution equation for dissipative charge current. Deriving the transport coefficients for particles obeying quantum statistics is another problem worth investigating. Moreover, since, as shown here, the thermodynamically-consistent incorporation of the realistic equation of state within the quasiparticle picture results in a significant modification of the transport coefficients, in particular bulk viscosity, it would be interesting to determine the impact of this change in a realistic higher-dimensional simulations. These studies may be especially interesting in the context of the recent findings concerning the importance of bulk viscosity in the evolution of matter in heavy-ion collisions [37]. We leave these questions for future studies.

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[1] P. Huovinen and P. V. Ruuskanen, Ann. Rev. Nucl. Part. Sci. **56**, 163 (2006), arXiv:nucl-th/0605008 [nucl-th].
[2] P. Romatschke, Int. J. Mod. Phys. **E19**, 1 (2010), arXiv:0902.3663 [hep-ph].
[3] C. Gale, S. Jeon, and B. Schenke, Int. J. Mod. Phys. **A28**, 1340011 (2013), arXiv:1301.5893 [nucl-th].
[4] A. Jaiswal and V. Roy, Adv. High Energy Phys. **2016**, 9623034 (2016), arXiv:1605.08694 [nucl-th].
[5] S. Jeon and U. Heinz, in *Quark-Gluon Plasma 5*, edited by X.-N. Wang (2016) pp. 131–187.
[6] E. Shuryak, *Heavy ion reaction from nuclear to quark matter. Proceedings, International School of Nuclear Physics, 25th Course, Erice, Italy, September 16-24, 2003*, Prog. Part. Nucl. Phys. **53**, 273 (2004), arXiv:hep-ph/0312227 [hep-ph].
[7] U. W. Heinz and P. F. Kolb, *Statistical QCD. Proceedings, International Symposium, Bielefeld, Germany, August 26-30, 2001*, Nucl. Phys. **A702**, 269 (2002), arXiv:hep-ph/0111075 [hep-ph].
[8] R. Ryblewski, J. Phys. **G40**, 093101 (2013).
[9] P. Danielewicz and M. Gyulassy, Phys. Rev. **D31**, 53 (1985).
[10] G. Policastro, D. T. Son, and A. O. Starinets, Phys. Rev. Lett. **87**, 081601 (2001), arXiv:hep-th/0104066 [hep-th].
[11] P. Kovtun, D. T. Son, and A. O. Starinets, Phys. Rev. Lett. **94**, 111601 (2005), arXiv:hep-th/0405231 [hep-th].
[12] A. Muronga, Phys. Rev. **C69**, 034903 (2004), arXiv:nucl-th/0309055 [nucl-th].
[13] R. Baier, P. Romatschke, and U. A. Wiedemann, Phys. Rev. **C73**, 064903 (2006), arXiv:hep-ph/0602249 [hep-ph].
[14] M. A. York and G. D. Moore, Phys. Rev. **D79**, 054011 (2009), arXiv:0811.0729 [hep-ph].
[15] A. El, Z. Xu, and C. Greiner, Phys. Rev. **C81**, 041901 (2010), arXiv:0907.4500 [hep-ph].
[16] G. S. Denicol, T. Koide, and D. H. Rischke, Phys. Rev. Lett. **105**, 162501 (2010), arXiv:1004.5013 [nucl-th].
[17] G. S. Denicol, H. Niemi, E. Molnar, and D. H. Rischke, Phys. Rev. **D85**, 114047 (2012), [Erratum: Phys. Rev.D91,no.3,039902(2015)], arXiv:1202.4551 [nucl-th].
[18] G. S. Denicol, E. Molnar, H. Niemi, and D. H. Rischke, Eur. Phys. J. **A48**, 170 (2012), arXiv:1206.1554 [nucl-th].
[19] A. Jaiswal, R. S. Bhalerao, and S. Pal, Phys. Lett. **B720**, 347 (2013), arXiv:1204.3779 [nucl-th].
[20] A. Jaiswal, Phys. Rev. **C87**, 051901 (2013), arXiv:1302.6311 [nucl-th].
[21] A. Jaiswal, R. S. Bhalerao, and S. Pal, Phys. Rev. **C87**, 021901 (2013), arXiv:1302.0666 [nucl-th].
[22] A. Jaiswal, Phys. Rev. **C88**, 021903 (2013), arXiv:1305.3480 [nucl-th].

[23] R. S. Bhalerao, A. Jaiswal, S. Pal, and V. Sreekanth, Phys. Rev. **C88**, 044911 (2013), arXiv:1305.4146 [nucl-th].
[24] R. S. Bhalerao, A. Jaiswal, S. Pal, and V. Sreekanth, Phys. Rev. **C89**, 054903 (2014), arXiv:1312.1864 [nucl-th].
[25] G. S. Denicol, S. Jeon, and C. Gale, Phys. Rev. **C90**, 024912 (2014), arXiv:1403.0962 [nucl-th].
[26] C. Chattopadhyay, A. Jaiswal, S. Pal, and R. Ryblewski, Phys. Rev. **C91**, 024917 (2015), arXiv:1411.2363 [nucl-th].
[27] Y. Kikuchi, K. Tsumura, and T. Kunihiro, Phys. Rev. **C92**, 064909 (2015), arXiv:1507.04894 [hep-ph].
[28] K. Tsumura, Y. Kikuchi, and T. Kunihiro, Phys. Rev. **D92**, 085048 (2015), arXiv:1506.00846 [hep-ph].
[29] P. Romatschke and U. Romatschke, Phys. Rev. Lett. **99**, 172301 (2007), arXiv:0706.1522 [nucl-th].
[30] P. Bozek, Phys. Rev. **C81**, 034909 (2010), arXiv:0911.2397 [nucl-th].
[31] H. Song, S. A. Bass, U. Heinz, T. Hirano, and C. Shen, Phys. Rev. Lett. **106**, 192301 (2011), [Erratum: Phys. Rev. Lett.109,139904(2012)], arXiv:1011.2783 [nucl-th].
[32] B. H. Alver, C. Gombeaud, M. Luzum, and J.-Y. Ollitrault, Phys. Rev. **C82**, 034913 (2010), arXiv:1007.5469 [nucl-th].
[33] B. Schenke, S. Jeon, and C. Gale, Phys. Rev. **C85**, 024901 (2012), arXiv:1109.6289 [hep-ph].
[34] C. Gale, S. Jeon, B. Schenke, P. Tribedy, and R. Venugopalan, Phys. Rev. Lett. **110**, 012302 (2013), arXiv:1209.6330 [nucl-th].
[35] J. Noronha-Hostler, G. S. Denicol, J. Noronha, R. P. G. Andrade, and F. Grassi, Phys. Rev. **C88**, 044916 (2013), arXiv:1305.1981 [nucl-th].
[36] F. G. Gardim, J. Noronha-Hostler, M. Luzum, and F. Grassi, Phys. Rev. **C91**, 034902 (2015), arXiv:1411.2574 [nucl-th].
[37] S. Ryu, J. F. Paquet, C. Shen, G. S. Denicol, B. Schenke, S. Jeon, and C. Gale, Phys. Rev. Lett. **115**, 132301 (2015), arXiv:1502.01675 [nucl-th].
[38] H. Niemi, K. J. Eskola, and R. Paatelainen, Phys. Rev. **C93**, 024907 (2016), arXiv:1505.02677 [hep-ph].
[39] C. Eckart, Phys. Rev. **58**, 267 (1940).
[40] L. Landau and E. Lifshitz, *Fluid Mechanics* (1959).
[41] W. Israel, Annals Phys. **100**, 310 (1976).
[42] J. M. Stewart, Proc. Roy. Soc. Lond. **357**, 59 (1977).
[43] W. Israel and J. M. Stewart, Annals Phys. **118**, 341 (1979).
[44] L. Lindblom, Annals Phys. **247**, 1 (1996), arXiv:gr-qc/9508058 [gr-qc].
[45] R. P. Geroch, J. Math. Phys. **36**, 4226 (1995).
[46] M. Strickland, *54th Cracow School of Theoretical Physics: QCD meets experiment: Zakopane, Poland, June 12-20, 2014*, Acta Phys. Polon. **B45**, 2355 (2014), arXiv:1410.5786 [nucl-th].
[47] M. P. Heller, R. A. Janik, and P. Witaszczyk, Phys. Rev. Lett. **108**, 201602 (2012), arXiv:1103.3452 [hep-th].
[48] A. Kurkela and Y. Zhu, Phys. Rev. Lett. **115**, 182301 (2015), arXiv:1506.06647 [hep-ph].
[49] Z. Fodor and S. D. Katz, (2009), arXiv:0908.3341 [hep-ph].
[50] A. Bazavov *et al.*, Phys. Rev. **D80**, 014504 (2009), arXiv:0903.4379 [hep-lat].
[51] S. Borsanyi, G. Endrodi, Z. Fodor, A. Jakovac, S. D. Katz, S. Krieg, C. Ratti, and K. K. Szabo, JHEP **11**, 077 (2010), arXiv:1007.2580 [hep-lat].
[52] A. Bazavov *et al.* (HotQCD), Phys. Rev. **D90**, 094503 (2014), arXiv:1407.6387 [hep-lat].
[53] S. Borsanyi, Z. Fodor, C. Hoelbling, S. D. Katz, S. Krieg, and K. K. Szabo, Phys. Lett. **B730**, 99 (2014), arXiv:1309.5258 [hep-lat].
[54] W. Florkowski, R. Ryblewski, and M. Strickland, Nucl. Phys. **A916**, 249 (2013), arXiv:1304.0665 [nucl-th].
[55] W. Florkowski, R. Ryblewski, and M. Strickland, Phys. Rev. **C88**, 024903 (2013), arXiv:1305.7234 [nucl-th].
[56] W. Florkowski, E. Maksymiuk, R. Ryblewski, and M. Strickland, Phys. Rev. **C89**, 054908 (2014), arXiv:1402.7348 [hep-ph].
[57] A. Jaiswal, R. Ryblewski, and M. Strickland, Phys. Rev. **C90**, 044908 (2014), arXiv:1407.7231 [hep-ph].
[58] G. S. Denicol, W. Florkowski, R. Ryblewski, and M. Strickland, Phys. Rev. **C90**, 044905 (2014), arXiv:1407.4767 [hep-ph].
[59] W. Florkowski, A. Jaiswal, E. Maksymiuk, R. Ryblewski, and M. Strickland, Phys. Rev. **C91**, 054907 (2015), arXiv:1503.03226 [nucl-th].
[60] H. B. Meyer, Phys. Rev. **D76**, 101701 (2007), arXiv:0704.1801 [hep-lat].
[61] H. B. Meyer, Phys. Rev. Lett. **100**, 162001 (2008), arXiv:0710.3717 [hep-lat].
[62] M. I. Gorenstein and S.-N. Yang, Phys. Rev. **D52**, 5206 (1995).
[63] P. Romatschke, Phys. Rev. **D85**, 065012 (2012), arXiv:1108.5561 [gr-qc].
[64] C. Sasaki and K. Redlich, Phys. Rev. **C79**, 055207 (2009), arXiv:0806.4745 [hep-ph].
[65] M. Bluhm, B. Kampfer, and K. Redlich, Phys. Rev. **C84**, 025201 (2011), arXiv:1011.5634 [hep-ph].
[66] L. Landau and E. Lifshitz, *Statistical Physics* (1975).
[67] V. Goloviznin and H. Satz, Z. Phys. **C57**, 671 (1993).
[68] E. Braaten and R. D. Pisarski, Nucl. Phys. **B337**, 569 (1990).
[69] S. Jeon, Phys. Rev. **D52**, 3591 (1995), arXiv:hep-ph/9409250 [hep-ph].
[70] S. Jeon and L. G. Yaffe, Phys. Rev. **D53**, 5799 (1996), arXiv:hep-ph/9512263 [hep-ph].
[71] P. Chakraborty and J. I. Kapusta, Phys. Rev. **C83**, 014906 (2011), arXiv:1006.0257 [nucl-th].
[72] M. Albright and J. I. Kapusta, Phys. Rev. **C93**, 014903 (2016), arXiv:1508.02696 [nucl-th].
[73] S. de Groot, W. van Leeuwen, and C. van Weert, *Relativistic Kinetic Theory: Principles and Applications* (1980).
[74] W. Florkowski, J. Hufner, S. P. Klevansky, and L. Neise, Annals Phys. **245**, 445 (1996), arXiv:hep-ph/9505407 [hep-ph].
[75] J. Anderson and H. Witting, Physica **74**, 466 (1974).
[76] M. Alqahtani, M. Nopoush, and M. Strickland, Phys. Rev. **C92**, 054910 (2015), arXiv:1509.02913 [hep-ph].
[77] M. Alqahtani, M. Nopoush, and M. Strickland, (2016), arXiv:1605.02101 [nucl-th].
[78] J.-B. Rose, J.-F. Paquet, G. S. Denicol, M. Luzum, B. Schenke, S. Jeon, and C. Gale, *Proceedings, 24th International Conference on Ultra-Relativistic Nucleus-Nucleus Collisions (Quark Matter 2014): Darmstadt, Germany, May 19-24, 2014*, Nucl. Phys. **A931**, 926 (2014), arXiv:1408.0024 [nucl-th].
[79] J. D. Bjorken, Phys. Rev. **D27**, 140 (1983).