Unintegrated double parton distributions

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We present the construction of unintegrated double parton distribution functions which include dependence on transverse momenta of partons. We extend the formulation which was used to obtain the single unintegrated parton distributions from the standard, integrated parton distribution functions. Starting from the homogeneous part of the evolution equations for the integrated double parton distributions, we construct the unintegrated double parton distributions as the convolutions of the integrated double distributions and the splitting functions, multiplied by the Sudakov form factors. We show that there exist three domains of external hard scales which require three distinct forms of the unintegrated double distributions. The additional transverse momentum dependence which arises through the Sudakov form factors leads to non-trivial correlations in the parton momenta. We also discuss the non-homogeneous contribution to the unintegrated double parton distributions, which arises due to the splitting of a single parton into daughter partons with high transverse momenta. We analyze two cases, the unfolding of the transverse momenta dependence from the last step of the evolution of two partons, and the case where the transverse momenta are generated directly from the single parton splitting.

Keywords: Quantum Chromodynamics, parton distributions, evolution equations, double parton scattering

I. INTRODUCTION

The Large Hadron Collider opened a completely new kinematic domain for exploring the dynamics of the strong interactions. At these very high energies the incoming hadrons are characterized by the large parton densities driven by the fast increase of the gluon density at low values of Bjorken x. Typically, for most hadron encounters only a single partonic interaction occurs. However, at large energies, it is also possible to have more than one partonic interaction per one collision of incoming hadrons. This is referred to as a multi-parton interaction. Such events were first observed by the AFS Colaboration at CERN [1] and followed by the measurements performed by the collaborations at the Tevatron collider [2–4]. Later, a systematic experimental analysis was performed at the Large Hadron Collider [5–7].

The theoretical description of single hard parton interactions is well established within the perturbative QCD. The standard approach is to use the collinear factorization [8, 9] with perturbatively calculable partonic cross sections and integrated parton distribution functions (PDFs) which obey the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equations [10–12]. Such factorization is well defined when the hard scale, like the transverse energy of the jet, invariant mass of the Drell-Yan pair or the mass of the produced heavy quark, is present in the process.

For the multiparton interactions, the theoretical description within the perturbative QCD is also possible in the presence of the sufficiently hard scales. The computation of double parton scattering (DPS) cross sections within the collinear framework makes use of the double parton distribution functions (DPDFs) [13–40]. Recently, a significant progress has been made towards a complete proof of the factorization theorem for the double parton interaction in the case of the double Drell-Yan production [41]. In the leading logarithmic approximation, the DPDFs obey QCD evolution equations similar to the DGLAP equations for the PDFs [13, 14, 17, 18, 23, 42], (see also [43, 44] for the analogous formulation of the evolution equations for double parton correlations inside jets).

The standard collinear approach with integrated PDFs may be, however, insufficient when trying to describe more exclusive processes, see for example [45]. In this case the more complete information about the kinematics of the partonic process should be included. This can be done by using the unintegrated parton distributions which in addition to the parton longitudinal momentum fractions also include their transverse momentum dependence.

The unintegrated parton distribution functions (UPDFs)[1] naturally appear in the small x formalism, where the so called k_T-factorization is utilized [46] with off-shell matrix elements and the unintegrated parton distributions. For example, the Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation [47–49] can be interpreted as an evolution equation in $\log x$ for the unintegrated parton distributions. The Catani-Ciafaloni-Fiorani-Marchesini (CCFM) equation [50–53]

[1] They belong to a general class of transverse momentum dependent parton distributions (TMDs).

is a further example of the evolution equation for the UPDFs, which in addition to the transverse momentum also depends on the hard scale of the process. Yet another formulation is the transverse momentum dependent (TMD) factorization, (for a comprehensive formulation see [54]), which is valid to the leading power in the hard scale.

A very useful approach to the UPDFs was formulated in [55–57], where the UPDFs were constructed from the integrated PDFs through the derivative of the latter additionally dressed with the Sudakov form factor. The inclusion of the Sudakov form factor leads to the emergence of the dependence on two scales, the transverse momentum of the parton and the hard scale. The hard scale plays the role of the cutoff in the angular ordering of the emitted soft gluons. This construction is relatively convenient as it allows for obtaining the UPDFs without actually solving separate equations (like the CCFM equation which is quite complicated) but rather using the standard integrated PDFs. The UPDFs obtained in this framework are widely used in phenomenology, where they are applied in the k_T factorization formalism together with the off-shell matrix elements, see [58] and references therein for recent analysis.

In this article, we extend the construction [55, 57] to the case of the unintegrated double parton distribution functions (UDPDFs)[2]. Starting from the evolution equations for the integrated double distribution functions (DPDFs) we recast them in the form that allows to extract the unintegrated versions of these distributions. We show that for the homogeneous part of the solution to these equations, the extension requires the convolution of the integrated DPDFs with splitting functions and multiplication by the appropriate Sudakov form factors. Since there are two hard scales in this case, we find that the form of the UDPDFs depends on the relation between the two hard scales. Also, we find that the cutoffs which regularize the real emission integrals and the Sudakov form factors induce nontrivial correlations between the longitudinal momenta of the two partons.

We also discuss the non-homogenous contribution to the UDPDFs which corresponds to the splitting of one parton into two daughter partons with large transverse momenta. We present the results of the unfolding of the transverse momenta dependence from the last step in the evolution of two partons in the non-homogeneous part of the solution to the evolution equations for the DPDFs. We also consider the contribution due to the parton splitting which includes transverse momentum dependence, derived in the light-front approach.

Our paper is organized as follows. In Sec. II we recapitulate the construction of the UPDFs presented in [55, 57] in both the Mellin space and the x-space. In Sec. III we recall the evolution equations for the integrated DPDFs and also show their formulation in the Mellin space. In Sec. IV we present the details of the construction of the UDPDFs for the homogeneous part of the solution the evolution equations for the DPDFs. We also briefly discuss the correlations between kinematic variables in the UDPDFs, induced by the regularization of the real emission integrals and the Sudakov form factors. In Sec. V we construct the non-homogeneous contribution to the UDPDFs, by first applying the construction performed for the homogeneous solution and also by the explicit derivation of the parton splitting term with the transverse momenta dependence on the light-front. Finally, in the last section we present the summary and conclusions.

II. UNINTEGRATED PARTON DISTRIBUTIONS

Let us first recapitulate the main points of the construction of the single *unintegrated parton distribution functions* proposed by Kimber-Martin-Ryskin (KMR) [55, 57]. The starting point are the DGLAP evolution equations for the single integrated parton distributions $D_a(x,\mu)$, where a denotes quark/antiquark flavors and also gluon, x is the longitudinal momentum fraction and μ is the scale for this distribution. The DGLAP equations with real and virtual parts separated read

$$\frac{\partial D_a(x,\mu)}{\partial \ln \mu^2} = \sum_{a'} \int_x^{1-\Delta} \frac{dz}{z}\, P_{aa'}(z,\mu)\, D_{a'}\Big(\frac{x}{z},\mu\Big) - D_a(x,\mu) \sum_{a'} \int_0^{1-\Delta} dzzP_{a'a}(z,\mu)\,. \tag{1}$$

The splitting functions $P_{aa'}$ can be computed order by order in perturbation theory and thus are given in powers of the running strong coupling constant, $\alpha_s(\mu)/(2\pi)$. In the leading logarithmic in μ^2 approximation we have

$$P_{aa'}(z,\mu) = \frac{\alpha_s(\mu)}{2\pi}\, P^{(0)}_{aa'}(z)\,, \tag{2}$$

where $P^{(0)}_{aa'}$ are the LO Altarelli-Parisi splitting functions. The upper limits in the divergent integrals in Eq. (1) are regularized by a parameter $\Delta < 1$ to be specified later. In the DGLAP equations $\Delta \to 0$ because singularities between

[2] They are also called double transverse momentum dependent distributions (DTMDs) in the current literature, see recent [59].

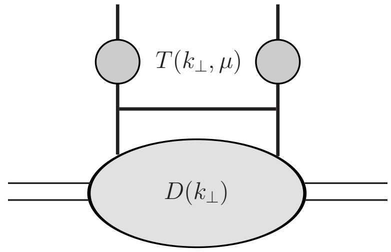

FIG. 1: Schematic representation of the UPDFs, Eq. (8). The longitudinal momentum structure is suppressed on this plot. The horizontal line denotes the real parton emission with the splitting functions P_{ab} and the circular blobs on the vertical lines indicate the Sudakov form factor.

real and virtual terms cancel, but we will keep Δ finite to be able to manipulate these equations. The first (real) term on the r.h.s of Eq. (1) can be interpreted as a number of partons which are emitted in the interval $\mu^2 \leq k_\perp^2 \leq \mu^2 + \delta\mu^2$. The second (virtual) term does not change the transverse momentum of the parton and therefore can be integrated as we shall show below.

Let us take for the factorization scale parton transverse momentum, $\mu = |\mathbf{k}_\perp| \equiv k_\perp$ and rewrite these equations in the following form

$$\frac{\partial D_a(x,k_\perp)}{\partial \ln k_\perp^2} + D_a(x,k_\perp) \sum_{a'} \int_0^{1-\Delta} dz z\, P_{a'a}(z,k_\perp) = \sum_{a'} \int_x^{1-\Delta} \frac{dz}{z}\, P_{aa'}(z,k_\perp)\, D_{a'}\Big(\frac{x}{z}, k_\perp\Big)\,. \tag{3}$$

After multiplying both sides of this equation by the Sudakov form factor[3],

$$T_a(Q,k_\perp) = \exp\left\{ - \int_{k_\perp^2}^{Q^2} \frac{dp_\perp^2}{p_\perp^2} \sum_{a'} \int_0^{1-\Delta} dz z P_{a'a}(z,p_\perp) \right\}, \tag{4}$$

where $k_\perp^2 \leq Q^2$, the l.h.s. can be written as a full derivative, and Eq. (3) reads

$$\frac{\partial}{\partial \ln k_\perp^2} \left[T_a(Q,k_\perp) D_a(x,k_\perp)\right] = T_a(Q,k_\perp) \sum_{a'} \int_x^{1-\Delta} \frac{dz}{z}\, P_{aa'}(z,k_\perp) D_{a'}\Big(\frac{x}{z}, k_\perp\Big)\,. \tag{5}$$

The Sudakov form factor is interpreted as the probability that the parton a with transverse momentum $k_\perp$ will not split into a pair of partons during the evolution in $p_\perp^2$ up to the scale Q^2. Integrating both sides of Eq. (5) over $k_\perp$ in the limits $Q_0 \leq Q$, where Q_0 is an initial scale for the DGLAP evolution, we find on the l.h.s.

$$\int_{Q_0^2}^{Q^2} \frac{dk_\perp^2}{k_\perp^2} \frac{\partial}{\partial \ln k_\perp^2} \left[T_a(Q,k_\perp) D_a(x,k_\perp)\right] = D_a(x,Q) - T_a(Q,Q_0) D_a(x,Q_0)\,, \tag{6}$$

since $T_a(Q,Q) = 1$. Thus, Eq. (5) takes the following form

$$D_a(x,Q) = T_a(Q,Q_0,) D_a(x,Q_0) + \int_{Q_0^2}^{Q^2} \frac{dk_\perp^2}{k_\perp^2} \left\{ T_a(Q,k_\perp) \sum_{a'} \int_x^{1-\Delta} \frac{dz}{z}\, P_{aa'}(z,k_\perp)\, D_{a'}\Big(\frac{x}{z}, k_\perp\Big) \right\}. \tag{7}$$

The first term on the r.h.s. corresponds to the absence of splitting during the evolution from Q_0 to Q while the second one describes a sequence of partonic emissions interlaced with the probabilities for no emissions. This constitutes the Monte Carlo scheme for generation of parton cascades.

The expression in the curly brackets in Eq. (7) defines the unintegrated parton distribution functions (UPDFs),

$$f_a(x,k_\perp,Q) \equiv T_a(Q,k_\perp) \sum_{a'} \int_x^{1-\Delta} \frac{dz}{z}\, P_{aa'}(z,k_\perp)\, D_{a'}\Big(\frac{x}{z}, k_\perp\Big)\,, \tag{8}$$

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[3] Due to relations between the LO splitting functions, one power of z under the integral can be removed at the price of introducing an overall factor 1/2 in the argument of the exponent.

defined for each flavor a (including gluon) for the transverse momenta in the range $Q_0 \le k_\perp \le Q$, which structure is shown in Fig. 1. The region below Q_0 merges into the non-perturbative domain and is effectively described by the initial distribution $D_a(x, Q_0)$ in the first term on the r.h.s. of Eq. (7). For small values of x, parton saturation effects become important in this region of transverse momenta and special attention is necessary in phenomenological approaches to the description of physical processes in this kinematic region, see e.g. [60]. The discussion of such effects, however, is beyond the scope of the present paper.

In order to fully fix the UPDFs, the cutoff parameter Δ in Eqs. (4) and (8) needs to be specified. In Ref. [55] the cutoff was set in the spirit of the DGLAP ordering of parton real emission in transverse momenta to

$$\Delta = \frac{k_\perp}{Q} \,. \tag{9}$$

Thus, from the upper integration limit, $x < (1-\Delta)$, the UPDFs are nonzero for $k_\perp < Q(1-x)$. The Sudakov form factor $T_a(Q, Q_0)$ in Eq. (7) is also regulated with the corresponding cutoff $\Delta_0 = Q_0/Q$.

The prescription was further modified in Ref. [56, 57] to account for the angular ordering in parton emissions in accord with the CCFM evolution scheme [50–53],

$$\Delta = \frac{k_\perp}{k_\perp + Q} \,. \tag{10}$$

In such a case, the nonzero values of the UDPFs are given for $k_\perp < Q(1/x - 1)$. The upper cutoff now is bigger than in the DGLAP scheme. This is particularly important for the small x values which allows for a smooth transition of transverse momenta in the CCFM scheme into the region $k_\perp \gg Q$, see Ref. [56, 57] for more details.

A. Alternative derivation

In this subsection we shall construct an alternative derivation of the unintegrated single parton density. The aim is to prepare the ground and methods for the construction of the unintegrated double parton distributions. The solution of the DGLAP equations (1) can be written in terms of the parton-to-parton evolution distributions $E_{ab}(x, \mu_0, \mu)$, which obey the following equation

$$\frac{\partial}{\partial \ln \mu^2} E_{ab}(x, \mu, \mu_0) = \sum_{a'} \int_x^1 \frac{dz}{z}\, P_{aa'}(z, \mu)\, E_{a'b}\Big(\frac{x}{z}, \mu, \mu_0\Big) - E_{ab}(x, \mu, \mu_0) \sum_{a'} \int_0^1 dz z P_{a'a}(z, \mu) \,, \tag{11}$$

with the initial condition

$$E_{ab}(x, \mu_0, \mu_0) = \delta_{ab}\, \delta(1-x) \,. \tag{12}$$

In the above we have regularized singularity of the splitting functions at $z = 1$ by introducing a small parameter ϵ, e.g. $P^{(0)}_{qq}(z) \sim 1/(1 - z + \epsilon)$ for $z \to 1$. These distributions generate the evolution of the PDFs

$$D_a(x, \mu) = \sum_b \int_x^1 \frac{dz}{z}\, E_{ab}\Big(\frac{x}{z}, \mu, \mu_0\Big)\, D_b(z, \mu_0) \,, \tag{13}$$

since the parton distributions obtained from this relation obey the standard DGLAP evolution equations. This can be easily proven by using the Mellin transform

$$\tilde{A}(n) = \int_0^1 dx\, x^n A(x) \,. \tag{14}$$

Using the above definition, Eq. (11) reads

$$\frac{\partial}{\partial \ln \mu^2} \tilde{E}_{ab}(n, \mu, \mu_0) = \sum_{a'} \tilde{P}_{aa'}(n, \mu)\, \tilde{E}_{a'b}(n, \mu, \mu_0) - \tilde{E}_{ab}(n, \mu, \mu_0) \sum_{a'} \int_0^1 dz z P_{a'a}(z, \mu) \,, \tag{15}$$

with the initial condition $\tilde{E}_{ab}(n, \mu_0, \mu_0) = \delta_{ab}$, while Eq. (13) is given by

$$\tilde{D}_a(n, \mu) = \sum_b \tilde{E}_{ab}(n, \mu, \mu_0)\, \tilde{D}_b(n, \mu_0) \,. \tag{16}$$

Multiplying both sides of Eq. (15) by $\tilde{D}_b(n,\mu_0)$ and summing over b, we obtain Eq. (1) in the Mellin moment space

$$\frac{\partial}{\partial \ln\mu^2}\tilde{D}_a(n,\mu) = \sum_{a'} \tilde{P}_{aa'}(n,\mu)\,\tilde{D}_{a'}(n,\mu) - \tilde{D}_a(n,\mu)\sum_{a'}\int_0^1 dzzP_{a'a}(z,\mu)\,. \tag{17}$$

To find the UPDFs, we set $\mu = k_\perp$ in Eq. (15) and multiply both sides by the Sudakov form factor,

$$T_a(Q,k_\perp) = \exp\left\{-\int_{k_\perp^2}^{Q^2}\frac{dp_\perp^2}{p_\perp^2}\sum_{a'}\int_0^1 dzzP_{a'a}(z,k_\perp)\right\}, \tag{18}$$

to obtain

$$\frac{\partial}{\partial \ln k_\perp^2}\left[T_a(Q,k_\perp)\tilde{E}_{ab}(n,k_\perp,\mu_0)\right] = T_a(Q,k_\perp)\sum_{a'}\tilde{P}_{aa'}(n,k_\perp)\,\tilde{E}_{a'b}(n,k_\perp,\mu_0)\,. \tag{19}$$

Integrating both sides of this equation over $k_\perp$ from $\mu_0 \equiv Q_0$ to Q, we find

$$\tilde{E}_{ab}(n,Q,Q_0) = T_a(Q,Q_0)\,\delta_{ab} + \int_{Q_0^2}^{Q^2}\frac{dk_\perp^2}{k_\perp^2}\,T_a(Q,k_\perp)\sum_{a'}\tilde{P}_{aa'}(n,k_\perp)\,\tilde{E}_{a'b}(n,k_\perp,Q_0)\,, \tag{20}$$

and using Eq. (16), we obtain

$$\tilde{D}_a(n,Q) = T_a(Q,Q_0)\tilde{D}_a(n,Q_0) + \int_{Q_0^2}^{Q^2}\frac{dk_\perp^2}{k_\perp^2}\,T_a(Q,k_\perp)\sum_{a'}\tilde{P}_{aa'}(n,k_\perp)\,\tilde{D}_{a'}(n,k_\perp)\,. \tag{21}$$

The expression under the integral in the above equation is the unintegrated parton distribution (8) in the Mellin moment space. Transforming it into the x space, we find the following equation

$$f_a(x,k_\perp,Q) = T_a(Q,k_\perp)\sum_{a'}\int_x^1\frac{dz}{z}\,P_{aa'}(z,k_\perp)\,D_{a'}\Big(\frac{x}{z},k_\perp\Big)\,, \tag{22}$$

which is equivalent to Eq. (8) after switching from the ϵ regularization of the splitting functions to the regularization with $(1-\Delta)$ in the upper integration limit, both in the above equation and in the Sudakov form factor (18).

III. DOUBLE PARTON DISTRIBUTIONS

We start this section from recalling the evolution equations for the integrated double parton distribution functions, $D_{a_1a_2}(x_2,x_2,Q_1,Q_2)$, following results of Ref. [21] appended by virtual corrections[4]. The evolution of the DPDFs can be cast in the following form

$$\begin{aligned} D_{a_1a_2}(x_1,x_2,Q_1,Q_2) = \sum_{a',a''}\Bigg\{&\int_{x_1}^{1-x_2}\frac{dz_1}{z_1}\int_{x_2}^{1-z_1}\frac{dz_2}{z_2}\,E_{a_1a'}\Big(\frac{x_1}{z_1},Q_1,Q_0\Big)E_{a_2a''}\Big(\frac{x_2}{z_2},Q_2,Q_0\Big)D_{a'a''}(z_1,z_2,Q_0,Q_0) \\ &+\int_{Q_0^2}^{Q^2_{\min}}\frac{dQ_s^2}{Q_s^2}\int_{x_1}^{1-x_2}\frac{dz_1}{z_1}\int_{x_2}^{1-z_1}\frac{dz_2}{z_2}\,E_{a_1a'}\Big(\frac{x_1}{z_1},Q_1,Q_s\Big)E_{a_2a''}\Big(\frac{x_2}{z_2},Q_2,Q_s\Big)D^{(sp)}_{a'a''}(z_1,z_2,Q_s)\Bigg\}\,, \end{aligned} \tag{23}$$

where $Q^2_{\min} = \min\{Q_1^2,Q_2^2\}$, and the distributions E_{ab} obey evolution equation (11). The integration limits take into account kinematic constraints $x_1,x_2 > 0$ and $x_1 + x_2 \le 1$.

[4] The DPDFs also depend on the transverse momentum vector $\mathbf{r}_\perp$, which we set to zero. For $\mathbf{r}_\perp = 0$, the DPDFs in the lowest order approximation are probabilities to find two partons with longitudinal momentum fractions $x_{1,2}$, see [27] for more details.

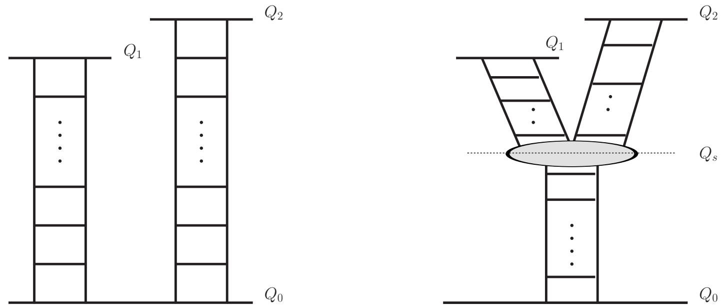

FIG. 2: Schematic illustration of the two contributions to the DPDFs (23). Left: homogeneous term; right: inhomogeneous term. It is understood that all the ladders are cut diagrams.

The first, homogenous, term on the r.h.s. of Eq. (23), is proportional to the double parton density and corresponds to the independent evolution of two partons from the initial scale Q_0 to Q_1 and from Q_0 to Q_2. The second, non-homogeneous, term contains the distribution

$$D^{(sp)}_{a'a''}(x_1, x_2, Q_s) = \frac{\alpha_s(Q_s)}{2\pi} \sum_a \frac{D_a(x_1 + x_2, Q_s)}{x_1 + x_2} \, P_{a\to a'a''}\left(\frac{x_1}{x_1 + x_2}\right), \tag{24}$$

which describes the splitting of the parton $a \to a'a''$. Notice the single PDFs, D_a, at the splitting scale Q_s along with the real emission LO splitting functions (2), $P_{a\to a'a''}(z) = P^{(0)}_{a'a}(z)$ on the r.h.s. . In the LO, the second parton flavor a'' is uniquely determined from the splitting $a \to a'$. The two contributions in Eq. (23) are schematicaly shown in Fig. 2.

The presented results can also be written in the Mellin moment space introducing the double Mellin transform

$$\tilde{A}(n_1, n_2) = \int_0^1 dx_1 \int_0^1 dx_2 \, x_1^{n_1} x_2^{n_2} \, \theta(1 - x_1 - x_2) \, A(x_1, x_2) \,. \tag{25}$$

Then Eq. (23) reads

$$\tilde{D}_{a_1a_2}(n_1, n_2, Q_1, Q_2) = \tilde{D}^{(h)}_{a_1a_2}(n_1, n_2, Q_1, Q_2) + \tilde{D}^{(nh)}_{a_1a_2}(n_1, n_2, Q_1, Q_2) \tag{26}$$

where the homogeneous part of the double distributions is given by

$$\tilde{D}^{(h)}_{a_1a_2}(n_1, n_2, Q_1, Q_2) = \sum_{a',a''} \tilde{E}_{a_1a'}(n_1, Q_1, Q_0) \, \tilde{E}_{a_2a''}(n_2, Q_2, Q_0) \, \tilde{D}_{a'a''}(n_1, n_2, Q_0, Q_0) \tag{27}$$

while the non-homogeneous one reads

$$\tilde{D}^{(nh)}_{a_1a_2}(n_1, n_2, Q_1, Q_2) = \int_{Q_0^2}^{Q^2_{\min}} \frac{dQ_s^2}{Q_s^2} \, \tilde{E}_{a_1a'}(n_1, Q_1, Q_s) \, \tilde{E}_{a_2a''}(n_2, Q_2, Q_s) \, \tilde{D}^{(sp)}_{a'a''}(n_1, n_2, Q_s) \tag{28}$$

and

$$\tilde{D}^{(sp)}_{a'a''}(n_1, n_2, Q_s) = \frac{\alpha_s(Q_s)}{2\pi} \sum_a \tilde{D}_a(n_1 + n_2, Q_s) \int_0^1 dz z^{n_1} (1 - z)^{n_2} P_{a\to a'a''}(z) \,. \tag{29}$$

In the case of equal scales, $Q_1 = Q_1 \equiv Q$, Eq. (26) is a solution in the Mellin moment space to the well known evolution equations [17, 18, 23] for double parton distributions. The proof of this fact is given in Appendix A.

IV. UNINTEGRATED DOUBLE PARTON DISTRIBUTIONS

In this section we shall define the unintegrated double parton distribution functions by essentially generalizing the procedure introduced in [55, 57] for the single PDFs, reviewed in Sec. II. In what follows, we shall discuss the homogeneous and non-homogeneous parts separately as their treatment in the presence of the transverse momentum dependence is rather different.

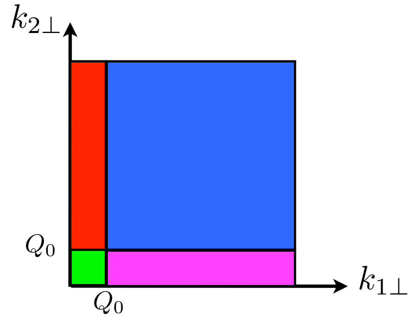

FIG. 3: Different regions of the transverse momenta contributing to Eq. (31). The smallest region with the transverse momenta below Q_0 corresponds to the first term in Eq. (31). In this region the transverse momenta have been integrated over in the corresponding expression. The regions for which either $k_{1\perp} \leq Q_0$ or $k_{2\perp} \leq Q_0$ correspond to the two subsequent terms in Eq. (31). In each case, the smaller transverse momentum has been integrated over. The largest region, with $k_{1\perp}, k_{2\perp} > Q_0$, corresponds to the last term in Eq. (31).

A. Homogeneous part in the Mellin space

Let us first concentrate on the homogeneous part of the double parton distributions (27). Substituting Eq. (20) with the regularized splitting functions there,

$$\tilde{D}^{(h)}_{a_1a_2}(n_1,n_2,Q_1,Q_2) = \sum_{a',a''} \left\{ T_{a_1}(Q_1,Q_0)\, \delta_{a_1a'} + \int_{Q_0^2}^{Q_1^2} \frac{dk^2_{1\perp}}{k^2_{1\perp}}\, T_{a_1}(Q_1,k_{1\perp}) \sum_b \tilde{P}_{a_1b}(n_1,k_{1\perp})\, \tilde{E}_{ba'}(n_1,k_{1\perp},Q_0) \right\}$$

$$\times \left\{ T_{a_2}(Q_2,Q_0)\, \delta_{a_2a''} + \int_{Q_0^2}^{Q_2^2} \frac{dk^2_{2\perp}}{k^2_{2\perp}}\, T_{a_2}(Q_2,k_{2\perp}) \sum_b \tilde{P}_{a_2b}(n_2,k_{2\perp})\, \tilde{E}_{ba''}(n_2,k_{2\perp},Q_0) \right\} \tilde{D}_{a'a''}(n_1,n_2,Q_0,Q_0)\,, \quad (30)$$

and multiplying term by term, we obtain

$$\tilde{D}^{(h)}_{a_1a_2}(n_1,n_2,Q_1,Q_2) = T_{a_1}(Q_1,Q_0)\, T_{a_2}(Q_2,Q_0)\, \tilde{D}_{a_1a_2}(n_1,n_2,Q_0,Q_0)$$

$$+ \int_{Q_0^2}^{Q_2^2} \frac{dk^2_{2\perp}}{k^2_{2\perp}} \left\{ T_{a_1}(Q_1,Q_0)\, T_{a_2}(Q_2,k_{2\perp}) \sum_b \tilde{P}_{a_2b}(n_2,k_{2\perp}) \tilde{D}^{(h)}_{a_1b}(n_1,n_2,Q_0,k_{2\perp}) \right\}$$

$$+ \int_{Q_0^2}^{Q_1^2} \frac{dk^2_{1\perp}}{k^2_{1\perp}} \left\{ T_{a_1}(Q_1,k_{1\perp})\, T_{a_2}(Q_2,Q_0) \sum_b \tilde{P}_{a_1b}(n_1,k_{1\perp}) \tilde{D}^{(h)}_{ba_2}(n_1,n_2,k_{1\perp},Q_0) \right\}$$

$$+ \int_{Q_0^2}^{Q_1^2} \frac{dk^2_{1\perp}}{k^2_{1\perp}} \int_{Q_0^2}^{Q_2^2} \frac{dk^2_{2\perp}}{k^2_{2\perp}} \left\{ T_{a_1}(Q_1,k_{1\perp})\, T_{a_2}(Q_2,k_{2\perp}) \sum_{b,c} \tilde{P}_{a_1b}(n_1,k_{1\perp})\, \tilde{P}_{a_2c}(n_2,k_{2\perp})\, \tilde{D}^{(h)}_{bc}(n_1,n_2,k_{1\perp},k_{2\perp}) \Big] \right\} \quad (31)$$

where we used definition (27) of the homogeneous double parton distributions on the rhs of the above equation. For example, the distribution in the second line reads

$$\tilde{D}^{(h)}_{a_1b}(n_1,n_2,Q_0,k_{2\perp}) = \sum_{a',a''} \delta_{a_1a'}\, \tilde{E}_{ba''}(n_2,k_{2\perp},Q_0)\, \tilde{D}_{a'a''}(n_1,n_2,Q_0,Q_0)\,, \quad (32)$$

since $\tilde{E}_{a_1a'}(n_1,Q_0,Q_0) = \delta_{a_1a'}$.

Similar to (31), but not identical, equation for the DPDFs was postulated in Ref. [26] in the x-space (for the transverse momentum vector $\mathbf{r}_\perp \equiv \Delta \neq 0$). It contains the first three terms of Eq. (31) with some modifications in the second and the third term. Namely, the Sudakov form factors indepndent of $k_\perp$ are missing, and the initial scale

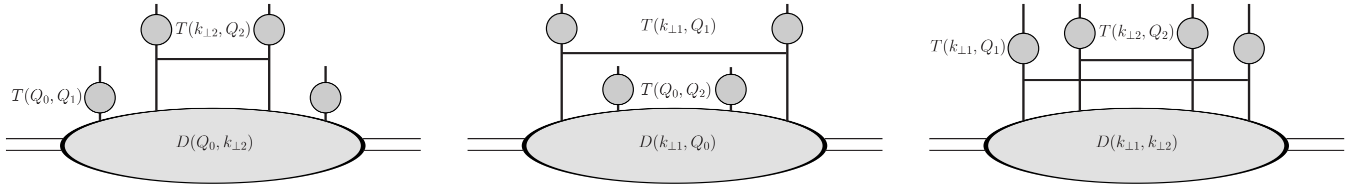

FIG. 4: Schematic representation of the UDPDFs for the three unintegrated DPDFs given by Eqs. (33)-(35), from left to right respectively. The longitudinal momentum structure is suppressed on these plots. The horizontal lines correspond to the real parton emission with the splitting functions and the circular blobs on the vertical lines indicate the Sudakov form factors.

Q_0 in the homogeneous double parton distributions is replaced by Q_1 and Q_2, respectively. This means that the full DGLAP evolution was performed for the corresponding partons. Thus, substituting the evolution operators (20) for this evolution, we find Eq. (31) with the fourth term multiplied by a factor of two. This observation suggests double counting in the equation proposed in Ref. [26].

The four terms in Eq. (31) are defined over four distinct regions of the transverse momenta, which are schematically shown in Fig. 3. The first term in the sum on the r.h.s. of Eq. (31) corresponds to the evolution of the two partons from the initial scale Q_0 to the hard scales, Q_1 and Q_2, without real parton emissions. This term is defined in the region of the lowest transverse momenta, $k_{1\perp}, k_{2\perp} \leq Q_0$, and does not depend on transverse momenta which have been integrated out.

The expressions in the curly brackets in Eq. (31) are *unintegrated double parton distribution functions* (UDPDFs), $\tilde{f}^{(h)}_{a_1 a_2}$, defined in the three remaining regions of transverse momenta. Thus, for $k_{1\perp} \leq Q_0$ and $k_{2\perp} > Q_0$, we find from the first integral

$$\tilde{f}^{(h)}_{a_1 a_2}(n_1, n_2, k_{2\perp}, Q_1, Q_2) = T_{a_1}(Q_1, Q_0)\, T_{a_2}(Q_2, k_{2\perp}) \sum_b \tilde{P}_{a_2 b}(n_2, k_{2\perp})\, \tilde{D}^{(h)}_{a_1 b}(n_1, n_2, Q_0, k_{2\perp})\,. \tag{33}$$

The dependence of the transverse momentum $k_{1\perp}$ is integrated over up to Q_0 in such a case and $k_{1\perp}$ is not present among the arguments of the defined function. The effect of such an integration is hidden in the integrated DPDFs on the r.h.s. taken at the scale Q_0 for the first parton. Similarly, for $k_{1\perp} > Q_0$ and $k_{2\perp} \leq Q_0$, we have from the the second integral

$$\tilde{f}^{(h)}_{a_1 a_2}(n_1, n_2, k_{1\perp}, Q_1, Q_2) = T_{a_1}(Q_1, k_{1\perp})\, T_{a_2}(Q_2, Q_0) \sum_b \tilde{P}_{a_1 b}(n_1, k_{1\perp})\, \tilde{D}^{(h)}_{b a_2}(n_1, n_2, k_{1\perp}, Q_0)\,. \tag{34}$$

Now the momentum $k_{2\perp}$ is integrated up to the scale Q_0 and only $k_{1\perp}$ dependence is present. Finally, for $k_{1\perp}, k_{2\perp} > Q_0$ the third integral gives the dependence on both transverse momenta,

$$\begin{aligned}\tilde{f}^{(h)}_{a_1 a_2}(n_1, n_2, k_{1\perp}, k_{2\perp}, Q_1, Q_2) &= T_{a_1}(Q_1, k_{1\perp})\, T_{a_2}(Q_2, k_{2\perp}) \\ &\times \sum_{b,c} \tilde{P}_{a_1 b}(n_1, k_{1\perp})\, \tilde{P}_{a_2 c}(n_2, k_{2\perp})\, \tilde{D}^{(h)}_{bc}(n_1, n_2, k_{1\perp}, k_{2\perp})\,.\end{aligned} \tag{35}$$

The three unintegrated DPDFs are schematically represented in Fig. 4.

In principle, all the regions of the transverse momenta need to be included for any configuration of the external hard scales Q_1 and Q_2. It is clear though, that some regions will be subdominant depending on the scales, due to the suppression originating from the Sudakov form factors. For example, the first term in Eq. (31) is going to be very small whenever any of the scales is much larger than Q_0.

B. Homogeneous part in the x-space

The corresponding expressions in the x-space can be easily found. For example, for Eq. (35) we obtain

$$f^{(h)}_{a_1a_2}(x_1,x_2,k_{1\perp},k_{2\perp},Q_1,Q_2) = T_{a_1}(Q_1,k_{1\perp})\,T_{a_2}(Q_2,k_{2\perp})$$

$$\times\sum_{b,c}\int_{x_1}^{1-x_2}\frac{dz_1}{z_1}\int_{x_2}^{1-z_1}\frac{dz_2}{z_2}P_{a_1b}\Big(\frac{x_1}{z_1},k_{1\perp}\Big)P_{a_2c}\Big(\frac{x_2}{z_2},k_{2\perp}\Big)D^{(h)}_{bc}(z_1,z_2,k_{1\perp},k_{2\perp})\,. \tag{36}$$

Similarly to the case of the single parton distributions, the integrals over $z_{1,2}$ need to be regularized since the splitting functions can be singular for $z_1 = x_1$ and $z_2 = x_2$. So far in order to be able to manipulate these equations we have implicitly assumed the regularization through the modification of the splitting function by the parameter ϵ, as indicated earlier. Following the original scheme presented in Sec. II A, we now introduce the regularization through the cutoff in the limits of the integrals over the longitudinal momenta. After changing the integration variables, $z_1 \to x_1/z_1$ and $z_2 \to x_2/z_2$, the singularities occur for $z_{1,2} = 1$, thus we change the upper integration limits form 1 to $1 \to (1-\Delta_{1,2})$ to finally find

$$f^{(h)}_{a_1a_2}(x_1,x_2,k_{1\perp},k_{2\perp},Q_1,Q_2) = T_{a_1}(Q_1,k_{1\perp})\,T_{a_2}(Q_2,k_{2\perp})$$

$$\times\sum_{b,c}\int_{\frac{x_1}{1-x_2}}^{1-\Delta_1}\frac{dz_1}{z_1}\int_{\frac{x_2}{1-x_1/z_1}}^{1-\Delta_2}\frac{dz_2}{z_2}P_{a_1b}(z_1,k_{1\perp})\,P_{a_2c}(z_2,k_{2\perp})\,D^{(h)}_{bc}\Big(\frac{x_1}{z_1},\frac{x_2}{z_2},k_{1\perp},k_{2\perp}\Big)\,. \tag{37}$$

The same regularization is necessary for the Sudakov form factors, T_{a_1} and T_{a_2}, with Δ_1 and Δ_2 respectively, see Eq. (8). Following the presentation of the single UPDFs, we choose

$$\Delta_i = \frac{k_{i\perp}}{Q_i}\,. \tag{38}$$

Applying the same procedure to the rest of the UDPDFs, we find for the distribution (33)

$$f^{(h)}_{a_1a_2}(x_1,x_2,k_{2\perp},Q_1,Q_2) = T_{a_1}(Q_1,Q_0)\,T_{a_2}(Q_2,k_{2\perp})$$

$$\times\sum_{b}\int_{\frac{x_2}{1-x_1}}^{1-\Delta_2}\frac{dz_2}{z_2}\,P_{a_2b}(z_2,k_{2\perp})\,D^{(h)}_{a_1b}\Big(x_1,\frac{x_2}{z_2},Q_0,k_{2\perp}\Big). \tag{39}$$

Similarly, we have for the distribution (34)

$$f^{(h)}_{a_1a_2}(x_1,x_2,k_{1\perp},Q_1,Q_2) = \,T_{a_1}(Q_1,k_{1\perp})\,T_{a_2}(Q_2,Q_0)$$

$$\times\sum_{b}\int_{\frac{x_1}{1-x_2}}^{1-\Delta_1}\frac{dz_1}{z_1}P_{a_1b}(z_1,k_{1\perp})\,D^{(h)}_{ba_2}\Big(\frac{x_1}{z_1},x_2,k_{1\perp},Q_0\Big)\,. \tag{40}$$

Remember that the lack of the transverse momenta, $k_{1\perp}$ or $k_{2\perp}$, among the arguments in the above formulas means that they were integrated over up to the scale Q_0.

Eqs. (37)-(40) constitute the main results of our analysis in the x-space. They define the homogeneous part of the unintegrated double parton distribution functions in three distinct domains of the hard scales shown in Fig. 3. Similarly to the single PDF case, the transverse momentum dependence is generated by the last step in the evolution where the distributions become dependent on the transverse momentum and the hard scales. The three unintegrated distributions were also discussed in [27] but only for real emission. Our results follow from a systematic derivation with virtual corrections included. Notice that in the convention adopted in this paper the defined UDPDFs are dimensionless quantities like the integrated DPDFs.

In the presented analysis we neglected the spin and color dependence of the double parton distributions and considered only the color singlet, spin averaged sector. For more information on this aspect see [27, 61] an reference therein. We also do not consider here the dependence on the momentum transfer, setting it to zero.

C. Correlations imposed by cutoffs

Let us analyze whether the double integration over longitudinal momentum fractions z_1 and z_2 in Eq. (36) imposes any restrictions on the choice of the cutoffs $\Delta_{1,2} > 0$. The integration over z_1 gives meaningful result if

$$1 - \Delta_1 > \frac{x_1}{1 - x_2} \qquad \Longrightarrow \qquad 0 < \Delta_1 < \frac{1 - x_1 - x_2}{1 - x_2}. \tag{41}$$

Notice that nonzero values of Δ_1 exist for any value of parton momentum fractions since $(1 - x_1 - x_2)/(1 - x_2) > 0$. On the other hand, the limits of the integration over z_2 should fulfill

$$1 - \Delta_2 > \frac{x_2}{1 - x_1/z_1} \qquad \Longrightarrow \qquad 0 < \Delta_2 < 1 - \frac{x_2}{1 - x_1/z_1}, \tag{42}$$

and the nonzero value of Δ_2 is allowed if

$$1 - \frac{x_2}{1 - x_1/z_1} > 0 \qquad \Longrightarrow \qquad z_1 > \frac{x_1}{1 - x_2}\ . \tag{43}$$

The last condition is always true, which implies that for any fixed values of $x_{1,2}$, the nonzero range of the cutoff values is possible. This means that with the choice (38), the transverse momenta of partons are bounded between zero and some maximal values which depend on $x_{1,2}$ and $Q_{1,2}$.

With given cutoffs $\Delta_{1,2}$, the nonzero UDPDFs are defined in a region of $x_{1,2}$ values which are smaller than that defined by the usual conditions, $x_{1,2} > 0$ and $x_1 + x_2 \leq 1$. In particular, Eq. (42) constrains the lower limit of the z_1 integration,

$$\frac{x_1}{1 - x_2/(1 - \Delta_2)} \ \leq\ z_1\ \leq\ 1 - \Delta_1, \tag{44}$$

which leads to the following condition

$$1 - \frac{x_1}{1 - \Delta_1} - \frac{x_2}{1 - \Delta_2} \geq 0. \tag{45}$$

The region defined by the above condition is indeed smaller than the standard one, $(1 - x_1 - x_2) \geq 0$. In view of these results, with the transverse momentum dependent cutoffs (38), the variables $(x_1, x_2, k_{1\perp}, k_{2\perp}, Q_1, Q_2)$ are strongly correlated in the UDPDFs.

V. NON-HOMOGENEOUS PART OF UDPDFS

We shall now turn to the discussion of the inhomogeneous term in the parton evolution. As we shall see, the inclusion of the transverse momentum dependence for this contribution is much more complicated than for the homogeneous part. This is because, there is another source of the transverse momentum dependence in the double parton distributions. The parent parton can perturbatively split into two daughter partons with transverse momenta $k_{1\perp}, k_{2\perp} \geq Q_0$. This mechanism is a source of parton correlations which is purely perturbative.

A. Transverse momenta from evolution of two partons

Let us consider the non-homogeneous part of Eq. (26) which describes the splitting contribution,

$$\tilde{D}^{(nh)}_{a_1a_2}(n_1, n_2, Q_1, Q_2) = \int_{Q_0^2}^{Q^2_{\min}} \frac{dQ_s^2}{Q_s^2} \sum_{a',a''} \tilde{E}_{a_1a'}(n_1, Q_1, Q_s)\, \tilde{E}_{a_2a''}(n_2, Q_2, Q_s)\, \tilde{D}^{(sp)}_{a'a''}(n_1, n_2, Q_s)\,, \tag{46}$$

where $Q^2_{\min} = \min\{Q_1^2, Q_2^2\}$. We see that there are two potential sources of the transverse momentum dependence in the this formula, from the splitting vertex itself, and from the evolution above the splitting vertex. In this section, we will discuss the latter possibility by applying the procedure developed for the homogeneous part of the UDPDFs. In that way we shall explicitly see the limits of the applicability of this formulation.

Due to the integration over Q_s^2, the splitting contribution (46) is sizable only in the case $Q_{1,2}^2 \gg Q_0^2$, the condition we consider from now on. Substituting Eq. (20) in Eq. (46) and multiplying the obtained expressions term by term, we find the formula similar to that for the homogeneous part (31),

$$\tilde{D}^{(nh)}_{a_1a_2}(n_1,n_2,Q_1,Q_2) = \int_{Q_0^2}^{Q^2_{\min}} \frac{dQ_s^2}{Q_s^2}\Bigg[T_{a_1}(Q_1,Q_s)\,T_{a_2}(Q_2,Q_s)\,\tilde{D}^{(sp)}_{a_1a_2}(n_1,n_2,Q_s)$$

$$+\int_{Q_s^2}^{Q_2^2}\frac{dk_{2\perp}^2}{k_{2\perp}^2}\Big\{T_{a_1}(Q_1,Q_s)\,T_{a_2}(Q_2,k_{2\perp})\sum_b \tilde{P}_{a_2b}(n_2,k_{2\perp})\sum_{a''}\tilde{E}_{ba''}(n_2,k_{2\perp},Q_s)\,\tilde{D}^{(sp)}_{a_1a''}(n_1,n_2,Q_s)\Big\}$$

$$+\int_{Q_s^2}^{Q_1^2}\frac{dk_{1\perp}^2}{k_{1\perp}^2}\Big\{T_{a_2}(Q_2,Q_s)\,T_{a_1}(Q_1,k_{1\perp})\sum_b \tilde{P}_{a_1b}(n_1,k_{1\perp})\sum_{a'}\tilde{E}_{ba'}(n_1,k_{1\perp},Q_s)\,\tilde{D}^{(sp)}_{a'a_2}(n_1,n_2,Q_s)\Big\}$$

$$+\int_{Q_s^2}^{Q_1^2}\frac{dk_{1\perp}^2}{k_{1\perp}^2}\int_{Q_s^2}^{Q_2^2}\frac{dk_{2\perp}^2}{k_{2\perp}^2}\Big\{T_{a_1}(Q_1,k_{1\perp})\,T_{a_2}(Q_2,k_{2\perp})\sum_{b,c}\tilde{P}_{a_1b}(n_1,k_{1\perp})\,\tilde{P}_{a_2c}(n_2,k_{2\perp})$$

$$\times\sum_{a',a''}\tilde{E}_{ba'}(n_1,k_{1\perp},Q_s)\,\tilde{E}_{ca''}(n_2,k_{2\perp},Q_s)\,\tilde{D}^{(sp)}_{a'a''}(n_1,n_2,Q_s)\Big\}\Bigg]\,. \tag{47}$$

Notice that the transverse momenta are confined to the perturbative region only, $k_{1\perp}, k_{2\perp} \geq Q_0$, see the blue rectangle in Fig. 3. Introducing the relation

$$\int_{Q_s^2}^{Q_i^2}\frac{dk_{i\perp}^2}{k_{i\perp}^2} = \int_{Q_0^2}^{Q_i^2}\frac{dk_{i\perp}^2}{k_{i\perp}^2}\,\theta(k_{i\perp}^2 - Q_s^2)\,, \qquad i = 1,2\,, \tag{48}$$

one can change the order of the integrations over Q_s^2 and transverse momenta $k_{i\perp}^2$ in Eq. (47). Thus, from the third integral in the square brackets we find the following non-homogenenous part of the UDPDFs,

$$\tilde{f}^{(nh)}_{a_1a_2}(n_1,n_2,k_{1\perp},k_{2\perp},Q_1,Q_2) = T_{a_1}(Q_1,k_{1\perp})\,T_{a_2}(Q_2,k_{2\perp})\sum_{b,c}\tilde{P}_{a_1b}(n_1,k_{1\perp})\,\tilde{P}_{a_2c}(n_2,k_{2\perp})$$

$$\times\int_{Q_0^2}^{Q^2_{\min}}\frac{dQ_s^2}{Q_s^2}\,\theta(k_{1\perp}^2-Q_s^2)\,\theta(k_{2\perp}^2-Q_s^2)\,\tilde{\mathcal{D}}^{(sp)}_{bc}(n_1,n_2,k_{1\perp},k_{2\perp},Q_s)\,, \tag{49}$$

where we defined new distributions

$$\tilde{\mathcal{D}}^{(sp)}_{bc}(n_1,n_2,k_{1\perp},k_{2\perp},Q_s) = \sum_{a',a''}\tilde{E}_{ba'}(n_1,k_{1\perp},Q_s)\,\tilde{E}_{ca''}(n_2,k_{2\perp},Q_s)\,\tilde{D}^{(sp)}_{a'a''}(n_1,n_2,Q_s)\,. \tag{50}$$

The new distributions have the same structure as the homogeneous distributions $\tilde{D}^{(h)}_{bc}$ in Eq. (35), corresponding to two independent DGLAP evolutions from the scale Q_s (where the collinear splitting of a single parent parton to two daughter partons occurs) up to the scales given by the transverse momenta. The initial conditions for such evolutions are given by the known distributions $\tilde{D}^{(sp)}_{a'a''}(n_1,n_2,Q_s)$, defined in Eq. (28).

The regularized expression for the distribution (49) in the x-space can be found in the same way as for the homogeneous part,

$$f^{(nh)}_{a_1a_2}(x_1,x_2,k_{1\perp},k_{2\perp},Q_1,Q_2) = T_{a_1}(Q_1,k_{1\perp})\,T_{a_2}(Q_2,k_{2\perp})$$

$$\times\int_{\frac{x_1}{1-x_2}}^{1-\Delta_1}\frac{dz_1}{z_1}\int_{\frac{x_2}{1-x_1/z_1}}^{1-\Delta_2}\frac{dz_2}{z_2}\sum_{b,c}P_{a_1b}(z_1,k_{1\perp})P_{a_2c}(z_2,k_{2\perp})$$

$$\times\int_{Q_0^2}^{Q^2_{\min}}\frac{dQ_s^2}{Q_s^2}\,\theta(k_{1\perp}^2-Q_s^2)\,\theta(k_{2\perp}^2-Q_s^2)\,\mathcal{D}^{(sp)}_{bc}\Big(\frac{x_1}{z_1},\frac{x_2}{z_2},k_{1\perp},k_{2\perp},Q_s\Big)\,, \tag{51}$$

where $\Delta_{1,2}$ are given by Eq. (38), and $\mathcal{D}^{(sp)}_{bc}$ are the distributions (50) transformed back to the x-space.

The three remaining terms in Eq. (47) correspond to the situation in which one or two partons from the splitting do not evolve. For example, if the first parton does not evolve, we find the following expression,

$$\tilde{f}^{(nh)}_{a_1a_2}(n_1,n_2,k_{2\perp},Q_1,Q_2) = T_{a_2}(Q_2,k_{2\perp})\sum_b \tilde{P}_{a_2b}(n_2,k_{2\perp}) \\ \times \int_{Q_0^2}^{Q^2_{\min}} \frac{dQ_s^2}{Q_s^2}\,\theta(k_{2\perp}^2-Q_s^2)\,T_{a_1}(Q_1,Q_s)\,\tilde{\mathcal{D}}^{(sp)}_{a_1b}(n_1,n_2,Q_s,k_{2\perp},Q_s) \tag{52}$$

where the new distributions on the r.h.s. now read,

$$\tilde{\mathcal{D}}^{(sp)}_{a_1b}(n_1,n_2,Q_s,k_{2\perp},Q_s) = \sum_{a',a''} \tilde{E}_{a_1a'}(n_1,Q_s,Q_s)\,\tilde{E}_{ba''}(n_2,k_{2\perp},Q_s)\,\tilde{D}^{(sp)}_{a'a''}(n_1,n_2,Q_s)\,, \tag{53}$$

since $\tilde{E}_{a_1a'}(n_1,Q_s,Q_s) = \delta_{a_1a'}$. Comparing Eq. (52) to its homogenenous counterpart (33), we see that in both expressions the transverse momentum $k_{1\perp}$ is not present. In Eq. (33), the momentum $k_{1\perp}$ is integrated out in the non-perturbative domain, $k_{1\perp} \le Q_0$,, thus, we may set the first parton on shell ($k_{1\perp}=0$) in the $k_\perp$-factorized cross sections with off-shell matrix elements. In the case of the distribution (52), however, the transverse momentum $k_{1\perp}$ is integrated out in the perturbative region, $Q_0 \le k_{1\perp} \le Q_s$. Therefore, such a procedure is no longer justified and the distributions (52) cannot be used in the $k_\perp$-factorized cross sections. The same conclusion is valid when the second parton or both partons do not evolve. In summary, only formula (51) in the x-space is acceptable for the UDPDFs in the non-homogenous case.

B. Parton splitting from light-front perturbation theory

In order to address the issues of the transverse momentum dependence coming directly from the perturbative splitting of a single parent parton into two daughter partons, we shall utilize the methods of the light-front perturbation theory.

Let us first start with the rederivation of the splitting term for the integrated parton densities using this framework. The definition of the integrated parton density using the light-front wave functions can be cast in the following form, (see [62])

$$D_a(x) = \frac{1}{x}\sum_n \int \frac{d^2\mathbf{k}_\perp}{2(2\pi)^3}\prod_{i=1}^{n-1}\frac{dy_i}{y_i}\frac{d^2\boldsymbol{\kappa}_{i\perp}}{2(2\pi)^3}\,|\Psi_n(\{y_i,\boldsymbol{\kappa}_{i\perp}\};x,\mathbf{k}_\perp,a)|^2(2\pi)^3\delta^{(2)}(\mathbf{k}_\perp+\sum_{i=1}^{n-1}\boldsymbol{\kappa}_{i\perp})\delta(1-x-\sum_{i=1}^{n-1}y_i)\ , \tag{54}$$

where $\Psi_n(\{y_i,\boldsymbol{\kappa}_{i\perp}\};x,\mathbf{k}_\perp,a)$ is the light-front wave function for n partons. Following [62] we shall use the convention where $\{y_i,\boldsymbol{\kappa}_{i\perp}\}$ are $n-1$ spectator partons with longitudinal momentum fractions y_i and transverse momenta $\boldsymbol{\kappa}_{i\perp}$. The density $D_a(x)$ is defined with respect to the parton of type a (where a could be gluon g, quark q or antiquark $\bar{q}$) with longitudinal momentum fraction x. The transverse momentum $\mathbf{k}_\perp$ of this parton is integrated out and therefore, the above definition is UV divergent and needs to be regulated as we shall see below. In the above definition we implicitly assumed the summation over the colors of the outgoing particles as well as their polarizations. As usual for the light-front calculation, we shall be working in the light cone gauge $A^+=0$.

To derive the splitting term contribution to the double parton density, let us focus from now on the gluon-gluon splitting, $g\to gg$; the other channels can be derived in the analogous way. This contribution is illustrated in Fig. 5, where we show the wave function Ψ_n in which one gluon with momentum $(x_3,\mathbf{k}_{3\perp})$ and polarization λ_3 undergoes the splitting into two daughter gluons with momenta $(x_1,\mathbf{k}_{1\perp})$ and $(x_2,\mathbf{k}_{2\perp})$, with the corresponding polarizations λ_1,λ_2. We recall that on the light-front, all of the particles are on-shell and longitudinal '+' and transverse momentum components are conserved while the '−' components are not. Using the rules of the light-front perturbation theory, see for example [63], we can write the contribution to the wave function Ψ_{n+1} as

$$\Psi^{AB}_{n+1}(\{y_i,\boldsymbol{\kappa}_{i\perp}\};x_1,\mathbf{k}_{1\perp},x_2,\mathbf{k}_{2\perp}) \;=\; V^{\alpha\beta\mu}(k_1,k_2,-k_3)\,\varepsilon^{\lambda_1 *}_\alpha(k_1)\,\varepsilon^{\lambda_2 *}_\beta(k_2)\,\varepsilon^{\lambda_3}_\mu(k_3) f^{ABC}\theta(k_1^+)\,\theta(k_2^+)\,\frac{1}{D_{n+1}} \\ \times\frac{1}{k_3^+}\,\Psi^C_n(\{y_i,\boldsymbol{\kappa}_{i\perp}\};x_3,\mathbf{k}_{3\perp})\ , \tag{55}$$

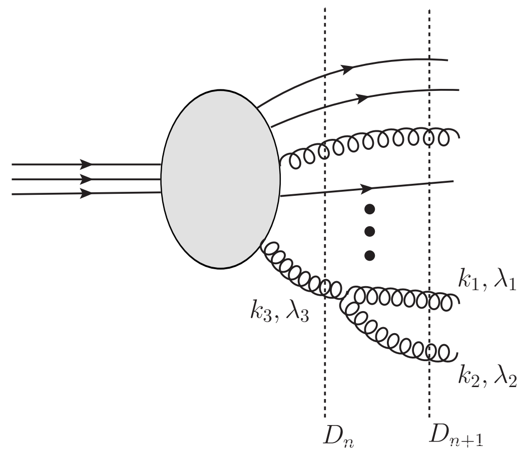

FIG. 5: Splitting contribution to the proton wave function in the light-front framework. The wave function Ψ_n is n component wave function of the initial hadron. The splitting gives contribution to the Ψ_{n+1} wave function. Vertical dashed lines denote light-front energy denominators.

where $V^{\alpha\beta\mu}(k_1, k_2, -k_3)$ is the triple gluon vertex (with all momenta outgoing) and $\varepsilon_\mu^\lambda(k)$ are polarization vectors defined as

$$\varepsilon_\mu^\lambda(k) = (0, \frac{2\vec{\varepsilon}_\perp^\lambda \cdot \mathbf{k}_\perp}{\eta \cdot k}, \vec{\varepsilon}_\perp^\lambda) \ . \tag{56}$$

In the above definition, the light-like vector $\eta = (0,1,0,0)$ and the two-dimensional transverse polarization vectors are defined as $\vec{\varepsilon}_\perp^\pm = \frac{1}{\sqrt{2}}(\pm 1, i)$. In the notation of the vectors we have used standard convention on the light-front where $k^\mu = (k^+, k^-, \mathbf{k}_\perp)$ with $k^\pm = k^0 \pm k^3$ and $\mathbf{k}_\perp = (k^1, k^2)$, thus $k^2 = k^+k^- - \mathbf{k}_\perp^2$. The light-front denominator D_{n+1} is defined as the difference between the light-front energies for the intermediate state and the initial state

$$D_{n+1} = k_1^- + k_2^- + \sum_{i=1}^{n-1} \kappa_i^- - P^- \ , \tag{57}$$

where P^- is the light-front energy of the incoming hadron. In Eq. (55) we also reinstated explicit colors of the gluons. We also introduce the following notation from [64]

$$v_{ij} = \vec{\varepsilon}_\perp^+ \cdot \left(\frac{\mathbf{k}_{i\perp}}{x_i} - \frac{\mathbf{k}_{j\perp}}{x_j} \right), \qquad v_{ij}^* = \vec{\varepsilon}_\perp^- \cdot \left(\frac{\mathbf{k}_{i\perp}}{x_i} - \frac{\mathbf{k}_{j\perp}}{x_j} \right), \tag{58}$$

where $x_i = k_i^+/P^+$. It is easy to see that the contraction of the triple gluon vertex with the polarization vectors leads to the following expressions (see Table 2 in [63])

$$V^{+\to++} = 2igx_3\, v_{21}, \qquad V^{+\to+-} = 2igx_1\, v_{32}^*, \qquad V^{+\to-+} = 2igx_2\, v_{13}^* \ . \tag{59}$$

Here, k_3 is incoming and k_1, k_2 are outgoing momenta, i.e. $k_3 \to k_1 k_2$. Let us first see how the splitting term in the collinear kinematics arises from these expressions. In the collinear case, one assumes strong ordering of the transverse momenta. Hence, $\mathbf{k}_{1\perp} \simeq -\mathbf{k}_{2\perp} \simeq \mathbf{k}_\perp$ and $k_\perp \gg k_{3\perp}$. In this approximation the light-front denominator is given by

$$D_{n+1} \ \simeq \ \frac{k_\perp^2 x_3}{x_1 x_2} \frac{1}{P^+} \ , \tag{60}$$

and the vertices in Eq. (59) reduce to

$$V^{+\to++} = -2igx_3^2\, \frac{\vec{\varepsilon}_\perp^+ \cdot \mathbf{k}_\perp}{x_1 x_2}, \qquad V^{+\to+-} = 2igx_1\, \frac{\vec{\varepsilon}_\perp^- \cdot \mathbf{k}_\perp}{x_2}, \qquad V^{+\to-+} = 2igx_2\, \frac{\vec{\varepsilon}_\perp^- \cdot \mathbf{k}_\perp}{x_1} \ . \tag{61}$$

Squaring the wave function, summing over the final state polarizations and colors, one obtains, see also [62],

$$|\Psi_{n+1}(\{y_i, \boldsymbol{\kappa}_{i\perp}\}; x_1, \mathbf{k}_{1\perp}; x_2, \mathbf{k}_{2\perp})|^2 = 8\pi\alpha_s\, \frac{x_1 x_2}{k_\perp^2 x_3^2}\, P_{gg}^{(0)}\left(\frac{x_1}{x_3}\right) |\Psi_n(\{y_i, \boldsymbol{\kappa}_{i\perp}\}; x_3, \mathbf{k}_{3\perp})|^2 \ . \tag{62}$$

Using this result one can re-derive the standard DGLAP evolution equation for the single parton distribution function as demonstrated in [62]. However, we are interested in the contribution to the double parton distribution function. Therefore, after integrating the wave function Ψ_{n+1} over the transverse momenta $\mathbf{k}_{1\perp}$ and $\mathbf{k}_{2\perp}$ and over the spectator momenta, but keeping both the longitudinal momentum fractions x_1 and x_2 fixed, we obtain the following contribution to the non-homogeneous part of the double integrated distribution function:

$$\begin{aligned}
D_{gg}^{nh}(x_1,x_2) \;=\; & \frac{1}{x_1x_2}\sum_n \int \frac{d^2\mathbf{k}_{1\perp}}{2(2\pi)^3}\int \frac{d^2\mathbf{k}_{2\perp}}{2(2\pi)^3}\prod_{i=1}^{n-1}\frac{dy_i}{y_i}\frac{d^2\boldsymbol{\kappa}_{i\perp}}{2(2\pi)^3} \\
& \times |\Psi_{n+1}(\{y_i,\boldsymbol{\kappa}_{i\perp}\};x_1,\mathbf{k}_{1\perp},x_2,\mathbf{k}_{2\perp})|^2(2\pi)^3\,\delta^{(2)}(\mathbf{k}_{1\perp}+\mathbf{k}_{2\perp}+\sum_{i=1}^{n-1}\boldsymbol{\kappa}_{i\perp})\,\delta(1-x_1-x_2-\sum_{i=1}^{n-1}y_i) = \\
& = \frac{1}{x_1x_2}\int\frac{d^2\mathbf{k}_\perp}{2(2\pi)^3}8\pi\alpha_s\frac{x_1x_2}{k_\perp^2x_3}P_{gg}^{(0)}\Big(\frac{x_1}{x_3}\Big)\frac{1}{x_3}\int\frac{d^2\mathbf{k}_{3\perp}}{2(2\pi)^3}\sum_n\prod_{i=1}^{n-1}\frac{dy_i}{y_i}\frac{d^2\boldsymbol{\kappa}_{i\perp}}{2(2\pi)^3}\,|\Psi_n(\{y_i,\boldsymbol{\kappa}_{i\perp}\};x_3,\mathbf{k}_{3\perp})|^2 \\
& \qquad \times(2\pi)^3\delta^{(2)}(\mathbf{k}_{3\perp}+\sum_{i=1}^{n-1}\boldsymbol{\kappa}_{i\perp})\,\delta(1-x_3-\sum_{i=1}^{n-1}y_i)\,, \qquad (63)
\end{aligned}$$

where we changed the integrated variables from $\mathbf{k}_{1\perp},\mathbf{k}_{2\perp}$ to the sum and the difference i.e. $\mathbf{k}_{3\perp}=\mathbf{k}_{1\perp}+\mathbf{k}_{2\perp}$ and $\mathbf{k}_\perp=\frac{1}{2}(\mathbf{k}_{1\perp}-\mathbf{k}_{2\perp})$. There are two integrals over the transverse momenta $\mathbf{k}_{3\perp}$ and $\mathbf{k}_\perp$ that need to be regulated. Since we are working in the collinear regime ($k_\perp \gg k_{3\perp}$) then we have that the $k_{3\perp}$ integral is regulated by the $k_\perp$ and it gives the integrated parton density

$$\frac{1}{x_1x_2}\int\frac{d^2\mathbf{k}_\perp}{2(2\pi)^3}\,8\pi\alpha_s\,\frac{x_1x_2}{k_\perp^2x_3}\,P_{gg}^{(0)}\Big(\frac{x_1}{x_3}\Big)D_g(x_3,k_\perp)\,. \qquad (64)$$

The integral over $\mathbf{k}_\perp$ in the above needs to be regulated as well with a UV cutoff. Introducing the scale μ, we finally obtain the splitting contribution to the double parton distribution function

$$\frac{\alpha_s}{2\pi}\int^{\mu^2}\frac{dk_\perp^2}{k_\perp^2}\frac{1}{x_3}P_{gg}^{(0)}\Big(\frac{x_1}{x_3}\Big)D_g(x_3,k_\perp)=\frac{\alpha_s}{2\pi}\frac{1}{x_1+x_2}P_{gg}^{(0)}\Big(\frac{x_1}{x_2+x_1}\Big)\int^{\mu^2}\frac{dk_\perp^2}{k_\perp^2}D_g(x_1+x_2,k_\perp)\,, \qquad (65)$$

where we used the fact that $x_3 = x_1 + x_2$. The left hand side of Eq. (63) can be interpreted as the integrated double parton density $D_{gg}(x_1,x_2)$ and thus the last equation gives the splitting contribution to the DPDF. The other channels can be obtained similarly. Differentiating with respect to μ we have that this gives the contribution to the inhomogeneous part of the evolution equation

$$\frac{\partial}{\partial\ln\mu^2}D_{a_1a_2}(x_1,x_2,\mu,\mu)=\frac{\alpha_s}{2\pi}\sum_a P_{a_1a}^{(0)}\Big(\frac{x_1}{x_1+x_2}\Big)\frac{D_a(x_1+x_2,\mu)}{x_1+x_2}\,, \qquad (66)$$

where we reinstated flavor indices a,a_1,a_2 to include other parton transitions. The above equation is equivalent to the non-homogeneous part of the evolution equations for the DPDFs, see Eq. (A2) in Appendix A.

C. Transverse momentum dependence in parton splitting

Let us now see, how the transverse momentum dependence can be introduced into the splitting. To this aim, we can go back to Eq. (63) and analyze the integrand of this expression. In addition we need to keep track of the transverse momenta after the splitting while still working in the strong ordering approximation. Thus we shall assume, $k_\perp \simeq k_{1\perp} \simeq k_{2\perp} \gg k_{3\perp}$. We obtain (for the gluon-gluon splitting case)[5]

$$f_{a_1a_2}(x_1,x_2,\mathbf{k}_{1\perp},\mathbf{k}_{2\perp})=\frac{\alpha_s}{2\pi}\frac{1}{x_1+x_2}\frac{k_{1\perp}^2k_{2\perp}^2}{k_{3\perp}^2k_\perp^2}\,P_{a_1a}^{(0)}\Big(\frac{x_1}{x_2+x_1}\Big)\,f_a(x_1+x_2,\mathbf{k}_{3\perp})\,. \qquad (67)$$

[5] In Eq. (67) there is an angular dependence in the transverse momenta $\mathbf{k}_{1\perp},\mathbf{k}_{2\perp}$. The unintegrated function with such a dependence is therefore integrated in the cross section with the measure $\frac{d^2\mathbf{k}_{1\perp}}{\pi k_{1\perp}^2}\frac{d^2\mathbf{k}_{2\perp}}{\pi k_{2\perp}^2}$. In the case when there is no angular dependence in the unintegrated function, like in Eq. (67), the measure needs to be taken as $\frac{dk_{1\perp}^2}{k_{1\perp}^2}\frac{dk_{2\perp}^2}{k_{2\perp}^2}$.

This contribution has been previously derived in [27], with some modifications which include the polarizations of the produced partons. In that case there is also a second parton distribution, the Boer-Mulders function [65, 66] which describes polarized partons in the unpolarized hadron. The contribution from it vanishes when the angular integrals are performed.

For practical applications we propose to utilize the formula (67) with the unintegrated PDFs modeled according to the KMR approach, discussed in Sec. II,

$$f_a(x_1+x_2, k_{3\perp}, Q) = T_a(Q, k_{3\perp}) \sum_{a'} \int_{x_1+x_2}^{1-\Delta} \frac{dz}{z} P_{aa'}(z, k_{3\perp})\, D_{a'}\!\left(\frac{x_1+x_2}{z}, k_{3\perp}\right). \tag{68}$$

In such a case, the unintegrated double distributions (67) become scale dependent with equal scales, $Q_1 = Q_2 = Q$,

$$f_{a_1a_2}(x_1, x_2, k_{1\perp}, k_{2\perp}, Q, Q) = \frac{\alpha_s}{2\pi} \frac{1}{x_1+x_2} \frac{k_{1\perp}^2 k_{2\perp}^2}{k_{3\perp}^2 k_\perp^2} P_{a_1a}^{(0)}\!\left(\frac{x_1}{x_2+x_1}\right) f_a(x_1+x_2, k_{3\perp}, Q)\ . \tag{69}$$

The reason for equal scales is that this formula only contains evolution of the unintegrated parton density up to a scale Q and then the splitting is treated with the transverse momentum dependence. The two partons from the splitting should evolve now. However, the initial partons have nonzero transverse momenta which may be from the perturbative region, $k_{1\perp}, k_{2\perp} \geq Q_0$. Thus, we should consider QCD radiation with transverse momentum dependent splitting functions, see e.g. [67, 68]. We postpone considering such a case for a future publication.

An important comment is in order here. The scale Q which appears on the right hand side in Eq. (69) is the scale which can be related to the cutoff on the transverse momentum $k_{3\perp}$. The assumption that we are making is that it is the same scale which appears on the left hand side of Eq. (69) for the double parton distribution function. This is motivated by the structure in the integrated form of the non-homogeneous part of Eq. (66). That is Eq. (69), when integrated over the transverse momentum, leads to Eq. (66).

D. Momentum transfer dependence

Let us briefly discuss here another very important issue, namely the momentum transfer dependence in double parton scattering. In general, in addition to the transverse momenta $\mathbf{k}_{1\perp}, \mathbf{k}_{2\perp}$ there is also a dependence on the momentum transfer $\mathbf{r}_\perp$ which can flow around in the double parton scattering diagrams. To take into account this dependence, would require generalization of the definition of the double parton density, see for example (63). This means that one would need to include different momenta in the amplitude and in the complex amplitude, as proposed for example in [20] and discussed more recently in [69]. That is, in general we should be using double generalized parton distribution functions in the description of the double parton scattering processes.

In many phenomenological applications this problem is avoided by taking the momentum transfer dependence into account through a form factor, see for example [20, 36, 37]. However, as pointed out for example in [26] and [24] this is not sufficient. The form factor may be justified for the homogeneous term, provided the two partons are evolving through the DGLAP equations from the non-perturbative wave function of the hadron in the initial state. In this scenario, at least in the case when the evolution is governed by the DGLAP equations, one may assume that the momentum transfer dependence (which in terms of the conjugate variable is related to the transverse distance of the partons) can be modelled via two-gluon non-perturbative form factor. This rests on the assumption that the hard scale in the DGLAP evolution is much larger than the non-perturbative scale which determines the size of the hadron.

This assumption is no longer valid for the inhomogeneous term. There, the perturbative splitting of one parton into two daughter partons can occur at scales much larger than the non-perturbative scale of the hadron. Thus the two daughter partons will be correlated on small distance scales, see [24, 26]. In [27] a more general formula for the contribution to the double parton density from the splitting, which includes the dependence on the momentum transfer variable, has been derived,

$$f_{a_1a_2}(x_1, x_2, \mathbf{k}_{1\perp}, \mathbf{k}_{2\perp}, \mathbf{r}_\perp; Q, Q) = \frac{\alpha_s}{2\pi} \frac{k_{1\perp}^2 k_{2\perp}^2}{\kappa_\perp^2} \frac{(\mathbf{k}_\perp + \frac{1}{2}\mathbf{r}_\perp)\cdot(\mathbf{k}_\perp - \frac{1}{2}\mathbf{r}_\perp)}{(\mathbf{k}_\perp + \frac{1}{2}\mathbf{r}_\perp)^2\,(\mathbf{k}_\perp - \frac{1}{2}\mathbf{r}_\perp)^2} \frac{f_a(x_1+x_2, \kappa_\perp, Q)}{x_1+x_2} P_{a_1a}^{(0)}\!\left(\frac{x_1}{x_1+x_2}\right), \tag{70}$$

which clearly reduces to the previous formula (69) when $\mathbf{r}_\perp \to 0$. Thus, the first step to include the momentum transfer in the double parton densities, would be to include it through the splitting like in Eq. (70). The QCD evolution of the two partons after the splitting should include the transverse momentum dependence, see [67, 68].

Still, the above formula is only a leading power approximation as the longitudinal and transverse momenta have been factorized. It is possible to go beyond the leading power approximation and derive splittings in more exact kinematics. This could be performed starting from the perturbative hadron wave function and considering exact kinematics in the splitting. This would certainly imply additional correlations between longitudinal and transverse components. We shall leave the systematic analysis of these improvements to a future work.

VI. CONCLUSIONS

We presented a construction of the unintegrated double parton distribution functions which depend on parton transverse momenta, $k_{1\perp}$ and $k_{2\perp}$, in addition to their longitudinal momentum fractions, x_1 and x_2, and factorization scales, Q_1 and Q_2. We follow the KMR method [55] to construct the unintegrated single parton distribution functions, which relies on unfolding the last step in the DGLAP evolution of the integrated PDFs. We found two contributions to the unintegrated DPDFs, corresponding to the possibility that the two partons originate either directly from the proton or from the perturbative splitting of a single parton. In the first case, the unintegrated DPDFs in the x-space are given by Eqs. (37)-(40). They correspond to four regions of transverse momenta, shown in Fig. 3.

The perturbative case with parton splitting is more involved. We analyzed two cases, the unfolding of the transverse momentum dependence from the last step in the DGLAP evolution of two partons, and the case where transverse momenta are generated directly from the single parton splitting into two partons. In the first case, we found that only formula (51) is acceptable for the unintegrated DPDFs from the point of view of the $k_\perp$-factorization of the double parton scattering cross sections. In the second case, we propose formula (69), which includes transverse momentum dependence generated from the perturbative splitting of one parton into two daughter partons. In that case, the KMR prescription is applied to the single PDF, in order to introduce the transverse momentum dependence, and then the splitting is treated by including the transverse momentum dependence. We kept the derivation in the strong ordering approximation to be consistent with the rest of the framework.

It should also be mentioned that in our discussion we neglected the spin and color dependence by considering the spin averaged and color singlet case. We also set the momentum transfer $\mathbf{r}_\perp$ to zero. In practical applications, such a dependence is usually modeled with an appropriate form factor, see [20, 36, 37]. This may be justified for the homogeneous term when the hard scales are much bigger than the non-perturbative scale which sets the size of the hadron. In a more refined approach, however, the $\mathbf{r}_\perp$ dependence should be treated with the transverse momentum dependent splitting kernels in the DGLAP evolution. In addition, for the inhomogeneous term, which is of the perturbative origin, the momentum transfer dependence should be explicitly included by computing the parton splitting in more exact kinematics. We postpone such studies to a future work.

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Appendix A: Proof

Eq. (26) with equal scales, $Q_1 = Q_2 \equiv Q$, is a solution to the evolution equations for the DPDFs in the leading logarithmic approximation (LLA). This means that the splitting functions in these equations are proportional to the strong coupling constant in the LLA, which can be absorbed in the definition of a new evolution parameter

$$t = \frac{6}{33-2n_f} \ln \frac{\ln(Q^2/\Lambda^2_{QCD})}{\ln(Q_0^2/\Lambda^2_{QCD})}\,, \tag{A1}$$

where n_f is the number of active quark flavors. In such a case the evolution equations read [17, 18, 23]

$$\begin{aligned}\frac{\partial}{\partial t}\tilde{D}_{a_1a_2}(n_1,n_2,t) \;&=\; \sum_{a'} \tilde{P}_{a_1a'}(n_1)\tilde{D}_{a'a_2}(n_1,n_2,t) - S_{a_1}\tilde{D}_{a_1a_2}(n_1,n_2,t) \\ &+\; \sum_{a'} \tilde{P}_{a_2a'}(n_2)\tilde{D}_{a_1a'}(n_1,n_2,t) - S_{a_2}\tilde{D}_{a_1a_2}(n_1,n_2,t) \;+\; \tilde{D}^{(sp)}_{a_1a_2}(n_1,n_2,t),\end{aligned} \tag{A2}$$

where $\tilde{P}_{aa'}(n)$ are the LLA splitting functions in the Mellin space (anomalous dimensions), $\tilde{D}^{(sp)}_{a_1a_2}$ is given by Eq. (29), and

$$S_a = \sum_{a'} \int_0^1 dzz\hat{P}_{a'a}(z) \,. \tag{A3}$$

Eq. (A2) can be written in the matrix form with respect to flavor indices,

$$\frac{\partial}{\partial t}\tilde{D}(n_1,n_2,t) = \mathcal{P}(n_1)\,\tilde{D}(n_1,n_2,t) + \tilde{D}(n_1,n_2,t)\,\mathcal{P}^\dagger(n_2) + \tilde{D}^{(sp)}(n_1,n_2,t)\,, \tag{A4}$$

where the matrices $\tilde{D} = (\tilde{D}_{ab})$, $\tilde{D}^{(sp)} = (\tilde{D}^{(sp)}_{ab})$ and $\mathcal{P} = (\tilde{P}_{ab} - S_a\,\delta_{ab})$. To solve Eq. (A4), we postulate the solution in the form

$$\tilde{D}(n_1,n_2,t) = \tilde{E}(n_1,t)\,\tilde{D}^0(n_1,n_2,t)\,\tilde{E}^\dagger(n_2,t). \tag{A5}$$

Eq. (A4) is fulfilled if

$$\frac{d\tilde{E}(n,t)}{dt} = \mathcal{P}(n)\,\tilde{E}(n,t)\;, \tag{A6}$$

with the initial condition $\tilde{E}(n,0) = 1$, and

$$\hat{E}(n_1,t)\frac{d\hat{D}^0(n_1,n_2,t)}{dt}\hat{E}^\dagger(n_2,t) = \tilde{D}^{sp}(n_1,n_2,t)\,. \tag{A7}$$

Notice that Eq. (A6) is equivalent to Eq. (15) after changing the evolution variable to t. Thus, $\hat{E}(n,t)$ is the parton-to-parton evolution distribution introduced in Section II A. The solution to Eq. (A6) reads

$$\tilde{E}(n) = \mathrm{e}^{\mathcal{P}(n)t}\,, \tag{A8}$$

therefore, it fulfills the relation

$$\tilde{E}(n,t_1)\tilde{E}(n,t_2) = \tilde{E}(n,t_1+t_2)\,, \tag{A9}$$

which can be used to write Eq. (A7) in the form

$$\frac{d\tilde{D}^0(n_1,n_2,t)}{dt} = \tilde{E}(n_1,-t)\,\tilde{D}^{(sp)}(n_1,n_2,t)\,\tilde{E}^\dagger(n_2,-t)\,. \tag{A10}$$

Its solution is given by

$$\tilde{D}^0(n_1,n_2,t) = \tilde{D}(n_1,n_2,0) + \int_0^t dt'\tilde{E}(n_1,-t')\,\tilde{D}^{(sp)}(n_1,n_2,t')\,\tilde{E}^\dagger(n_2,-t')\,, \tag{A11}$$

where $\tilde{D}(n_1,n_2,0)$ is an initial condition. Substituting (A11) into (A5), we find the final form of the solution,

$$\tilde{D}(n_1,n_2,t) \;=\; \tilde{E}(n_1,t)\,\tilde{D}(n_1,n_2,0)\,\tilde{E}^\dagger(n_2,t) + \int_0^t dt'\,\tilde{E}(n_1,t-t')\,\tilde{D}^{(sp)}(n_1,n_2,t')\,\tilde{E}^\dagger(n_2,t-t')\,, \tag{A12}$$

which is equivalent to relation (26).

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