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Phys. Rev. D **95**, 025012 — Published 17 January 2017

DOI: 10.1103/PhysRevD.95.025012

On the Question of a Possible Infrared Zero in the Beta Function of the Finite-N Gross-Neveu Model

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We investigate whether the beta function of the finite-N Gross-Neveu model, as calculated up to the four-loop level, exhibits evidence for an infrared zero. As part of our analysis, we calculate and analyze Padé approximants to this beta function and evaluate effects of scheme dependence. From our study, we find that in the range of coupling where the perturbative calculation of the four-loop beta function is reliable, it does not exhibit robust evidence for an infrared zero.

I. INTRODUCTION

The Gross-Neveu (GN) model [1] is a quantum field theory in $d = 2$ spacetime dimensions with an N-component massless fermion ψ_j, $j = 1, ..., N$, defined by the path integral

$$Z = \int \prod_x [\mathcal{D}\psi][\mathcal{D}\bar{\psi}] \, e^{i\int d^2x \, \mathcal{L}} \ , \tag{1.1}$$

with the Lagrangian density [2]

$$\mathcal{L} = i\bar{\psi}\partial\!\!\!/\psi + \frac{g}{2}(\bar{\psi}\psi)^2 \ . \tag{1.2}$$

This model is of interest because it exhibits, albeit in a lower-dimensional, non-gauge-theory context, some properties of quantum chromodynamics (QCD), namely asymptotic freedom, dynamical symmetry breaking of a certain chiral symmetry, and the formation of a massive bound state of fermions. These properties were shown by an exact solution of the model in [1] in an $N \to \infty$ limit that enabled Gross and Neveu to obtain nonperturbative information about the theory. A semiclassical calculation of the bound-state spectrum of the model was carried out in [3].

The Gross-Neveu model has also been studied at finite N, where it is not, in general, exactly solvable. In these studies, one again makes use of a property that the model shares with QCD, namely asymptotic freedom, which allows one to carry out reliable perturbative calculations at high Euclidean energy/momentum scales μ in the deep ultraviolet (UV), where the running four-fermion coupling, $g(\mu)$, approaches zero. In this context, there is an interesting and fundamental question: how does this running coupling $g(\mu)$ change as the scale μ decreases from the deep UV to the infrared (IR) limit at $\mu = 0$? This change of $g(\mu)$ as a function of μ is described by the renormalization group (RG) [4] and the associated beta function, $\beta = dg/dt$, where $dt = d\ln\mu$. The asymptotic freedom property is equivalent to the fact that β is negative in the vicinity of the origin, $g = 0$, so that this point is a UV fixed point (UVFP) of the renormalization group. As μ decreases from the UV toward the IR, several different types of behavior of a theory are, *a priori*, possible. One is that the (perturbatively calculated) beta function has no IR zero, so that as μ decreases, $g(\mu)$ eventually increases beyond the range where perturbative methods can be used to study its RG evolution. An alternative possibility is that β has an IR zero at sufficiently small coupling so that it can be studied using perturbative methods. An exact IR zero of β would be an IR fixed point (IRFP) of the renormalization group. In the $N \to \infty$ limit used in [1] to solve the model, the resultant beta function (given below in Eq. (2.9)) does not exhibit any IR zero. Ref. [5] calculated $1/N$ corrections to the $N \to \infty$ limit in the Gross-Neveu model and excluded the presence of an IR zero to this order. However, to our knowledge, there has not been an analysis of the beta function of the GN model for finite N to higher-loop order to address the question of whether it exhibits evidence for an infrared fixed point.

In this paper we shall carry out this analysis of the beta function of the finite-N Gross-Neveu model to address and answer the question of whether this function exhibits an IR zero. We shall investigate the beta function to the highest loop order to which it has been calculated, namely four loops, making use of a recent computation of the four-loop term in Ref. [6].

This paper is organized as follows. In Section II we review some background information about the Gross-Neveu model. In Section III we carry out our analysis of the beta function of the finite-N Gross-Neveu model up to the four-loop level. In Section IV we extend this analysis using Padé approximants. Section V contains an analysis of the effect of scheme transformations on the beta function. In Section VI we comment further on the large-N limit. Our conclusions are given in Section VII.

II. SOME RELEVANT BACKGROUND ON THE GROSS-NEVEU MODEL

Here we briefly review some relevant background concerning the Gross-Neveu model. We first comment on some notation. In Ref. [1], the coefficient in front of the $(\bar{\psi}\psi)^2$ operator was written as a squared coupling, which we denote as $(g_{GN}^2/2)$, while many subsequent works have written it as $g/2$, so one has

$$g \equiv g_{GN}^2 \ . \tag{2.1}$$

The analysis of the model in [1] made use of a functional integral identity to express the path integral as the $m \to \infty$ limit of a path integral containing an auxiliary real scalar field ϕ with a mass m and a Yukawa interaction

$$\mathcal{L}_Y = g_{GN} m [\bar{\psi}\psi]\phi \ . \tag{2.2}$$

Since ϕ is a real field, the hermiticity of $\mathcal{L}_Y$ implies that g_{GN} must be real, which, in conjunction with Eq. (2.1), implies that g must be non-negative:

$$g \ge 0 \ . \tag{2.3}$$

For $d = 2$ (as more generally, for any even spacetime dimension), one can define a product of Dirac gamma matrices, denoted γ_5, that satisfies the anticommutation relation $\{\gamma_5, \gamma_\mu\} = 0$ for all γ_μ. This γ_5 matrix also satisfies $\gamma_5^2 = 1$ and $\gamma_5^\dagger = \gamma_5$. (An explicit representation is $\gamma_0 = \sigma_1$, $\gamma_1 = \sigma_2$, with $\gamma_0\gamma_1 = i\gamma_5 = i\sigma_3$, where σ_j are the Pauli matrices.) One can then define chiral projection operators $P_{L,R} = (1/2)(1 \pm \gamma_5)$. As usual, one then defines left and right chiral components of the fermion field as $\psi_L = P_L\psi$ and $\psi_R = P_R\psi$.

The Gross-Neveu model is invariant under a discrete global $\mathbb{Z}_2$ group generated by the identity and the chiral transformation

$$\psi \to \gamma_5 \psi \ . \tag{2.4}$$

This discrete chiral transformation (2.4) takes $\bar{\psi}\psi \to -\bar{\psi}\psi$, and hence this $\mathbb{Z}_2$ symmetry forbids (i) a mass term in the Lagrangian (1.2) and (ii) the generation of a nonzero condensate $\langle \bar{\psi}\psi \rangle$. This is true to all (finite) orders of perturbation theory.

The Gross-Neveu model is also invariant under the continuous global (cg) symmetry group

$$G_{cg} = \mathrm{U}(N) \tag{2.5}$$

defined by the transformation

$$\psi \to U\psi \ , \tag{2.6}$$

where $U \in \mathrm{U}(N)$ (so $\bar{\psi} \to \bar{\psi}U^\dagger$). In terms of the chiral components of the fermion field, the continuous global symmetry transformation (2.6) is $\psi_L \to U\psi_L$, $\psi_R \to U\psi_R$. In contrast to the discrete γ_5 symmetry, the continuous symmetry G_{cg} leaves the operator $\bar{\psi}\psi$ invariant [7].

An exact solution of the theory was obtained in [1] in the limit $N \to \infty$ and $g_{GN} \to 0$ with the product

$$\lambda \equiv g_{GN}^2 N \equiv gN \tag{2.7}$$

a fixed and finite function of μ. We shall denote this as the LN limit (i.e., the large-N limit with the condition (2.7) imposed). In this limit, there is a nonperturbative generation of a nonzero bilinear fermion condensate, $\langle \bar{\psi}\psi \rangle$, dynamically breaking the discrete $\mathbb{Z}_2$ chiral symmetry. In this limit, there is also the formation of a massive bound state of fermions.

The beta function for g_{GN} is

$$\beta_{GN} = \frac{dg_{GN}}{dt} \ , \tag{2.8}$$

where $dt = d\ln\mu$. (The μ dependence of the coupling will often be suppressed in the notation.) In the LN limit, this beta function is [1, 8]

$$\beta_{GN} = -\frac{g_{GN}\lambda}{2\pi} \ . \tag{2.9}$$

The fact that this beta function is negative is an expression of the asymptotic freedom of the theory. This beta function does not exhibit any zero away from the origin, i.e., any infrared zero. However, since the calculation in [1] was performed in the LN limit, this leaves open the possibility that at finite N, there could be an IR zero in the beta function that would disappear in the LN limit. We discuss this LN limit further in Section VI below.

III. BETA FUNCTION FOR GENERAL N

Although the Gross-Neveu model is not, in general, solvable away from the LN limit, there has also been interest over the years in analyzing it for finite N. In terms of the coupling g, the beta function of the finite-N GN model is

$$\beta = \frac{dg}{dt} \ , \tag{3.1}$$

where, as before, $dt = d\ln\mu$. For our purposes, it will be convenient to introduce a variable a that includes the factor $1/(2\pi)$ resulting from Feynman integrals in $d = 2$ dimensions, namely

$$a = \frac{g}{2\pi} = \frac{g_{GN}^2}{2\pi} \ . \tag{3.2}$$

The model defined by the Lagrangian of Eq. (1.2) can be generalized with the addition of further four-fermion operators [1, 9]. The regularization and renormalization of the Gross-Neveu model has been carried out in this more general context [9]-[13], [6].

As was true of other theories, such as the nonlinear σ model [14], one may consider this model in spacetime dimension $d > 2$. At finite N, the model is not renormalizable for $d > 2$, since the Maxwellian dimension of a four-fermion operator is $2(d-1)$, which is larger than d if $d > 2$. As in the case of the nonlinear σ model [14], in the $N \to \infty$ limit, one can still solve the model and study its properties. Alternatively, for finite N, one can regard it as a low-energy effective field theory. With this generalization and $d \gtrsim 2$, β has the form

$$\begin{aligned} \beta &= g\Big[d-2+\sum_{\ell=1}^{\infty} b_\ell \Big(\frac{g}{2\pi}\Big)^\ell \Big] \\ &= 2\pi a\Big[d-2+\sum_{\ell=1}^{\infty} b_\ell \, a^\ell \Big] , \end{aligned} \tag{3.3}$$

where $b_\ell a^\ell$ is the ℓ-loop term. The n-loop ($n\ell$) beta function, denoted $\beta_{n\ell}$, is obtained by the replacement of $\ell = \infty$ by $\ell = n$ in Eq. (3.3). Early discussions of the GN model for $d > 2$ include [1] and [10]; for more recent work see, e.g., [6], [15], and, for condensed-matter applications, [16], and references therein. In this paper, aside from some comments in Section VI, we will restrict ourselves to the Gross-Neveu model in $d = 2$, where g is dimensionless.

The $\ell = 1$ and $\ell = 2$ loop terms in β are independent of the scheme used for regularization and renormalization, while the terms at loop order $\ell \ge 3$ are scheme-dependent. The beta function was calculated up to two-loop level in [11], with the results

$$b_1 = -2(N-1) \tag{3.4}$$

and

$$b_2 = 2(N-1) \ . \tag{3.5}$$

(See also [17] for a two-loop calculation in a related Thirring model.) The fact that b_1 in Eq. (3.4) is negative means that in $d = 2$, this theory is asymptotically free for any finite $N > 1$ as well as in the $N \to \infty$ limit considered in [1].

The three-loop coefficient, b_3, was calculated in [12, 13] in the commonly used scheme with dimensional regularization and modified minimal subtraction, denoted $\overline{\rm MS}$ [18], yielding the result

$$b_3 = \frac{(N-1)(2N-7)}{2} \ . \tag{3.6}$$

Recently, the four-loop coefficient, b_4 has been calculated, again in the $\overline{\rm MS}$ scheme, to be [6]

$$b_4 = \frac{1}{3}(N-1)\Big[-2N^2-19N+24-6(11N-17)\zeta_3\Big] \ , \tag{3.7}$$

where $\zeta_s = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function.

We comment on the dependence of the beta function coefficients on N. The property that these coefficients all contain a factor of $(N-1)$ is a consequence of the fact that for $N = 1$ the GN model is equivalent to the massless abelian Thirring model [19], which has an identically zero beta function [20, 21]. Note that this statement about the beta function of the Thirring model is scheme-independent; if a beta function vanishes in one scheme, then it vanishes in all other schemes reached by acceptable (nonsingular) scheme transformations [22]. It follows that all of the coefficients b_ℓ contain a factor of $(N-1)$. Therefore, it is only necessary to analyze the beta function of the Gross-Neveu model for $N > 1$, where it is nonvanishing, and we will thus restrict to the physical integral values $N \ge 2$ henceforth. We next discuss how the b_ℓ depend on N in the relevant range $N > 1$. For this discussion, we consider N to be extended from the positive integers to the real numbers. The three-loop coefficient b_3 is a monotonically increasing function of N that is negative for $N < 7/2$, vanishes for $N = 7/2$, and is positive for $N > 7/2$. Thus, for physical, integral values, $b_3 < 0$ if $N = 2$ or $N = 3$ and $b_3 > 0$ if $N \ge 4$. The coefficient b_4 is negative for large N and is positive for N in the interval

$$N_{b4z,m} < N < N_{b4z,p} \ , \tag{3.8}$$

where the subscript $b4z$ stands for "b_4 zero" and

$$N_{b4z,(p,m)} = \frac{-19-66\zeta_3 \pm \sqrt{553+3324\zeta_3+4356\zeta_3^2}}{4} \tag{3.9}$$

with (p,m) corresponding to the $\pm$ sign. These have the values $N_{b4z,m} = -50.616$ and $N_{b4z,p} = 1.448$ to the given floating-point accuracy. Thus, in the relevant range $N > 1$ under consideration here, b_4 is negative.

We proceed to investigate the question of whether the beta function for the Gross-Neveu model at finite N exhibits evidence for an infrared zero. We denote an IR zero of the n-loop beta function $\beta_{n\ell}$ as $a_{IR,n\ell}$, and the corresponding value of g as $g_{IR,n\ell} = 2\pi a_{IR,n\ell}$. This IR zero of beta is a zero for positive a closest to the origin (if there is such a zero), which one would thus reach as μ decreases from the deep UV at large μ to the IR at small μ and a increases from 0. At the two-loop level, $\beta_{2\ell}$ has an IR zero at

$$a_{IR,2\ell} = -\frac{b_1}{b_2} = 1 \ , \tag{3.10}$$

i.e., $g_{IR,2\ell} = 2\pi$. Note that this value is independent of N. To judge whether this constitutes convincing evidence of an IR zero in the beta function, it is necessary to determine if higher-loop calculations confirm it. We next carry out this task.

At the three-loop level, the condition that $\beta_{3\ell} = 0$ away from the origin is the quadratic equation $b_1 + b_2 a + b_3 a^2 = 0$. This has two solutions,

$$a = \frac{2[-1 \pm \sqrt{2(N-3)}\ ]}{2N-7} \ . \tag{3.11}$$

If $N < 3$, then these solutions are complex and hence unphysical. If $N = 3$, these roots coincide, so that $a_{IR,3\ell} = 2$, i.e., $g_{IR,3\ell} = 4\pi$. For $N \ge 3$, there is only one physical root, namely

$$a_{IR,3\ell} = \frac{2[-1 + \sqrt{2(N-3)}\ ]}{2N-7} \ . \tag{3.12}$$

However, this is not, in general, close to the two-loop zero of the beta function at $a_{IR,2\ell} = 1$. Furthermore, while

$a_{IR,2\ell} = 1$ is independent of N, $a_{IR,3\ell}$ has a completely different behavior as a function of N; it decreases monotonically with N in the interval $N \ge 3$ over which it is physical and approaches zero asymptotically like

$$a_{IR,3\ell} \sim \sqrt{\frac{2}{N}} - \frac{1}{N} + O\Big(\frac{1}{N^{3/2}}\Big) \quad \text{as } N \to \infty \ . \quad (3.13)$$

At the four-loop level, the condition that $\beta_{4\ell} = 0$ away from the origin is the cubic equation

$$b_1 + b_2 a + b_3 a^2 + b_4 a^3 = 0 \ . \quad (3.14)$$

The nature of the roots of this equation is determined by the discriminant,

$$\Delta_3 = b_2^2 b_3^2 - 27 b_1^2 b_4^2 - 4(b_1 b_3^3 + b_4 b_2^3) + 18 b_1 b_2 b_3 b_4 \ . \quad (3.15)$$

This discriminant is negative for the relevant range $N \ge 2$ (indeed, it is negative for all real N). This implies that Eq. (3.14) has one real root and a pair of complex-conjugate roots. The real root is negative and hence is unphysical, since it violates the positivity requirement (2.3). Moreover, since it is negative, it is clearly incompatible with the values of $a_{IR,2\ell}$ and $a_{IR,3\ell}$, which are positive (discarding the unphysical complex value of $a_{IR,3\ell}$ at $N = 2$). We therefore do not label this root as $a_{IR,4\ell}$, but instead as $a_{rt,4\ell}$, where rt stands simply for the real root of Eq. (3.14). We find that the magnitude of $a_{rt,4\ell}$ decreases toward zero monotonically as N increases in the relevant interval $N \ge 2$, with the asymptotic behavior

$$a_{rt,4\ell} \sim -\frac{3^{1/3}}{N^{2/3}} + \frac{1}{2N} + O\Big(\frac{1}{N^{4/3}}\Big) \quad \text{as } N \to \infty \ . \quad (3.16)$$

We list the values of $a_{IR,2\ell}$, $a_{IR,3\ell}$, and $a_{rt,4\ell}$ in Table I for N from 2 to 10 and for three representative larger values, $N = 100$, 300, and 10^3.

In our discussion above, we had stated that in order to judge whether the result for $a_{IR,2\ell}$ constitutes convincing evidence of an IR zero in the beta function, it is necessary to determine if higher-loop calculations confirm it. A necessary condition for the reliability of a perturbative calculation is that if one calculates some quantity to a given loop order, then there should not be a large fractional change in this quantity if one computes it to one higher order in the loop expansion. This condition applies, in particular, to the calculation of a putative zero of the beta function. Quantitatively, in order for the perturbative calculation of the IR zero of a beta function to be reliable, it is necessary that the fractional difference

$$\frac{|a_{IR,(n-1)\ell} - a_{IR,n\ell}|}{\frac{1}{2}[a_{IR,(n-1)\ell} + a_{IR,n\ell}]} \quad (3.17)$$

should be reasonably small and should tend to decrease with increasing loop order, n. As is evident both from our analytic formulas and from the numerical results listed in Table I, this necessary condition is not satisfied in the present case.

TABLE I: Values of $a_{IR,2\ell}$, $a_{IR,3\ell}$, and $a_{rt,4\ell}$ for the beta function of the Gross-Neveu model, as a function of N. Here, the three-loop and four-loop coefficients b_3 and b_4 are calculated in the $\overline{\rm MS}$ scheme. If $N = 2$, then the zeros of $\beta_{3\ell}$ at nonzero a form an unphysical complex (cmplx) pair. As indicated, all of the values of $a_{rt,4\ell}$ are negative and hence unphysical. See text for further details.

N	$a_{IR,2\ell}$	$a_{IR,3\ell}$	$a_{rt,4\ell}$
2	1	cmplx	-0.573
3	1	2.000	-0.370
4	1	0.828	-0.302
5	1	0.667	-0.264
6	1	0.580	-0.239
7	1	0.522	-0.220
8	1	0.481	-0.205
9	1	0.448	-0.194
10	1	0.422	-0.184
100	1	0.134	-0.0567
300	1	0.0788	-0.0295
10^3	1	0.0438	-0.0138

The reason for this is clear from a plot of the beta functions $\beta_{n\ell}$ at loop orders $n = 2$, $n = 3$, and $n = 4$. This shows that the IR zero in the two-loop beta function occurs at a value of a that is too large for the perturbative calculation to be reliable. In Figs. 1 and 2 we plot the two-loop, three-loop, and four-loop beta functions for the Gross-Neveu model as functions of a for two illustrative values of N, namely $N = 3$ and $N = 10$. As is evident from these plots, the beta function does not satisfy the necessary criterion for the reliability of a calculation of an IR zero. For the IR zero of the two-loop beta function at $a_{IR,2\ell} = 1$ to be reliable, one requires that the curves for the three-loop and four-loop beta functions should agree approximately with the curve for the two-loop beta function for $a \simeq 1$, and that these higher-loop beta functions should thus have respective IR zeros that are close to the two-loop zero at $a_{IR,2\ell} = 1$. But this is not the case; for $N = 3$, $\beta_{3\ell}$ has a double zero at the larger value, $a_{IR,3\ell} = 2$ and then goes negative again, while $\beta_{4\ell}$ has no IR zero in the physical region, $a > 0$. For $N = 10$ the three-loop beta function $\beta_{3\ell}$ vanishes at a smaller value of a than $a = 1$ (and this value, $a_{IR,3\ell}$ decreases as N increases), while the four-loop beta function $\beta_{4\ell}$ again has no IR zero in the physical region, $a > 0$. The behavior illustrated for $N = 10$ is generic for other values of $N \ge 4$. Indeed, the curves for these beta functions at loop order $n = 2,\ 3,\ 4$ only agree with each other close to the origin, and deviate strongly from each other before one gets to values of a where a zero occurs. Specifically, for $N = 3$, $\beta_{2\ell}$ and $\beta_{3\ell}$ only agree with each other for a up to about 0.5, while $\beta_{4\ell}$ deviates from these lower-loop beta functions as a increases beyond approximately 0.2. As N increases, these deviations occur for smaller a. Thus, for

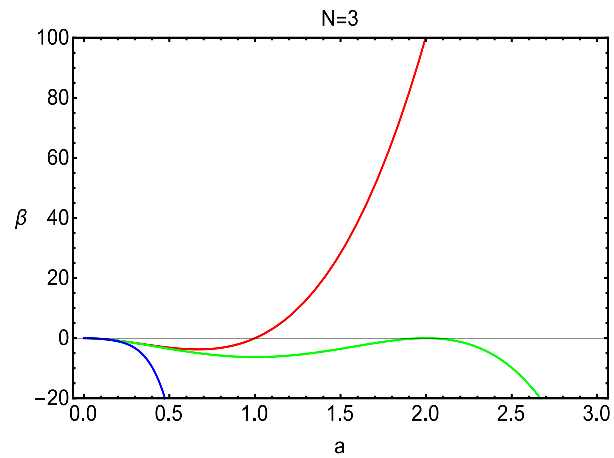

FIG. 1: Plot of the n-loop β function $\beta_{a,n\ell}$ of the Gross-Neveu model as a function of a for $N = 3$ and (i) $n = 2$ (red), (ii) $n = 3$ (green), and (iii) $n = 4$ (blue) (colors in online version). At $a = 0.5$, going from bottom to top, the curves are $\beta_{4\ell}$, $\beta_{3\ell}$, and $\beta_{2\ell}$.

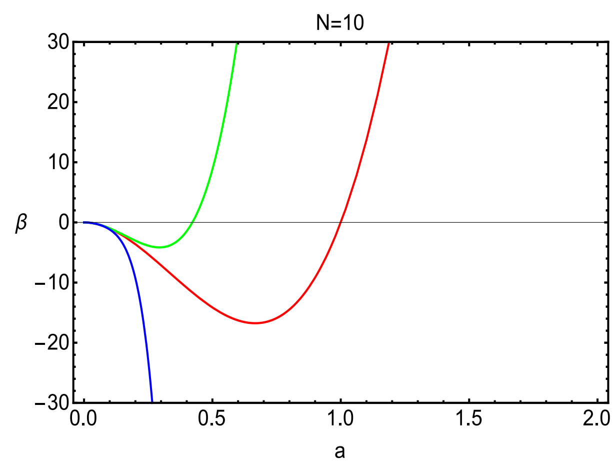

FIG. 2: Plot of the n-loop β function $\beta_{a,n\ell}$ of the Gross-Neveu model as a function of a for $N = 10$ and (i) $n = 2$ (red), (ii) $n = 3$ (green), and (iii) $n = 4$ (blue) (colors in online version). At $a = 0.2$, going from bottom to top, the curves are $\beta_{4\ell}$, $\beta_{2\ell}$, and $\beta_{3\ell}$.

$N = 10$, $\beta_{2\ell}$ and $\beta_{3\ell}$ only agree with each other for a up to roughly 0.15, while $\beta_{4\ell}$ deviates from these lower-loop beta functions as a increases beyond about 0.08.

These results are similar to what was found in a search for a UV zero in the beta function of an IR-free theory, namely the O(N) $\lambda|\vec{\phi}|^4$ scalar field theory in $d = 4$ spacetime dimensions [23]. In that theory, although the two-loop beta function exhibits a UV zero, higher-loop calculations up to five-loop order for general N and up to six-loop order for $N = 1$ do not confirm the two-loop result, and the reason was found to be that the two-loop UV zero occurs at too large a value of the quartic coupling for the two-loop perturbative calculation to be applicable and reliable.

IV. ANALYSIS WITH PADÉ APPROXIMANTS

In this section we carry out a further investigation of a possible IR fixed point in the renormalization-group flow for the Gross-Neveu model by calculating and analyzing Padé approximants (PAs) to the beta function at three-loop and four-loop level. Since we are interested in a possible zero of the beta function away from the origin, it will be convenient to deal with a reduced (rd) beta function,

$$\beta_{rd} \equiv \frac{\beta}{2\pi b_1 a^2} = 1 + \frac{1}{b_1}\sum_{\ell=2}^{\infty} b_\ell a^{\ell-1} \ . \qquad (4.1)$$

The n-loop reduced beta function with $n \geq 2$, denoted $\beta_{rd,n\ell}$, is obtained from Eq. (4.1) by replacing $\ell = \infty$ by $\ell = n$ as the upper limit in the summand. This n-loop reduced beta function is thus a polynomial of degree $n-1$ in a. The $[p,q]$ Padé approximant to this polynomial is the rational function

$$[p,q]_{\beta_{rd,n\ell}} = \frac{1 + \sum_{j=1}^{p} n_j x^j}{1 + \sum_{k=1}^{q} d_k\, x^k} \qquad (4.2)$$

with

$$p + q = n - 1 \ , \qquad (4.3)$$

where the n_j and d_k are a-independent coefficients of the respective polynomials in the numerator and denominator of $[p,q]_{\beta_{rd,n\ell}}$. (Our notation follows [24].) Hence, at a given n-loop order, there are n Padé approximants that one can calculate, namely

$$\{\ [n-k, k-1]_{\beta_{rd,n\ell}}\ \} \quad \text{with } 1 \leq k \leq n \ . \qquad (4.4)$$

These provide rational-function approximations of the series expansion for $\beta_{rd,n\ell}$ that fits this series to the loop order n. As in our earlier work, e.g., [25, 26], these provide an alternate approach to investigating zeros of a beta function.

We shall label one of the p zeros of a $[p,q]_{\beta_{rd,n\ell}}$ Padé approximant as $[p,q]_{zero}$ and one of the q poles of this approximant as $[p,q]_{pole}$; in each case, the value of n is given by Eq. (4.3) as $n = p + q + 1$. At the n-loop level, the Padé approximant $[n-1,0]_{\beta_{rd,n\ell}}$ is equal to the reduced n-loop beta function $\beta_{rd,n\ell}$ itself, which we have already analyzed in the previous section, and the PA $[0,n-1]_{\beta_{rd,n\ell}}$ has no zeros, and hence is not useful for our study. Hence, at the n-loop level, we focus on the $n-2$ PAs $[p,q]_{\beta_{rd,n\ell}}$ with $[p,q] = [n-k, k-1]$ having $2 \leq k \leq n-1$.

At the $n = 3$ loop level, we thus consider the $[1,1]_{\beta_{rd,3\ell}}$ Padé approximant. This is

$$[1,1]_{\beta_{rd,3\ell}} = \frac{1 + \left(\frac{b_2}{b_1} - \frac{b_3}{b_2}\right)a}{1 - \left(\frac{b_3}{b_2}\right)a} = \frac{1 - \left(\frac{2N-3}{4}\right)a}{1 - \left(\frac{2N-7}{4}\right)a} \ . \qquad (4.5)$$

where the coefficients b_1, b_2, and b_3 were given in Eqs. (3.4)-(3.6) above. This [1,1] PA has a zero at

$$[1,1]_{zero} = \frac{4}{2N-3} \tag{4.6}$$

and a pole at

$$[1,1]_{pole} = \frac{4}{2N-7} \ . \tag{4.7}$$

The $a = [1,1]_{pole}$ is not relevant, since if $N = 2$ or 3, it has the respective negative and hence unphysical values $-4/3$ and -4, while for $N \ge 4$, it lies farther from the origin than the zero. This is clear from the fact that the difference

$$[1,1]_{pole} - [1,1]_{zero} = \frac{16}{(2N-3)(2N-7)} \tag{4.8}$$

is positive for this range $N \ge 4$. Since the $[1,1]_{pole}$ lies farther from the origin than $[1,1]_{zero}$, the coupling $a = a(\mu)$ never reaches the pole as μ decreases from large values in the UV to $\mu = 0$ and thus $a(\mu)$ increases from 0 to $[1,1]_{zero}$. We list the values of the zero of the $[1,1]_{\beta_{rd,3\ell}}$ Padé approximant in Table II. For $N \ge 3$, the value of $a = [1,1]_{zero}$ is smaller than $a_{IR,3\ell}$ and decreases more rapidly to zero as $N \to \infty$ than $a_{IR,3\ell}$. If $N = 3$, the comparison cannot be made, since $a_{IR,3\ell}$ is complex. Thus, this analysis of the [1,1] Padé approximant to the reduced three-loop beta function, $\beta_{rd,3\ell}$ yields further evidence against a (reliably calculable) IR zero in the beta function up to the three-loop level.

At the $n = 4$ loop level, there are two Padé approximants to analyze, namely $[2,1]_{\beta_{rd,4\ell}}$ and $[1,2]_{\beta_{rd,4\ell}}$. We calculate

$$[2,1]_{\beta_{rd,4\ell}} = \frac{1 + \left(\frac{b_2}{b_1} - \frac{b_4}{b_3}\right)a + \left(\frac{b_3}{b_1} - \frac{b_2 b_4}{b_1 b_3}\right)a^2}{1 - \frac{b_4}{b_3}a} \ , \tag{4.9}$$

where the coefficients b_n were given in Eqs. (3.4)-(3.7). The zeros of the numerator occur at $a = [2,1]_{zero,(i,ii)}$, where

$$[2,1]_{zero,(i,ii)} = \frac{b_2 b_3 - b_1 b_4 \pm \left[b_1^2 b_4^2 + b_2^2 b_3^2 - 4 b_1 b_3^3 + 2 b_1 b_2 b_3 b_4\right]^{1/2}}{2(b_2 b_4 - b_3^2)} \ . \tag{4.10}$$

and the subscripts i and ii correspond to the $\pm$ sign in front of the square root. It is straightforward to substitute the explicit expressions for the coefficients b_2, b_3, and b_4 in Eq. (4.10), but the resultant expressions for these quadratic roots in terms of the explicit coefficients b_n, $1 \le n \le 4$ are somewhat lengthy, so we do not display them. The pole of the $[2,1]_{\beta_{rd,4\ell}}$ PA occurs at $a = [2,1]_{pole}$, where

$$[2,1]_{pole} = \frac{b_3}{b_4} = -\frac{3(2N-7)}{2[2N^2 + 19N - 24 + 6(11N-17)\zeta_3]} \ . \tag{4.11}$$

TABLE II: Values of $[1,1]_{zero}$ from [1,1] Padé approximant to the reduced three-loop beta function, $\beta_{rd,3\ell}$, and $[2,1]_{zero,i}$ from the [2,1] Padé approximant to the four-loop beta function, $\beta_{rd,4\ell}$. See text for further details.

N	$[1,1]_{zero}$	$[2,1]_{zero,i}$
2	4.000	0.940
3	1.333	0.998
4	0.800	0.999
5	0.571	0.992
6	0.444	0.982
7	0.364	0.9725
8	0.308	0.963
9	0.267	0.953
10	0.235	0.943
100	0.0203	0.683
300	0.00670	0.615
10^3	0.00200	0.585

If one has a series expansion of a function that contains n_{zero} zeros and n_{pole} poles, and one calculates $[r,s]$ Padé approximants to this series with $r > n_{zeros}$ and $s > n_{poles}$, the approximants typically exhibit sets of nearly coincident zero-pole pairs in addition to fitting the actual zeros and poles of the function (e.g., see [24, 26]). These nearly coincident zero-pole pairs may thus be ignored. This happens in the present case. For example, for $N = 3$, the $[2,1]_{\beta_{rd,4\ell}}$ PA has a zero at $a = 0.99773$, a zero at $a = 0.009015$ and a pole at $a = 0.009015$, and similarly for other values of N. In Table II we list the first zero, denoted $[2,1]_{zero,i}$, as a function of N.

We calculate the $[1,2]_{\beta_{rd,4\ell}}$ Padé approximant to be

$$[1,2]_{\beta_{rd,4\ell}} = \frac{1 + \left[\frac{b_1^2 b_4 + b_2^3 - 2b_1 b_2 b_3}{b_1(b_2^2 - b_1 b_3)}\right]a}{1 + \left(\frac{b_1 b_4 - b_2 b_3}{b_2^2 - b_1 b_3}\right)a + \left(\frac{b_3^2 - b_2 b_4}{b_2^2 - b_1 b_3}\right)a^2} \ . \tag{4.12}$$

The two poles of the $[1,2]_{\beta_{rd,4\ell}}$ approximant occur at $a = [1,2]_{pole,(i,ii)}$, where

$$[1,2]_{pole,(i,ii)} = \frac{b_1b_4 - b_2b_3 \pm \Big[b_1^2b_4^2 - 3b_2^2b_3^2 + 4b_1b_3^3 + 4b_2^3b_4 - 6b_1b_2b_3b_4\Big]^{1/2}}{2(b_2b_4 - b_3^2)} \ . \tag{4.13}$$

The zero of this approximant occurs at $a = [1,2]_{zero}$, where

$$[1,2]_{zero} = \frac{b_1(b_1b_3 - b_2^2)}{b_1^2b_4 + b_2^3 - 2b_1b_2b_3} = -\frac{3(2N-3)}{2[2N^2+13N-9+6(11N-17)\zeta_3]} \ . \tag{4.14}$$

Both of the poles $[1,2]_{pole,i}$ and $[1,2]_{pole,ii}$ are negative. Furthermore, we find that this approximant has nearly coincident zero-pole pairs, which thus can both be ignored. For example, for $N=3$, the zero occurs at $a=-0.027540$ while one of the poles occurs at the nearly equal value, $a=-0.027556$, and the other pole is at $a=-0.97919$. Similar results hold for other values of N, i.e., the $[1,2]_{\beta_{rd,4\ell}}$ PA has a nearly coincident zero-pole pair (at negative a) together with a second unphysical pole at negative a.

As we have discussed, the four-loop beta function has a negative real root, in strong contrast with the two-loop and three-loop beta functions. At this four-loop level, the [1,2] PA does not exhibit any true zero, but only a zero that is nearly coincident with a pole and hence can be identified as an artifact. The [2,1] PA yields a zero, but it is at a completely different value than the only real root of the actual four-loop beta function, $a_{rt,4\ell}$. Thus, our analysis of the [2,1] and [1,2] Padé approximants to the four-loop (reduced) beta function yield further evidence against a robust IR zero in this four-loop beta function.

V. ANALYSIS USING SCHEME TRANSFORMATIONS

Since the coefficients b_ℓ with $\ell \ge 3$ in the beta function are scheme-dependent, it is necessary to check that the conclusions from our analysis of the beta function with b_3 and b_4 calculated in the $\overline{\rm MS}$ scheme are robust with respect to scheme transformations. To begin, we study scheme transformations that are designed to remove higher-loop terms in the beta function. We first review some relevant background. In [22], formulas were derived for the coefficients b'_ℓ resulting from a general scheme transformation $f(a')$ of the form

$$a = a' f(a') \ . \tag{5.1}$$

Since a scheme transformation has no effect in the case of a free field theory, $f(a')$ satisfies the condition that $f(0)=1$. Expressing $f(a')$ as a power series in a', one has

$$f(a') = 1 + \sum_{s=1}^{s_{max}} k_s (a')^s \ , \tag{5.2}$$

where the k_s are constants and s_{max} may be finite or infinite. It follows that the Jacobian of this transformation, $J = da/da'$ satisfies the condition $J(0)=1$ and has the expansion

$$J = 1 + \sum_{s=1}^{s_{max}} (s+1)k_s(a')^s \ . \tag{5.3}$$

Then in the transformed scheme, the coefficients of the three-loop and four-loop terms in the beta function are [22]

$$b'_3 = b_3 + k_1b_2 + (k_1^2 - k_2)b_1 \ , \tag{5.4}$$

$$b'_4 = b_4 + 2k_1b_3 + k_1^2b_2 + (-2k_1^3 + 4k_1k_2 - 2k_3)b_1 \ , \tag{5.5}$$

and so forth for higher b'_ℓ.

In [22] a set of conditions was given that should be obeyed by a nonpathological scheme transformation. Condition C_1 was that the scheme transformation must map a physical (real, positive) a to a real positive a', since a map that yields a negative or complex value of a' would violate the unitarity of the theory. As condition C_2, we required that the scheme transformation should preserve perturbativity, and hence should not map a small or moderate value of a to an excessively large value of a' or vice versa. Condition C_3 stated that the Jacobian J should not vanish or diverge, since otherwise the transformation would be singular. More generally, if J were to become too small or too large, it could lead to a violation of condition C_2. Finally, condition C_4 was that if a beta function exhibited a zero at a sufficiently small value as to be perturbatively reliable, then a scheme transformation should not alter this property. Ref. [22] also gave the first explicit scheme transformation to set $b'_\ell = 0$ for $\ell \ge 3$, at least in the local vicinity of the origin, but it also showed that this does not, in general, work to remove these higher-loop terms at a point located away from the origin, i.e., an IR zero in an asymptotically free theory or a UV zero in an IR-free theory. The reason, as shown in [22] and [27], if one attempts to apply such a scheme transformation to remove these higher-loop terms at a point away from the origin, then the transformation violates one or more of the conditions C_1-C_4 for acceptability. As in [27], we denote the scheme transformation presented in [22] (with $s_{max}=m$) that removes the coefficients in the beta function up to loop order $\ell=m+1$, at least near the origin, as $S_{R,m}$.

We proceed with our analysis with the $S_{R,m}$ scheme transformation. The $S_{R,2}$ transformation has [22]

$$k_2 = \frac{b_3}{b_1} \tag{5.6}$$

and the $S_{R,3}$ transformation has this k_2 and

$$k_3 = \frac{b_4}{2b_1} \ . \tag{5.7}$$

We begin by determining whether the scheme transformation $S_{R,2}$ can be applied in the relevant region of a where we need to apply it to set $b_3'=0$ and thus remove the three-loop term in the beta function. Since the (scheme-independent) two-loop value is $a_{IR,2\ell} = a'_{IR,2\ell}=1$, the relevant region is in the neighborhood of $a=1$. This $S_{R,2}$ transformation is defined by Eq. (5.2) with $s_{max}=2$ and k_2 given by Eq. (5.6). If the application of this $S_{R,2}$ transformation in the vicinity of $a=$ were possible, then it would follow from Eq. (5.5) that $b_4'=b_4$. For $S_{R,2}$, Eq. (5.1) is

$$S_{R,2} \implies a = a'[1+k_2(a')^2] = a'\Big[1+\frac{b_3}{b_1}(a')^2\Big] \ . \tag{5.8}$$

Solving Eq. (5.8) for a', we obtain three roots, and we require that at least one of these should be a physical (real, positive) value for a in the relevant range of values comparable to $a_{IR,2\ell}=1$. We find that this necessary condition, C_1, is not satisfied. Instead, two of the solutions of Eq. (5.8) for a' form a complex-conjugate pair, while the third is negative. For example, for $a=a_{IR,2\ell}=1$ and $N=4$, the three solutions for a' are $1.191\pm 0.509i$ and -2.383, while for $N=10$, the three solutions for a' are $0.4125\pm 0.450i$ and -0.825. The Jacobian also exhibits pathological behavior; J is given by

$$\begin{aligned} S_{R,2} \implies J &= 1+3k_2(a')^2 = 1+\frac{3b_3}{b_1}(a')^2 \\ &= 1-\frac{3(2N-7)}{4}\,(a')^2 \ . \end{aligned} \tag{5.9}$$

For $a_{IR,2\ell}=a'_{IR,2\ell}=1$, $J=(25-6N)/4$, which decreases through zero as N (continued to the real numbers) increases through the value $N=25/6$, violating condition C_3. It is therefore not possible to use this scheme transformation to remove the three-loop term in the beta function in the region of a where we are trying to do this, namely the neighborhood of the (scheme-independent) value $a=a_{IR,2\ell}=1$.

We can also investigate whether the scheme transformation $S_{R,3}$ is physically acceptable to be applied in the relevant range of values of a, namely $a=a_{IR,2\ell}=1$. This transformation is defined by Eq. (5.2) with $s_{max}=3$ and k_2 and k_3 given by Eqs. (5.6) and (5.7):

$$\begin{aligned} S_{R,3} \implies a &= a'[1+k_2(a')^2+k_3(a')^3] \\ &= a'\Big[1+\frac{b_3}{b_1}(a')^2+\frac{b_4}{2b_1}(a')^3\Big] \ . \end{aligned} \tag{5.10}$$

The Jacobian for this transformation is

$$\begin{aligned} S_{R,3} \implies J &= 1+3k_2(a')^2+4k_3(a')^3 \\ &= 1+\frac{3b_3}{b_1}(a')^2+\frac{2b_4}{b_1}(a')^3 \ . \end{aligned} \tag{5.11}$$

With this $S_{R,3}$ scheme transformation we find that for the relevant range of $a\simeq 1$, J can deviate excessively far from unity, violating condition C_1. For example, for $a=1$ and $N=10$, we find that $J=339.8$, much larger than unity.

One can also apply the various scheme transformations that we have devised in [22]-[29] to the beta function calculated in the $\overline{\rm MS}$ scheme and compare the resulting value(s) of the zero(s) of the beta function with the value(s) obtained at the three-loop and four-loop level in the $\overline{\rm MS}$ scheme. Our general analyses in [22]-[29] (see also [30]) have shown that, for moderate values of the parameters determining these scheme transformations, the resultant values of the zero(s) are similar to those obtained in the original $\overline{\rm MS}$ scheme. In particular, the negative, unphysical value of $a_{rt,4\ell}$ will still be present in the transformed scheme.

Summarizing this section, we have shown that our conclusion, that the beta function of the finite-N Gross-Neveu model, calculated up to four-loop order, does not exhibit an IR zero, is robust with respect to scheme transformations.

VI. COMPARISON WITH RESULTS IN THE LN LIMIT AND BEHAVIOR FOR $d>2$

In this section we discuss how the conventional perturbative beta function reduces in the LN limit, and we also comment on some properties of the theory for spacetime dimension $d>2$. From Eq. (2.7), the quantity that remains finite and nonzero in the LN limit is $\lambda=gN$, and hence the corresponding beta function that is finite in this limit is

$$\beta_\lambda = \frac{d\lambda}{dt} = \lim_{LN} N\frac{dg}{dt} = \lim_{LN} N\beta \ . \tag{6.1}$$

With the limit $N\to\infty$ having been taken, β_λ has the series expansion, for $d \gtrsim 2$, with $\epsilon_d = d-2$,

$$\beta_\lambda = \lambda\Big[\epsilon_d + \sum_{\ell=1}^{\infty}\hat b_\ell \xi^\ell\Big] \ , \tag{6.2}$$

where

$$\xi = \lim_{LN} Na = \frac{\lambda}{2\pi} \tag{6.3}$$

and

$$\hat{b}_\ell = \lim_{LN} \frac{b_\ell}{N^\ell} \ . \tag{6.4}$$

Here we have used the fact that $b_\ell a^\ell = \hat{b}_\ell \xi^\ell$. We find

$$\hat{b}_1 = -2 \tag{6.5}$$

and

$$\hat{b}_\ell = 0 \quad \text{for } \ell \ge 2 \ . \tag{6.6}$$

The latter result follows from the fact that the structure of the bubble graphs in the calculation of b_ℓ in, e.g., the $\overline{\text{MS}}$ scheme, means that, for $\ell \ge 2$, b_ℓ is a polynomial in N of degree $\ell-1$. Although the b_ℓ with $\ell \ge 3$ are scheme-dependent, this property is maintained by scheme transformations that are finite in the LN limit [22]. Hence, for $\ell \ge 2$, $\lim_{LN} b_\ell/N^\ell = 0$, which is the result given in Eq. (6.6). Similarly, although $\hat{b}_\ell$ with $\ell \ge 3$ are, in general, scheme-dependent, if they are zero in one scheme, such as the $\overline{\text{MS}}$ scheme, then they are also zero in any other scheme reached by a scheme transformation function that is finite in the LN limit [22]. It follows that in the LN limit, with $d = 2+\epsilon \gtrsim 2$,

$$\beta_\lambda = \lambda[\epsilon - 2\xi] = \lambda\Big[\epsilon - \frac{\lambda}{\pi}\Big] \ . \tag{6.7}$$

Hence,

$$d = 2 \implies \beta_\lambda = -\frac{\lambda^2}{\pi} \ , \tag{6.8}$$

with only the UV zero in this beta function at $\lambda = 0$, and no IR zero. We can relate this to the beta function that was calculated in [1] in the LN limit. From Eqs. (2.1) and (3.1), we have

$$\beta = \frac{dg}{dt} = 2g_{GN}\frac{dg_{GN}}{dt} = 2g_{GN}\beta_{GN} \ . \tag{6.9}$$

Explicitly, in the LN limit, from Eqs. (6.8) and (2.1),

$$\beta_\lambda = -\frac{\lambda^2}{\pi} = -\lim_{LN}\frac{g_{GN}^4 N^2}{\pi} \ . \tag{6.10}$$

Combining Eqs. (6.1), (6.9), and (6.10) yields $\beta_{GN} = -g_{GN}^3 N/(2\pi) = -g_{GN}\lambda/(2\pi)$, in agreement with Eq. (2.9) above, or equivalently, Eq. (3.7) in Ref. [1]. This agreement was guaranteed, since the LN limit is a special limit of the result for finite N. Accordingly, our finding that there is no robust evidence for an IR zero in the finite-N beta function of the $(d = 2)$ Gross-Neveu model is, *a fortiori*, in agreement with the fact that in the LN limit, the beta function β_λ in Eq. (6.8) (equivalently, β_{GN} in Eq. (2.9) above), does not exhibit an IR zero.

If $d > 2$, then for small λ, the GN theory is IR-free, with an IR zero of β_λ at the origin, $\lambda = 0$, and a UV zero of β_λ at

$$\lambda_{UV} = \pi\epsilon \quad \text{for } d \gtrsim 2, \quad \text{LN limit} \ , \tag{6.11}$$

which is a UV fixed point of the renormalization group. This is closely analogous to the result found from an exact solution of the O(N) nonlinear σ model (NLσ M) in $d = 2+\epsilon$ dimensions in the $N \to \infty$ limit [14]. In that theory, denoting the analogous finite coupling in this limit as

$$x = \lim_{N\to\infty} N\lambda_{NL\sigma M} \ , \tag{6.12}$$

the exact solution yielded the beta function, for $d \gtrsim 2$,

$$\beta_x = \frac{dx}{dt} = x\Big[\epsilon - \frac{x}{2\pi}\Big] \ . \tag{6.13}$$

Thus, this nonlinear sigma model is, like the GN model in $d \gtrsim 2$, IR-free with a UV fixed point at

$$x_{UV} = 2\pi\epsilon \ . \tag{6.14}$$

VII. CONCLUSIONS

The Gross-Neveu model in $d = 2$ spacetime dimensions has long been of value as an asymptotically free theory which is exactly solvable in the LN limit, i.e., $N \to \infty$ with gN finite, exhibits nonperturbative fermion mass generation and associated dynamical chiral symmetry breaking. In this paper we have considered the finite-N Gross-Neveu model. We have addressed and answered a fundamental question about the UV to IR evolution of this model, as embodied in the beta function, namely whether this beta function exhibits evidence for an IR zero. For the purpose of our study, we have analyzed the beta function to the highest-loop order to which it has been calculated, namely the four-loop order. Our study used a combination of three methods, namely a direct analysis of the three-loop and four-loop beta functions, a study of Padé approximants, and a study of the effect of scheme transformations. We find that in the range of coupling where the perturbative calculation of the four-loop beta function is reliable, it does not exhibit robust evidence for an infrared zero.

Acknowledgments

This research was supported in part by the Danish National Research Foundation grant DNRF90 to CP^3-Origins at SDU (T.A.R.) and by the U.S. NSF Grant NSF-PHY-16-1620628 (R.S.)

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