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Effective pion mass term and the trace anomaly

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Recently, we developed an effective theory of pions and a light dilatonic meson for gauge theories with spontaneously broken chiral symmetry that are close to the conformal window. The pion mass term in this effective theory depends on an exponent y. We derive the transformation properties under dilatations of the renormalized fermion mass, and use this to rederive $y = 3 - \gamma_m^*$, where γ_m^* is fixed-point value of the mass anomalous dimension at the sill of the conformal window. This value for y is consistent with the trace anomaly of the underlying near-conformal gauge theory.

I. INTRODUCTION

In Ref. [1] we developed a low-energy effective action of pions and a dilatonic meson, which are the pseudo Nambu-Goldstone bosons for approximate chiral and scale symmetries, respectively, in near-conformal gauge theories that still undergo dynamical chiral symmetry breaking, and in which the scale symmetry is broken by the trace anomaly. The effective theory is organized in terms of a systematic power counting in

$$p^2/\Lambda^2 \ \sim \ m/\Lambda \ \sim \ 1/N \ \sim \ |n_f - n_f^*| \ \sim \ \delta \ , \tag{1.1}$$

where δ stands for the small expansion parameter. As in the usual chiral lagrangian, m is the fermion mass (we assume a common mass for all flavors for simplicity), and p^2 is a shorthand for the product of two external momenta, while Λ is the scale associated with chiral symmetry breaking. We invoke the Veneziano limit $N \to \infty$, where the number of colors $N_c = N$ tends to infinity in proportion to the number of fundamental representation flavors N_f [2]. Here $n_f = N_f/N_c$, and n_f^* is the critical value where the conformal window is entered in the Veneziano limit,

$$n_f^* = \lim_{N_c \to \infty} \frac{N_f^*(N_c)}{N_c} \ . \tag{1.2}$$

It is defined in terms of $N_f^*(N_c)$, which, in turn, is the smallest number of flavors for which the $SU(N_c)$ theory is infrared conformal.

The effective action is constructed in terms of the usual chiral source field $\chi_{ij}(x)$, where $i, j = 1, 2, \ldots, N_f$ are flavor indices, and the "dilaton" source field $\sigma(x)$, as well as effective fields for the pions and the dilatonic meson. After setting these sources equal to their expectation values, $\chi_{ij}(x) = m\delta_{ij}$ and $\sigma(x) = 0$, the leading-order lagrangian of the effective theory is [1]

$$\mathcal{L} = \mathcal{L}_\pi + \mathcal{L}_\tau + \mathcal{L}_m + \mathcal{L}_d \ , \tag{1.3}$$

where

$$\mathcal{L}_\pi \ = \ \frac{f_\pi^2}{4} \, e^{2\tau} \mathrm{tr} \left(\partial_\mu \Sigma^\dagger \partial_\mu \Sigma \right) \ , \tag{1.4}$$

$$\mathcal{L}_\tau \ = \ \frac{f_\tau^2}{2} \, e^{2\tau} (\partial_\mu \tau)^2 \ , \tag{1.5}$$

$$\mathcal{L}_m \ = \ -\frac{f_\pi^2 B_\pi m}{2} \, e^{y\tau} \mathrm{tr} \left(\Sigma + \Sigma^\dagger \right) \ , \tag{1.6}$$

$$\mathcal{L}_d \ = \ f_\tau^2 B_\tau \, e^{4\tau} (c_0 + c_1 \tau) \ . \tag{1.7}$$

Here $\Sigma(x) \in SU(N_f)$ is the usual non-linear field describing the pions, while $\tau(x)$ is the effective field of the dilatonic meson; f_π, f_τ, B_π, B_τ and $c_{0,1}$ are low-energy constants.

In Ref. [1] we argued that the exponent y in Eq. (1.6) is given by

$$y = 3 - \gamma_m^* \ , \tag{1.8}$$

where γ_m^* is the mass anomalous dimension at the sill of the conformal window in the Veneziano limit.[1] Here we give a more complete derivation of this result, and explore its relation to the trace anomaly. This new derivation was motivated by the observation, made in Ref. [3], that y may be determined by matching the divergences of the dilatation current in the underlying and effective theories.[2]

The power counting (1.1) hinges on the *assumption* that, near the chiral symmetry breaking scale Λ, the beta function of the renormalized 't Hooft coupling $\tilde{\alpha}_r \equiv g^2 N/(16\pi^2)$ behaves like [1, 4]

$$\beta(\tilde{\alpha}_r(\Lambda)) = O(n_f - n_f^*) + O(1/N) \ . \tag{1.9}$$

This relation expresses the fact that the theory is on the verge of developing an infrared attractive fixed point (a situation that is sometimes referred to as "emergent" scale invariance). The systematic expansion in $n_f - n_f^*$ derives from this central dynamical assumption.

In order that the effective action will manifestly exhibit the expansion in $n_f - n_f^*$ we have to choose the renormalization scale μ such that Eq. (1.9) is applicable. In other words, we need $\mu \sim \Lambda$; we must renormalize the microscopic theory near the scale where chiral symmetry breaking takes place. Since the microscopic and the effective theories depend on the same set of external sources, the mass parameter occurring in Eq. (1.6) is therefore the renormalized mass, $m = m_r(\mu)$, with the renormalization scale μ chosen as above. As we will show in Sec. II, Eq. (1.8) is then a direct consequence of the transformation properties of $m_r(\mu)$ under dilatations.[3] In Sec. III we explore the matching procedure proposed in Ref. [3], finding that this procedure reproduces Eq. (1.8) as well. Sec. IV contains our conclusions.

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[1] More precisely, $\gamma_m^* = \gamma_m(\tilde{\alpha}_*)$, where $\tilde{\alpha}_*$ is infrared fixed-point 't Hooft coupling at the sill of the conformal window.

[2] However, we disagree with the actual result for y claimed in Ref. [3].

[3] We refer to Ref. [1] for a general discussion of the dilatation transformation properties of the effective theory, both with and without the $\sigma(x)$ source field.

II. DILATATION TRANSFORMATION OF THE RENORMALIZED MASS

We start from the bare lagrangian of the microscopic theory, using dimensional regularization. After a rescaling of the bare gauge and fermion fields by the bare coupling g_0, the d-dimensional action is

$$S = \int d^d x \, \frac{\mu_0^{(d-4)}}{\hat{g}_0^2} \left(\mathcal{L}_k + \mathcal{L}_m \right) \ , \tag{2.1}$$

where

$$\mathcal{L}_k \ = \ \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \overline{\psi}_i \not{D} \psi_i \ , \tag{2.2}$$

$$\mathcal{L}_m \ = \ m_0 \overline{\psi}_i \psi_i \ , \tag{2.3}$$

and m_0 is the bare mass. To expose the fact that g_0 is dimensionful for $d \neq 4$ we substituted

$$g_0 = \hat{g}_0 \, \mu_0^{2-d/2} \ , \tag{2.4}$$

so that $\hat{g}_0$ is dimensionless for any d. As the only dimensionful parameter in the massless bare action, μ_0 may be interpreted as an ultraviolet cutoff scale.

A dilatation transformation acts on the fields and parameters occurring in the bare lagrangian according to their canonical dimension,

$$A_\mu(x) \ \to \ \lambda A_\mu(\lambda x) \ , \tag{2.5a}$$

$$\psi(x) \ \to \ \lambda^{3/2} \psi(\lambda x) \ , \qquad \overline{\psi}(x) \ \to \ \lambda^{3/2} \overline{\psi}(\lambda x) \ , \tag{2.5b}$$

$$m_0 \ \to \ \lambda m_0 \ , \tag{2.5c}$$

$$\mu_0 \ \to \ \lambda \mu_0 \ , \tag{2.5d}$$

$$\hat{g}_0 \ \to \ \hat{g}_0 \ . \tag{2.5e}$$

It is easy to check that the d-dimensional action is invariant under this transformation. Here we take m_0 and μ_0 to be global spurions, with transformation rules that make the bare action (2.1) invariant.[4]

We next proceed to the renormalized parameters. The renormalized coupling g_r is defined as usual via

$$Z_g(\epsilon; g_r) g_r = \mu^{-\epsilon} g_0 = (\mu_0/\mu)^\epsilon \, \hat{g}_0 \ , \tag{2.6}$$

[4] The formulation in terms of spurion *fields*, and its relation with Eqs. (2.1) and (2.5), will be discussed in Sec. III below. For our purposes here the global spurions m_0 and μ_0 are sufficient.

and the renormalized mass m_r by

$$m_0 = m_r \, Z_m(\epsilon; g_r) \ . \tag{2.7}$$

Here $\epsilon = 2 - d/2$, and in the second equality of Eq. (2.6) we have used Eq. (2.4). We use a mass-independent renormalization scheme, which implies that all Z factors have a series expansion in $1/\epsilon$ and in g_r^2. It follows that the renormalized coupling itself, as well as all the Z factors, depend on μ and μ_0 only through their ratio μ/μ_0. In particular, the renormalized mass satisfies the renormalization-group equation

$$\frac{\partial \log m_r}{\partial \log \mu} = -\gamma_m = -\frac{\partial \log Z_m}{\partial \log \mu} = \frac{\partial \log Z_m}{\partial \log \mu_0} \ , \tag{2.8}$$

where in the last equality we have used that $Z_m = Z_m(\mu/\mu_0)$.

The dilatation transformation rule of the renormalized mass is obtained as follows. What needs to be calculated is the response of the renormalized mass to the transformation (2.5), which is applied to the bare theory while holding fixed the physical scale represented by the renormalization scale μ. Letting $m_0(\lambda) = \lambda m_0$ and $\mu_0(\lambda) = \lambda \mu_0$, the transformation rule of $m_r = m_r(\lambda)$ under an infinitesimal dilatation is obtained by differentiating Eq. (2.7) with respect to $\log \lambda$,

$$\frac{1}{Z_m} \frac{\partial m_0}{\partial \log \lambda} = \frac{\partial m_r}{\partial \log \lambda} + \frac{m_r}{Z_m} \frac{\partial \mu_0}{\partial \log \lambda} \frac{\partial Z_m}{\partial \mu_0} \ . \tag{2.9}$$

The derivative with respect to μ_0 can be traded with a derivative with respect to μ with the help of Eq. (2.8). It follows that

$$m_r = \frac{m_0}{Z_m} = \frac{1}{Z_m} \frac{\partial m_0}{\partial \log \lambda} = \frac{\partial m_r}{\partial \log \lambda} - m_r \gamma_m \ , \tag{2.10}$$

or

$$\frac{\partial m_r}{\partial \log \lambda} = (1 + \gamma_m) m_r \ . \tag{2.11}$$

As explained in the introduction, in the effective theory we are expanding in $n_f - n_f^*$. In the limit $n_f \nearrow n_f^*$ (and $N \to \infty$) the 't Hooft coupling behaves like $\tilde{\alpha}_r(\Lambda) \to \tilde{\alpha}_*$, where $\tilde{\alpha}_*$ is the location of the infrared fixed point at the sill of the conformal window, in the Veneziano limit. For small $|n_f - n_f^*|$, $\tilde{\alpha}_r(\Lambda)$ is close to $\tilde{\alpha}_*$, and $\gamma_m(\tilde{\alpha}_r(\Lambda)) = \gamma_m^*$ up to corrections of order $n_f - n_f^*$. In the leading-order effective action we thus have $\gamma_m = \gamma_m^*$. Since this is a constant, we can integrate Eq. (2.11) in closed form, obtaining the transformation rule under a finite dilatation

$$m_r \to \lambda^{1+\gamma_m^*} m_r \ . \tag{2.12}$$

The dilatation transformation rules of the effective fields are [1]

$$\Sigma(x) \;\to\; \Sigma(\lambda x) \;, \tag{2.13a}$$

$$\tau(x) \;\to\; \tau(\lambda x) + \log\lambda \;. \tag{2.13b}$$

Using Eqs. (2.12) and (2.13) it is easy to check that $\mathcal{L}_m$ is invariant under dilatations if and only if y is given by Eq. (1.8). This is our main result.

III. RELATION TO THE TRACE ANOMALY

It was recently observed in Ref. [3] that the relation of y with the mass anomalous dimension may be inferred via the following alternative procedure. One first obtains the divergence of the dilatation current $\partial_\mu S_\mu$ by applying a suitable differential operator to the action. This is done separately in the (bare) microscopic theory and in the effective theory. One then requires that *the same* differential operator will yield $\partial_\mu S_\mu$ in both cases, following the general requirement that correlation functions obtained by differentiating the partition functions of the microscopic and of the effective theories will match.[5] In particular, the same differential operator that yields $\partial_\mu S_\mu$ in the microscopic theory should therefore also reproduce $\partial_\mu S_\mu$ at the level of the effective theory. In Ref. [3] it was claimed that the outcome of this consistency requirement is that $y = 3$, which is in conflict with the value we derived in Sec. II. Here we will show that also this procedure leads to $y = 3 - \gamma_m^*$, in disagreement with Ref. [3]. The key point is that, before it can be applied to the effective theory, the differential operator needs to be expressed in terms of the renormalized mass.

We begin by coupling the bare action to local sources, which is done by replacing Eq. (2.1) with

$$S = \int d^d x \, \frac{(\mu_0 e^{\sigma(x)})^{(d-4)}}{\hat{g}_0^2} \left(\mathcal{L}_k + \mathcal{L}_{src}\right) \;, \tag{3.1}$$

where now

$$\mathcal{L}_{src} = \frac{1}{2}\Big(\chi_{ij}\,\overline{\psi}_i(1+\gamma_5)\psi_j + \chi^*_{ji}\,\overline{\psi}_i(1-\gamma_5)\psi_j\Big) \;. \tag{3.2}$$

The dilatation transformation rules of the dynamical bare fields remain the same as before,

[5] Note that S_μ itself does not renormalize, because it is equal to $x_\nu T_{\mu\nu}$ with $T_{\mu\nu}$ the conserved energy-momentum tensor.

whereas the transformation rules of the source fields and the parameters are given by

$$\chi(x) \;\to\; \lambda\chi(\lambda x) \;, \tag{3.3a}$$

$$\sigma(x) \;\to\; \sigma(\lambda x) + \zeta \log\lambda \;, \tag{3.3b}$$

$$\mu_0 \;\to\; \lambda^{(1-\zeta)}\mu_0 \;, \tag{3.3c}$$

$$\hat{g}_0 \;\to\; \hat{g}_0 \;. \tag{3.3d}$$

Notice the freedom to choose the parameter ζ, which follows from the redundancy between μ_0 and the constant mode of the $\sigma(x)$ field, which we will denote as σ_0.

There are two variants of the matching procedure. One can obtain $\partial_\mu S_\mu(x)$ via suitable differentiations with respect to the local sources; or one can obtain the integrated version $\int d^dx\, \partial_\mu S_\mu(x)$, for which we may set the local sources to the constant values $\sigma(x) = \sigma_0$, $\chi_{ij}(x) = m_0\delta_{ij}$. Here we will choose the second variant.

The integrated divergence $\int d^dx\, \partial_\mu S_\mu(x)$ is obtained by applying an infinitesimal dilatation to the dynamical fields only. Since S is invariant when the dilatation is applied to both fields and sources or parameters, it follows that

$$\int d^dx\, \partial_\mu S_\mu = -\hat{\delta} S \;, \tag{3.4}$$

where the differential operator on the right-hand side is

$$\hat{\delta} = \zeta \frac{\partial}{\partial\sigma_0}\bigg|_{m_0,\mu_0} + (1-\zeta)\mu_0 \frac{\partial}{\partial\mu_0}\bigg|_{m_0,\sigma_0} + m_0 \frac{\partial}{\partial m_0}\bigg|_{\mu_0,\sigma_0} \;, \tag{3.5}$$

and we have indicated explicitly which parameters are held fixed during each differentiation. Because the bare action depends only on $e^{\sigma_0}\mu_0$, the only change in Eqs. (2.6) and (2.7) is that now g_r and all the Z factors are functions of the combination

$$\frac{e^{\sigma_0}\mu_0}{\mu} \;. \tag{3.6}$$

It follows that the derivative $\partial/\partial\sigma_0$ is interchangeable with $\partial/\partial\log\mu_0$. We thus have the alternative forms,

$$\hat{\delta} \;=\; \mu_0 \frac{\partial}{\partial\mu_0}\bigg|_{m_0,\sigma_0} + m_0 \frac{\partial}{\partial m_0}\bigg|_{\mu_0,\sigma_0} \;=\; \frac{\partial}{\partial\sigma_0}\bigg|_{m_0,\mu_0} + m_0 \frac{\partial}{\partial m_0}\bigg|_{\mu_0,\sigma_0} \;, \tag{3.7}$$

which correspond to $\zeta = 0$ and $\zeta = 1$ in Eq. (3.5).

As explained in the introduction, the effective theory depends on the renormalized mass m_r (or, more generally, on the renormalized chiral source $\chi_r(x)$). In order to be able to compare the action of $\hat{\delta}$ on the bare microscopic action and on the effective action, we must first trade the bare mass for the renormalized mass. One has

$$\begin{aligned} \mu_0 \frac{\partial}{\partial \mu_0}\bigg|_{m_0,\sigma_0} &= \mu_0 \frac{\partial}{\partial \mu_0}\bigg|_{m_r,\sigma_0} + \mu_0 \frac{\partial m_r}{\partial \mu_0}\bigg|_{m_0,\sigma_0} \frac{\partial}{\partial m_r}\bigg|_{\mu_0,\sigma_0} \\ &= \mu_0 \frac{\partial}{\partial \mu_0}\bigg|_{m_r,\sigma_0} + \gamma_m m_r \frac{\partial}{\partial m_r}\bigg|_{\mu_0,\sigma_0} , \end{aligned} \tag{3.8}$$

where the last step follows from Eq. (2.8).[6] In addition, Eq. (2.7) implies that

$$m_0 \frac{\partial}{\partial m_0}\bigg|_{\mu_0} = m_r \frac{\partial}{\partial m_r}\bigg|_{\mu_0} , \tag{3.9}$$

when acting on the microscopic action. Putting it together gives

$$\hat{\delta} = \mu_0 \frac{\partial}{\partial \mu_0}\bigg|_{m_r,\sigma_0} + (1+\gamma_m) m_r \frac{\partial}{\partial m_r}\bigg|_{\mu_0,\sigma_0} \tag{3.10a}$$

$$= \frac{\partial}{\partial \sigma_0}\bigg|_{m_r,\mu_0} + (1+\gamma_m) m_r \frac{\partial}{\partial m_r}\bigg|_{\sigma_0,\mu_0} . \tag{3.10b}$$

In the effective theory, the leading-order expressions for the Noether current S_μ and its divergence were calculated in Ref. [1]. As first observed in Ref. [3], the leading-order $\partial_\mu S_\mu$ may also be obtained as follows. Starting from the leading-order effective action S_{eff} which depends on the dilaton source $\sigma(x)$ and the renormalized chiral source $\chi_{ij,r}(x)$ as detailed in Sec. 3.2 of Ref. [1], we let $\sigma(x) = \sigma_0$ and $\chi_{ij,r}(x) = \delta_{ij} m_r$, and find that the (integrated) divergence of S_μ is equal to

$$\int d^4x \, \partial_\mu S_\mu = -\hat{\delta}_{\text{eff}} S_{\text{eff}} , \tag{3.11}$$

where

$$\hat{\delta}_{\text{eff}} = \frac{\partial}{\partial \sigma_0}\bigg|_{m_r} + (4-y) m_r \frac{\partial}{\partial m_r}\bigg|_{\sigma_0} , \tag{3.12}$$

and σ_0 is set equal to zero in the end. The requirement that the effective theory match the microscopic theory thus implies that the differential operators $\hat{\delta}$ and $\hat{\delta}_{\text{eff}}$ must be the

[6] The term proportional to γ_m was overlooked in Ref. [3].

same. Comparing Eqs. (3.10b) and (3.12) (and remembering that the effective theory does not depend explicitly on μ_0) shows that this agreement will be reached provided that

$$4 - y = 1 + \gamma_m \ . \tag{3.13}$$

Finally, taking the renormalization scale to be as described in the introduction, we reproduce $\gamma_m = \gamma_m^*$ to leading order in the power counting, and, with that, Eq. (1.8) as well.

The manipulations we have carried out in this section are closely related to the original derivation of the trace anomaly in Ref. [5], which we will refer to as "CDJ." At the starting point, CDJ introduces a parameter a (called the loop expansion parameter), and multiplies the bare lagrangian by $1/a$. If we again rescale the bare fields as we did in Sec. II, so that the bare gauge coupling appears as an overall factor $1/g_0^2$ in front of the lagrangian density, we reach the equivalence

$$\frac{1}{ag_0^2} \Leftrightarrow \frac{e^{(d-4)\sigma_0}}{g_0^2} = \frac{(\mu_0 e^{\sigma_0})^{d-4}}{\hat{g}_0^2} \ , \tag{3.14}$$

where the left-hand side refers to CDJ, and the right-hand side to Eq. (3.1) (with $\sigma(x) = \sigma_0$). We already observed that our Z factors depend on μ, μ_0 and σ_0 only through the variable (3.6). If we consider the dependence on $\hat{g}_0$ as well, our Z factors depend only on the variable

$$\frac{1}{\hat{g}_0^2}\left(\frac{e^{\sigma_0}\mu_0}{\mu}\right)^{d-4} \ . \tag{3.15}$$

Correspondingly, CDJ observes that their Z factors depend only on

$$\frac{\mu^{4-d}}{ag_0^2} \ . \tag{3.16}$$

At a key step in the argument, CDJ then trades the derivatives with respect to a (holding μ fixed) with derivatives with respect to the renormalization scale μ. Evidently, what we have done is completely analogous, taking derivatives with respect to μ_0 (or with respect to σ_0), which are the variables that play the role of a in our setting, and trading them for derivatives with respect to μ. This close correspondence has to exist, because, as we have seen, acting on the bare action with the differential operator $\hat{\delta}$ generates the integral of $\partial_\mu S_\mu$. In particular, the term proportional to γ_m in Eq. (3.10) corresponds directly to the term proportional to γ_m in the trace anomaly.

IV. CONCLUSION

In this note we provided a more complete discussion of the relation between the parameter y in the pion-mass term of the effective lagrangian (1.3) and the mass anomalous dimension γ_m, thereby confirming the result already conjectured in Ref. [1]. We traced the incorrect result obtained in Ref. [3] to the difference between the behavior of bare and renormalized sources under scale transformations. We also pointed out that our result is required for consistency with the expression for the trace anomaly in the underlying gauge theory [5].

In principle, the scale at which we renormalize the microscopic theory can be chosen arbitrarily. Imagine that we are very close to the gaussian fixed point at $g_0 = 0$, by taking the renormalization scale μ higher and higher. We may then use the one-loop expression for the mass anomalous dimension, so that γ_m is linear in the renormalized 't Hooft coupling $\tilde{\alpha}_r(\mu)$. We see that γ_m becomes arbitrarily small if we take μ arbitrarily large, and, eventually, $y = 3 - \gamma_m(\tilde{\alpha}_r(\mu))$ will approach $y = 3$. This, of course, is merely a reflection of the fact that the theory is asymptotically free. However, if we want to define our effective theory (1.3) as the leading term in a systematic expansion in $|n_f - n_f^*| \sim \delta$, we are forced to choose the renormalization scale near the chiral symmetry breaking scale, and, consequently, the difference $3 - y = \gamma_m^*$ is of order one.

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