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Phys. Rev. D **95**, 014007 — Published 9 January 2017

DOI: 10.1103/PhysRevD.95.014007

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Abstract

The scaling behavior of the order parameter at the chiral phase transition, the so called magnetic equation of state, of strongly interacting matter is studied within effective models. We explore universal and non-universal structures near the critical point. These include the scaling functions, the leading corrections to scaling and the corresponding size of the scaling window as well as their dependence on an external symmetry breaking field. We consider two models in the mean-field approximation, the quark-meson (QM) and the Polyakov loop extended quark-meson (PQM) models, and compare their critical properties with a purely bosonic theory, the $O(N)$ linear sigma (LS) model in the $N \to \infty$ limit. In these models the order parameter scaling function is found analytically using the high temperature expansion of the thermodynamic potential. The effects of a gluonic background on the non-universal scaling parameters are studied within the PQM model.

I. INTRODUCTION

Spontaneous chiral symmetry breaking and its restoration at finite temperature and density is an essential ingredient in our understanding of the phase structure of strongly interacting matter and hence a key problem in QCD [1–3].

In the limit of massless light quark flavors, the chiral phase transition in Quantum Chromodynamics (QCD) was conjectured to be of second order, in the O(4) universality class [4]. Current lattice QCD (LQCD) simulations at physical up, down and strange quark masses show that at vanishing and small baryon density the transition from a hadron gas to a quark gluon plasma is a smooth crossover [5]. Moreover, lattice studies of the scaling properties of the chiral order parameter are consistent with the conjectured O(4) symmetry and indicate that the scaling violations are fairly small for physical quark masses [6–9]. Consequently, quantities that are sensitive to chiral criticality, are expected to exhibit characteristic properties governed by the universal singular part of the free energy density. The magnetic equation of state, which reveals the scaling of the chiral order parameter as a function of the reduced temperature and the quark masses, is a key quantity in this context [10]. We note, however, that the issue whether the chiral transition of QCD exhibits O(N) scaling is quite subtle. Indeed, several studies suggest that in the chiral limit the transition could be first order [11–13].

The critical properties of QCD are often studied in effective models that share the chiral symmetry of the QCD Lagrangian and exhibit spontaneous breaking of this symmetry in vacuum. Popular models include the quark-meson (QM) model [14] and its Polyakov loop extended version (PQM) [15–23], the $O(N)$ linear sigma (LS) model [24–31] as well as the Nambu-Jona–Lasinio (NJL) model [32–38]. In the chiral limit, all these models undergo a second order phase transition of the $O(4)$ universality class. Conseqently, they belong to the same universality class as QCD.

The non-zero u and d quark masses break the chiral symmetry explicitly. However, for small masses, the dynamics is by and large determined by the underlying second order phase transition, while the non-zero quark masses act as a weak perturbation. Clearly, even at small masses, there is no phase transition in a strict sense. In a cross-over region, the order parameter decreases smoothly from a large value at small temperatures and densities to a very small but finite one at high temperatures and densities. The melting of the order

parameter near the critical point is captured by the magnetic equation of state.

The value of the light quark mass up to which the critical fluctuations of the underlying second order phase transition dominate the physics near the pseudocritical point is model dependent and consequently non-universal. As noted above, LQCD calculations suggest [6–9], that for physical quark masses the critical behavior of the chiral condensate is well approximated by the O(N) scaling magnetic equation of state. This indicates that the scaling window of the QCD chiral crossover transition extends more or less to the physical light quark masses.

In functional renormalization group (FRG) [39–42] studies of the QM model, it was shown [43] that at the physical pion mass, the behavior of the condensate is not well described by the universal scaling function, in spite of the fact that in the chiral limit this theory belongs to the O(4) universality class. The different critical behavior of QCD and the QM model within the FRG approach is linked to the scaling breaking terms in the magnetic equation of state, which are non-universal.

In this paper, we explore the critical properties of the chiral order parameter and the magnetic equation of state in QCD-like models. We focus on their universal and non-universal structure near the critical point. This includes a derivation of the scaling functions and leading scaling violating corrections. In particular, we systematically study the dependence of the magnetic equation of state on an external symmetry breaking field, and assess the size of the critical region. For transparency, we consider the QM and PQM models in the mean-field approximation, where only fermionic fluctuations are accounted for, and confront their critical properties with a purely bosonic theory, the $O(N)$ linear sigma model. In the mean-field approximation to the QM model as well as in the $N \to \infty$ limit of the LS model, the calculation of the magnetic equation of state is carried out analytically by employing the high temperature expansion. The effects of the gluonic background on the non-universal scaling parameters are assessed in the PQM model.

We stress that, although these models do not reproduce the expected scaling behavior of QCD on a quantitative level, they provide a transparent framework for exploring chiral criticality. Moreover, this study yields new insight into possible patterns of scaling violation exhibited by the magnetic equation of state.

The paper is organized as follows. In Sec. II we briefly summarize the theory of second order phase transitions and introduce the magnetic equation of state. In Sec. III A the mag-

netic equation of state is discussed within Landau-theory. The effective Landau coefficients are obtained in Sec. III B for the QM model. The LS model and its magnetic equation of state are introduced in Sec. III D. In Sec. IV we compare results for magnetic equation of state in different models and discuss the non-universal corrections. In the final section, we present a summary and conclusions.

II. UNIVERSALITY AND SCALING

In the scaling theory of phase transitions, the free energy density $f(T,H)$ is, in the vicinity of a second order critical point, split into a singular scaling part $f_s(T,H)$ and a regular part. In a given universality class, the singular part has a universal structure [10].

For a given temperature T and external field H, one introduces the scaling variables,

$$\overline{t} = \frac{t}{t_0} = \frac{T - T_c}{T_c t_0}, \quad h = \frac{H}{H_0 h_0}, \quad z = \frac{\overline{t}}{h^{1/(\beta\delta)}}, \tag{1}$$

where T_c is the critical temperature and t_0, H_0, h_0 are appropriately chosen constants. In terms of these variables, the scaling part of the free energy has the universal form

$$f_s(T,H) = F_0 \, h^{1+1/\delta} f_f\left(z\right). \tag{2}$$

The scaling of the order parameter $\langle\sigma\rangle$ is obtained from Eq. (2)

$$\langle\sigma\rangle = \frac{\partial}{\partial H} f_s(T,H) = \sigma_0 h^{1/\delta} f_G(z). \tag{3}$$

In Eqs. (2,3) F_0 and σ_0 are again appropriately chosen constants. The functions f_f, f_G and the critical exponents are universal, as they do not depend on the details of the model, but only on its universality class. The scaling function f_G has the following asymptotic properties: $f_G(0) = 1$, and $\lim_{z\to-\infty} \frac{f_G(z)}{(-z)^\beta} = 1$.

From the scaling function, one arrives at the following well know scaling properties of the order parameters on the coexistence line ($T < T_c, H = 0$) and at the pseudo-critical point ($T = T_c, H > 0$),

$$\langle\sigma\rangle = \begin{cases} \sigma_0 h^{1/\delta} = \sigma_0 \left(\frac{H}{H_0 h_0}\right)^{1/\delta}, & T = T_c,\ H > 0 \\ \sigma_0 \left(-\overline{t}\right)^\beta \quad = \sigma_0 \left(\frac{(-t)}{t_0}\right)^\beta, & H = 0,\ T < T_c \end{cases}. \tag{4}$$

The normalization constants t_0 and h_0 are determined by these equations, once σ_0 and H_0 are specified. We choose $\sigma_0 = \langle\sigma\rangle_{T=0} = f_\pi \approx 93$ MeV and $H_0 = m_\pi^2 f_\pi \approx 1.77 \cdot 10^6$ MeV^3.

The scaling of the order parameter susceptibility in the vicinity of the critical point is obtained from Eq. (3)

$$\chi_\sigma = \frac{\partial\langle\sigma\rangle}{\partial H} = \chi_0 h^{1/\delta-1}\left(f_G(z) - \frac{z f_G'(z)}{\beta}\right) \equiv \chi_0 h^{1/\delta-1} f_\chi(z). \tag{5}$$

Consequently, the maximum of the susceptibility is located at a fixed value of $z = z_p$, independently of the external field H. From this, it follows that the pseudo-critical temperature at a finite external field is given by [6]

$$\frac{T_p(H) - T_c}{T_c} = \frac{z_p}{z_0}\left(\frac{H}{H_0}\right)^{1/(\beta\delta)}, \tag{6}$$

where $z_0 = h_0^{1/(\beta\delta)}/t_0$ is a non-universal parameter.

The width of the cross-over region can be defined from the susceptibility of the order parameter χ_σ. The universal part of χ_σ is a peaked function with a width of Δz, which depends only on the universality class. Thus, the width of the cross-over region in temperature, which for a given external field given by

$$\frac{\Delta T}{T_c} = \frac{\Delta z}{z_0}\left(\frac{H}{H_0}\right)^{1/(\beta\delta)}, \tag{7}$$

depends on the non-universal parameter z_0.

III. MODELING MAGNETIC EQUATION OF STATE

The scaling theory of phase transitions provides definite predictions for the critical properties of various thermodynamic observables. These are characterized by the critical exponents of the corresponding universality class, which are ingrained in the scaling free energy. In particular, the order parameter is given by the magnetic equation of state $\langle\sigma\rangle/(\sigma_0 h^{1/\delta}) = x(z,h)$, which in the critical region collapses to the universal scaling function $x = f_G(z)$, introduced in Eq. (3). However, sufficiently far away from the critical point, corrections to the universal scaling become significant and $x(z,h)$ deviates from $f_G(z)$. The size of the scaling region is not universal, and hence model dependent.

In the following, we focus on the QCD chiral phase transition and discuss the scaling properties of the chiral order parameter near the critical point. We consider effective models

belonging to the universality class of the QCD chiral transition, and compute leading order corrections to the scaling curve in the magnetic equation of state. We analyze scaling violations induced by finite quark masses in the context of recent LQCD findings, which indicate that for a physical value of the pion mass, QCD lies in the scaling regime of the underlying second order phase transition. We consider the QM and PQM models in the mean-field approximation and a purely bosonic theory, the $O(N)$ linear sigma model, in the $N \to \infty$ limit. We examine, to what extent the scaling violating terms are compatible with QCD for a physical value of the pion mass.

In the next section we consider the Landau theory of second order phase transitions and construct the corresponding magnetic equation of state as a baseline for a quantitative description of the QM and PQM models. We then go beyond the mean-field approximation and study the magnetic equation of state and deviation from scaling in the large N limit of the $O(N)$ linear sigma model, using the high temperature expansion.

A. The Landau theory

In mean-field theory, second order phase transitions are generically described by Landau theory [44]. There, the effective potential is a polynomial in the order parameter σ, with coefficients that are analytic functions of the temperature T. Assuming a symmetry under reflections, $\sigma \to -\sigma$, the effective potential is, apart from a symmetry breaking term proportional to the external field H, an even polynomial in σ, and reads

$$\mathcal{L}(T, H; \sigma) = a(t)\frac{\sigma^2}{2} + b(t)\frac{\sigma^4}{4} + c(t)\frac{\sigma^6}{6} + d(t)\frac{\sigma^8}{8} + \cdots - H\sigma. \tag{8}$$

Here the T-dependent coefficients are parmeterized as polynomials in the reduced temperature $t = \frac{T}{T_c} - 1$ and have the form

$$\begin{aligned} a(t) &= a_1 t + a_2 t^2 + a_3 t^3 + \ldots, \\ b(t) &= b_0 + b_1 t + b_2 t^2 + \ldots, \\ c(t) &= c_0 + c_1 t + \ldots, \\ d(t) &= d_0 + \ldots. \end{aligned} \tag{9}$$

For a given value of T and H, the order parameter is given by the location of the minimum

of $\mathcal{L}(T, H; \sigma)$. This is determined by solving the gap equation

$$\frac{\partial \mathcal{L}}{\partial \sigma} = a(t) \langle \sigma \rangle + b(t) \langle \sigma \rangle^3 + c(t) \langle \sigma \rangle^5 + d(t) \langle \sigma \rangle^7 + \cdots = H. \tag{10}$$

For vanishing external field and $b(t) > 0$ and $c(t), d(t) \geq 0$, there is a second order phase transition at $T = T_c$, where $a(t)$ vanishes. Around the corresponding critical point, the order parameter exhibits the following scaling properties

$$\langle \sigma \rangle = \begin{cases} \left(\frac{H}{b_0}\right)^{1/\delta}, & T = T_c,\ H > 0 \\ \left(\frac{a_1(-t)}{b_0}\right)^{\beta}, & H = 0,\ T < T_c \end{cases}, \tag{11}$$

with $\delta = 3$ and $\beta = 1/2$, respectively. Comparing Eq. (11) with the general scaling behavior of the order parameter in Eq. (4) one can extract t_0 and h_0 in Landau theory,

$$h_0 = \frac{b_0 \sigma_0^3}{H_0}, \qquad t_0 = \frac{b_0 \sigma_0^2}{a_1}. \tag{12}$$

We note that both h_0 and t_0 depend on the normalization of the order parameter, while h_0 depends also on the normalization of the external field. Thus, a comparison of these parameters between different models has to be done with care. The model dependence can be reduced by considering the combination $z_0 = h_0^{2/3}/t_0 = \frac{a_1 b_0^{2/3}}{b_0 H_0^{2/3}}$, which is independent of σ_0. Hence, one can compare the value of z_0 with other approaches, provided the normalization of the external field H_0 is known.

The mean-field magnetic equation of state is obtained from Landau's thermodynamic potential, by introducing the scaling variables

$$\begin{aligned} x &= \frac{\sigma/\sigma_0}{h^{1/3}} = \sigma \left(\frac{b_0}{H}\right)^{1/3}, \\ z &= \frac{\bar{t}}{h^{2/3}} = \frac{a_1 t}{b_0} \left(\frac{b_0}{H}\right)^{2/3}, \end{aligned} \tag{13}$$

where $x > 0$ and z can take any real value. The variable z can be used to map out the phase diagram of the system. Thus, $|z| \ll 1$ corresponds to a system near the critical point $T = T_c$, while $z \ll -1$ refers to the phase with broken symmetry and $z \gg 1$ to the one where the symmetry is restored.

Using Eq. (13), we express the reduced temperature and the order parameter in terms of x, z and H,

$$t = \frac{b_0 z}{a_1} \left(\frac{H}{b_0}\right)^{2/3} = \frac{z}{z_0} \left(\frac{H}{H_0}\right)^{2/3}, \qquad \sigma = x \left(\frac{H}{b_0}\right)^{1/3}. \tag{14}$$

The gap equation Eq. (10) can be expressed in terms of the scaling variables x, z

$$\left(x(x^2+z)-1\right)+\left(\frac{H}{b_0}\right)^{2/3}\left(\frac{c_0}{b_0}x^5+\frac{b_1}{a_1}x^3z+\frac{a_2b_0}{a_1^2}xz^2\right)+\mathcal{O}\left(\left(\frac{H}{b_0}\right)^{4/3}\right)=0. \tag{15}$$

The solution of the gap equation yields the magnetic equation of state $x = x(z, h)$ in Landau theory.

The universal scaling curve of mean-field theory is obtained by taking the limit $H \to 0$ in the gap equation. In this limit only the terms grouped in the first parenthesis in Eq. (15) survive, while the terms proportional to $(H/b_0)^{2/3}$ provide the leading scaling violation. Since the latter depend on the parameters of the model, introduced in Eq. (9), they are non-universal and consequently model dependent. Thus, to quantify the deviations from universal scaling, we must specify the coefficients in the Landau effective potential. This will be done in the next section in the QM and PQM models. However, qualitative features of the scaling violation can be extracted from general considerations. We can distinguish three asymptotic regimes:

- $z \to -\infty$: In this limit the scaling curve behaves as $x(z) \simeq \sqrt{-z}$, and the sign of the scaling violation is determined by the sign of $(c_0/b_0 - b_1/a_1 + a_2b_0/a_1^2)$.

- $z = 0$: At this point the scaling curve goes through $x = 1$, and the sign of c_0 determines the sign of the deviations.

- $z \to \infty$: In this limit the scaling curve behaves as $x \simeq 1/z$, and the sign of the correction is determined by the sign of a_2.

The discussion of the asymptotic behavior of the leading order scaling violation shows that, depending the model parameters, the sign of the deviation from the scaling curve can change as a function of z and can therefore cross the universal curve at several points. Indeed, Eq. (15) shows that in Landau-theory, the first order correction vanishes at points (x, z) where

$$\frac{c_0}{b_0}+\frac{b_1}{a_1}\left(\frac{z}{x^2}\right)+\frac{a_2b_0}{a_1^2}\left(\frac{z}{x^2}\right)^2=0. \tag{16}$$

The roots of the second order equation are $z = \alpha^{\pm}x^2$, where

$$\alpha^{\pm}=-\frac{a_1}{2a_2b_0}\left(b_1\pm\sqrt{b_1^2-4a_2c_0}\right). \tag{17}$$

Substitution of the roots into the universal curve yields the coordinates of the crossing points

$$z_c^{\pm} = \frac{\alpha^{\pm}}{(1+\alpha^{\pm})^{2/3}}, \qquad x_c^{\pm} = (1+\alpha^{\pm})^{-1/3}. \tag{18}$$

If α^+ and α^- are not real, or both of them are real and smaller than (-1), then the magnetic equation of state does not cross the universal scaling function, to leading order in H/b_0. On the other hand, if the coefficients $\alpha^{\pm}$ are real, and only one of them is larger than (-1), there is one crossing point. Finally, if both solutions are real and larger than (-1), then there are two crossing points.

B. Quark-meson model in the mean-field approximation

The quark-meson (QM) model is widely employed as a low energy effective theory of QCD, because it shares an important characteristic with QCD, namely spontaneous breaking of chiral symmetry in vacuum and its restoration at finite temperature. The elementary fields in this model are the quark $q = \{u, d\}$ and meson $\phi = \{\sigma, \vec{\pi}\}$ fields, with the Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \sigma\right)^2 + \frac{1}{2}\left(\partial_\mu \vec{\pi}\right)^2 - U_m(\sigma, \vec{\pi}) + \overline{q}\left(i\gamma^\mu \partial_\mu - g\left(\sigma + i\gamma_5 \vec{\tau}\vec{\pi}\right)\right) q. \tag{19}$$

Here the mesonic potential is given by

$$U_m(\sigma, \vec{\pi}) = \frac{\lambda}{4}\left(\sigma^2 + \vec{\pi}^2 - v^2\right)^2 - H\sigma. \tag{20}$$

At low temperatures, chiral symmetry is spontaneously broken, and the σ field gains a non-zero expectation value [45]. In the chiral limit, $H \to 0$, there is a second order phase transition at the temperature $T = T_c$. Ignoring the fluctuations of the meson fields, the transition is governed by mean-field dynamics, corresponding to the $O(4)$ universality class in four dimensions. In this approximation, the dynamics of the chiral symmetry breaking can be mapped onto a Landau effective potential $\Omega(T, \mu; \sigma)$, which is a polynomial in the order parameter σ. The effect of vacuum and thermal fluctuations of the fermion fields are accounted for in the effective potential [20]

$$\begin{aligned}\Omega(T, \mu; \sigma) =& U_m(\sigma, 0) - \frac{N_c N_f}{8\pi^2} g^4 \sigma^4 \ln\left(\frac{g\sigma}{M}\right) \\ &- \frac{N_c N_f}{\pi^2}\int_0^\infty dp p^2 T\left(\ln\left(1 + e^{(\mu - \epsilon_q(p,\sigma))/T}\right) + \ln\left(1 + e^{(-\mu - \epsilon_q(p,\sigma))/T}\right)\right),\end{aligned} \tag{21}$$

where $\epsilon_q(p,\sigma) = \sqrt{p^2 + g^2\sigma^2}$. The values of the parameters λ, v^2 and H are determined by choosing the pion mass m_π, the sigma mass m_σ, as well as the pion decay constant f_π in vacuum. The Yukawa coupling g is set by the constituent quark mass in vacuum. In the chiral limit, $H \to 0$, the pion is a true Goldstone boson, with a vanishing vacuum mass. The parameter M is an arbitrary renormalization scale. Modifications of M can be absorbed by redefining λ and v^2.

As pointed out in Ref. [20], both the vacuum and thermal contributions contain a non-analytic term in σ, which cancel at non-zero temperature. This cancellation is crucial for obtaining a second order chiral transition in the chiral limit. Close to the critical point, where $H \approx 0$ and $T \approx T_c$, the order parameter σ is very small. Consequently, the contribution of the thermal fermion loop to the Landau effective potential can be obtained in the high temperature expansion [46, 47]. We thus obtain the Landau free energy density

$$\Omega(T,H;\sigma) = a(t)\frac{\sigma^2}{2} + b(t)\frac{\sigma^4}{4} + c(t)\frac{\sigma^6}{6} + \cdots - H\sigma, \tag{22}$$

where the coefficients are functions of the reduced temperature t and the input parameters

$$\begin{aligned} a(t) &= \left(\frac{N_c N_f g^2 T_c^2 (2t+t^2)}{6}\right), \\ b(t) &= \frac{m_\sigma^2 - m_\pi^2}{2f_\pi^2} + \frac{N_c N_f g^4}{2\pi^2}\left(\gamma_E - \ln\left(\frac{\pi T_c(1+t)}{g f_\pi}\right)\right), \\ c(t) &= -\frac{7\zeta(3) N_c N_f g^6}{32\pi^4 T_c^2 (1+t)^2}. \end{aligned} \tag{23}$$

In the mean-field approximation, the second order chiral phase transition appears at

$$T_c = \sqrt{\frac{3\left(m_\sigma^2 - 3m_\pi^2\right)}{N_c N_f g^2} + \frac{3g^2 f_\pi^2}{2\pi^2}}. \tag{24}$$

In this approximation, the QM model is a particular realization of Landau theory. Using the coefficients of the effective potential, Eq. (23), we can, following the discussion in the previous section, compute h_0, t_0 and z_0 and explicitly determine the magnetic equation of state given in Eq. (15), including the leading order scaling violating term.

C. Polyakov loop extended quark-meson model

The chiral QM model is an effective realization of the chiral sector of QCD. However, because the local $SU(N_c)$ invariance of QCD is replaced by a global symmetry in the model,

color confinement is lost. Nevertheless, the confining properties of QCD can be approximately accounted for by including the expectation value of the Polyakov loop

$$\Phi = \frac{1}{N_c} \langle \mathrm{Tr}_c \, L(\vec{x}) \rangle \,, \quad \overline{\Phi} = \frac{1}{N_c} \left\langle \mathrm{Tr}_c \, L^\dagger(\vec{x}) \right\rangle , \tag{25}$$

with

$$L(\vec{x}) = \mathcal{P} \exp \left(i \int_0^\beta d\tau A_4(\vec{x}, \tau) \right) , \tag{26}$$

in a low-energy chiral effective model, like the QM model [15, 19, 21, 34, 48, 49]. Here $A_4 = iA_0$ is the temporal component of the Euclidean gluon field, $\beta = 1/T$ and $\mathcal{P}$ denotes path ordering. Thus, the Polyakov loop extended quark-meson (PQM) model effectively combines both the chiral symmetry and confinement of QCD.

The Lagrangian of the PQM model reads

$$\mathcal{L} = \overline{q} \left(i\gamma^\mu D_\mu - g(\sigma + i\gamma_5 \vec{\tau}\vec{\pi}) \right) q + \frac{1}{2} \left(\partial_\mu \sigma \right)^2 + \frac{1}{2} \left(\partial_\mu \vec{\pi} \right)^2 - U_m(\sigma, \vec{\pi}) - \mathcal{U}(\Phi, \overline{\Phi}). \tag{27}$$

The coupling between the effective gluon field and quarks is implemented through the covariant derivative, $D_\mu = \partial_\mu + iA_\mu$, where the spatial components of the gluon field are neglected, i.e. $A_\mu = \delta_{\mu 0} A_0$. Here $\mathcal{U}(\Phi, \overline{\Phi})$ is the potential for the thermal expectation value of the Polyakov loop.

The thermodynamic potential is in the PQM model given by [20]

$$\begin{aligned} \Omega(\sigma, \Phi, \overline{\Phi}; T, \mu) = U_m(\sigma, 0) - \frac{N_c N_f}{8\pi^2} g^4 \sigma^4 \ln \left(\frac{g\sigma}{M} \right) \\ + \Omega_f^{th}(T, \mu; \sigma, \Phi, \overline{\Phi}) + \mathcal{U}(\Phi, \overline{\Phi}; T) \end{aligned} \tag{28}$$

where the meson and vacuum fermion contributions are the same as in the QM model in Eq. (21), whereas the thermal fermionic contribution is modified due to coupling of quarks to the Polyakov loop background

$$\Omega_f^{th}(T, \mu; \sigma, \Phi, \overline{\Phi}) = -\frac{N_f T}{\pi^2} \int_0^\infty dp p^2 \left(\ln g^{(+)}(T, \mu; \sigma, \Phi, \overline{\Phi}) + \ln g^{(-)}(T, \mu; \sigma, \Phi, \overline{\Phi}) \right) \tag{29}$$

with

$$\begin{aligned} g^{(+)}(T, \mu; \sigma, \Phi, \overline{\Phi}) &= 1 + 3\Phi e^{-(E_q - \mu)/T} + 3\overline{\Phi} e^{-2(E_q - \mu)/T} + e^{-3(E_q - \mu)/T}, \\ g^{(-)}(T, \mu; \sigma, \Phi, \overline{\Phi}) &= g^{(+)}(T, -\mu; \sigma, \overline{\Phi}, \Phi). \end{aligned} \tag{30}$$

Clearly, by taking the limit $\Phi \to 1$ in Eq. (29), one recovers the fermion part of the effective potential of the QM model, Eq. (21). The gluon potential, $\mathcal{U}(\Phi, \overline{\Phi}; T)$, is constructed so

as to respect the $Z(N_c)$ global symmetry, with parameters chosen to reproduce the thermodynamics of pure lattice gauge theory [35, 37, 50]. We use the potential obtained in Ref. [37]

$$\frac{\mathcal{U}(\Phi,\overline{\Phi};T)}{T^4} = -\frac{1}{2}a(T)\left(\Phi\overline{\Phi}\right) + b(T)\log M_H(\Phi,\overline{\Phi}) \tag{31}$$

with

$$M_H(\Phi,\overline{\Phi}) = 1 - 6\Phi\overline{\Phi} + 4\left(\Phi^3 + \overline{\Phi}^3\right) - 3\left(\Phi\overline{\Phi}\right)^2. \tag{32}$$

The temperature dependent coefficients are given by

$$a(T) = a_0 + a_1\left(\frac{T_0}{T}\right) + a_2\left(\frac{T_0}{T}\right)^2, \quad b(T) = b_3\left(\frac{T_0}{T}\right)^3 \tag{33}$$

with the parameters

$$a_0 = 3.51, \quad a_1 = -2.47, \quad a_2 = 15.2, \quad b_3 = -1.75, \quad T_0 = 270\ \text{MeV}. \tag{34}$$

In the mean-field approximation, the expectation value of σ and of the Polyakov loop, Φ and $\overline{\Phi}$ are determined by requiring that the thermodynamic potential is stationary [51]

$$\frac{\partial\Omega}{\partial\sigma} = \frac{\partial\Omega}{\partial\Phi} = \frac{\partial\Omega}{\partial\overline{\Phi}} = 0. \tag{35}$$

The model parameters are fixed by requiring that the vacuum physics is reproduced, as indicated in Sec. III B for the QM model.

In the chiral limit, this model exhibits second order chiral phase transition with mean-field exponents. Near the critical point, the thermodynamic potential is a polynomial in the order parameter σ, as in Eq. (8), with coefficients that can be extracted from Eq. (28) using the high temperature expansion [47]. In this case, however, the coefficients of the potential depend on the expectation value of the Polyakov-loop and cannot be obtained in closed form. Thus, for the PQM model the critical temperature, the coefficients of the Landau potential and the magnetic equation of state are computed numerically.

D. *O(N)* linear sigma model in the large *N* limit

In the preceeding sections we have introduced the Landau effective action and two chiral effective models, which allow us to explore various aspects of the magnetic equation of state in the mean-field approximation. In this section we turn to the $O(N)$ symmetric linear sigma

(LS) model, where the thermodynamic potential and the scaling properties near the critical point can be computed exactly in the large N limit. The LS model in (3+1) dimensions is described by the Euclidean action

$$S_E = \int d^4x \left(\frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2}(\partial_\mu \pi_i)^2 + \frac{1}{2}m^2(\sigma^2 + \pi_i^2) + \frac{\lambda}{N}\left(\sigma^2 + \pi_i^2\right)^2 - \frac{\sqrt{N}}{2}H\sigma \right), \quad (36)$$

where the subscript μ denotes a direction in Euclidean space-time while i is an index in flavor space, spanned by the N-component vectors $\{\sigma, \pi_i\}$. The N dependent factors in Eq. (36) are introduced for later convenience.

For vanishing external field H, the action is invariant under rotations in the N dimensional flavor space. For negative values of m^2, this symmetry is spontaneously broken in the vacuum and the N-tuple $\{\sigma, \pi_i\}$ acquires a non-zero expectation value, a condensate. The coordinates in flavor space are chosen such that the condensate is in the σ direction. Consequently, the $N-1$ remaining fields, π_i, have a vanishing expectation value. The condensate $\langle\sigma\rangle$ is an order parameter of the spontaneously broken $O(N)$ symmetry and the shifted field $\sigma' = \sigma - \langle\sigma\rangle$ represents the fluctuations of the σ field about its expectation value.

To determine the thermodynamic potential density $\omega[T]$, we employ the $2PI$ formalism [52, 53]. The $2PI$ functional for the theory yields

$$\omega[T; \langle\sigma\rangle, G_\pi, G_\sigma] = \frac{T}{V}\Bigg(S_E[\langle\sigma\rangle] + \frac{N-1}{2}\mathrm{Tr}\ln G_\pi^{-1} + \frac{N-1}{2}\mathrm{Tr}\left(\left(D_\pi^{-1} - G_\pi^{-1}\right)G_\pi\right) + \frac{1}{2}\mathrm{Tr}\ln G_\sigma^{-1} + \frac{1}{2}\mathrm{Tr}\left(\left(D_\sigma^{-1} - G_\sigma^{-1}\right)G_\sigma\right) - \Gamma_2[\langle\sigma\rangle, G_\pi, G_\sigma]\Bigg), \quad (37)$$

where D_σ, D_π, G_σ and G_π are the bare and dressed propagators of the sigma and pion fields, respectively. Moreover, $\Gamma_2[\langle\sigma\rangle, G_\pi, G_\sigma]$ denotes the sum of all possible $2PI$ diagrams (with dressed propagators), while V and T are the volume and the temperature of the system. Finally, the trace in Eq. (37) is given by

$$\mathrm{Tr} = T\sum_n \int \frac{d^3k}{(2\pi)^3}, \quad (38)$$

where the sum is over the Matsubara frequencies.

The physical values of the dressed propagator and the expectation value of the field are determined by the stationarity conditions

$$\frac{\delta\omega}{\delta\langle\sigma\rangle} = 0, \qquad \frac{\delta\omega}{\delta G_\pi} = 0, \qquad \frac{\delta\omega}{\delta G_\sigma} = 0. \quad (39)$$

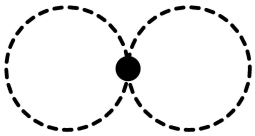

FIG. 1. The only contribution to Γ_2 in the $1/N$ expansion of the $O(N)$ sigma model for $N \to \infty$. The dashed lines are pion propagators, whereas the filled dot depicts the four-point vertex, with coupling strength λ/N.

The second and the third equations are just the Dyson equation for the pion and sigma fields respectively,

$$G^{-1} = D^{-1} - 2\,\frac{\delta\Gamma_2}{\delta G} = D^{-1} + \Sigma, \tag{40}$$

where $-2\,\delta\Gamma_2/\delta G$ is identified with the self-energy Σ. A tractable self-consistent scheme for calculating the thermodynamic potential starting from Eq. (37) is defined by a choice of the set of $2PI$ diagrams contributing to Γ_2 (for details see ref. [52]).

For convenience we simplify our notation by introducing $\phi \equiv 2\langle\sigma\rangle/\sqrt{N}$. With this choice the inverse bare Euclidean propagators are given by

$$D_\sigma^{-1}(\mathbf{k}, n) = \mathbf{k}^2 + \omega_n^2 + m^2 + 3\lambda\phi^2, \tag{41}$$

$$D_\pi^{-1}(\mathbf{k}, n) = \mathbf{k}^2 + \omega_n^2 + m^2 + \lambda\phi^2, \tag{42}$$

where $\omega_n = 2n\pi T$ is the Matsubara frequency and $\mathbf{k}$ denotes the momentum in the spatial direction. The bare mass-square of the σ and π fields are given by $m^2 + 3\lambda\phi^2$ and $m^2 + \lambda\phi^2$, respectively.

In the $O(N)$ linear sigma model, at $N \to \infty$, the contributions of sigma loops to $\Gamma_2[\langle\sigma\rangle, G_\pi, G_\sigma]$ are suppressed, due to $1/N$ factor introduced in the four-point coupling in the action. Consequently, to leading order in $1/N$, the only relevant contribution to the $2PI$ diagrams is the two-pion loop diagram shown in Fig. 1

$$\Gamma_2[\phi, G_\pi] = -N\lambda(\mathrm{Tr}\, G_\pi)^2. \tag{43}$$

This in turn yields the pion self-energy

$$\Sigma_\pi(\mathbf{k}, n) = 4\lambda \mathrm{Tr}\, G_\pi. \tag{44}$$

Since the fluctuations of the σ field are neglected in the large N limit, the thermodynamic potential density depends only on the condensate $\langle\sigma\rangle$ and on the dressed pion propagator

G_π. These are then determined by the first two equations of Eq. (39). Up to leading order in N the potential reads

$$\omega[T;\phi,G_\pi] = U[\phi] + \frac{N}{2}\mathrm{Tr}\ln G_\pi^{-1} + \frac{N}{2}\mathrm{Tr}\left(\left(D_\pi^{-1} - G_\pi^{-1}\right)G_\pi\right) + N\lambda\left(\mathrm{Tr}G_\pi\right)^2, \tag{45}$$

with

$$U[\phi] = \frac{N}{4}\left(\frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 - H\phi\right). \tag{46}$$

The Dyson equation in this approximation yields

$$G_\pi^{-1}(\mathbf{k},n) = D_\pi^{-1}(\mathbf{k},n) + \Sigma_\pi(\mathbf{k},n) = \mathbf{k}^2 + \omega_n^2 + m^2 + \lambda\phi^2 + 4\lambda\mathrm{Tr}G_\pi. \tag{47}$$

This is a self-consistent equation for the self-energy or equivalently for the renormalized pion mass. This is readily seen by rewriting the propagator in the compact form

$$G_\pi(\mathbf{k},n) = \left(\mathbf{k}^2 + \omega_n^2 + M_\pi^2\right)^{-1}, \tag{48}$$

where the pion mass M_π is a solution of the equation

$$M_\pi^2 = m^2 + \lambda\phi^2 + 4\lambda\mathrm{Tr}G_\pi(M_\pi). \tag{49}$$

The boson loop integral $\mathrm{Tr}\,G_\pi$ can conveniently be expressed in terms of the logarithmic term in Eq. (45)

$$\begin{aligned}\mathrm{Tr}(G_\pi) &= \frac{\partial}{\partial M_\pi^2}\mathrm{Tr}\ln(G_\pi^{-1}).\\ &= \frac{\partial}{\partial M_\pi^2}T\sum_n\int\frac{d^3k}{(2\pi)^3}\ln\left(k^2 + (2n\pi T)^2 + M_\pi^2\right)\\ &= \frac{\partial}{\partial M_\pi^2}\int\frac{d^3k}{(2\pi)^3}\left(\sqrt{k^2+M_\pi^2} + 2T\ln\left(1 - e^{-\sqrt{k^2+M_\pi^2}/T}\right)\right).\end{aligned} \tag{50}$$

The first term is the UV divergent vacuum contribution, while the second term is the finite temperature contribution. Using dimensional regularization, we retain only the finite part of the vacuum integral, following [27, 54–56]. The renormalized vacuum contribution is then given by

$$\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\sqrt{k^2+M_\pi^2}} \to \frac{1}{(4\pi)^2}\left(M_\pi^2\ln\frac{M_\pi^2}{\mu^2} - M_\pi^2 + \mu^2\right), \tag{51}$$

where μ is an arbitrary renormalization scale. We choose $\mu = m_\pi = 138\,\mathrm{MeV}$, which simplifies the formulas somewhat.

Stationarity of the functional given in Eq. (45) yields hence the following system of equations for the renormalized pion mass M_π and the order parameter ϕ,

$$\begin{aligned} H &= M_\pi^2 \phi, \\ M_\pi^2 &= m^2 + \lambda\phi^2 + 4\lambda \mathrm{Tr} G_\pi. \end{aligned} \tag{52}$$

The model parameters λ, m^2 and H are chosen so as to reproduce the vacuum pion and sigma mass, as well as the pion decay constant. These conditions yield the following constraints [27]:

$$\begin{aligned} H &= m_\pi^2 f_\pi, \qquad \lambda = \frac{m_\sigma^2 - m_\pi^2}{2f_\pi^2}, \\ m^2 &= -\frac{m_\sigma^2 - 3m_\pi^2}{2} - \frac{\lambda}{4\pi^2}\left(m_\pi^2 \ln\frac{m_\pi^2}{\mu^2} - m_\pi^2 + \mu^2\right) = -\frac{m_\sigma^2 - 3m_\pi^2}{2}, \end{aligned} \tag{53}$$

where in the last equality we used $\mu = m_\pi$. To derive the magnetic equation of state for this model, we again apply the high temperature expansion [46].

1. *The magnetic equation of state of the LS model*

Near the critical point, i.e. where $H \approx 0$, $\phi \approx 0$ and the pion mass $M_\pi \approx 0$, we can expand $\mathrm{Tr}(G_\pi)$ in powers of M_π/T by using the high temperature expansion,

$$\mathrm{Tr}(G_\pi) = \left(\frac{T^2}{12} + \frac{\mu^2}{16\pi^2}\right) - \frac{T}{4\pi}M_\pi + \frac{\ln\left(\frac{4\pi T}{\mu}\right) - \gamma_E}{8\pi^2}M_\pi^2 + \mathcal{O}\left(M_\pi^4\right). \tag{54}$$

Here γ_E is the Euler-Mascheroni constant.

By substituting the leading term in Eq. (54) into the gap equation, Eq. (52), with $M_\pi = \phi = 0$, one finds the critical temperature for the second order transition [27]

$$m^2 + 4\lambda\left(\frac{T_c^2}{12} + \frac{\mu^2}{16\pi^2}\right) = 0 \to T_c = \sqrt{3}\, f_\pi^{ch}, \tag{55}$$

where

$$f_\pi^{ch} = f_\pi\left(\frac{m_\sigma^2 - 3m_\pi^2}{m_\sigma^2 - m_\pi^2} - \frac{1}{4\pi^2}\frac{m_\pi^2}{f_\pi^2}\right)^{1/2} \tag{56}$$

is the pion decay constant in the chiral limit. In Eq. (56) we have inserted the renormalization scale $\mu = m_\pi$. Using Eq. (52) we also find that for $H = 0$ and $M_\pi = 0$ the order parameter in the broken phase is given by

$$\phi = \sqrt{\frac{T_c^2 - T^2}{3}}. \tag{57}$$

Thus, near the critical point the order parameter scales as $\phi \sim (T_c - T)^{1/2} \sim (-t)^{1/2}$, with the critical exponent $\beta = 1/2$.

In order to obtain the exponent δ and the magnetic equation equation of state, we retain only the leading (linear) term in M_π in the second equation in Eq. (52). This leads to the system of equations

$$\begin{aligned} H &= M_\pi^2 \phi, \\ 0 &= \lambda \left(\frac{T_c^2 - T^2}{3} - \phi^2 + \frac{T}{\pi} M_\pi \right). \end{aligned} \tag{58}$$

Consequently, at $T = T_c$, we find

$$\phi \sim H^{1/5}, \quad \text{and} \quad M_\pi \sim H^{2/5}. \tag{59}$$

Thus, the critical exponent $\delta = 5$, as in the spherical model in three spatial dimension [10, 57]. We note that in four dimensions, the model yields $\beta = 1/2$ and $\delta = 3$, as in the mean-field case.

We are now ready to derive the magnetic equation of state, including the leading order scaling violating term. By eliminating the pion mass in Eq. (52) and using the high temperature expansion of the one-loop self-energy given in Eq. (54), one arrives at the gap equation

$$\frac{T_c}{\pi}(1+t) \left(\frac{H}{\phi} \right)^{1/2} = \phi^2 + \frac{T_c^2}{3}(2t + t^2) + \frac{3\,\alpha}{4\,\pi^2} \frac{H}{\phi}, \tag{60}$$

which is valid near the critical point, where t, H and ϕ are small. Here we introduced the short-hand notation

$$\alpha = \frac{2}{3} \left[\ln \left(\frac{4\pi T_c}{\mu} \right) - \gamma_E \right] - \frac{4\,\pi^2}{3\,\lambda} \tag{61}$$

and neglected the temperature dependence of the logarithm, which yields only terms of higher order in the scaling violating field. In analogy with Eqs. (1) and (13), we introduce the scaling fields z, x and h by means of

$$H = h\, h_0\, H_0, \qquad t = z\, t_0\, h^{2/5}, \qquad \phi = x\, \phi_0\, h^{1/5}. \tag{62}$$

The constants h_0 and t_0 are determined by the normalization conditions

$$x(z=0) = 1, \qquad \lim_{z \to -\infty} x(z)/(-z)^\beta = 1, \tag{63}$$

which are equivalent to Eq. (11). One finds

$$h_0 = \frac{\phi_0^5\,\pi^2}{H_0\,T_c^2}, \qquad t_0 = \frac{3\,\phi_0^2}{2\,T_c^2}, \tag{64}$$

which depend explicitly on the normalization scale ϕ_0, while the ratio

$$z_0 = \frac{h_0^{2/5}}{t_0} = \frac{2}{3}\left(\frac{\pi^4 T_c^6}{H_0^2}\right)^{1/5} \tag{65}$$

depends, as expected, only on H_0.

The gap equation, expressed in terms of the scaling variables, is now obtained by squaring Eq. (60) and consistently retaining terms up to order $h^{2/5}$,

$$\left[x(x^2+z)^2 - 1\right] + h^{2/5}\,t_0\left[(x^5 - 1 + \alpha)(x^2+z)\right] = 0. \tag{66}$$

In the limit $h \to 0$, only the first term in square brackets in Eq. (66) survives. This yields the universal scaling magnetic equation of state for the $O(N)$ linear sigma model in the $N \to \infty$ limit. More generally, for non-zero h, the solution of Eq. (66) yields the magnetic equation of state, including the leading scaling violation.

The subleading term in Eq. (66) is not unique, since it may be modified by using the leading order (scaling) magnetic equation of state. The form given here was obtained by eliminating terms with non-integer powers of x as well as those involving higher powers than linear in z. Another form of this term leads to a modified magnetic equation of state for non-zero h. However, the difference is of higher order, i.e. at least of order $h^{4/5}$. Clearly, other forms of the leading scaling violating term in Eq. (15) can be obtained in an analogous manner. We note that the non-uniqueness of the leading symmetry breaking term does not affect the location of the possible crossing points, discussed in Sec. III A.

IV. MODEL DEPENDENCE OF SCALING PROPERTIES OF THE ORDER PARAMETER

In the preceding section we have computed the magnetic equation of state in three different models. The quark-meson (QM) model and its Polyakov loop extended counterpart, the PQM model, were both evaluated in the mean-field approximation. Consequently, the corresponding scaling functions coincide and are given by the solution of the gap equation

$$x(x^2+z) = 1. \tag{67}$$

The corresponding equation in the $O(N)$ linear sigma model in the large N limit, differs from Eq. (67), and reads

$$x(x^2 + z)^2 = 1. \tag{68}$$

In Fig. 2 we show the magnetic equations of state, given by the solutions $x = f_G(z)$ of Eqs. (67) and (68). In the broken phase the universal curves are close to each other, while in the restored phase, they differ considerably. On a qualitative level, this behavior can be understood by considering the structure of the magnetic equation of state in the asymptotic regions $z \to \pm\infty$. For large negative z, the scaling function $f_G(z)$ is of the form $(-z)^\beta$, whereas for positive z it asymptotically approaches $z^{-\gamma}$. Since in both models the critical exponent $\beta = 1/2$, the two universal curves are very similar for $z < 0$. On the other hand, $\gamma = 1$ in the mean-field QM and PQM models and $\gamma = 1$ in the large N linear sigma model. This difference is clearly reflected in the magnetic equation of state in the restored phase, i.e. for $z > 0$.

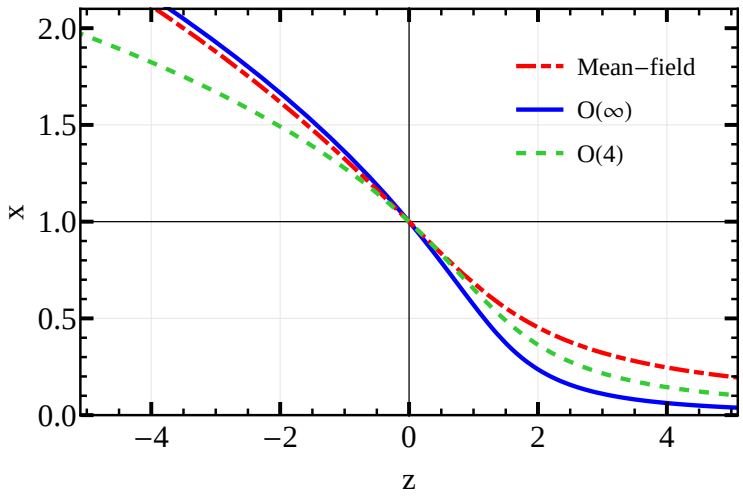

FIG. 2. The scaling function of the order parameter in mean-field models, the $O(N)$ linear sigma model in the $N \to \infty$ limit, and in the $O(4)$ universality class.

For comparison, the scaling equation of state of the $O(4)$ universality class, obtained in lattice simulations [58], is also shown. There are clear differences between the model results and the $O(4)$ universality class. Again, the characteristics can be understood in terms of the values of the γ and β exponents.

Since the QM and PQM models belong to the $O(4)$ universality class [59], differences

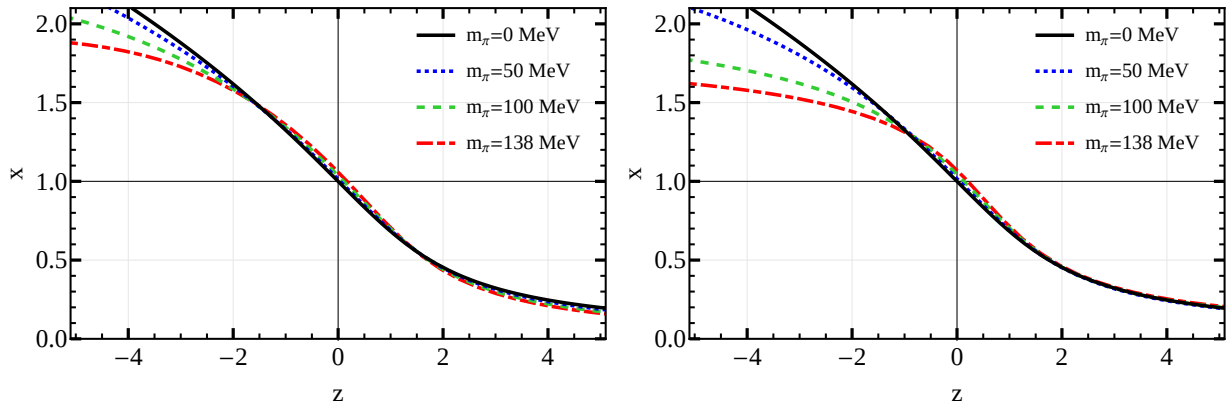

FIG. 3. The magnetic equation of state for the QM (left) and PQM (right) models. The black line corresponds to the universal scaling curve, which coincides in these two models. In both models the sigma mass was fixed to $m_\sigma = 400$ MeV, whereas the constituent quark mass in the vacuum was set to $m_q = 300$ MeV.

between the scaling properties of these models and the $O(4)$ universality class, seen in Fig. 2, will disappear when the effect of fluctuations is properly included in the thermodynamic potential.

Recent LQCD studies [9] of the chiral phase transition with (2+1)-flavors indicate that the scaling violation seen in the QCD magnetic equation of state remains moderate up to physical values of the light quarks masses. Moreover, the non-universal parameters h_0 and t_0 for QCD were determined. In this section we assess the scaling violation in the models presented above, and compare the non-universal parameters extracted in the models with the lattice QCD results.

In Figs. 3 and 4 we show the scaling behavior of the order parameter for several values of the symmetry breaking term $H/H_0 = (m_\pi/m_{\pi\ phys})^2$. The leading order corrections to the scaling functions of the QM, PQM and $O(N)$ models were discussed in the preceding section.

A comparison of the (mean-field) scaling properties of the order parameter in the QM and PQM models, shown in Fig. 3, shows that the coupling of quarks to the Polyakov loop enhances the scaling violation. This is particularly apparent in the broken phase, where the Polyakov loop expectation value differs appreciably from unity. Nevertheless, up to the physical pion mass, both models are still in the scaling regime of the underlying second order

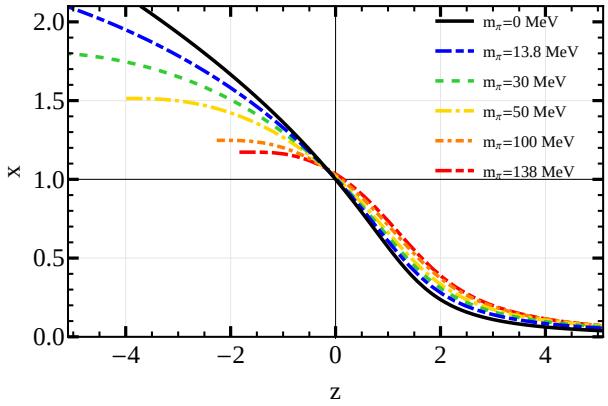

FIG. 4. The magnetic equation of state for the $O(N)$ linear sigma model in the $N \to \infty$ limit. The black line corresponds to the universal scaling curve of the $O(N = \infty)$ universality class. The end points of the curves at negative z values correspond to $T = 0$. In this calculation, $m_\sigma = 400$ MeV.

Model	H_0	σ_0	t_0	h_0	z_0
QM $m_\sigma = 400\,\mathrm{MeV}$	$m_\pi^2 f_\pi$	f_π	0.34	6.99	10.64
QM $m_\sigma = 800\,\mathrm{MeV}$	$m_\pi^2 f_\pi$	f_π	0.30	13.57	19.10
PQM $m_\sigma = 400\,\mathrm{MeV}$	$m_\pi^2 f_\pi$	f_π	0.073	5.26	41.50
LS $m_\sigma = 400\,\mathrm{MeV}$	$m_\pi^2 f_\pi$	f_π	0.74	2.22	1.85
LS $m_\sigma = 800\,\mathrm{MeV}$	$m_\pi^2 f_\pi$	f_π	0.57	1.69	2.18
QM FRG [43]	1	1	23.86 GeV/T_c	346.4 GeV^3	2.69
Lattice (p4) $N_\tau = 4$ [7]	$\frac{m_\pi^2 f_\pi m_s}{m_l}$	$\frac{T_c^4 f_\pi}{m_s \langle\overline{\psi}\psi\rangle_l^{T=0}}$	0.00407	0.00295	53.92
Lattice (p4) $N_\tau = 8$ [7]	$\frac{m_\pi^2 f_\pi m_s}{m_l}$	$\frac{T_c^4 f_\pi}{m_s \langle\overline{\psi}\psi\rangle_l^{T=0}}$	0.00271	0.00048	27.27

TABLE I. Comparison of the non-universal constants t_0, h_0 and z_0 in different theories. The normalization of the order parameter σ_0 and the external field H_0 differs, thus the t_0 and h_0 values cannot be directly compared between different models. The z_0 column contains converted values to our normalization convention, thus the results of different models can be directly compared.

phase transition, as also found in LQCD.

In the scaling plot of the QM model, shown in Fig. 3, there are two distinct points, where the curves for different values of the pion mass cross. This behavior was anticipated in our discussion of Landau theory in Sec. III A. By substituting the coefficients of the Landau

thermodynamic potential obtained in the QM model, given in Eq. (23), into Eqs. (17,18), we obtain the following (z, x)-coordinates of the crossing points

$$C_1 = \{-1.31, 1.42\}, \qquad C_2 = \{1.50, 0.55\}, \tag{69}$$

in agreement with the numerical results shown in Fig. 3. Obviously, the crossing points are independent of the pion mass only as long as the sub-leading scaling violating terms are negligible.

In the PQM model, shown in the right panel of Fig. 3, the location of the crossing points depends on the strength of the symmetry breaking field for values of the pion mass below the physical one. This indicates that in the PQM model, the convergence of the expansion in powers of the symmetry breaking field in Eq. (15) is worse than in the QM model. The above behavior can be linked to the coupling of quarks with the Polyakov loop. In the low temperature phase, the quark fluctuations are suppressed by the Polyakov loop, which results in a weaker dependence of the chiral condensate on the temperature. Consequently, close to the critical point, the chiral restoration as a function of temperature in the PQM model is sharper than in the QM model. This implies, that the size of the scaling window is reduced, and that deviations from scaling are larger in the PQM than in the QM model.

The difference in strength of the scaling-violation found in the QM and PQM models, is even more pronounced in the $O(N)$ sigma model. As shown in Fig. 4, the $O(N)$ model exhibits stronger deviations from the universal scaling curve than the QM and PQM models for the corresponding strength of the symmetry breaking field. Indeed, the scaling of the order parameter in the $O(N)$ model is preserved only for a very weak external field and the deviations from the universal line are substantial for the physical value of the pion mass. The qualitative differences in the universal scaling curves and in the strength of the scaling violation indicate that fluctuations of the meson fields, not accounted for in the mean-field models, play an important rôle in the determination of the magnetic equation of state.

In spite of the fact, that deviations from the universal scaling curve are large in the $O(N)$ sigma model, the lines with different pion masses cross at a unique point. This suggests that, close to the critical temperature, the sub-leading corrections in the magnetic equation of state are negligible up to the physical value of the pion mass. Applying the procedure discussed in the previous section, one finds, that (z, x)-coordinates of this crossing point appear at $(-0.15, 1.06)$, in agreement with the numerical results shown in Fig. 4.

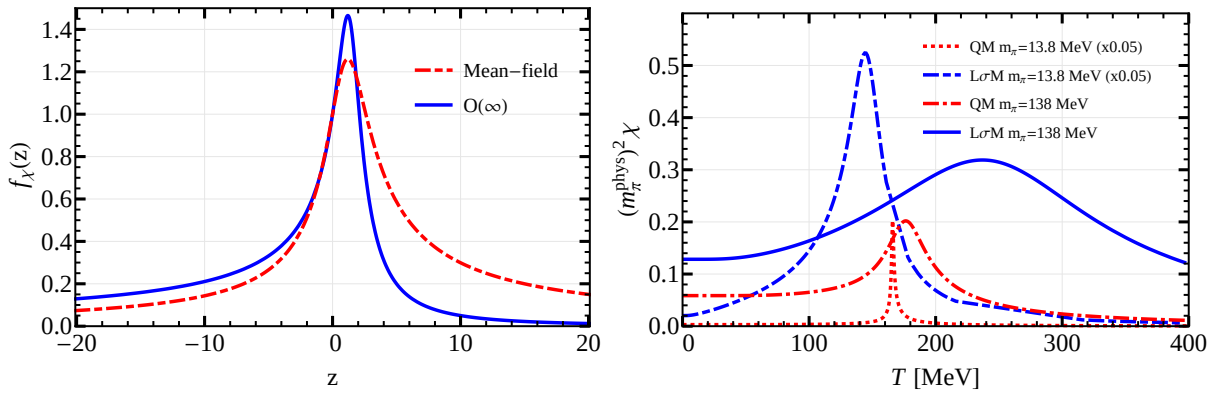

FIG. 5. Left figure: The universal scaling part of the chiral susceptibility in the QM model calculated within the mean-field dynamics and in the $O(N)$ LS model obtained in the $N \to \infty$ limit within high temperature expansion. Right figure: The chiral susceptibilities in the LS model in the $N \to \infty$ limit and in the QM model under mean-field approximation calculated at physical and at ten times lower pion mass.

The strong violation of scaling obtained in this model is consistent with previous studies within the FRG approach [43]. However, contrary to the FRG results of [43], we do not observe the approximate scaling of the order parameter for pion masses ~ 100 MeV to a non-universal line for $z > -1$.

A. Scaling violation and non-universal parameters

In previous sections we studied the leading order corrections to the magnetic equation of state and scaling functions in the mean-field approximation to the QM and PQM models. Clearly such a calculation cannot reproduce the universal properties of the $O(4)$ criticality expected in QCD (in three dimensions). However, this can be achieved by systematically including fluctuations of the meson fields e.g. within the FRG approach. In this context we note that the mean-field approximation does reproduce the universal properties of the $O(4)$ model in four dimensions. The difference between the mean-field approach and $O(4)$, or equivalently between $O(4)$ in three and four dimensions, is illustrated in Fig. 2 on the level of the scaling functions. The scaling function for the $O(N \to \infty)$ sigma model, which belongs to another universality class in three dimensions, is also shown in Fig. 2. In spite of these differences, the mean-field models and the $(1/N)$-expansion of the $O(N)$ model

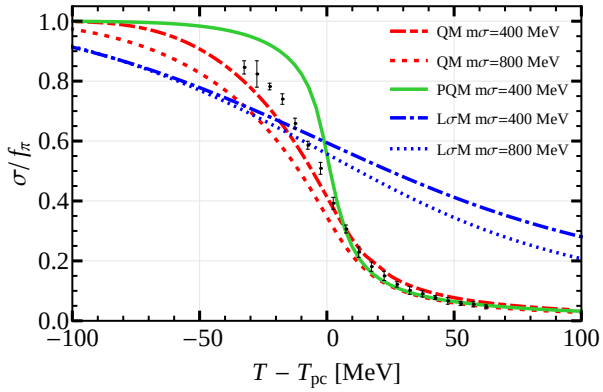

FIG. 6. The chiral order parameters in the QM and LS models calculated with different inputs for the vacuum sigma mass. The data points are lattice results from Ref. [60]. The pseudocritical temperature T_{pc} for these data is taken as the inflection point of the order parameter.

allow us to explore the scaling violation in a transparent framework and to illustrate general features of the magnetic equation of state, which are expected to be independent of the universality class.

The differences in the strength of the scaling violation seen in Figs. 3 and 4 are connected with very different values of the non-universal parameters: t_0, h_0 and z_0. This is seen in Tab. I, where we summarize their values in the present model calculations and in the previous studies of the magnetic equation of state within the FRG approach, as well as, in $(2+1)$-flavor LQCD. In the QM model and $O(N=\infty)$ linear sigma model, explicit expressions for h_0 and t_0 are given in Eqs. (12) and (64). In the PQM model these constants were obtained numerically, by fitting the order parameter to the asymptotic scaling laws Eq. (4).

Clearly, the values of these non-universal parameters, are not only model dependent, but are also influenced by the normalization convention of the external field and the order parameter. The constant z_0, however, does not depend on the choice of the normalization of the order parameter. The values of z_0 given in Tab. I can be directly compared between different models, since they were re-computed with the same normalization of the external field.

From Tab. I it is clear, that the z_0 values obtained in the mean-field models are roughly compatible with the lattice results. On the other hand, the effective models with bosonic fluctuations yield a much smaller z_0 We note that here we are comparing 2-flavor model calculations with (2+1)-flavor lattice QCD simulations. Such a comparison makes sense,

since the strange quark remains massive at the chiral transition and hence contributes only to the regular part of the free-energy. This leads to small reduction of the chiral transition temperature, but has only a minor effect on the critical properties. Thus, for the purposes of this exploratory study, the neglect of the strange quark is reasonable.

As discussed in Sec. II, the non-universal parameter z_0 influences the critical properties of relevant observables in the cross-over regime. In particular, as shown in Eqs. (6) and (7), the constant z_0 determines the width of the transition region and the peak position of the order parameter susceptibility χ. In Fig. 5 we show the chiral susceptibility, computed in the QM and in $O(N \to \infty)$ linear sigma models. The left-hand figure shows the universal part of the chiral susceptibility, whereas the right one depicts the temperature dependence of χ for $m_\pi/m_{\pi\,\mathrm{phys}} = 0.1$ and 1. Although the scaling functions of these models correspond to different universality classes, they are quantitatively rather similar. This, however is not the case for $\chi(T, m_\pi)$, since owing to the difference in the values of z_0, the crossover region in the LS model is considerably wider and the shift in the pseudocritical temperature with increasing pion mass is larger than in the QM model.

Moreover, due to comparable values of z_0 in the mean-field models and in LQCD, the melting of the chiral condensate in LQCD should be better described by the QM and PQM model than by the LS model. This is indeed seen on Fig. 6, where we compare the LQCD data with model results. Models, where bosonic fluctuations are included [43], such as the LS model in the $N \to \infty$ limit, yield a much smaller value of z_0 than LQCD calculations. Hence, the reduction of the chiral condensate at the cross-over transition is much smoother than in LQCD. Such a broadening of the transition region, when meson fluctuations are included, was observed also in the QM and PQM model mean-field and FRG calculations of Ref. [19].

By comparing different model results obtained in the present studies, together with previous FRG findings in the PQM model and LQCD results, we could confirm the rôle of the parameter z_0, which, using Eq. (7), determines the width of the transition in physical units. Again, this correlation is independent of the universality class, modulo minor variations in z_p. Thus, theories with a large z_0, roughly comparable with the lattice results, exhibit a relatively narrow transition region, as found in LQCD. This is the case for the QM and, in particular, for the PQM models in the mean field approximation. On the other hand, theories with a small value of z_0, like the QM model with mesonic fluctuations included as

well as the O(N) sigma model at large N, exhibit a much smoother transition. Consequently, a viable effective model for the critical chiral dynamics of QCD should exhibit a value for z_0 comparable to that obtained in LQCD.

Moreover, we find that the scaling window depends on the parameter h_0 and on the non-singular background, which in the Landau model, to leading order, is determined by the sixth-order coupling c_0. Thus, we conclude that in a given model the scaling window can be tuned to agree with lattice QCD by varying the non-universal parameter h_0 and the strength of the effective sixth order coupling.

We note that adjusting model parameters to lattice QCD results for certain non-universal quantities does not guarantee that other non-universal quantities are reproduced by the model. However, for modelling the effect of critical fluctuations at the chiral transition, the width of the transition region and the size of the scaling window are, besides the universality class, the most important criteria for discriminating between models. Our study indicates how effective models can be tuned so that these key quantities are reproduced.

V. CONCLUSION

We have discussed universal and non-universal aspects of the chiral phase transition and the corresponding magnetic equation of state in different effective models of QCD. The critical properties of the quark-meson (QM) and Polyakov loop extended quark-meson (PQM) models were explored in the mean-field approximation, where only fermionic fluctuations are accounted for, and compared to those of a purely bosonic theory, the $O(N)$ linear sigma (LS) model in the $N \to \infty$ limit. In the QM and LS models the magnetic equation of state was computed analytically within the high temperature expansion. The effects of a gluonic background on the non-universal scaling parameters were assessed within the PQM model.

We have analyzed the scaling violation at non-zero quark masses in the context of recent LQCD results, which indicate that, at a physical pion mass, QCD lies in the scaling regime of the underlying second order phase transition. We showed that to understand the chiral critical properties of QCD it is not enough to have a model in the same universality class, but the model under question should approach criticality in a similar manner. We quantified this with dimensionless, non-universal parameters t_0, h_0 and z_0 that connect the physical quark mass and temperature scales with the dimensionless scaling variables of the universality

class. We found that these non-universal quantities differ significantly from model to model and this influences the size of the order parameter scaling window.

In the QM and PQM models, the scaling violating contributions to the order parameter were found to remain small up to the physical pion mass. This is in qualitative agreement with the scaling behavior found in LQCD at the chiral cross-over transition.

On the other hand, in the $O(N)$ LS model, which we solved in the $N \to \infty$ limit, we found that the fluctuations of the meson fields yield a much stronger scaling violation than that obtained in the QM and PQM models, which in the mean-field approximation accounts only for fluctuations of fermions. In particular, we observed that the order parameter in the LS model follows the universal scaling law only for very small values of the pion mass. Consequently, at physical pion mass, the chiral condensate in the LS model exhibits substantial violation of the universal scaling law.

A very different scaling behavior of these models was linked to very different values of the non-universal scaling parameter z_0. In the QM and PQM model, z_0 was found to be roughly compatible with that obtained in LQCD, whereas in the LS model this parameter is almost an order of magnitude smaller. The value of z_0 is also reflected in the width of the cross-over transition and the shift in the peak position of the chiral susceptibility with increasing pion mass. This analysis indicates that models where bosonic fluctuations are accounted for, tend to have a small z_0, a broad peak in the chiral susceptibility and a narrow critical region.

From general considerations in Landau theory we have obtained a connection between distinctive features of the scaling violation and specific properties of the coefficients of the effective potential. This provides a framework for discussing general characteristics of the scaling violation in terms of the model parameters and in particular to understand how the scaling function approaches the universal scaling curve. We found that, depending on the temperature dependence of the coefficients of the effective potential, the magnetic equation of state may exhibit a non-trivial structure with common crossing points for different values of the symmetry breaking filed H. We have quantified these properties in the QM, PQM and LS models. In the QM model, there are two distinct points on the universal scaling line, where to leading order in H, all curves cross. In the LS model, there is only one such point, while in the PQM model the crossing of the universal scaling line is H-dependent already for pion masses well below its physical value. We presented a straightforward interpretation

of these features, based on general considerations derived within Landau theory.

Finally, we argued that the critical properties of QCD, namely the width of the chiral transition and the size of the scaling window, can be reproduced by tuning the non-universal parameters z_0, h_0 and the strength of the effective sixth-order coupling. Our calculations indicate that this is indeed the case in the mean-field models and in the large N linear sigma model considered.

ACKNOWLEDGMENTS

The work of B.F. and K.R. was partly supported by the Extreme Matter Institute EMMI. K. R. also acknowledges partial supports of the Polish Science Center (NCN) under Maestro grant DEC-2013/10/A/ST2/00106, and the U.S. Department of Energy under Grant No. DE-FG02-05ER41367. W.T. is grateful to GSI for the hospitality during the Summer Student Programme. G. A. acknowledges the support of the Hessian LOEWE initiative through the Helmholtz International Center for FAIR (HIC for FAIR).

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